Nonaxisymmetric Three-Dimensional Stagnation-Point Flow and Heat Transfer on a Flat Plate

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*The existing solutions of Navier–Stokes and energy equations in the literature regarding the three-dimensional problem of stagnation-point flow either on a flat plate or on a cylinder are only for the case of axisymmetric formulation. The only exception is the study of three-dimensional stagnation-point flow on a flat plate by Howarth (1951, "The Boundary Layer in Three-Dimensional Flow—Part II: The Flow Near Stagnation Point," Philos. Mag., **42**, pp. 1433–1440), which is based on boundary layer theory approximation and zero pressure assumption in direction of normal to the surface. In our study the nonaxisymmetric three-dimensional steady viscous stagnation-point flow and heat transfer in the vicinity of a flat plate are investigated based on potential flow theory, which is the most general solution. An external fluid, along z-direction, with strain rate a impinges on this flat plate and produces a two-dimensional flow with different components of velocity on the plate. This situation may happen if the flow pattern on the plate is bounded from both sides in one of the directions, for example x-axis, because of any physical limitation. A similarity solution of the Navier–Stokes equations and energy equation is presented in this problem. A reduction in these equations is obtained by the use of appropriate similarity transformations. Velocity profiles and surface stress-tensors and temperature profiles along with pressure profile are presented for different values of velocity ratios, and Prandtl number.*
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1 Introduction

There are many three-dimensional axisymmetric solutions for Navier–Stokes and energy equations regarding the problem of stagnation-point flow and heat transfer in the vicinity of a flat plate or a cylinder. Fundamental three-dimensional axisymmetric studies in which the nonlinearity is removed by superposition of fundamental exact solutions that lead to nonlinear coupled ordinary differential equations by separation of coordinate variables are uniform shear flow over a flat plate in which the flow is induced by a plate oscillating in its own plane beneath a quiescent fluid by Stokes [1], two-dimensional stagnation-point flow by Hiemenz [2], and the flow induced by a disk rotating in its own plane by Von Karman [3]. The works in which nonlinearity is readily superposed but still axisymmetric are flow over a flat plate with uniform normal suction by Griffith and Meredith [4], three-dimensional stagnation-point flow by Homann [5]. The same kind of work but on a cylinder is the axisymmetric stagnation flow on a circular cylinder by Wang [6]. Further three-dimensional axisymmetric exact solutions to the Navier–Stokes equations are obtained by superposition of the uniform shear flow on a body oscillating or translating in its own plane, with or without suction. The examples are superposition of two-dimensional and three-dimensional stagnation-point flows by Howarth [7], Reshotko [8], and Libby [9]. The ones using superposition of stagnation flow are by Gersten [10] and Papenfuss [11]. Also, the three-dimensional axisymmetric solution for a fluid oscillating about a nonzero mean flow parallel to a flat plate with uniform suction was given by Stuart [12]. More three-dimensional axisymmetric solutions with superposition of stagnation-point flow on a flat plate oscillating in its own plate, and also consideration of the case where the plate is stationary and the stagnation stream is made to oscillate was done by Glauert [13], uniform shear flow aligned with out flowing two-dimensional stagnation-point flow has been presented by Stuart [14], and uniform flow along a flat plate with time-dependent suction and included periodic oscillations of the external stream was studied by Kelly [15]. In addition, Gorla [16] has studied unsteady laminar axisymmetric stagnation flow over a circular cylinder, nonsimilar axisymmetric stagnation flow on a moving cylinder [17], transient response behavior of an axisymmetric stagnation flow on a circular cylinder due to time-dependent free stream velocity [18], and unsteady viscous flow in the vicinity of an axisymmetric stagnation-point on a cylinder [19]. Additionally, suppression of turbulence in wall-bounded flows by high-frequency spanwise oscillations has been studied by Jung et al. [20], axisymmetric stagnation flow toward a moving plate by Wang [21], and axisymmetric stagnation-point flow impinging on a transversely oscillating plate with suction by Weidman and Mahalingam [22]. Studies under the same category as above but with a rotating body include superposition of uniform suction at the boundary of a rotating disk by Stuart [23], shear flow over a rotating plate by Wang [24], and radial stagnation flow on a rotating cylinder with uniform transpiration by Cunning et al. [25].

Three-dimensional axisymmetric studies considering exact solutions of the Navier–Stokes equations along with energy equation include the problems of heat transfer in an axisymmetric stagnation flow on a cylinder by Gorla [26], axisymmetric stagnation-point flow and heat transfer of a viscous fluid on a moving cylinder with time-dependent axial velocity and uniform transpiration by Saleh and Rahimi [27], and axisymmetric stagnation-point flow and heat transfer of a viscous fluid on a rotating cylinder with time-dependent angular velocity and uniform transpiration by Rahimi and Saleh [28], and similarity solution of nonaxisymmetric heat transfer in stagnation-point flow on a cylinder with simultaneous axial and rotational movements by Rahimi and Saleh [29].

In this study the nonaxisymmetric three-dimensional steady viscous stagnation-point flow and heat transfer in the vicinity of a flat plate are investigated. A similarity solution of the Navier–Stokes equations and energy equation is derived in this problem. A reduction in these equations is obtained by use of these appropriate similarity transformations [30]. The obtained coupled ordinary differential equations are solved using numerical techniques. Velocity profiles and surface stress-tensors and temperature profiles along with pressure profile are presented for different values of impinging fluid strain rate, different forms of jet arrangements, and Prandtl number.

2 Problem Formulation

Flow is considered in Cartesian coordinates (x,y,z) with corresponding velocity components (u,v,w), Fig. 1. This figure represents a three-dimensional surface which is the boundary of a po-

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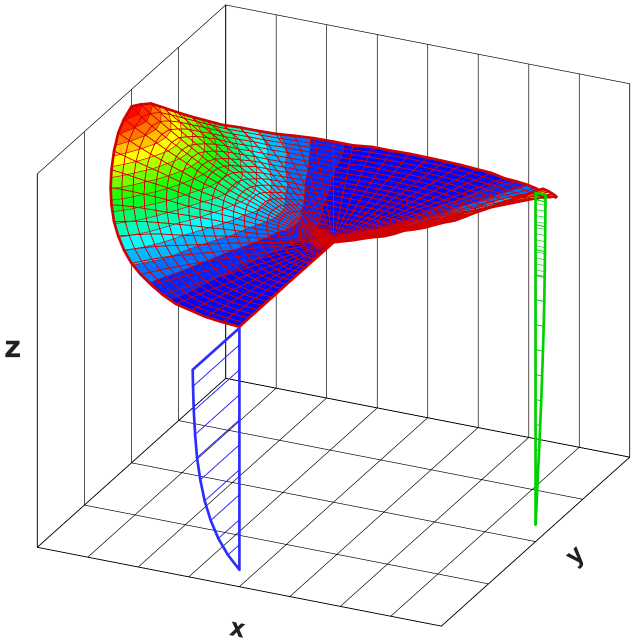

Fig. 1 Three-dimensional stream surface and velocity profiles

tential region, and with the region of rapid changes of velocity components for all $0<\lambda\leq 1$, in which λ is the coefficient that indicates the difference between the velocity components in x- and y-directions. The velocity components in these directions are the same if $\lambda=1$, the axisymmetric case. We consider the laminar steady incompressible flow and heat transfer of a viscous fluid in the neighborhood of the stagnation-point on a flat plate located in the plane $z=0$. An external fluid along the z-direction with strain rate a impinges on this flat plate and produces a two-dimensional flow with different components of velocity on the plate. This situation may be encountered if the flow pattern on the plate is bounded in one of the directions, for example on the x-axis, because of physical limitations. The three-dimensional, steady Navier–Stokes and energy equations in Cartesian coordinates are used in which p, ρ, ν, and α are the fluid pressure, density, kinematic viscosity, and thermal diffusivity.

3 Self-Similar Solution

3.1 Fluid Flow Solution. An inviscid solution of governing equations valid far above the plane is given by

$$U=a\lambda x, \quad 0<\lambda\leq 1 \tag{1}$$

$$V=ay \tag{2}$$

$$W=-a(\lambda+1)z \tag{3}$$

$$p=p_0-\tfrac{1}{2}\rho a^2[\lambda^2x^2+y^2+(\lambda+1)^2z^2] \tag{4}$$

p_0 is stagnation pressure.

A reduction in the Navier–Stokes equations is sought by the following coordinate separation in which the solution of the viscous problem inside the boundary layer is obtained by composing the inviscid and viscous parts of the velocity components as the following:

$$u=a\lambda x f'(\eta), \quad 0<\lambda\leq 1 \tag{5}$$

$$v=ay[f'(\eta)+g'(\eta)] \tag{6}$$

$$w=-\sqrt{a\nu}[(\lambda+1)f(\eta)+g(\eta)] \tag{7}$$

$$\eta=\sqrt{a/\nu}z \tag{8}$$

in which the terms involving $f(\eta)$ and $g(\eta)$ in Eqs. (5)–(7) comprise the Cartesian similarity form for steady stagnation-point flow, and prime denotes differentiation with respect to η. Note that the boundary layer is defined here as the edge of the points where its velocity is 99% of its corresponding potential velocity. Transformations (5)–(7) satisfy continuity automatically, and their insertion into momentum equations yields a coupled system of ordinary differential equations in terms of $f(\eta)$ and $g(\eta)$, and an expression for the pressure:

$$f'''+[(\lambda+1)f+g]f''+\lambda[1-(f')^2]=0 \tag{9}$$

$$g'''+[(\lambda+1)f+g]g''-[g'+2f']g'-(1-\lambda)[(f')^2-1]=0 \tag{10}$$

$$p=p_0-\tfrac{1}{2}\rho a^2(\lambda^2x^2+y^2)-\rho a\nu\left\{\tfrac{1}{2}[(\lambda+1)f+g]^2+\lambda f'+(f'+g')-(\lambda+1)-\gamma\eta(\lambda+1)\right\} \tag{11}$$

$\gamma=\lim_{\eta\to\infty} g(\eta)=$constant. This constant is obtained after solving Eqs. (9) and (10). The boundary conditions are

$$\eta=0{:}f=0, \quad f'=0, \quad g=0, \quad g'=0 \tag{12}$$

$$\eta\to\infty{:}f'=1, \quad g'=0 \tag{13}$$

Note that when $\lambda=1$, the case of axisymmetric three-dimensional results are obtained [5]. When $\lambda=0$, the results are the same as a two-dimensional problem.

3.2 Heat Transfer Solution. To transform the energy equation into a nondimensional form for the case of defined wall temperature, we introduce

$$\theta=\frac{T(\eta)-T_\infty}{T_w-T_\infty} \tag{14}$$

Making use of Eqs. (5)–(8), the energy equation may be written as

$$\theta''+\Pr[g+(1+\lambda)f]\theta'=0 \tag{15}$$

with the boundary conditions as

$$\eta=0{:}\theta=1 \tag{16}$$

$$\eta\to\infty{:}\theta=0 \tag{17}$$

where $\Pr=\nu/\alpha$ is the Prandtl number, and prime indicates differentiation with respect to η.

Note that for Pr=1, the thickness of the fluid boundary layer and heat boundary layer become the same, and therefore this concept is proved by reaching Eq. (15) from Eq. (9) through substitution of $\theta=f'$.

Eqs. (9), (10), and (15) are solved numerically using a shooting method trial and error and based on the Runge–Kutta algorithm, and the results are presented for selected values of λ and Pr in Secs. 4 and 5. Since Eqs. (9) and (10) are coupled, we guess a value for $g(\eta)$ function first and solve Eq. (9) for $f(\eta)$. Then Eq. (10) is integrated and a new value of $g(\eta)$ is obtained, which is used to solve Eq. (9) again. This procedure is repeated until the differences between the results are less than 0.00001.

4 Shear-Stress

The shear-stress at the wall surface is calculated from:

$$\tau=\mu\left(\frac{\partial u}{\partial z}\boldsymbol{e}_x+\frac{\partial v}{\partial z}\boldsymbol{e}_y\right)_{z=0} \tag{18}$$

where μ is the fluid viscosity. Using the transformation equations (5)–(8), the shear-stress at the flat plate surface becomes

$$\tau=\rho\nu^{1/2}a^{3/2}(\lambda^2x^2f''^2+y^2(f''+g'')^2)^{1/2} \tag{19}$$

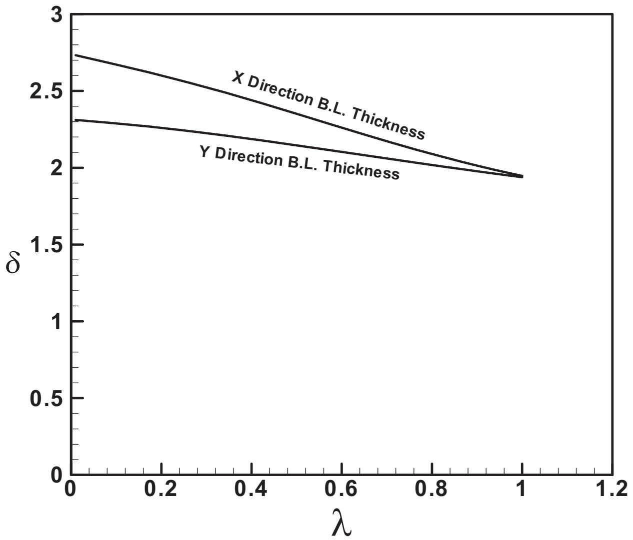

Fig. 2 Boundary layer thickness versus variation of velocity ratio

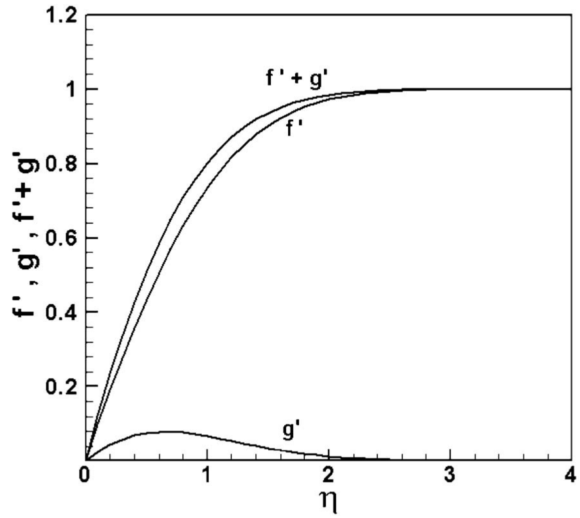

Fig. 4 Typical *u* and *v* velocity components for λ=0.5

The wall shear stress is presented for different values of λ in the Presentation of Results.

5 Presentation of Results

In this section, the solution of the self-similar equations (9), (10), and (15) along with the surface shear-stresses for different values of velocity ratios and Prandtl numbers are presented.

The boundary layer thickness in the two directions on the flat plate versus the velocity ratio is presented in Fig. 2. This thickness is larger in the x-direction compared with the y-direction, because of the difference of the velocity components in these directions. The difference of the boundary layer thickness in directions x and y decreases as λ increases until the value of unity where these two layers meet each other. From Fig. 2, the following relations can be obtained for the boundary layer thickness versus the ratio of the velocities in potential flow:

$$\delta_x = -0.75\lambda + 2.75$$
$$\delta_y = -0.35\lambda + 2.35 \quad (20)$$

Comparing these results with the ones in Howarth [7], the difference between the boundary layer thickness in x-direction is 2% and in y-direction is 18%, which is because of the inablity of his approximation solution method.

Figures 3 and 4 present the profiles of f', g', and $f'+g'$ for different values of velocity ratio λ. The smaller the λ, the bigger g' and therefore the difference between the velocity components is larger. For $\lambda \to 1$, then $g' \to 0$ and the two velocity components become the same.

Figures 5 and 6 depict the f and g profiles, and therefore w-component of velocity versus velocity ratio. The bigger the λ, the larger the absolute value of the w-component of the velocity, as expected. This component of velocity, which is the penetration of momentum into the boundary layer in the z-direction, changes abruptly as λ increases. This is because the boundary layer increases faster as λ gets larger, and therefore there is need for more penetration of the momentum and hence this component of velocity gets bigger.

The temperature profiles for different values of velocity ratio and selected values of Prandtl numbers are presented in Figs. 7 and 8. The increase in velocity ratio and increase in Prandtl number both cause the decrease in the temperature profile. It is also noted that for $\lambda \to 1$ and Pr=1, the temperature boundary layer is obtained the same as the velocity boundary layer.

Figure 9 presents the change in shear-stress on the flat plate surface in terms of velocity ratio λ. The following relations can be deduced from this plot

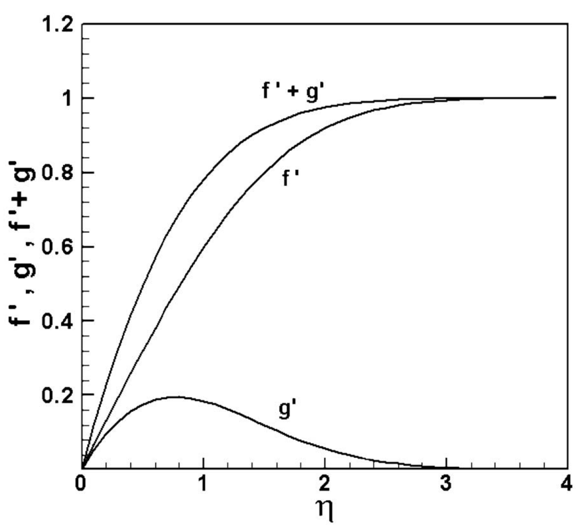

Fig. 3 Typical *u* and *v* velocity components for λ=0.1

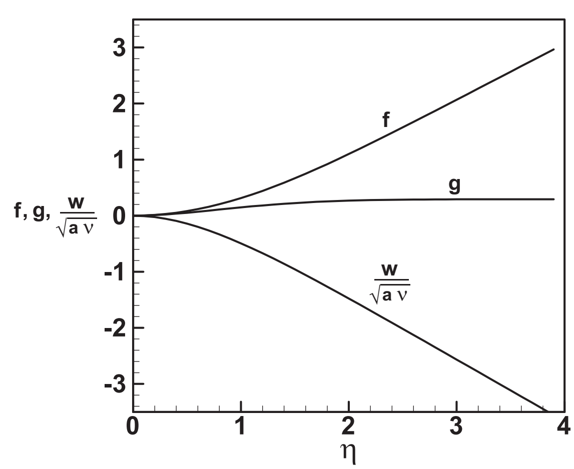

Fig. 5 Typical *w*-component of velocity for λ=0.1

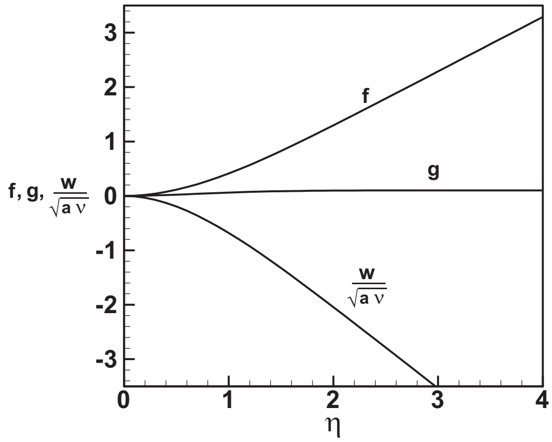

Fig. 6 Typical *w*-component of velocity for $\lambda = 0.50$

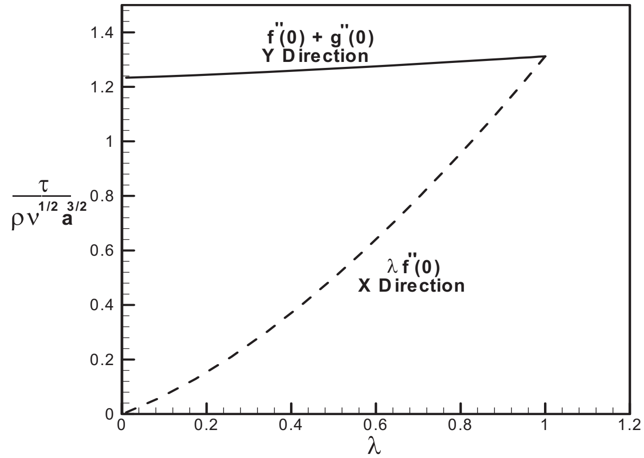

Fig. 9 Surface shear-stress components on the flat plate

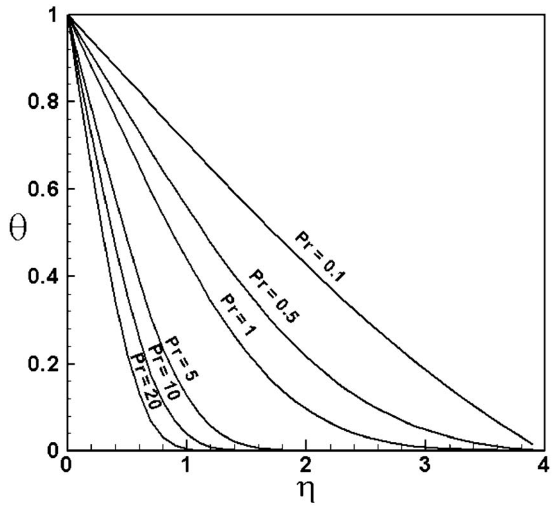

Fig. 7 Temperature profile for the case of $\lambda = 0.1$ and different Pr values

$$\tau_x = \lambda^{1.55} + 0.3\lambda$$
$$\tau_y = 0.1\lambda^{1.1} - 0.03\lambda + 1.235 \tag{21}$$

As $\lambda \to 0$, the stress tensor in *x*-direction tends to zero but note that $\lambda = 0$ does not represent a physical situation. In Howarth's work [7], for the case of the velocity aspect ratio (in our study, λ) equal to zero, the shear-stress in *x*-direction is calculated to be an amount equal to 0.570, though it must be close to zero physically, as it has been shown by τ_x relation obtained above. This shows again the inability of his solution method and the error approximation it brings about.

Pressure profiles inside the boundary layer are shown in Fig. 10 for selected values of λ. From these profiles it can be seen that with an increase in velocity ratio in *x*- and *y*-directions and tending toward the symmetric situation, the variation in pressure inside the boundary layer increases because λ affects velocity directly, and the pressure changes with velocity in power form.

6 Conclusions

The most general solution of the Navier–Stokes equations and energy equation for nonaxisymmetric three-dimensional stagnation-point flow and heat transfer on a flat plate has been presented in this paper based on potential flow theory. This task

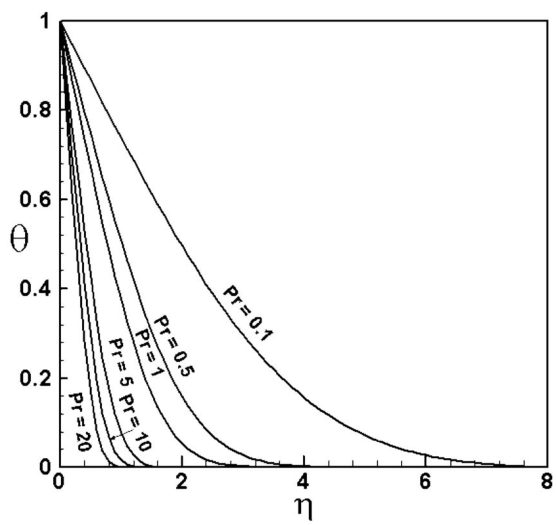

Fig. 8 Temperature profile for the case of $\lambda = 0.50$ and different Pr values

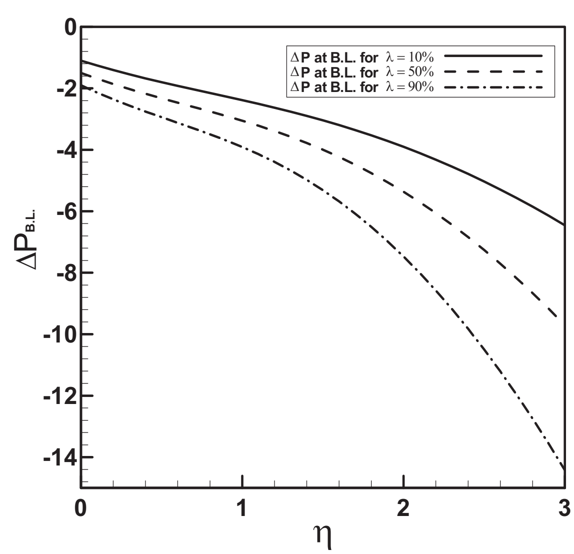

Fig. 10 Pressure profiles for selected values of λ

has been accomplished by choosing appropriate similarity transformations and reduction in these governing equations to a system of coupled ordinary differential equations and subsequent numerical integration. Velocity components, temperature profiles, pressure change, and surface stress tensor have been presented for selected values of velocity ratios and Prandtl numbers. This solution represents many physical situations, including the stagnation-point problem in which the flow pattern on the plate is bounded from both sides in one direction because of any physical limitation.

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