Nonclassicality Criteria from Phase-Space Representations and Information-Theoretical Constraints Are Maximally Inequivalent

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We consider two celebrated criteria for defining the nonclassicality of bipartite bosonic quantum systems, the first stemming from information theoretic concepts and the second from physical constraints on the quantum phase space. Consequently, two sets of allegedly classical states are singled out: (i) the set $\mathcal{C}$ composed of the so-called classical-classical (CC) states—separable states that are locally distinguishable and do not possess quantum discord; (ii) the set $\mathcal{P}$ of states endowed with a positive P representation (P-classical states)—mixtures of Glauber coherent states that, e.g., fail to show negativity of their Wigner function. By showing that $\mathcal{C}$ and $\mathcal{P}$ are almost disjoint, we prove that the two defining criteria are maximally inequivalent. Thus, the notions of classicality that they put forward are radically different. In particular, generic CC states show quantumness in their P representation, and vice versa, almost all P-classical states have positive quantum discord and, hence, are not CC. This inequivalence is further elucidated considering different applications of P-classical and CC states. Our results suggest that there are other quantum correlations in nature than those revealed by entanglement and quantum discord.

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The question of whether a quantum system exhibits a behavior without a classical analogue has been of interest since the early days of quantum mechanics. Considering bosonic systems, a major framework for attacking this question was established more than half a century ago, stemming from the notions of quantum phase space and quasiprobability distributions [1,2]. There, physical constraints expressing classical behavior impose criteria of nonclassicality that have been experimentally tested in a variety of quantum systems [3–5]. On the other hand, in the last two decades nonclassical correlations have been the subject of a renewed interest, mainly due to the general belief that they are a fundamental resource for quantum information processing. Within this perspective, a different approach to nonclassicality has emerged, which is based on the information-theoretic aspects of quantum correlations. In particular, rigorous criteria to define nonclassicality of correlations have been put forward [6–9], giving rise to well-established concepts like entanglement or quantum discord.

Here, we compare these two approaches, investigating in particular whether physical constraints emerging from the former can bring new insight in the assessment of quantum correlations beyond the purely information-theoretic aspects of the latter. We have found that this is indeed the case: the notion of nonclassical correlations springing from physical considerations on the quantum phase space is inequivalent to that emerging from information-theoretic arguments. In a sense, that will be specified in the following: these two notions of nonclassicality are maximally inequivalent. This, in particular, suggests that there are other quantum correlations in nature than those revealed by entanglement and quantum discord.

Nonclassicality in the phase space.—The uncertainty relations make the notion of phase space in quantum mechanics problematic. Following the seminal investigations of Wigner [1], an abundance of quantum mechanical phase-space quasidistributions were introduced, ranging from the Husimi function to the Glauber-Sudarshan P function [10]. Besides the fundamental aspect, investigations on quasidistributions boosted the development of efficient theoretical tools in various fields of modern physics, e.g., quantum optics and quantum chemistry [10,11]. These functions cannot, however, be interpreted as probability distributions over a classical phase space because for some quantum states they may be negative or singular. Consistently, it is commonly accepted that such features underpin a good notion of nonclassicality. Supporting this interpretation, fundamental links between quasiprobability functions and the notions of nonlocality [12] and contextuality [13] have been recognized.

In this framework, possibly the most accepted definition of nonclassicality has been introduced by Glauber in terms of the P function [2]. For concreteness, let us consider the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ of a bipartite system made of two modes a and b of a bosonic field ($[a, a^\dagger] = [b, b^\dagger] = 1$). Considering $\alpha, \beta \in \mathbb{C}$, let us denote with $|\alpha\rangle$ and $|\beta\rangle$ the Glauber coherent states of the systems, that is, the eigenstates of the annihilation operators ($a|\alpha\rangle = \alpha|\alpha\rangle$ and $b|\beta\rangle = \beta|\beta\rangle$). Any state ϱ of the system can be expressed in terms of a diagonal mixture of coherent states: $\varrho = \int d^2\alpha d^2\beta P(\alpha, \beta)|\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta|$, where $P(\alpha, \beta)$ is the P function of ϱ. When the P function is a well-behaved probability density function, then ϱ can be expressed as a statistical

mixture of coherent states [14]. Thus, we have the following classicality criterion:

Criterion P (P-classical states).—A state of a bipartite bosonic system is P classical if it can be written as

$$\varrho_p = \iint_{\mathbb{C}} d^2\alpha d^2\beta P(\alpha, \beta)|\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta|, \qquad (1)$$

where $P(\alpha, \beta)$ is a positive, nonsingular, and normalized function. This criterion represents the most conservative notion of nonclassicality in the quasiprobability setting, since when the P function is well behaved so are all other quasiprobabilities. The success of using quasiprobabilities to characterize the quantumness of a state or its space-time correlations [15] is also, loosely speaking, related to their ability to capture the difficulty in generating and manipulating quantum states. In particular, in quantum optics the easiest states to generate in a lab are thermal states, characterized by a well-behaved P function. On the other hand, coherent, squeezed, photon-subtracted, photon-added, and number states, characterized by increasingly ill-behaved P functions, happen to be much more difficult to generate. In this sense, the P function captures the physical constraints of producing increasingly more-quantum states. Notice, however, that different coherent states are not orthogonal; hence, even when $P(\alpha, \beta)$ behaves like a true probability density, it does not describe probabilities of mutually exclusive events.

Nonclassicality and information theory.—The first rigorous attempt to address the classification of quantum correlations from an information theoretical viewpoint was pioneered by Werner [6], who put on firm basis the elusive concept of quantum entanglement [16,17]. A state of a bipartite system is called entangled if it cannot be written as follows:

$$\varrho_{AB} = \sum p_k \sigma_{Ak} \otimes \sigma_{Bk}, \qquad (2)$$

where σ_{Ak} and σ_{Bk} are generic density matrices describing the states of the two subsystems. The definition above has an immediate operational interpretation: unentangled (separable) states can be prepared by local operations and classical communication between the two parties. One might have thought that such classical information exchange cannot bring any quantum character to the correlations in the state. In this sense, separability has often been regarded as synonymous with classicality in this information theoretical framework.

On the other hand, as it has been extensively discussed in the last decade [7,8,18–20], this may not be the case. An entropic measure of correlations—quantum discord—has been introduced as the mismatch between the quantum analogues of two classically equivalent expressions of the mutual information. For pure entangled states, quantum discord coincides with the entropy of entanglement. However, quantum discord can be different from zero also for (mixed) separable states. In other words, classical communication can give rise to quantum correlations. This can be understood by considering that the states σ_{Ak} and σ_{Bk} in Eq. (2) may be physically indistinguishable, and thus, not all the information about them can be locally retrieved. This phenomenon has no classical counterpart, thus accounting for the quantumness of the correlations in separable states with positive discord. Few explicit formulas have been derived for the quantum discord of some states [18,21,22], and more general entropic measures of nonclassical correlations have been also discussed [23]. Discord finds an operational meaning in terms of quantum state merging [24], and its role has been studied in quantum information processing with mixed states, where there are computational and communication tasks which are seemingly impossible to achieve classically and yet can be attained using little or no entanglement [25–27]. More recently, monogamy properties of discord have been investigated [28], and it has been shown that quantum correlations in separable states may be activated into distillable entanglement [29]. Discord is also related to the minimum entanglement generated between system and apparatus in a partial measurement process [30].

Remarkably, even states with zero discord can show nonclassical correlations. In order to see this effect in detail, let us recall that discord is asymmetric in the two modes and that a bipartite state with zero A discord can be written in the form $\varrho_{AB} = \sum_k p_k|\theta_k\rangle\langle\theta_k| \otimes \sigma_{Bk}$, where the $|\theta_k\rangle$'s form an orthonormal basis, and the σ_{Bk}'s are a set of generic nonorthogonal states. These states—dubbed quantum-classical states—cannot be cloned locally (locally broadcasted), despite having zero discord [9]. This security against local broadcasting is not featured by any correlated state of a classical system, thus revealing the quantumness of this type of zero discord states. The set of states that can be locally broadcasted has been shown to be equivalent to a set of states called classical-classical (CC) [9]. Any member of such set can be written as

$$\varrho_{AB} = \sum_{ks} p_{ks}|\theta_k\rangle\langle\theta_k| \otimes |\eta_s\rangle\langle\eta_s|, \qquad (3)$$

where $|\theta_k\rangle$ and $|\eta_s\rangle$ are bases for the Hilbert spaces of the two subsystems. These states are now commonly regarded as purely classical correlated states [29]. The reason for this is based on information theoretic arguments. All the information encoded in a CC state can be locally retrieved and stored in a classical register. Indeed, states appearing in Eq. (3) are perfectly distinguishable by local quantum measurements. In this sense, CC states simply accommodate the joint probability p_{ks} in a quantum formalism, thus putting forward the most conservative notion of nonclassicality in an information-theoretical setting. However, we will show in the following that also this class of states can exhibit quantum correlations that cannot be featured by systems that admit a classical description in the quantum phase space.

Definition (3) was introduced in the context of finite-dimensional systems, and it needs to be slightly generalized

in order to fully take into account some subtleties of bosonic systems for which there exist bases that are unitarily inequivalent [31]. Considering $x, y \in \mathbb{R}$, let us denote with $|x\rangle$ and $|y\rangle$ two generic bases of A and B, respectively. We introduce the following classical criterion:

Criterion C (classical-classical states).—A state of a bipartite bosonic system is CC if it can be written as

$$\varrho_c = \iint_{\mathbb{R}} dxdy F(x, y)|x\rangle\langle x| \otimes |y\rangle\langle y|, \tag{4}$$

and $F(x, y)$ is a positive, nonsingular, and normalized function. Notice that, in general, the joint probability distribution $F(x, y)$ spans over a continuous set. Clearly, one recovers Eq. (3) if $F(x, y)$ is nonzero only over a discrete set.

Number correlated states.—In the following we show that the foregoing criteria of nonclassicality are maximally inequivalent. However, before proceeding with a formal proof, let us discuss a specific example. Consider the two-mode P-classical states introduced in Eq. (1) and define the observable $O_D = a^\dagger a - b^\dagger b$, which detects the difference between the number of quanta of the two modes. Since for coherent states $\langle z|a^\dagger a|z\rangle = |z|^2$ and $\langle z|(a^\dagger a)^2|z\rangle = |z|^4 + |z|^2$, for any P-classical state (different from the vacuum) we have

$$\Delta O_D^2 = |\alpha_0|^2 + |\beta_0|^2 + \mathrm{Tr}C \geq |\alpha_0|^2 + |\beta_0|^2 > 0, \tag{5}$$

where α_0, β_0, and C are the mean values and the covariance matrix of $P(\alpha, \beta)$, respectively. The observable O_D detects correlations between the number of quanta in the two modes. The above inequality captures the intuition behind the idea that the behavior of a classical state should be that of a mixture of coherent states: each mode has a fluctuating number of quanta, and the difference should fluctuate accordingly. In other words, for a classical two-mode system the amount of intensity correlations between two modes is bounded.

Let us now consider the two modes prepared in the state $\varrho_{nc} = \sum_n p_n |n\rangle\langle n| \otimes |n\rangle\langle n|$, where $a^\dagger a|n\rangle = n|n\rangle$. This is the state generated by, say, a pair of machine guns, each producing a random but equal number of bullets n according to the distribution p_n. The state ϱ_{nc} is separable and, according to the terminology introduced above, CC. Yet it shows perfect correlations in the number of quanta. Actually, the product states $|n\rangle\langle n| \otimes |n\rangle\langle n|$ are the projectors over the degenerate eigenspace of O_D with eigenvalue zero. In other words, for any choice of the distribution $\{p_n\}$ we have $\Delta O_D^2 = 0$ for ϱ_{nc}, which in turn violates the inequality Eq. (5). Thus, the family of number correlated states ϱ_{nc} gives an example of states that obey criterion C while violating criterion P. We will now proceed to prove that the two criteria are not only inequivalent, but that their inequivalence is maximal. Specifically, we will show that generic states obeying criterion P violate criterion C and vice versa.

Generic P-classical states are not CC.—Consider the following property of any CC state (necessary condition for CC states): any two states of system A conditioned to a measurement on B commute. This can be seen by considering the definition in Eq. (4) and applying any positive-operator valued measure on B. It immediately follows that any state of A conditioned on any outcome at B will remain diagonal in the original basis. Thus, all possible conditioned states of A will mutually commute.

Consider now a generic P-classical state and the following two convenient conditioned states of A: $\varrho_A = \mathrm{Tr}_B[\varrho_p] = \int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|$, and $\varrho_0 = \mathrm{Tr}_B[\varrho_p|0\rangle\langle 0|] = \int d^2\alpha P_0(\alpha)|\alpha\rangle\langle\alpha|$, where $P(\alpha) = \int d^2\beta P(\alpha, \beta)$, $P_0(\alpha) = \int d^2\beta P(\alpha, \beta)e^{-|\beta|^2}$, and $|0\rangle\langle 0|$ is the vacuum. Calculating the commutator between the above states and evaluating it on the vacuum, one has

$$\langle 0|[\varrho_A, \varrho_0]|0\rangle = \int d^2\alpha d^2\alpha' P(\alpha)P_0(\alpha')e^{-|\alpha|^2}e^{-|\alpha'|^2}(e^{\alpha\bar{\alpha}'} - \text{c.c.}). \tag{6}$$

Imposing that the commutator above is identical to zero yields the following nontrivial constraint on the P function $P(\alpha, \beta)$: $\int d^2\alpha d^2\alpha' d^2\beta d^2\beta' P(\alpha, \beta)P(\alpha', \beta') \times e^{-|\alpha|^2}e^{-|\alpha'|^2}e^{-|\beta'|^2}(e^{\alpha\bar{\alpha}'} - \text{c.c.}) = 0$. A generic (well-behaved) P function does not satisfy the above constraint. This, in turn, implies that almost all P-classical states are not CC. Equivalently, generic P-classical states violate criterion C. Notice that the proof works as well for A-discord states, thus showing that almost all P-classical states have positive discord.

Generic CC states are not P classical.—We first need to show that the set $\mathcal{P}$ of single mode $\mathcal{P}$-classical states is nowhere dense in the bosonic space. By definition, $\mathcal{P}$ is nowhere dense if its closure $\bar{\mathcal{P}}$ has no interior points. Denoting the frontier of $\mathcal{P}$ (namely, the set of its accumulation points) by $\partial\mathcal{P}$, one has that $\bar{\mathcal{P}} = \mathcal{P} \cup \partial\mathcal{P}$. The P function of any operator $\delta \in \partial\mathcal{P}$ must be positive since it is the limit of positive functions. In addition, it cannot be singular everywhere in the phase space, given that it is the limit of normalizable functions. As a consequence, any operator $\bar{\varrho} \in \bar{\mathcal{P}}$ is such that its P function is positive and not everywhere singular. Let us now show that no $\bar{\varrho}$ can be an interior point of $\bar{\mathcal{P}}$. First, given any $\bar{\varrho}$, denote by $\bar{\alpha}$ a point in the phase space where the P function of $\bar{\varrho}$ is nonsingular [i.e., $P_{\bar{\varrho}}(\bar{\alpha}) < \infty$]. Then, define a convenient perturbation of $\bar{\varrho}$: $\varrho = (1 - \epsilon)\bar{\varrho} + \epsilon D(\bar{\alpha})\varrho_1 D^\dagger(\bar{\alpha})$, where $0 < \epsilon < 1$, $D(\bar{\alpha}) = \exp[\bar{\alpha}a^\dagger - \bar{\alpha}^* a]$ is the displacement operator, and $\varrho_1 = |1\rangle\langle 1|$ is a single excitation state. One has that the P function of ϱ is given by $P_\varrho(\alpha) = (1 - \epsilon)P_{\bar{\varrho}}(\alpha) + \epsilon P_{\varrho_1}(\alpha - \bar{\alpha})$. Since the P function of the single excitation state is negative and singular at the origin, one has that $P_\varrho(\alpha)$ is nonpositive (and singular in $\bar{\alpha}$). For what is shown above, this means that (for any ϵ) $\varrho \notin \bar{\mathcal{P}}$; hence, $\bar{\varrho}$ is not an interior point of $\bar{\mathcal{P}}$. Since this holds true for any $\bar{\varrho}$, one has that $\bar{\mathcal{P}}$ has no interior points. As a consequence, P is nowhere dense in the space of single mode bosonic systems.

Consider now the set $\mathcal{P}_2$ of two-mode P classical states. Based on the above considerations, one can show that P classical states $\varrho_p \in \mathcal{P}_2$ are nowhere dense in the set $\mathcal{C}$ of CC states. First, recall that the partial trace of any P-classical state is a P-classical state (necessary condition for P-classical states). This implies that the partial trace of any $\bar{\varrho}_p \in \bar{\mathcal{P}}_2$ must have a non-negative P function. Then, using the same arguments as above (technical details are omitted), one can build a CC state ϱ' that, despite being an infinitesimal perturbation of $\bar{\varrho}_p$, does not belong to $\bar{\mathcal{P}}_2$. This implies that $\bar{\mathcal{P}}_2$ has no interior point in $\mathcal{C}$; hence, P-classical states are nowhere dense in the set of CC states. Equivalently, generic CC states violate criterion P.

Discussion.—The foregoing arguments show that the set of states simultaneously obeying criteria P and C is negligible, both in a metrical and topological sense [32]. In other words, the two criteria considered here put forward two radically different notions of classicality of correlations. Criterion C looks at the correlations between the information of A and B, as encoded in their states and regardless of the quantumness of the states themselves, whereas criterion P takes into account physical constraints on those as well. Referring to the example of number correlated states ϱ_{nc}, creating Fock states with the same number of quanta does correspond to establishing quantum correlations between the modes, irrespective of the fact that the information needed to perform this action may be of purely classical (local) origin. It has been often argued that a suitable quantity to reveal quantum correlations in bipartite systems, beyond the presence of entanglement, should be related to the joint information carried by the state. For example, quantum discord focuses on this and can be used to assess states for application in quantum communication. On the other hand, from a fundamental physical point of view, discord (and information-theoretical quantities, more generally) appears unable to account for the very physical constraints involved in the establishment of correlations. Ultimately, this means that allegedly classical correlations established between systems prepared in states with no classical analogue are quantum in nature.

Operationally, the fact that P-classical states violate criterion C allows us to use them as an experimentally cheap resource in communication protocols that require security against local broadcasting. On the other hand, the nonclassicality of CC states like ϱ_{nc} may find an operational characterization in terms of conditional measurements. Consider a generic bipartite state and perform a measurement described by the positive-operator valued measure $\{\Pi_x\}$ on one mode, say mode 1. If the state is P classical then the P function of the conditional state $\varrho_{px} = \mathrm{Tr}_1[\varrho_p \Pi_x \otimes \mathbb{I}]/p_x$ may be written as

$$P(\beta) = \frac{1}{p_x} \int d^2\alpha P(\alpha, \beta) \langle \alpha | \Pi_x | \alpha \rangle.$$

This is a well-behaved probability density function, and thus, the state ϱ_{px} is classical. In other words, only states violating criterion P may lead to the conditional generation of genuine quantum states with no classical analogue [33,34].

Conclusions.—In the last two decades, the fruitful exchange of notions between information science and quantum physics led to the emergence of radically new concepts and applications. The slogan "information is physical" [35] has become increasingly popular, emphasizing the role of physical constraints in quantum information processing [36]. Our results reinforce this position; however, they also present an unusual case in which the information-theoretical and physical perspectives appear fundamentally conflicting. Specifically, by addressing the notion of nonclassicality as it emerges from physical considerations, we have shown that there exist other genuinely quantum correlations than those revealed by information-theoretic arguments. This indicates that the slogan should be complemented by a second part illustrating that information-theoretic considerations cannot substitute physical constraints, thus suggesting that "information is physical, and physics is not merely information."

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