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# NONCOMMUTATIVE BURKHOLDER/ROSENTHAL INEQUALITIES

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We investigate martingale inequalities in noncommutative $L^p$-spaces associated with a von Neumann algebra equipped with a faithful normal state. We prove the noncommutative analogue of the classical Burkholder inequality on the conditioned (or little) square function and extend the noncommutative Burkholder–Gundy inequalities from *Comm. Math. Phys.* **189** (1997) 667–698 to this nontracial setting. We include several related results.

**0. Introduction.** Inspired by mathematical physics, noncommutative (or quantum) probability is, today, an independent field of mathematical research. We refer to the recent books [27, 30], the successive conference proceedings [2] for the interplay between mathematical physics, noncommutative probability and classical (i.e., commutative) probability, to the books [17, 18, 10] about the almost sure convergence of noncommutative martingales, and to [28] for the connection with harmonic analysis. A further example of the fruitful interaction of probability and operator algebras is the fast developing theory of free probability introduced by Voiculescu in the beginning of the 1980s; see [44] and [45]. Finally, we should point out that noncommutative probability is intimately related to the recent theory of operator spaces, developed mainly during the last decade (cf. [11, 32]).

In this paper our main attention is on classical martingale inequalities and their noncommutative counterparts. In the classical probability theory, Burkholder and his coauthors developed powerful tools of martingale transforms, maximal functions and stopping times which are well established in the modern theory of stochastic processes (cf. [4] and the references given therein). We should emphasize that it is often highly nontrivial and requires additional functional analytic or combinatorial insight to transfer classical martingale inequalities to the noncommutative setting. Indeed, most of the stopping time arguments are no longer available. It is well known that martingale inequalities are closely related to problems in harmonic analysis (cf. [5, 37]). Indeed, the noncommutative version of Stein's inequality is a key building block in the approach towards the noncommutative Burkholder–Gundy inequalities in [34]. Carlen and Krée [7] obtained, independently and almost at the same time, results related to those in [34] on the Itô–Clifford integral. Very shortly after [34], Biane and Speicher [3] developed the stochastic analysis on Wigner space (= free probability space).

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Pisier [33] extended some results in [34] to a more general setting. Very recently, the noncommutative analogue of the classical Doob inequality has been established in [19].

We continue this line of research by establishing the noncommutative analogue of the Rosenthal/Burkholder inequality. Interested in new examples for $\mathcal{L}_p$ spaces in the theory of Banach spaces, Rosenthal [36] established an inequality for the $p$-norm of independent mean-zero random variables. Aware of this inequality, Burkholder (cf. [4] and [6]) generalized this inequality to the context of martingales as follows. Let $2 \le p < \infty$ and $(\mathcal{F}_k)$ be a filtration on a probability space $(\Omega, \mu)$. Given $x \in L^p$, the conditional expectations $(E_k)$ and the martingale differences are given by

$$E_k(x) = E(x|\mathcal{F}_k) \quad \text{and} \quad d_k = d_k(x) = E_k(x) - E_{k-1}(x).$$

Then the $L^p$-norm of $x$ satisfies

$$\|x\|_p \sim_{c_p} \left(\sum_k \|d_k\|_p^p\right)^{1/p} + \left\|\left(\sum_k E_{k-1}(d_k^2)\right)^{1/2}\right\|_p .$$

The second term on the right-hand side above is called the conditioned square function. Rosenthal's inequality for independent mean-zero random variables is a special case where $d_k = f_k$ and $E_{k-1}(d_k^2) = \|f_k\|_2^2$ is just a scalar. In Section 5, we prove the noncommutative analogue of Burkholder's inequality by replacing conditional expectations onto the subalgebras generated by filtrations by the conditional expectations onto an increasing sequence of von Neumann subalgebras of a given von Neumann algebra (using the appropriate noncommutative analogue of the conditioned square function). In Section 6, we extend these results to the case $1 < p \le 2$. In this range, the appropriate formulation is motivated by the $K$-functional in interpolation theory. This might be new even in the commutative case. In a subsequent paper [21], we will show that the noncommutative version of Rosenthal's original inequality has far reaching applications in particular to the $L^p$-norms of random matrices. Namely, for a matrix $(f_{ij})_{i,j=1}^n$ of independent mean-zero random variables and $2 \le p < \infty$,

$$\big(\mathbb{E}\|(f_{ij})_{i,j=1}^n\|_p^p\big)^{1/p} \sim_{c_p} \left(\sum_{i,j=1}^n \|f_{ij}\|_p^p\right)^{1/p} + \left(\sum_{i=1}^n \left(\sum_{j=1}^n \|f_{ij}\|_2^2\right)^{p/2}\right)^{1/p} + \left(\sum_{j=1}^n \left(\sum_{i=1}^n \|f_{ij}\|_2^2\right)^{p/2}\right)^{1/p}.$$

Ever since the discovery of type III$_\lambda$-factors realized as infinite tensor products of $2 \times 2$-matrices, martingale theory is important in operator algebras and in particular for noncommutative $L^p$-spaces. As in many other applications it is therefore natural to assume that a $W^*$-noncommutative probability space $(\mathcal{M}, \varphi)$

is given by a von Neumann algebra $\mathcal{M}$ and a normal faithful state $\varphi$ (or even a weight, but we will concentrate on states in this paper). Given an increasing sequence of von Neumann subalgebras $(\mathcal{M}_n)$ of $\mathcal{M}$, the existence of normal conditional expectations from $\mathcal{M}$ onto $\mathcal{M}_n$ is guaranteed if the $\mathcal{M}_n$'s are invariant under the modular group of the given state $\varphi$. We will consider noncommutative martingale inequalities in this setting, and in particular, prove the corresponding Burkholder inequality. The pattern of our proof of this latter inequality is similar to that set up in [34]. However, the results in [34] were only obtained for the tracial case. In Section 3, we indicate the necessary modifications needed to extend the Burkholder–Gundy inequality to the nontracial case, thereby extending these results. Let us mention that this approach still only provides exponential estimates for the constants. Due to very recent results by Randrianantoanina [35] the order of constants is now better understood in the tracial case, see also the forthcoming publication [22]. Using a very deep crossed product argument of Haagerup [15] (unfortunately unpublished), one can deduce the nontracial case from the tracial one. This alternative approach seems to be the right one in terms of constants. In this paper (accomplished before Randrianantoanina's results) we pursue a different strategy and show that the abstract Haagerup $L^p$-space provides the right framework for understanding algebraic properties of (invariant) conditional expectations and their extensions to $L^p$-spaces; see Section 2 for details.

In Section 4, we give a description of the dual of the Hardy space $H^p$ and the connection to BMO spaces, revealing the relations between the various martingale inequalities. This part is closely related to results in [19] and in fact the underlying duality concept for conditioned square functions is central in both papers, in particular for the dual form of Burkholder's inequality for $1 < p \leq 2$; see Section 6.

Section 7 contains a norm inequality on conditional expectations on $L^p$ for $p < 1$. This inequality is closely linked to the above mentioned dual form of the noncommutative Doob inequalities in [19].

The last section contains a brief discussion of the nonfaithful case, motivated by natural examples of invariant finite dimensional subalgebras in $B(\ell^2)$. In these cases the conditional expectations can no longer be assumed to be faithful and the dual Burkholder inequality turns out to be wrong in general. However, for $p \geq 2$, the results can be deduced from the faithful case.

In the subsequent paper [21], we will present the noncommutative Rosenthal inequalities (which are consequences of the noncommutative Burkholder inequality), and various applications, especially those to the linear structure of symmetric subspaces of noncommutative $L^p$-spaces.

**1. Preliminaries.** We use standard notation in operator algebras. We refer to [23, 31, 38–40] for modular theory, to [14, 42] for the Haagerup noncommutative $L^p$-spaces. Let us recall some basic facts about these spaces and fix the relevant notation used throughout this paper. Let $\mathcal{M}$ be a $\sigma$-finite von Neumann algebra and $\varphi$ a distinguished normal faithful state on $\mathcal{M}$. Let $\sigma_t = \sigma_t^{\varphi}$, $t \in \mathbb{R}$, denote the one parameter modular automorphism group of $\mathbb{R}$ on $\mathcal{M}$ associated with $\varphi$. We consider the crossed product $\mathcal{R} = \mathcal{M} \rtimes_\sigma \mathbb{R}$. We recall briefly the definition of $\mathcal{R}$. If $\mathcal{M}$ acts on a Hilbert space $H$, $\mathcal{R}$ is a von Neumann algebra acting on $L^2(\mathbb{R}, H)$, generated by the operators $\pi(x)$, $x \in \mathcal{M}$, and the operators $\lambda(s)$, $s \in \mathbb{R}$, defined by the following conditions: for every $\xi \in L^2(\mathbb{R}, H)$ and $t \in \mathbb{R}$,

$$\pi(x)(\xi)(t) = \sigma_{-t}(x)\xi(t) \quad \text{and} \quad \lambda(s)(\xi)(t) = \xi(t-s).$$

Note that $\pi$ is a normal faithful representation of $\mathcal{M}$ on $L^2(\mathbb{R}, H)$. Thus we may identify $\mathcal{M}$ with $\pi(\mathcal{M})$. Then the one parameter modular automorphism group $\{\sigma_t\}_{t\in\mathbb{R}}$ is given by

$$\sigma_t(x) = \lambda(t)x\lambda(t)^*, \qquad x \in \mathcal{M},\ t \in \mathbb{R}.$$

There is a dual action $\{\widehat{\sigma}_t\}_{t\in\mathbb{R}}$ of $\mathbb{R}$ on $\mathcal{R}$. This is a one parameter automorphism group of $\mathbb{R}$ on $\mathcal{R}$, implemented by the unitary representation $\{W(t)\}_{t\in\mathbb{R}}$ of $\mathbb{R}$ on $L^2(\mathbb{R}, H)$:

$$\widehat{\sigma}_t(x) = W(t)xW(t)^*, \qquad t \in \mathbb{R},\ x \in \mathcal{R},$$

where

$$W(t)(\xi)(s) = e^{-its}\xi(s), \qquad \xi \in L^2(\mathbb{R}, H), \qquad t, s \in \mathbb{R}.$$

Note that the dual action $\widehat{\sigma}_t$ is also uniquely determined by the following conditions

$$\widehat{\sigma}_t(x) = x \quad \text{and} \quad \widehat{\sigma}_t(\lambda(s)) = e^{-ist}\lambda(s), \qquad x \in \mathcal{M},\ s, t \in \mathbb{R}.$$

Thus $\mathcal{M}$ is invariant under $\{\widehat{\sigma}_t\}_{t\in\mathbb{R}}$. In fact, $\mathcal{M}$ is exactly the space of the fixed points of $\{\widehat{\sigma}_t\}_{t\in\mathbb{R}}$, namely,

$$\mathcal{M} = \{x \in \mathcal{R} : \widehat{\sigma}_t(x) = x,\ \forall t \in \mathbb{R}\}.$$

Recall that the crossed product $\mathcal{R}$ is semifinite (cf.[31]), and moreover there is a canonical normal semifinite faithful (abbreviated as nsf) trace satisfying

$$\tau \circ \widehat{\sigma}_t = e^{-t}\tau, \qquad t \in \mathbb{R}.$$

Any normal positive functional $\omega$ on $\mathcal{M}$ induces a dual nsf weight $\widetilde{\omega}$ on $\mathcal{R}$ which admits a Radon–Nikodym derivative with respect to $\tau$. In particular, the dual weight $\widetilde{\varphi}$ of our distinguished state has a Radon–Nikodym derivative $D$ with respect to $\tau$. In this paper, $D$ will be *exclusively* reserved to denote this derivative. Then

$$\widetilde{\varphi}(x) = \tau(Dx), \qquad x \in \mathcal{R}_+.$$

Recall that $D$ is an invertible positive selfadjoint operator on $L^2(\mathbb{R}, H)$, affiliated with $\mathcal{R}$, and that the regular representation $\lambda(t)$ above is given by

$$\lambda(t) = D^{it}, \qquad t \in \mathbb{R}.$$

Now we are ready to define the Haagerup noncommutative $L^p$-spaces. Let $L^0(\mathcal{R},\tau)$ denote the topological $*$-algebra of all operators on $L^2(\mathbb{R},H)$ measurable with respect to $(\mathcal{R},\tau)$. Then the Haagerup $L^p$-space $0<p\le\infty$ is defined as

$$L^p(\mathcal{M},\varphi)=\{x\in L^0(\mathcal{R},\tau):\widehat{\sigma}_t(x)=e^{-t/p}x,\ \forall\, t\in\mathbb{R}\}.$$

It is clear that $L^p(\mathcal{M},\varphi)$ is a vector subspace of $L^0(\mathcal{R},\tau)$, invariant under the $*$-operation. The algebraic structure of $L^p(\mathcal{M},\varphi)$ is inherited from that of $L^0(\mathcal{R},\tau)$. Let $x\in L^p(\mathcal{M},\varphi)$, and let $x=u|x|$ be its polar decomposition, where $|x|=(x^*x)^{1/2}$ is the modulus of $x$. Then $u\in\mathcal{M}$ and $|x|\in L^p(\mathcal{M},\varphi)$. Recall that

$$L^\infty(\mathcal{M},\varphi)=\mathcal{M}\quad\text{and}\quad L^1(\mathcal{M},\varphi)=\mathcal{M}_*.$$

The latter equality is understood as follows. As mentioned previously, for any $\omega\in\mathcal{M}_*^+$, the dual weight $\widetilde{\omega}$ has a Radon–Nikodym derivative with respect to $\tau$, denoted by $h_\omega$:

$$\widetilde{\omega}(x)=\tau(h_\omega x),\qquad x\in\mathcal{R}_+.$$

Then

$$h_\omega\in L^0(\mathcal{R},\tau)\quad\text{and}\quad\widehat{\sigma}_t(h_\omega)=e^{-t}h_\omega,\qquad t\in\mathbb{R}.$$

Thus $h_\omega\in L^1(\mathcal{M},\varphi)_+$. This correspondence between $\mathcal{M}_*^+$ and $L^1(\mathcal{M},\varphi)_+$ extends to a bijection between $\mathcal{M}_*$ and $L^1(\mathcal{M},\varphi)$. Then for any $\omega\in\mathcal{M}_*$, if $\omega=u|\omega|$ is its polar decomposition, the corresponding $h_\omega\in L^1(\mathcal{M},\varphi)$ admits the polar decomposition

$$h_\omega=u|h_\omega|=uh_{|\omega|}.$$

Thus we may define a norm on $L^1(\mathcal{M},\varphi)$ by

$$\|h_\omega\|_1=|\omega|(1)=\|\omega\|_*,\qquad \omega\in\mathcal{M}_*.$$

In this way, $L^1(\mathcal{M},\varphi)=\mathcal{M}_*$ isometrically. Now let $0<p<\infty$. Since $x\in L^p(\mathcal{M},\varphi)$ iff $|x|^p\in L^1(\mathcal{M},\varphi)$, we define

$$\|x\|_p=\||x|^p\|_1^{1/p},\qquad x\in L^p(\mathcal{M},\varphi).$$

Then $\|\cdot\|_p$ is a norm (resp. a $p$-norm) on $L^p(\mathcal{M},\varphi)$ for $1\le p<\infty$ (resp. $0<p<1$). Equipped with $\|\cdot\|_p$, $L^p(\mathcal{M},\varphi)$ becomes a Banach space or a quasi-Banach space, according to whether $1\le p<\infty$ or $0<p<1$. Clearly,

$$\|x\|_p=\|x^*\|_p=\||x|\|_p,\qquad x\in L^p(\mathcal{M},\varphi).$$

It is well known that $L^p(\mathcal{M},\varphi)$ is independent of $\varphi$ up to isometry (see [42]). Thus, following Haagerup, we will use the notation $L^p(\mathcal{M})$ for the abstract Haagerup $L^p$-space $L^p(\mathcal{M},\varphi)$. As usual, for $1\le p<\infty$ the dual space of $L^p(\mathcal{M})$ is $L^{p'}(\mathcal{M})$, $1/p+1/p'=1$. To describe this duality, we use the distinguished linear functional on $L^1(\mathcal{M})$, called *trace* and denoted by tr, which is given by

$$\operatorname{tr}(x) = \omega_x(1), \qquad x \in L^1(\mathcal{M}),$$

where $\omega_x \in \mathcal{M}_*$ is the unique normal functional associated with $x$ by the above identification between $\mathcal{M}_*$ and $L^1(\mathcal{M})$. Then tr is a continuous functional on $L^1(\mathcal{M})$ satisfying

$$|\operatorname{tr}(x)| \leq \operatorname{tr}(|x|) = \|x\|_1, \qquad x \in L^1(\mathcal{M}).$$

The usual Hölder inequality also holds for these noncommutative $L^p$-spaces. Let $0 < p, q, r \leq \infty$ such that $1/r = 1/p + 1/q$. Then

$$x \in L^p(\mathcal{M}) \quad \text{and} \quad y \in L^q(\mathcal{M}) \quad \Rightarrow \quad xy \in L^r(\mathcal{M}) \quad \text{and} \quad \|xy\|_r \leq \|x\|_p \|y\|_q.$$

In particular, for any $1 \leq p \leq \infty$ we have

$$|\operatorname{tr}(xy)| \leq \|xy\|_1 \leq \|x\|_p \|y\|_{p'}, \qquad x \in L^p(\mathcal{M}),\ y \in L^{p'}(\mathcal{M}).$$

Thus, $(x, y) \mapsto \operatorname{tr}(xy)$ defines a duality between $L^p(\mathcal{M})$ and $L^{p'}(\mathcal{M})$, with respect to which

$$\big(L^p(\mathcal{M})\big)^* = L^{p'}(\mathcal{M}) \text{ isometrically}, \qquad 1 \leq p < \infty.$$

The functional tr on $L^1(\mathcal{M})$ plays the rôle of a trace. Indeed, it satisfies the following *tracial* property

$$\operatorname{tr}(xy) = \operatorname{tr}(yx), \qquad x \in L^p(\mathcal{M}),\ y \in L^{p'}(\mathcal{M}).$$

Moreover, our distinguished state $\varphi$ can be recovered from tr (recalling that $D$ is the Radon–Nikodym derivative of $\widetilde{\varphi}$ with respect to $\tau$), namely,

$$\varphi(x) = \operatorname{tr}(Dx), \qquad x \in \mathcal{M}. \tag{1.1}$$

All properties described above will be repeatedly used throughout this paper without any reference.

In this paper, all notation introduced previously will be kept fixed, unless explicitly indicated otherwise. For our development, we will need some more preliminaries on the Haagerup $L^p$-spaces. Let $\mathcal{M}_a$ be the family of analytic vectors in $\mathcal{M}$. Recall that $x \in \mathcal{M}_a$ iff the function $t \mapsto \sigma_t(x)$ extends to an analytic function from $\mathbb{C}$ to $\mathcal{M}$. Then $\mathcal{M}_a$ is a $w^*$-dense $*$-subalgebra of $\mathcal{M}$ (cf. [31]).

LEMMA 1.1. *Let* $0 < p < \infty$, $0 \leq \theta \leq 1$. *Then*:

(i) $D^{(1-\theta)/p} \mathcal{M}_a D^{\theta/p} = \mathcal{M}_a D^{1/p}$;

(ii) $\mathcal{M}_a D^{1/p}$ *is dense in* $L^p(\mathcal{M})$.

PROOF. (i) Let $x \in \mathcal{M}_a$. Then

$$x D^{1/p} = D^{(1-\theta)/p}\big[D^{-(1-\theta)/p} x D^{(1-\theta)/p}\big] D^{\theta/p}$$
$$= D^{(1-\theta)/p}[\sigma_{i(1-\theta)/p}(x)] D^{\theta/p} \in D^{(1-\theta)/p} \mathcal{M}_a D^{\theta/p}$$

whence $\mathcal{M}_a D^{1/p} \subset D^{(1-\theta)/p} \mathcal{M}_a D^{\theta/p}$. The reverse inclusion can be proved in a similar way.

(ii) Since $D \in L^1(\mathcal{M})$, by the Hölder inequality, $\mathcal{M}_a D^{1/p} \subset L^p(\mathcal{M})$. To show the density of $\mathcal{M}_a D^{1/p}$ in $L^p(\mathcal{M})$, we first consider the case $1 \le p < \infty$. Let $y \in (L^p(\mathcal{M}))^* = L^{p'}(\mathcal{M})$ such that

$$\mathrm{tr}(x D^{1/p} y) = 0, \qquad x \in \mathcal{M}_a.$$

Note that $D^{1/p} y \in L^1(\mathcal{M})$. Thus, by the $w^*$-density of $\mathcal{M}_a$ in $\mathcal{M}$, we deduce $D^{1/p} y = 0$. Since $D^{1/p}$ is injective, we obtain $y = 0$, and so $\mathcal{M}_a D^{1/p}$ is dense in $L^p(\mathcal{M})$.

Now, suppose $1/2 \le p < 1$. Let $x \in L^p(\mathcal{M})$, and write its polar decomposition: $x = u|x|$. Then we can write $x = yz$, where $y = u|x|^{1/2}$ and $z = |x|^{1/2}$. Since $y, z \in L^{2p}(\mathcal{M})$ and $2p \ge 1$, by (i) and the preceding part already proved, there are $y_n, z_n \in \mathcal{M}_a$ such that

$$\lim_{n\to\infty} \|D^{1/2p} y_n - y\|_{2p} = 0, \qquad \lim_{n\to\infty} \|z_n D^{1/2p} - z\|_{2p} = 0.$$

Then $D^{1/2p} y_n z_n D^{1/2p} \in D^{1/2p} \mathcal{M}_a D^{1/2p}$, and by the Hölder inequality,

$$\lim_{n\to\infty} \|D^{1/2p} y_n z_n D^{1/2p} - x\|_p = 0.$$

Therefore, by (i), we deduce the desired density in the case $1/2 \le p < 1$. Iterating this procedure, we obtain the density of $\mathcal{M}_a D^{1/p}$ in $L^p(\mathcal{M})$ for all $0 < p < \infty$. □

We will need some spaces formed of sequences in $L^p(\mathcal{M})$, as introduced in [34] (see also [26]) in the case where the state $\varphi$ is tracial. Let $a = (a_n)_{n\ge 0} \subset L^p(\mathcal{M})$ be a finite sequence (i.e., only finitely many terms of $a$ are not zero). It is clear that $(\sum_{n\ge 0} |a_n|^2)^{1/2} \in L^p(\mathcal{M})$. Put

$$\|a\|_{L^p(\mathcal{M};\ell^2_c)} = \left\| \left( \sum_{n\ge 0} |a_n|^2 \right)^{1/2} \right\|_p .$$

As in the case where $\varphi$ is tracial, we easily see that $\|\cdot\|_{L^p(\mathcal{M};\ell^2_c)}$ defines a norm (if $p \ge 1$) on the family of all finite sequences in $L^p(\mathcal{M})$. To justify this, let $B(\ell^2)$ denote the space of all bounded operators on $\ell^2$ and Tr be the usual trace on $B(\ell^2)$. Consider the von Neumann tensor product $\mathcal{M} \otimes B(\ell^2)$, equipped with the tensor product weight $\varphi \otimes \mathrm{Tr}$. Note that $\varphi \otimes \mathrm{Tr}$ is no longer a state, but an nsf weight. All the previous discussion about the Haagerup $L^p$-spaces associated with a state is still valid in the case of nsf weights. However, if one wishes, for what follows one may keep oneself in the case of states, simply by considering $B(\ell_n^2)$ instead of $B(\ell^2)$ with an arbitrary positive integer $n$. Let $\psi = \varphi \otimes \mathrm{Tr}$. Then the corresponding one parameter automorphism group is

$$\sigma_t^{\psi} = \sigma_t \otimes \mathrm{id}_{B(\ell^2)}, \qquad t \in \mathbb{R}.$$

Thus, it follows that

$$[\mathcal{M} \otimes B(\ell^2)] \rtimes_{\sigma^\psi} \mathbb{R} = (\mathcal{M} \rtimes_\sigma \mathbb{R}) \otimes B(\ell^2) = \mathcal{R} \otimes B(\ell^2).$$

The canonical nsf trace $\nu$ on $[\mathcal{M} \otimes B(\ell^2)] \rtimes_{\sigma^\psi} \mathbb{R}$ is the tensor product $\tau \otimes \mathrm{Tr}$ (recalling that $\tau$ is the canonical trace on $\mathcal{R}$). Let $L^p(\mathcal{M} \otimes B(\ell^2))$ denote the Haagerup $L^p$-space associated with $\varphi \otimes \mathrm{Tr}$. Observe that the distinguished tracial functional on $L^1(\mathcal{M} \otimes B(\ell^2))$ is equal to $\mathrm{tr} \otimes \mathrm{Tr}$. Consequently, $(x, y) \mapsto \mathrm{tr} \otimes \mathrm{Tr}(xy)$ defines a duality between $L^p(\mathcal{M} \otimes B(\ell^2))$ and $L^{p'}(\mathcal{M} \otimes B(\ell^2))$ for $1 \le p < \infty$.

Elements in $L^p(\mathcal{M} \otimes B(\ell^2))$ can be considered as matrices with entries in $L^p(\mathcal{M})$, and $L^p(\mathcal{M})$ can be identified as an isometric subspace of $L^p(\mathcal{M} \otimes B(\ell^2))$ via the following map

$$x \mapsto \begin{pmatrix} x & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}, \qquad x \in L^p(\mathcal{M}).$$

In the same way, any finite sequence $a = (a_n)_{n \ge 0} \subset L^p(\mathcal{M})$ can be regarded as a column matrix in $L^p(\mathcal{M} \otimes B(\ell^2))$:

$$a \mapsto T(a) = \begin{pmatrix} a_0 & 0 & \cdots \\ a_1 & 0 & \cdots \\ \vdots & \vdots & \end{pmatrix}.$$

Then

$$\|a\|_{L^p(\mathcal{M};\ell_c^2)} = \||Ta|\|_{L^p(\mathcal{M} \otimes B(\ell^2))} = \|Ta\|_{L^p(\mathcal{M} \otimes B(\ell^2))}.$$

Therefore, $\|\cdot\|_{L^p(\mathcal{M};\ell_c^2)}$ defines a norm for $p \ge 1$ and a quasi-norm for $0 < p < 1$ on the family of all finite sequences in $L^p(\mathcal{M})$. The corresponding completion, for $0 < p < \infty$, is denoted by $L^p(\mathcal{M};\ell_c^2)$. Then, $L^p(\mathcal{M};\ell_c^2)$ is identified, via the above map $T$, with a closed subspace of $L^p(\mathcal{M} \otimes B(\ell^2))$, called the column subspace of $L^p(\mathcal{M} \otimes B(\ell^2))$. For $p = \infty$, we denote by $L^\infty(\mathcal{M};\ell_c^2)$ the Banach space of (possible infinite) sequences in $L^\infty(\mathcal{M})$ such that $\sum_n a_n^* a_n$ converges in the $w^*$-topology. Thus $L^\infty(\mathcal{M};\ell_c^2)$ is isometric to the column subspace of $L^\infty(\mathcal{M} \otimes B(\ell^2))$ consisting of column matrices.

Similarly, given a finite sequence $a = (a_n) \subset L^p(\mathcal{M})$, we set

$$\|a\|_{L^p(\mathcal{M};\ell_r^2)} = \left\| \left( \sum_{n \ge 0} |a_n^*|^2 \right)^{1/2} \right\|_p .$$

This is again a norm or a quasi-norm according to whether $p \geq 1$ or $p < 1$. The corresponding completion (relative to the $w^*$-topology in the case $p = \infty$) is denoted by $L^p(\mathcal{M}; \ell_r^2)$. Then, $L^p(\mathcal{M}; \ell_r^2)$ is naturally identified with the row subspace of $L^p(\mathcal{M} \otimes B(\ell^2))$.

The above tensor product argument shows the following Hölder-type inequality, which will be frequently used in this paper. Let $0 < p, q, s \leq \infty$ such that $1/s = 1/p + 1/q$. Then for any finite sequences $a = (a_n)_{n\geq 0} \in L^p(\mathcal{M}; \ell_c^2)$ and $b = (b_n)_{n\geq 0} \in L^p(\mathcal{M}; \ell_c^2)$,

$$\left\| \sum_{n\geq 0} a_n^* b_n \right\|_s \leq \|a\|_{L^p(\mathcal{M};\ell_c^2)} \|b\|_{L^q(\mathcal{M};\ell_c^2)}. \tag{1.2}$$

The same inequality holds with $L^p(\mathcal{M}; \ell_r^2)$ instead of $L^p(\mathcal{M}; \ell_c^2)$. In particular, if $1 \leq p < \infty$ and $q = p'$, the conjugate index of $p$, we get

$$\left| \sum_{n\geq 0} \mathrm{tr}(a_n^* b_n) \right| \leq \|a\|_{L^p(\mathcal{M};\ell_c^2)} \|b\|_{L^{p'}(\mathcal{M};\ell_c^2)}, \tag{1.3}$$
$$a \in L^p(\mathcal{M}; \ell_c^2),\ b \in L^{p'}(\mathcal{M}; \ell_c^2).$$

This last inequality yields a natural anti-linear duality between $L^p(\mathcal{M}; \ell_c^2)$ and $L^{p'}(\mathcal{M}; \ell_c^2)$. As a consequence of Lemma 1.2 below, the dual space of $L^p(\mathcal{M}; \ell_c^2)$ is indeed $L^{p'}(\mathcal{M}; \ell_c^2)$ with respect to this anti-linear duality.

The following lemma might be known to specialists. We include a proof for the sake of completeness. Note that it is obvious in the tracial case (i.e., when $\varphi$ is tracial).

LEMMA 1.2. *Let $1 \leq p \leq \infty$. For any finite matrix $x = (x_{ij})_{i,j\geq 0} \in L^p(\mathcal{M} \otimes B(\ell^2))$ we define*

$$P(x) = \begin{pmatrix} x_{00} & 0 & \cdots \\ x_{10} & 0 & \cdots \\ \vdots & \vdots & \end{pmatrix},$$

*that is, $P(x)$ is the matrix whose first column is that of $x$ and all others are $0$. Then $P$ extends to a contractive projection from $L^p(\mathcal{M} \otimes B(\ell^2))$ onto $L^p(\mathcal{M}; \ell_c^2)$. Consequently, $L^p(\mathcal{M}; \ell_c^2)$ is one-complemented in $L^p(\mathcal{M} \otimes B(\ell^2))$. Similarly, $L^p(\mathcal{M}; \ell_r^2)$ is one-complemented in $L^p(\mathcal{M} \otimes B(\ell^2))$.*

PROOF. This is easy (and well known for $p = \infty$ and $p = 1$). More precisely, we consider the projection $e = (1 \otimes e_{00})$ and note that

$$L^p(\mathcal{M} \otimes B(\ell^2))e = L^p(\mathcal{M}; \ell_c^2).$$

Clearly, the map $T(x) = xe$ is a projection onto $L^p(\mathcal{M}; \ell_c^2)$. □

We record the following immediate consequence of (1.3) and Lemma 1.2.

COROLLARY 1.3. *Let $1 \le p < \infty$, and let $p'$ be the conjugate index of $p$. Then*

$$\big(L^p(\mathcal{M};\ell_c^2)\big)^* = L^{p'}(\mathcal{M};\ell_c^2) \quad \textit{and} \quad \big(L^p(\mathcal{M};\ell_r^2)\big)^* = L^{p'}(\mathcal{M};\ell_r^2)$$

*isometrically. The anti-linear duality is given by $(a,b) \mapsto \sum_{n\ge 0} \mathrm{tr}(b_n^* a_n)$.*

**2. Conditional expectations.** Let $\mathcal{M}, \varphi, \sigma_t$ be fixed as in the previous section. Let $\mathcal{N} \subset \mathcal{M}$ be a von Neumann subalgebra. By von Neumann subalgebras we mean unital $w^*$-closed $*$-subalgebras. Assume $\mathcal{N}$ is invariant under $\{\sigma_t\}_{t\in\mathbb{R}}$, that is,

$$\sigma_t(\mathcal{N}) \subset \mathcal{N}, \qquad t \in \mathbb{R}. \tag{2.1}$$

It is well known (cf. [40]) that (2.1) is equivalent to the existence of a (unique) normal conditional expectation $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ such that

$$\varphi \circ \mathcal{E} = \varphi. \tag{2.2}$$

This conditional expectation $\mathcal{E}$ commutes with $\{\sigma_t\}_{t\in\mathbb{R}}$ (cf. [9]):

$$\mathcal{E} \circ \sigma_t = \sigma_t \circ \mathcal{E}, \qquad t \in \mathbb{R}. \tag{2.3}$$

Now let $\psi = \varphi|_{\mathcal{N}}$ be the restriction of $\varphi$ to $\mathcal{N}$. (2.1) implies that the modular automorphism group associated with $\psi$ is the restriction of $\sigma_t$ to $\mathcal{N}$, namely,

$$\sigma_t^{\psi} = \sigma_t\big|_{\mathcal{N}} \qquad \forall t \in \mathbb{R}.$$

It follows that the crossed product $\mathcal{S} = \mathcal{N} \rtimes_{\sigma^\psi} \mathbb{R}$ is a von Neumann subalgebra of $\mathcal{R} = \mathcal{M} \rtimes_\sigma \mathbb{R}$. Let $\nu$ be the canonical nsf trace on $\mathcal{S}$. Then $\nu$ is equal to the restriction of $\tau$ to $\mathcal{S}$ (recalling that $\tau$ is the canonical trace on $\mathcal{R}$). Observe that the conditional expectation $\mathcal{E}$ extends to a normal faithful conditional expectation $\tilde{\mathcal{E}}$ from $\mathcal{R}$ onto $\mathcal{S}$, satisfying $\tau \circ \tilde{\mathcal{E}} = \tau$, that is, $\nu \circ \tilde{\mathcal{E}} = \tau$. Let $\tilde{\varphi}$ and $\tilde{\psi}$ be the dual weights of $\varphi$ and $\psi$ respectively. Then $\tilde{\psi} \circ \tilde{\mathcal{E}} = \tilde{\varphi}$, and by [9],

$$(D\tilde{\varphi} : D\tau)_t = \lambda(t) = (D\tilde{\psi} : D\nu)_t, \qquad t \in \mathbb{R}.$$

Therefore, the Radon–Nikodym derivative of $\tilde{\psi}$ with respect to $\nu$ is equal to $D$, the Radon–Nikodym derivative of $\tilde{\varphi}$ with respect to $\tau$.

The discussion above shows, in particular, that $L^0(\mathcal{S}, \nu)$ is naturally identified with a subspace of $L^0(\mathcal{R}, \tau)$. Then, since $\sigma_t^\psi = \sigma_t|_{\mathcal{N}}$ ($t \in \mathbb{R}$), the space $L^p(\mathcal{N}) = L^p(\mathcal{N}, \psi)$ can be naturally isometrically identified with a subspace of $L^p(\mathcal{M})$, $0 < p < \infty$. In the sequel we will not distinguish between $\varphi, \sigma_t, \tau$ and their respective restrictions.

It is well known that in the tracial case, the conditional expectation $\mathcal{E}$ extends to a contractive projection from $L^p(\mathcal{M})$ onto $L^p(\mathcal{N})$ for any $1 \le p < \infty$, which still is positive and has the modular property $\mathcal{E}(axb) = a\mathcal{E}(x)b$ for all $a, b \in \mathcal{N}$, $x \in L^p(\mathcal{M})$. In view of the commutative theory these are very desirable properties and important tools in the investigation of martingales. In our first attempt to formulate these properties using Kosaki's (or Terp's) interpolation spaces, these properties seemed inexplicable or appeared in a rather awkward algebraic formulation involving the modular automorphism group of $\varphi$. This is due to the fact that the interpolated $L^p$-spaces do not have positive cones and that either the left or the right module action (or even both actions) of $\mathcal{M}$ on the interpolated $L^p$-spaces are difficult to describe. We refer to [1, 8, 16, 13] for some related results in this direction. Moreover, for $0 < p < 1$, $L^p$ itself and the extension of the conditional expectation $\mathcal{E}$ to a large dense subspace of $L^p$ can not be recovered by this interpolation approach. Since these tools turned out to be crucial, we are forced to work in the context of Haagerup $L^p$-spaces. The rest of this section is devoted to establishing the usual algebraic properties of conditional expectations in this setting.

Recall that $\mathcal{M}_a$ denotes the family of all analytic vectors of $\mathcal{M}$. For $0 \le \theta \le 1$ and $0 < p < \infty$ we define $\mathcal{E}_{p,\theta} : \mathcal{M}_a D^{1/p} \to \mathcal{M}_a D^{1/p}$ by

$$\mathcal{E}_{p,\theta}(D^{(1-\theta)/p} x D^{\theta/p}) = D^{(1-\theta)/p} \mathcal{E}(x) D^{\theta/p}, \qquad x \in \mathcal{M}_a.$$

LEMMA 2.1. *Let $0 < p < \infty$ and $0 \le \theta, \eta \le 1$. Then $\mathcal{E}_{p,\theta} = \mathcal{E}_{p,\eta}$.*

PROOF. It suffices to consider the case $\eta = 1$. Let $a \in \mathcal{M}_a D^{1/p}$. Then (see the proof of Lemma 1.1)

$$a = D^{(1-\theta)/p} x D^{\theta/p} = \sigma_{i(1-\theta)/p}(x) D^{1/p} \qquad \text{for some } x \in \mathcal{M}_a.$$

Thus

$$\begin{aligned} \mathcal{E}_{p,1}(a) &= \mathcal{E}_{p,1}\big(\sigma_{i(1-\theta)/p}(x) D^{1/p}\big) \\ &= \mathcal{E}\big(\sigma_{i(1-\theta)/p}(x)\big) D^{1/p} \\ &= \sigma_{i(1-\theta)/p}(\mathcal{E}(x)) D^{1/p} \qquad \text{[by (2.3)]} \\ &= D^{(1-\theta)/p} \mathcal{E}(x) D^{\theta/p} = \mathcal{E}_{p,\theta}(a). \end{aligned}$$

This completes the proof. □

Lemma 2.1 enables us to drop the subscript $\theta$ from $\mathcal{E}_{p,\theta}$. Thus we will denote $\mathcal{E}_{p,\theta}$ by $\mathcal{E}_p$.

LEMMA 2.2. *For any $1 \le p < \infty$, $\mathcal{E}_p$ extends to a contractive projection from $L^p(\mathcal{M})$ onto $L^p(\mathcal{N})$.*

PROOF. First consider the case $p = 1$. Let $x \in \mathcal{M}_a$ and $y \in \mathcal{N}$. Then by (1.1) and (2.2),

$$\begin{aligned} \operatorname{tr}\big(y \mathcal{E}_1(xD)\big) &= \operatorname{tr}\big(y \mathcal{E}(x) D\big) \\ &= \varphi\big(y \mathcal{E}(x)\big) = \varphi\big(\mathcal{E}(yx)\big) \\ &= \varphi(yx) = \operatorname{tr}(yxD). \end{aligned}$$

Therefore,

$$\left|\operatorname{tr}\left(y\mathcal{E}_1(xD)\right)\right| \le \|y\|_\infty \|xD\|_1,$$

whence

$$\|\mathcal{E}_1(xD)\|_1 \le \|xD\|_1, \qquad x \in \mathcal{M}_a.$$

By Lemma 1.1, $\mathcal{M}_a D$ is dense in $L^1(\mathcal{M})$. Thus, $\mathcal{E}_1$ extends to a contraction on $L^1(\mathcal{M})$. It is then clear that $\mathcal{E}_1$ is a projection of range equal to $L^1(\mathcal{N})$.

Next, assume $1 < p < \infty$. Let $x \in \mathcal{M}_a$ and $y \in L^{p'}(\mathcal{N})$ $(1/p + 1/p' = 1)$. Then $D^{1/p}y \in L^1(\mathcal{N})$. Thus, by Lemma 1.1, there is $y_n \in \mathcal{N}$ such that

$$\lim_{n\to\infty} y_n D = D^{1/p} y \qquad \text{in } L^1(\mathcal{N}).$$

Therefore, it follows that

$$\begin{aligned}\operatorname{tr}\big(\mathcal{E}_p(xD^{1/p})y\big) &= \operatorname{tr}\big(\mathcal{E}(x)D^{1/p}y\big)\\ &= \lim_{n\to\infty} \operatorname{tr}\big(\mathcal{E}(x)y_n D\big)\\ &= \lim_{n\to\infty} \operatorname{tr}\big(\mathcal{E}(xy_n)D\big)\\ &= \lim_{n\to\infty} \varphi\big(\mathcal{E}(xy_n)\big)\\ &= \lim_{n\to\infty} \varphi(xy_n) = \operatorname{tr}(xD^{1/p}y).\end{aligned}$$

This implies, as before in the case $p = 1$, that $\mathcal{E}_p$ extends to a contractive projection from $L^p(\mathcal{M})$ onto $L^p(\mathcal{N})$. □

REMARK. It is easy to see that $\mathcal{E}_2$ is the orthogonal projection from $L^2(\mathcal{M})$ onto $L^2(\mathcal{N})$, and that $\mathcal{E}_{p'}$ is the adjoint of $\mathcal{E}_p$.

In order to simplify the notation in the sequel, we will use the same letter $\mathcal{E}$ to denote the family $\{\mathcal{E}_p\}$ (thus drop the subscript $p$ from $\mathcal{E}_p$). This should not cause any ambiguity. During the proof of Lemma 2.1, we have proved the following equality, which will be repeatedly used later:

$$\operatorname{tr}(\mathcal{E}(x)) = \operatorname{tr}(x), \qquad x \in L^1(\mathcal{M}). \tag{2.4}$$

We now show that the conditional expectations on $L^p(\mathcal{M})$ posses all the usual algebraic properties.

PROPOSITION 2.3. (i) *Let* $1 \le p \le \infty$ *and* $x \in L^p(\mathcal{M})$. *Then*

$$(\mathcal{E}(x))^* = \mathcal{E}(x^*) \quad \textit{and} \quad x \ge 0 \Rightarrow \mathcal{E}(x) \ge 0.$$

(ii) *Let* $1 \le p, q, r \le \infty$ *such that* $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \le 1$. *Then*

$$\mathcal{E}(axb) = a\mathcal{E}(x)b, \qquad a \in L^p(\mathcal{N}),\ b \in L^q(\mathcal{N}),\ x \in L^r(\mathcal{M}).$$

(iii) *Let* $0 < p \le \infty$ *and* $x \in \mathcal{M}_a D^{1/p}$. *Then* $\mathcal{E}(x)^*\mathcal{E}(x) \le \mathcal{E}(x^*x)$. *Consequently, if* $p \ge 2$, $\mathcal{E}(x)^*\mathcal{E}(x) \le \mathcal{E}(x^*x)$ *for all* $x \in L^p(\mathcal{M})$.

PROOF. (i) Let $x \in \mathcal{M}_a$. Then, by Lemma 2.1,

$$\big(\mathcal{E}(xD^{1/p})\big)^* = \big(\mathcal{E}(x)D^{1/p}\big)^* = D^{1/p}\mathcal{E}(x^*) = \mathcal{E}(D^{1/p}x^*) = \mathcal{E}\big((xD^{1/p})^*\big).$$

Therefore, by Lemma 1.1 and the continuity of $\mathcal{E}$, we get the first part of (i). We delay the second part to the end of the proof of (ii).

(ii) First consider the case $p = q = \infty$. If $r = \infty$, we go back to the classical case in $\mathcal{M}$. Assume $r < \infty$. Then by Lemma 1.1, it suffices to consider $x = x'D^{1/r}$ with $x' \in \mathcal{M}_a$. Let $a, b \in L^\infty(\mathcal{N}) = \mathcal{N}$. Then

$$\mathcal{E}(ax) = \mathcal{E}(ax'D^{1/r}) = \mathcal{E}(ax')\ D^{1/r} = a\mathcal{E}(x')D^{1/r} = a\mathcal{E}(x).$$

On the other hand, by the first part of (i) above,

$$\big(\mathcal{E}(xb)\big)^* = \mathcal{E}(b^*x^*) = b^*\mathcal{E}(x^*) = b^*(\mathcal{E}(x))^* = \big(\mathcal{E}(x)b\big)^*,$$

whence $\mathcal{E}(xb) = \mathcal{E}(x)b$. Thus (ii) is proved in the case $p = q = \infty$.

Next assume $p < \infty$ and $q < \infty$ (the case where one of $p, q$ is finite and another infinite can be treated similarly). Again, by Lemma 1.1, it suffices to consider $a = a'D^{1/p}$, $b = D^{1/q}b'$ with $a', b' \in \mathcal{N}$. Then, by the case $p = q = \infty$ already proved, we get

$$\mathcal{E}(axb) = a'\mathcal{E}(D^{1/p}xD^{1/q})b'.$$

Thus, it remains to show

$$\mathcal{E}(D^{1/p}xD^{1/q}) = D^{1/p}\mathcal{E}(x)D^{1/q}, \qquad x \in L^r(\mathcal{M}).$$

We do this by considering $r < \infty$ and $r = \infty$ separately. If $r < \infty$, again by Lemma 1.1, we may assume $x = x'D^{1/r}$ with $x' \in \mathcal{M}_a$. Then, by Lemma 2.1,

$$\mathcal{E}(D^{1/p}xD^{1/q}) = D^{1/p}\mathcal{E}(x')D^{1/r+1/q} = D^{1/p}\mathcal{E}(x)D^{1/q},$$

as desired. For $r = \infty$, we use duality. Let $y \in L^{s'}(\mathcal{N})$, where $s$ is determined by $\frac{1}{s} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$ and $s'$ the conjugate index of $s$. Note that $s < \infty$; so by what we have already proved,

$$\mathcal{E}(yD^{1/p}xD^{1/q}) = y\mathcal{E}(D^{1/p}xD^{1/q}).$$

Thus, by (2.4),

$$\begin{aligned} \operatorname{tr}\big(y\mathcal{E}(D^{1/p}xD^{1/q})\big) &= \operatorname{tr}(yD^{1/p}xD^{1/q}) \\ &= \operatorname{tr}(D^{1/q}yD^{1/p}x) \\ &= \operatorname{tr}\big[\mathcal{E}(D^{1/q}yD^{1/p}x)\big]. \end{aligned}$$

Now $D^{1/q}yD^{1/p} \in L^1(\mathcal{N})$. Hence

$$\mathcal{E}(D^{1/q}yD^{1/p}x) = D^{1/q}yD^{1/p}\mathcal{E}(x).$$

Therefore

$$\mathrm{tr}\big[y\mathcal{E}(D^{1/p}xD^{1/q})\big] = \mathrm{tr}\big(yD^{1/p}\mathcal{E}(x)D^{1/q}\big), \qquad y \in L^{s'}(\mathcal{N})$$

whence

$$\mathcal{E}(D^{1/p}xD^{1/q}) = D^{1/p}\mathcal{E}(x)D^{1/q}.$$

This completes the proof of (ii). Now we come back to the second part of (i). Let $x \in L^p(\mathcal{M})$, $x \geq 0$. Then by (ii) and (2.4),

$$\mathrm{tr}\,\big(y\mathcal{E}(x)\big) = \mathrm{tr}\,\big(\mathcal{E}(yx)\big) = \mathrm{tr}(yx) \geq 0, \qquad y \in L^{p'}(\mathcal{N}),\ y \geq 0.$$

It follows that $x \geq 0$.

(iii) This is well known for $p = \infty$. Now let $x = aD^{1/p}$. Then

$$\mathcal{E}(x)^*\mathcal{E}(x) = D^{1/p}\mathcal{E}(a)^*\mathcal{E}(a)D^{1/p} \leq D^{1/p}\mathcal{E}(a^*a)D^{1/p} = \mathcal{E}(x^*x).$$

The second assertion of (iii) then follows by density. □

In the rest of this section, we introduce noncommutative martingales that we will deal with in the subsequent sections. Let $\mathcal{M}, \varphi, \{\sigma_t\}$ be fixed as before. Let $\{\mathcal{M}_n\}_{n\geq 0}$ be an increasing filtration of von Neumann subalgebras of $\mathcal{M}$ such that $\bigcup_{n\geq 0}\mathcal{M}_n$ is $w^*$-dense in $\mathcal{M}$. We assume that every $\mathcal{M}_n$ is invariant under $\{\sigma_t\}_{t\in\mathbb{R}}$ [i.e., (2.1) holds with $\mathcal{N} = \mathcal{M}_n$ for every $n \geq 0$]. Then by the preceding discussion, for each $n \geq 0$, there is a normal faithful conditional expectation $\mathcal{E}_n$ from $\mathcal{M}$ onto $\mathcal{M}_n$ such that $\varphi \circ \mathcal{E}_n = \varphi$. We have

$$\mathcal{E}_m\mathcal{E}_n = \mathcal{E}_n\mathcal{E}_m = \mathcal{E}_{\min(m,n)}, \qquad m, n \geq 0. \tag{2.5}$$

By Proposition 2.3, each $\mathcal{E}_n$ induces a contractive projection from $L^p(\mathcal{M})$ onto $L^p(\mathcal{M}_n)$ for all $1 \leq p \leq \infty$. Then all the notions of noncommutative martingales from [34] can be transferred to the present setting without any modification. For instance, a noncommutative $L^p$-martingale is a sequence $x = (x_n)_{n\geq 0} \subset L^p(\mathcal{M})$ such that $\mathcal{E}_m(x_n) = x_m$ for all $0 \leq m \leq n$. This implies that the sequence $x$ is adapted, that is, $x_n \in L^p(\mathcal{M}_n)$ for all $n \geq 0$. Set $\|x\|_p = \sup_n \|x_n\|_p$. If $\|x\|_p < \infty$, $x$ is said to be bounded. The difference sequence of $x$ is $dx = (dx_n)_{n\geq 0}$, where $dx_n = x_n - x_{n-1}$ for $n \geq 0$ (with $x_{-1} = 0$, by convention).

REMARK. Let $1 \leq p \leq \infty$ and $x_\infty \in L^p(\mathcal{M})$. Define $x_n = \mathcal{E}_n(x_\infty), n \geq 0$. Then $x = (x_n)_{n\geq 0}$ is a bounded $L^p$-martingale and $\lim_{n\to\infty} x_n = x_\infty$ in $L^p(\mathcal{M})$ (with respect to the $w^*$-topology in the case $p = \infty$). Conversely, for $1 < p < \infty$, using the uniform convexity of $L^p(\mathcal{M})$ we deduce that any bounded $L^p$-martingale $x$ converges to an element $x_\infty$ in $L^p(\mathcal{M})$, and thus is of this form. As usual, we often identify a martingale with its final value, whenever the latter exists.

We close this section by two conventions used throughout this paper. First, unless explicitly indicated otherwise, $\mathcal{M}, \varphi, \{\sigma_t\}$, $\{\mathcal{M}_n\}_{n\geq 0}$ and $\{\mathcal{E}_n\}_{n\geq 0}$ will be fixed as before, except in Section 8, where we will consider nonfaithful conditional expectations; all noncommutative martingales will be with respect to $\{\mathcal{M}_n\}_{n\geq 0}$. Second, letters like $\alpha_p, \beta_p \dots$ will denote absolute positive constants, which depend only on $p$ and may change from line to line.

**3. Noncommutative Burkholder–Gundy inequalities.** This section is devoted to extending the noncommutative Burkholder–Gundy inequalities proved in [34] to the case of Haagerup $L^p$-spaces. We first recall the two square functions introduced in [34]. Let $x$ be an $L^p$-martingale. We define

$$S_{c,n}(x) = \left(\sum_{k=0}^{n} |dx_k|^2\right)^{1/2} \quad \text{and} \quad S_{r,n}(x) = \left(\sum_{k=0}^{n} |dx_k^*|^2\right)^{1/2}.$$

If $dx \in L^p(\mathcal{M}; \ell_c^2)$ [equivalently, if $\sup_{n\geq 0} \|S_{c,n}(x)\|_p < \infty$], we set

$$S_c(x) = \left(\sum_{k=0}^{\infty} |dx_k|^2\right)^{1/2}.$$

Similarly, if $dx \in L^p(\mathcal{M}; \ell_r^2)$, set

$$S_r(x) = \left(\sum_{k=0}^{\infty} |dx_k^*|^2\right)^{1/2}.$$

Then $S_c(x)$ and $S_r(x)$ are elements in $L^p(\mathcal{M})$. Let $1 \leq p < \infty$. Define $\mathcal{H}_c^p(\mathcal{M})$ [resp. $\mathcal{H}_r^p(\mathcal{M})$] to be the space of all $L^p$-martingales $x$ with respect to the filtration $(\mathcal{M}_n)_{n\geq 0}$ such that $dx \in L^p(\mathcal{M}; \ell_c^2)$ [resp. $dx \in L^p(\mathcal{M}; \ell_r^2)$], and set

$$\|x\|_{\mathcal{H}_c^p(\mathcal{M})} = \|dx\|_{L^p(\mathcal{M};\ell_c^2)} \quad \text{and} \quad \|x\|_{\mathcal{H}_r^p(\mathcal{M})} = \|dx\|_{L^p(\mathcal{M};\ell_r^2)}.$$

From the discussion in Section 1, $\|\cdot\|_{\mathcal{H}_c^p(\mathcal{M})}$ and $\|\cdot\|_{\mathcal{H}_r^p(\mathcal{M})}$ are two norms, for which $\mathcal{H}_c^p(\mathcal{M})$ and $\mathcal{H}_r^p(\mathcal{M})$ become Banach spaces. Note that if $x \in \mathcal{H}_c^p(\mathcal{M})$,

$$\|x\|_{\mathcal{H}_c^p(\mathcal{M})} = \sup_{n\geq 0} \|S_{c,n}(x)\|_p = \|S_c(x)\|_p,$$

and similarly for $\mathcal{H}_r^p(\mathcal{M})$. Then we define the Hardy spaces of noncommutative martingales as

$$\mathcal{H}^p(\mathcal{M}) = \mathcal{H}_c^p(\mathcal{M}) + \mathcal{H}_r^p(\mathcal{M}) \qquad \text{for } 1 \leq p < 2$$

equipped with the norm

$$\|x\| = \inf\{\|y\|_{\mathcal{H}_c^p(\mathcal{M})} + \|z\|_{\mathcal{H}_r^p(\mathcal{M})} : x = y + z,\ y \in \mathcal{H}_c^p(\mathcal{M}),\ z \in \mathcal{H}_r^p(\mathcal{M})\};$$

and

$$\mathcal{H}^p(\mathcal{M}) = \mathcal{H}_c^p(\mathcal{M}) \cap \mathcal{H}_r^p(\mathcal{M}) \qquad \text{for } 2 \leq p < \infty$$

equipped with the norm

$$\|x\| = \max\big\{\|x\|_{\mathcal{H}_c^p(\mathcal{M})}, \|x\|_{\mathcal{H}_r^p(\mathcal{M})}\big\}.$$

Now we can transfer the main results in Section 2 of [34] to the present setting.

THEOREM 3.1. *Let $1 < p < \infty$. Let $x = (x_n)_{n\geq 0}$ be an $L^p$-martingale with respect to $\{\mathcal{M}_n\}_{n\geq 0}$. Then $x$ is bounded in $L^p(\mathcal{M})$ iff $x$ belongs to $\mathcal{H}^p(\mathcal{M})$; moreover, if this is the case,*

$$(\mathrm{BG}_p) \qquad \alpha_p^{-1}\|x\|_{\mathcal{H}^p(\mathcal{M})} \leq \|x\|_p \leq \beta_p\|x\|_{\mathcal{H}^p(\mathcal{M})}.$$

Identifying bounded $L^p$-martingales with their limits, we may reformulate Theorem 3.1 as follows.

COROLLARY 3.2. *Let $1 < p < \infty$. Then $\mathcal{H}^p(\mathcal{M}) = L^p(\mathcal{M})$ with equivalent norms.*

The noncommutative Stein inequality in [34] also holds now.

THEOREM 3.3. *Let $1 < p < \infty$. Define the map $Q$ on all finite sequences $a = (a_n)_{n\geq 0}$ in $L^p(\mathcal{M})$ by $Q(a) = (\mathcal{E}_n a_n)_{n\geq 0}$. Then*

$$(\mathrm{S}_p) \quad \|Q(a)\|_{L^p(\mathcal{M};l_c^2)} \leq \gamma_p\|a\|_{L^p(\mathcal{M};l_c^2)}, \qquad \|Q(a)\|_{L^p(\mathcal{M};\ell_r^2)} \leq \gamma_p\|a\|_{L^p(\mathcal{M};\ell_r^2)}.$$

*Thus $Q$ extends to a bounded projection on $L^p(\mathcal{M}; l_c^2)$ and $L^p(\mathcal{M}; \ell_r^2)$; consequently, $\mathcal{H}^p(\mathcal{M})$ is complemented in $L^p(\mathcal{M}; l_c^2) + L^p(\mathcal{M}, \ell_r^2)$ or $L^p(\mathcal{M}; l_c^2) \cap L^p(\mathcal{M}; \ell_r^2)$ according to $1 < p \leq 2$ or $2 \leq p < \infty$.*

Our proof below for Theorems 3.1 and 3.3 follows the same pattern as in [34]. The only modifications which require a new justification are the interpolation arguments used in [34]. Although interpolation arguments cannot directly be applied for Haagerup $L^p$-spaces, a direct application of the three lines lemma as in Lemma 1.2 is possible in all the modifications we need here. Of course, this idea is not new (see, e.g., [42] for the proof of the Clarkson inequality), and it will be used several times in the sequel.

SKETCH OF THE PROOF OF THEOREMS 3.1 AND 3.3. Below, we indicate the places in the proof of Theorem 2.1 and Theorem 2.3 in [34] which require modifications. The reader will easily be able to complete the omitted details.

(i) *Lemmas* 1.1, 2.6 *and* 2.7 *in* [34] *are still valid.* In the state case, Lemma 2.6 in [34] is proved by the tensor product argument given in Section 1. Let us indicate the modifications in the interpolation arguments of the proofs in Lemma 1.1 and Lemma 2.7 of [34] exemplary for Lemma 2.7. This means we have to prove the following inequality. Given $2 \le p \le \infty$ and a finite $L^p$-martingale $x = (x_n)_{n\ge 0}$ then

$$\left(\sum_{n\ge 0} \|dx_n\|_p^p\right)^{1/p} \le 2^{1-2/p}\|x\|_p. \tag{3.1}$$

This is trivial for $p = \infty$ as well as for $p = 2$ [since for any $L^2$-martingale $x$, $(dx_n)_{n\ge 0}$ is an orthogonal sequence in $L^2(\mathcal{M})$]. Now assume $2 < p < \infty$. Let $x$ be a finite $L^p$-martingale with $\|x\|_p \le 1$. Then there is an $n$ such that $x_k = x_n$ for all $k \ge n$. Choose a finite sequence $b = (b_k)_{0\le k\le n} \subset L^{p'}(\mathcal{M})$ $(1/p + 1/p' = 1)$ such that

$$\sum_{0\le k\le n} \|b_k\|_{p'}^{p'} \le 1 \quad \text{and} \quad \left(\sum_{k\ge 0} \|dx_k\|_p^p\right)^{1/p} = \sum_{0\le k\le n} \operatorname{tr}(b_k dx_k).$$

By approximation and in view of Kosaki's results [24], we may assume $b_k = B_k(\frac{2}{p})D^{1/p'}$ and the $B_k$ are continuous functions with values in $\mathcal{M}$ defined on the strip $S = \{z \in \mathbb{C} : 0 \le \operatorname{Re} z \le 1\}$, analytic in the interior, such that

$$\sup_t \max\left\{\sum_{0\le k\le n} \|B_k(it)D\|_1, \left(\sum_{0\le k\le n} \|B_k(1+it)D^{1/2}\|_2^2\right)^{1/2}\right\} \le 1.$$

Similarly, we can assume $x_n = D^{1/p}X(\frac{2}{p})$ and $X$ is an analytic function on the strip with values in $\mathcal{M}_n$ such that

$$\sup_t \max\{\|X(it)\|_\infty, \|D^{1/2}X(1+it)\|_2\} \le 1.$$

Then we consider the analytic function

$$F(z) = \sum_{0\le k\le n} \operatorname{tr}\left[B_k(z)D(\mathcal{E}_k - \mathcal{E}_{k-1})X(z)\right].$$

Using the Hölder inequality and the case $p = \infty$, we deduce

$$\begin{aligned} |F(it)| &\le \sum_{0\le k\le n} \|B_k(it)D\|_1 \|(\mathcal{E}_k - \mathcal{E}_{k-1})X(it)\|_\infty \\ &\le 2\|X(it)\|_\infty \le 2 \end{aligned}$$

for all $t \in \mathbb{R}$. Similarly, $|F(1+it)| \le 1$ for all $t \in \mathbb{R}$. Thus, by the three lines lemma,

$$|F(2/p)| = \sum_{0\le k\le n} \operatorname{tr}(b_k dx_k) \le 2^{1-2/p},$$

whence (3.1) by the choice of the $(b_k)$'s.

(ii) *The inequality* $(\mathrm{S}_p)$ *can be interpolated.* By this we mean that if $1 < p_0 < p_1 < \infty$, then the validity of $(\mathrm{S}_{p_0})$ and $(\mathrm{S}_{p_1})$ implies that of $(\mathrm{S}_p)$ for all $p_0 < p < p_1$. Fix a finite sequence $a = (a_n) \in L^p(\mathcal{M}; l_c^2)$ of norm $\leq 1$. Let $x$ be the finite column matrix represented by $a$ (see Lemma 1.2). Choose a row matrix $y \in L^{p'}(\mathcal{M} \otimes B(\ell_n^2))$ of norm 1 such that

$$\|a\|_{L^p(\mathcal{M};l_c^2)} = \|x\|_{L^p(\mathcal{M}\otimes B(\ell_n^2))} = \mathrm{tr} \otimes \mathrm{Tr}(yx) = \sum_{k=1}^{n} \mathrm{tr}(y_k a_k).$$

We could still work with the state $\varphi_n = \varphi \otimes \frac{\mathrm{Tr}}{n}$ and therefore apply Kosaki's results with the density $D_n = D \otimes 1$. Let $\theta$ be determined by $1/p = (1-\theta)/p_0 + \theta/p_1$. Again by approximation (and complementation of the column subspace), we can assume $a_k = A_k(\theta)D^{1/p}$ and

$$\max\left\{\left\|\left(A_k(it)D^{1/p_0}\right)\right\|_{L^{p_0}(\mathcal{M},\ell_c^2)}, \left\|\left(A_k(1+it)D^{1/p_1}\right)\right\|_{L^{p_1}(\mathcal{M},\ell_c^2)}\right\} \leq 1$$

for all $t \in \mathbb{R}$. Similarly, $y = (y_1, \ldots, y_n)$ can be assumed to satisfy $y_k = D^{1/p'} Y_k(\theta)$ and

$$\max\left\{\left\|\left(D^{1/p_0'}Y_k(it)\right)\right\|_{L^{p_0'}(\mathcal{M},\ell_r^2)}, \left\|\left(D^{1/p_1'}Y_k(1+it)\right)\right\|_{L^{p_1'}(\mathcal{M},\ell_r^2)}\right\} \leq 1$$

for all $t \in \mathbb{R}$. Then, we consider

$$F(z) = \sum_{k=1}^{n} \mathrm{tr}\left(DY_k(z)\mathcal{E}_k\left(A_k(z)\right)\right).$$

For any $t \in \mathbb{R}$, by the Hölder inequality, $(\mathrm{S}_{p_0})$ and Lemma 1.2, we deduce

$$|F(it)| \leq \left\|\left(D^{1/p_0'}Y_k(it)\right)\right\|_{L^{p_0'}(\mathcal{M},\ell_r^2)} \left\|\left(\mathcal{E}_k\left(A_k(it)\right)D^{1/p_0}\right)\right\|_{L^{p_0}(\mathcal{M},\ell_c^2)} \leq \gamma_{p_0}.$$

Similarly, $|F(1+it)| \leq \gamma_{p_1}$ for all $t \in \mathbb{R}$. Therefore, by the three lines lemma,

$$|F(\theta)| \leq \gamma_{p_0}^{1-\theta}\gamma_{p_1}^{\theta}.$$

Since $F(\theta) = \mathrm{tr} \otimes \mathrm{Tr}(yx)$ and by the choice of $y$, we deduce $(\mathrm{S}_p)$ with $\gamma_p \leq \gamma_{p_0}^{1-\theta}\gamma_{p_1}^{\theta}$.

(iii) *The first inequality of* $(\mathrm{BG}_p)$ *can be interpolated.* This can be done by combining the arguments in the last step and those in the proof of (3.1).

(iv) $(\mathrm{BG}_p)$ *implies* $(\mathrm{S}_p)$. The proof given in [34] works as well in the present setting. However, we prefer to give a slightly different proof of this fact with a better estimate for the constant $\gamma_p$ in $(\mathrm{S}_p)$ in terms of $\alpha_p$ and $\beta_p$ in $(\mathrm{BG}_p)$. Let $1 < p < \infty$. Suppose $(\mathrm{BG}_p)$ holds. We will show $(\mathrm{S}_p)$ holds as well. To this end, fix a finite sequence $a = (a_k)_{0 \leq k \leq n} \subset L^p(\mathcal{M})$. We consider the tensor product $(\mathcal{M}, \varphi) \otimes (\mathcal{N}, \sigma)$, where $\mathcal{N} = B(l_{n+1}^2)$ and $\sigma = (n+1)^{-1}\mathrm{Tr}$ is the normalized trace on $B(l_{n+1}^2)$. Note that $\psi = \varphi \otimes \sigma$ is a normal state on $\mathcal{M} \otimes \mathcal{N}$ (see Section 1).

Let $\tilde{\mathcal{E}}_k = \mathcal{E}_k \otimes \mathrm{id}_{\mathcal{N}}$ denote the conditional expectation of $\mathcal{M} \otimes \mathcal{N}$ with respect to $\tilde{\mathcal{M}}_k = \mathcal{M}_k \otimes \mathcal{N}$. Then we have ($\mathrm{BG}_p$) for all martingales relative to the filtration $(\mathcal{M}_k \otimes \mathcal{N})_{k\geq 0}$. Now set

$$A_k = (n+1)^{1/p} a_k \otimes e_{k,0}, \qquad 0 \leq k \leq n.$$

Let $(r_n)_{n\geq 0}$ be the sequence of Rademacher functions on $[0, 1]$. Then for any $t \in [0, 1]$,

$$\begin{aligned} \|Q(a)\|_{L^p(\mathcal{M};l_c^2)} &= \left\| \sum_{k=0}^{n} \tilde{\mathcal{E}}_k\big(r_k(t)A_k\big)\right\|_p \\ &= \left\| \sum_{k=0}^{n} \tilde{\mathcal{E}}_n\big(r_k(t)A_k\big) - \sum_{k=0}^{n-1}\sum_{j=k}^{n-1}(\tilde{\mathcal{E}}_{j+1} - \tilde{\mathcal{E}}_j)\big(r_k(t)A_k\big)\right\|_p \\ &\leq \|a\|_{L^p(\mathcal{M};l_c^2)} + \left\| \sum_{j=0}^{n-1}(\tilde{\mathcal{E}}_{j+1} - \tilde{\mathcal{E}}_j)\left(\sum_{k=0}^{j} r_k(t)A_k\right)\right\|_p. \end{aligned}$$

Let

$$f = \sum_{k=0}^{n-1} r_k A_k.$$

Now we consider the filtration

$$\tilde{\mathcal{M}}_0 \otimes \mathcal{F}_0,\ \tilde{\mathcal{M}}_1 \otimes \mathcal{F}_0,\ \tilde{\mathcal{M}}_1 \otimes \mathcal{F}_1,\ \tilde{\mathcal{M}}_2 \otimes \mathcal{F}_1,\ \tilde{\mathcal{M}}_2 \otimes \mathcal{F}_2, \ldots$$

where $\mathcal{F}_j$ is the $\sigma$-field generated by $\{r_0, \ldots, r_j\}$. Denoting by $(df_j)_{j\geq 0}$ the difference sequence of $f$ with respect to this filtration, we have

$$\sum_{j=0}^{n-1}(\tilde{\mathcal{E}}_{j+1} - \tilde{\mathcal{E}}_j)\left(\sum_{k=0}^{j} r_k(t)A_k\right) = \sum_{j=0}^{n-1} df_{2j+1}.$$

Now note that ($\mathrm{BG}_p$) implies the unconditionality of all martingale differences with constant majorized by $\alpha_p\beta_p$. Therefore, it follows that

$$\left\|\sum_{j=0}^{n-1} df_{2j+1}\right\|_p \leq \alpha_p\beta_p \left\|\sum_{j=0}^{n-1} df_j\right\|_p = \alpha_p\beta_p\|f\|_p = \alpha_p\beta_p\|a\|_{L^p(\mathcal{M};\ell_c^2)}.$$

Combining the preceding inequalities, we obtain ($\mathrm{S}_p$) with $\gamma_p \leq (1 + \alpha_p\beta_p)$. This estimate is better than that in [34], which is $\gamma_p \leq (\alpha_p\beta_p)^3$. □

REMARK. The arguments in [33] also work for martingale differences considered here. Note that Pisier's proof is written with respect to a tracial state, but replacing $\tau$ by tr, the same combinatorial arguments carry through, because we still have $\|x\|_{2n}^{2n} = \mathrm{tr}((x^*x)^n)$ and the $2n$-orthogonality $\mathrm{tr}(d_{f(1)} \cdots d_{f(2n)})$

is satisfied for injective functions. Hence, we deduce that the constant $\beta_p$ in $(\mathrm{BG}_p)$ is of order $O(p)$ for even integers $p$. However, using the recent results of Radrianantoanina [35] in combination with [15], it follows that $\gamma_p \leq C\max\{p, p'\}$. We refer to the forthcoming paper [22] for the fact that these constants are optimal. The same approach also provides good (or optimal) constants $\beta_p \leq Cp$, $\alpha_p \leq Cp$ for $p \geq 2$.

We end this section by the noncommutative Khintchine inequalities.

THEOREM 3.4. *There are absolute constants $\beta$ and $\alpha$ with the following property. Let $(r_n)_{n\geq 0}$ be the sequence of Rademacher functions on $[0,1]$. Let $1 \leq p < \infty$. Let $a = (a_n)_{n\geq 0}$ be a finite sequence in $L^p(\mathcal{M})$.*

(i) *If $2 \leq p < \infty$,*

$$\|a\|_{L^p(\mathcal{M};\ell_c^2)\cap L^p(\mathcal{M};\ell_r^2)} \leq \left(\int_0^1 \left\|\sum_{n\geq 0} r_n(t)a_n\right\|_p^2 dt\right)^{1/2} \leq \beta\sqrt{p}\|a\|_{L^p(\mathcal{M};\ell_c^2)\cap L^p(\mathcal{M};\ell_r^2)}.$$

(ii) *If $1 \leq p < 2$,*

$$\alpha\|a\|_{L^p(\mathcal{M};\ell_c^2)+L^p(\mathcal{M};\ell_r^2)} \leq \left(\int_0^1 \left\|\sum_{n\geq 0} r_n(t)a_n\right\|_p^2 dt\right)^{1/2} \leq \|a\|_{L^p(\mathcal{M};\ell_c^2)+L^p(\mathcal{M};\ell_r^2)}.$$

In the tracial case, this theorem is contained in [25, 26, 32]. The nontracial case can be reduced to the tracial one in virtue of [15] (as mentioned above) or using a direct proof based on [26]. Note that Theorem 3.1 implies Theorem 3.4 for $1 < p < \infty$ but with a worse constant. Also note that the first inequality in (i) and the second in (ii) above are easy to check (cf. [26]). Finally, we should point out that the theorem in the Appendix of [26] yields the first inequality in (ii) for $p = 1$. We omit all details. The subsequent paper [21] contains more related inequalities.

**4. A description of the dual of $\mathcal{H}^p$, $1 \leq p < 2$.** In the Appendix of [34], the classical Fefferman duality between $H^1$ and *BMO* was extended to the noncommutative martingale setting (with a trace). This result was further extended to the nontracial case in [19]. In fact, the duality result proved in [19] describes more generally the dual of $\mathcal{H}^p$ for every $1 \leq p < 2$. Using this duality, we can show that the second inequality of $(\mathrm{BG}_p)$ holds for $p = 1$ (as the tracial case in [34]), and moreover, the constant $\beta_p$ there remains bounded for $1 < p < 2$.

We begin with a notation introduced in [19]. Let $1 \le p \le \infty$, and let $(a_n)_{n\ge 0}$ be a finite sequence of positive elements in $L^p(\mathcal{M})$. Set (with $p'$ the index conjugate to $p$)

$$\Big\| \sup_n a_n \Big\|_p = \sup\Big\{ \sum_{n\ge 0} \mathrm{tr}(a_n b_n) : b_n \in L^{p'}(\mathcal{M}), b_n \ge 0, \Big\| \sum_{n\ge 0} b_n \Big\|_{p'} \le 1 \Big\}.$$

We should call the reader's attention to the fact that $\sup_n a_n$ does not make any sense in the noncommutative setting, and the above $\| \sup_n a_n\|_p$ is just a (useful and suggestive) notation.

Now let $2 < q \le \infty$. We define $L_c^q\mathcal{MO}(\mathcal{M})$ (mean oscillation in $L^q$ in the column sense) as the space of all martingale difference sequences $(d_k)$ in $L^q$ such that the sequence $x = (x_n)_{n\ge 0}$ defined by $x_n = \sum_{k=1}^n d_k$ satisfies

$$\|x\|^2_{L_c^q\mathcal{MO}(\mathcal{M})} = \sup_{m\ge 0} \Big\| \sup_{0\le n\le m} \mathcal{E}_n\big(|x_m - x_{n-1}|^2\big)\Big\|_{q/2} < \infty.$$

Note that

$$\mathcal{E}_n\big(|x_m - x_{n-1}|^2\big) = \mathcal{E}_n\bigg(\sum_{k=n}^m |d_k|^2\bigg). \tag{4.1}$$

One can check that $\|\cdot\|_{L_c^q\mathcal{MO}(\mathcal{M})}$ is a norm, which makes $L_c^q\mathcal{MO}(\mathcal{M})$ a Banach space. Similarly, we define $L_r^q\mathcal{MO}(\mathcal{M})$ as the space of all $x$ such that $x^* \in L_c^q\mathcal{MO}(\mathcal{M})$, equipped with the norm

$$\|x\|_{L_r^q\mathcal{MO}(\mathcal{M})} = \|x^*\|_{L_c^q\mathcal{MO}(\mathcal{M})}.$$

Finally, we set

$$L^q\mathcal{MO}(\mathcal{M}) = L_c^q\mathcal{MO}(\mathcal{M}) \cap L_r^q\mathcal{MO}(\mathcal{M})$$

equipped with the intersection norm

$$\|x\|_{L_c^q\mathcal{MO}(\mathcal{M})} = \max\big\{\|x\|_{L_c^q\mathcal{MO}(\mathcal{M})}, \|x\|_{L_r^q\mathcal{MO}(\mathcal{M})}\big\}.$$

If $q = \infty$, all these spaces $L^\infty\mathcal{MO}(\mathcal{M}) = \mathcal{BMO}(\mathcal{M})$, $L_c^\infty\mathcal{MO}(\mathcal{M}) = \mathcal{BMO}_c(\mathcal{M})$, $L_r^\infty\mathcal{MO}(\mathcal{M}) = \mathcal{BMO}_r(\mathcal{M})$ coincide with those introduced in [34] (at least in the tracial case).

Any $y \in L_c^q\mathcal{MO}(\mathcal{M})$ defines a linear functional $\xi_y$ on the family of all finite $L^p$-martingales as follows ($p = q'$ being conjugate to $q$). Let $x$ be a finite $L^p$-martingale, say $x_n = x_m$ for all $n \ge m$. Then $\xi_y(x) = \mathrm{tr}(y_m^* x_m)$. Clearly, $\xi_y(x) = \mathrm{tr}(y_n^* x_n)$ for all $n \ge m$. Thus we can write

$$\xi_y(x) = \lim_{n\to\infty} \mathrm{tr}(y_n^* x_n). \tag{4.2}$$

The following result shows that $\xi_y$ extends to a continuous functional on $\mathcal{H}_c^p(\mathcal{M})$, and conversely, any continuous functional on $\mathcal{H}_c^p(\mathcal{M})$ is given by some $\xi_y$. One part of this result is a special application of [19] Proposition 4.2 to martingale difference sequences. It is the noncommutative analogue of a classical result in commutative martingale theory (cf. [12]).

THEOREM 4.1. *Let $1 \le p < 2$ and $q = p'$ the index conjugate to $p$.*

(i) *Let $y \in L^q_c\mathcal{MO}(\mathcal{M})$. Then $\xi_y$ defined by (4.2) for all finite $L^p$-martingales $x$ extends to a continuous linear functional on $\mathcal{H}^p_c(\mathcal{M})$.*

(ii) *Conversely, any $\xi \in (\mathcal{H}^p_c(\mathcal{M}))^*$ is given as above by some $y \in L^q_c\mathcal{MO}(\mathcal{M})$. Moreover,*

$$\lambda_p^{-1}\|y\|_{L^q_c\mathcal{MO}(\mathcal{M})} \le \|\xi_y\|_{(\mathcal{H}^p_c(\mathcal{M}))^*} \le \sqrt{2}\|y\|_{L^q_c\mathcal{MO}(\mathcal{M})}, \tag{4.3}$$

*where $\lambda_p > 0$ is a constant depending only on $p$ and $\lambda_p = O(1)$ as $p \to 1$. Consequently, $(\mathcal{H}^p_c(\mathcal{M}))^* = L^q_c\mathcal{MO}(\mathcal{M})$ with equivalent norms.*

(iii) *The same duality holds between $\mathcal{H}^p_r(\mathcal{M})$, $\mathcal{H}^p(\mathcal{M})$ and $L^q_r\mathcal{MO}(\mathcal{M})$, $L^q\mathcal{MO}(\mathcal{M})$, respectively,*

$$\big(\mathcal{H}^p_r(\mathcal{M})\big)^* = L^q_r\mathcal{MO}(\mathcal{M}) \quad \textit{and} \quad \big(\mathcal{H}^p(\mathcal{M})\big)^* = L^q\mathcal{MO}(\mathcal{M}).$$

PROOF. (iii) follows from (i) and (ii) by standard arguments. (i) and the second inequality of (4.3) were proved in [19]. Thus it remains to show (ii) and the first inequality in (4.3). Suppose $\xi \in (\mathcal{H}^p_c(\mathcal{M}))^*$ with norm $\le 1$. Then by the Hahn-Banach theorem, $\xi$ extends to a continuous functional on $L^p(\mathcal{M}, \ell^2_c)$ of the same norm. Thus by Corollary 1.3, there exists a sequence $(b_n) \in L^q(\mathcal{M}, \ell^2_c)$ such that

$$\Big\|\sum_{n\ge 0}|b_n|^2\Big\|_{q/2} \le 1 \quad \text{and} \quad \xi(x) = \sum_{n\ge 0} b_n^* dx_n \qquad \forall\, x \in \mathcal{H}^p_c(\mathcal{M}).$$

Let $y$ be the $L^q$-martingale given by $dy_0 = \mathcal{E}_0(b_0)$ and $dy_n = \mathcal{E}_n(b_n) - \mathcal{E}_{n-1}(b_n)$ for all $n \ge 1$. Then for any finite $L^p$-martingale $x$, $\xi(x) = \xi_y(x)$. Thus it remains to show $y \in L^q_c\mathcal{MO}$ and to find a bound for $\|y\|_{L^q_c\mathcal{MO}(\mathcal{M})}$. This is done as follows. If $k-1 \ge n \ge 0$, by (2.5) and Proposition 2.3(ii)

$$\mathcal{E}_n[\mathcal{E}_k b_k^*\mathcal{E}_{k-1}b_k] = \mathcal{E}_n[\mathcal{E}_{k-1}(\mathcal{E}_k b_k^*\mathcal{E}_{k-1}b_k)] = \mathcal{E}_n[\mathcal{E}_{k-1}b_k^*\mathcal{E}_{k-1}b_k];$$

similarly,

$$\mathcal{E}_n[\mathcal{E}_{k-1}b_k^*\mathcal{E}_k b_k] = \mathcal{E}_n[\mathcal{E}_{k-1}b_k^*\mathcal{E}_{k-1}b_k].$$

Here and in the sequel we will skip brackets in the use of $\mathcal{E}_n$ whenever this is possible. It then follows that if $k-1 \ge n \ge 0$,

$$\begin{aligned}\mathcal{E}_n\big[|dy_k|^2\big] &= \mathcal{E}_n[(\mathcal{E}_k b_k - \mathcal{E}_{k-1}b_k)^*(\mathcal{E}_k b_k - \mathcal{E}_{k-1}b_k)]\\ &= \mathcal{E}_n[\mathcal{E}_k b_k^*\mathcal{E}_k b_k - \mathcal{E}_{k-1}b_k^*\mathcal{E}_{k-1}b_k]\\ &\le \mathcal{E}_n[\mathcal{E}_k b_k^*\mathcal{E}_k b_k] \le \mathcal{E}_n|b_k|^2,\end{aligned} \tag{4.4}$$

where for the last inequality we used Proposition 2.3(iii). Now let $s$ be the conjugate index of $q/2$, and let $(a_n)$ be a finite sequence of positive elements in $L^s(\mathcal{M})$ such that

$$\Big\| \sum_{n\geq 0} a_n \Big\|_s \leq 1.$$

Fix a positive integer $m$. By (4.1), (4.4), (2.4) and Proposition 2.3(ii),

$$\begin{aligned}
&\sum_{0\leq n\leq m} \operatorname{tr}\big[\mathcal{E}_n|y_m - y_{n-1}|^2 a_n\big] \\
&\quad = \sum_{0\leq n\leq m} \operatorname{tr}\bigg[\bigg(\mathcal{E}_n \sum_{n\leq k\leq m} |dy_k|^2\bigg) a_n\bigg] \\
&\quad \leq \sum_{0\leq n\leq m} \operatorname{tr}\big[(\mathcal{E}_n|dy_n|^2)a_n\big] + \sum_{0\leq n\leq m} \operatorname{tr}\bigg[\bigg(\mathcal{E}_n \sum_{n+1\leq k\leq m} |b_k|^2\bigg) a_n\bigg] \\
&\quad = \sum_{0\leq n\leq m} \operatorname{tr}\big[|dy_n|^2 \mathcal{E}_n a_n\big] + \sum_{0\leq n\leq m} \operatorname{tr}\bigg[\sum_{n+1\leq k\leq m} |b_k|^2 \mathcal{E}_n a_n\bigg] \\
&\quad = I + II.
\end{aligned}$$

We majorize $I$ and $II$ separately. In order to estimate $I$ we use the elementary inequality $(a-b)^*(a-b) \leq 2(a^*a + b^*b)$ and $\mathcal{E}_n(b_n^*)\mathcal{E}_n(b_n) \leq \mathcal{E}_n(b_n^* b_n)$ and deduce from Proposition 2.3 and (2.4)

$$\begin{aligned}
I &\leq 2\sum_{0\leq n\leq m} \operatorname{tr}\big[|\mathcal{E}_n b_n|^2 \mathcal{E}_n a_n\big] + 2\sum_{1\leq n\leq m} \operatorname{tr}\big[|\mathcal{E}_{n-1} b_{n-1}|^2 \mathcal{E}_n a_n\big] \\
&\leq 2\sum_{0\leq n\leq m} \operatorname{tr}\big[|b_n|^2 \mathcal{E}_n a_n\big] + 2\sum_{1\leq n\leq m} \operatorname{tr}\big[|b_{n-1}|^2 \mathcal{E}_{n-1} a_n\big] \\
&\leq 2\operatorname{tr}\bigg[\bigg(\sum_{0\leq n\leq m} |b_n|^2\bigg)\bigg(\sum_{0\leq n\leq m} \mathcal{E}_n a_n\bigg)\bigg] \\
&\qquad + 2\operatorname{tr}\bigg[\bigg(\sum_{1\leq n\leq m} |b_n|^2\bigg)\bigg(\sum_{0\leq n\leq m} \mathcal{E}_{n-1} a_n\bigg)\bigg] \\
&\leq 2\Big\| \sum_{0\leq n\leq m} \mathcal{E}_n a_n \Big\|_s + 2\Big\| \sum_{1\leq n\leq m} \mathcal{E}_{n-1} a_n \Big\|_s .
\end{aligned}$$

As for the second term $II$, we have

$$\begin{aligned} II &= \operatorname{tr}\Bigg[\sum_{1\le k\le m}|b_k|^2 \sum_{k-1\le n\le m}\mathcal{E}_n a_n\Bigg] \\ &\le \operatorname{tr}\Bigg[\Bigg(\sum_{1\le k\le m}|b_k|^2\Bigg)\Bigg(\sum_{0\le n\le m}\mathcal{E}_n a_n\Bigg)\Bigg] \\ &\le \Bigg\|\sum_{0\le n\le m}\mathcal{E}_n a_n\Bigg\|_s. \end{aligned}$$

However, by the dual form of the noncommutative Doob inequality proved in [19],

$$\max\Bigg\{\Bigg\|\sum_{0\le n\le m}\mathcal{E}_n a_n\Bigg\|_s, \Bigg\|\sum_{0\le n\le m}\mathcal{E}_{n-1} a_n\Bigg\|_s\Bigg\} \le \lambda_s \Bigg\|\sum_{0\le n\le m} a_n\Bigg\|_s,$$

where the constant $\lambda_s$ remains bounded when $s$ is away from $\infty$, that is, when $p$ is away from 2 [see the end of Section 7 for a simple proof of this inequality in the (easier) range $1 \le s \le 2$]. Combining the preceding inequalities, we obtain

$$\sum_{0\le n\le m}\operatorname{tr}\big[\mathcal{E}_n|y_m - y_{n-1}|^2 a_n\big] \le 5\lambda_s$$

whence the desired result on $y$. Thus we have finished the proof of the theorem. □

Combining Theorem 4.1 with Corollary 3.2, we see that $L^q\mathcal{MO}(\mathcal{M}) = L^q(\mathcal{M})$ with equivalent norms for any $2 < q < \infty$. In particular,

$$\|a\|_{L^q\mathcal{MO}(\mathcal{M})} \le \delta_q\|a\|_q \qquad \forall\, a \in L^q(\mathcal{M}). \tag{4.5}$$

The constant $\delta_q$ obtained in this way goes to $\infty$ as $q \to \infty$. However, if $a = (x_n)$ is an $L^\infty(\mathcal{M})$-martingale and if $0 \le m \le n$, then the triangle inequality from [19], Proposition 18, implies

$$\|\mathcal{E}_n|x_m - x_{n-1}|^2\|^{1/2} \le \|\mathcal{E}_n|x_m|^2\|^{1/2} + \|\mathcal{E}_n|x_{n-1}|^2\|^{1/2} \le 2\|a\|_\infty.$$

Therefore,

$$\|a\|_{\mathcal{BMO}(\mathcal{M})} \le 2\|a\|_\infty, \qquad a \in L^\infty(\mathcal{M}). \tag{4.6}$$

Thus we are tempted to interpolate (4.5) for some fixed $q$ and (4.6) in the hope of getting a constant $\delta_q$ in (4.5) which remains bounded as $q \to \infty$. Fortunately, this is possible, as shown by the following result, which is the noncommutative analogue of the classical Fefferman–Stein inequality on the sharp function.

PROPOSITION 4.2. *Let $2 < q \le \infty$. Then there is a constant $\delta_q > 0$ with $q = O(1)$ as $q \to \infty$ such that* (4.5) *holds.*

PROOF. It suffices to consider large $q$, say $4 \le q \le \infty$. Using a highly nontrivial interpolation argument as in [29] (see also [20] for the nontracial case), we deduce from (4.5) for $q = 4$ and (4.6) that (4.5) holds with

$$\delta_q \le 2\delta_4^{4/q}.$$

For more details, see [29] and [20]. □

COROLLARY 4.3. *Let $1 \le p \le 2$. Then $\mathcal{H}^p(\mathcal{M}) \subset L^p(\mathcal{M})$ and there is an absolute constant $\beta > 0$ such that*

$$\|x\|_p \le \beta \|x\|_{\mathcal{H}^p(\mathcal{M})}, \qquad x \in \mathcal{H}^p.$$

*Consequently, the constant $\beta_p$ in* ($\mathrm{BG}_p$) *remains bounded for $1 < p \le 2$.*

PROOF. This follows immediately from Theorem 4.1 and Proposition 4.2 for $p$ close to 1. On the other hand, for $p$ close to 2, this is nothing but ($\mathrm{BG}_p$). □

REMARK. In the case $p = 1$, Corollary 4.3 is the corollary in the Appendix of [34] (for the tracial case). We profit of this opportunity to point out a gap in the corollary in the appendix of [34]. It is stated that

$$\|x\|_1 \le \sqrt{2}\, \|dx\|_{L^1(\mathcal{M};\ell_c^2)+L^1(\mathcal{M};\ell_r^2)} \qquad \forall x \in \mathcal{H}^1(\mathcal{M}).$$

However, the proof there does not give this. In fact, we do not know whether this inequality holds (even with some constant instead of $\sqrt{2}$).

We close this section with a problem concerning the constant in the inequality reverse to (4.5). As already observed above, for $2 < q < \infty$ there is a constant $\lambda'_q > 0$ such that

$$\|a\|_q \le \lambda'_q \|a\|_{L^q\mathcal{MO}(\mathcal{M})}, \qquad a \in L^q\mathcal{MO}(\mathcal{M}). \tag{4.7}$$

In the commutative case the optimal order is $O(q)$. By duality $\alpha_{q'} \le Cq^2$ implies $\lambda'_q \le Cq^2$; see [22] for more details.

PROBLEM. *What is the optimal order of $\lambda'_q$ in* (4.7) *as $q \to \infty$?*

**5. Noncommutative Burkholder inequalities: *p* ≥ 2.** This section and the next one are devoted to the noncommutative analogue of the classical Burkholder inequalities on conditioned square functions. In this section, we focus on the range $2 \le p < \infty$ and finite martingales where this inequality is exactly the noncommutative analogue of the classical Burkholder inequality (cf. [4, 6, 12]). In the next section we will consider the case $1 < p < 2$ (and general martingales).

We begin by introducing the conditioned square functions. Let $2 \le p < \infty$ and $x = (x_n)_{n\ge 0}$ a finite $L^p$-martingale. Set

$$s_c(x) = \left(\sum_{n\ge 0} \mathcal{E}_{n-1}|dx_n|^2\right)^{1/2} \quad \text{and} \quad s_r(x) = \left(\sum_{n\ge 0} \mathcal{E}_{n-1}|dx_n^*|^2\right)^{1/2}.$$

Here and till the end of Section 6 we set $\mathcal{E}_{-1} = \mathcal{E}_0$. These are the conditioned square functions. For notational convenience, we also set

$$s_d(x) = \left(\sum_{n\ge 0} |dx_n|^p\right)^{1/p}.$$

The result of this section is the following noncommutative Burkholder inequality for $2 \le p < \infty$.

THEOREM 5.1. *Let $2 \le p < \infty$. Then for any finite $L^p$-martingale $x$ we have*

$$\delta_p^{-1} s_p(x) \le \|x\|_p \le \eta_p s_p(x), \tag{$\mathrm{B}_p$}$$

*where $s_p(x) = \max\{\|s_d(x)\|_p, \|s_c(x)\|_p, \|s_r(x)\|_p\}$.*

For the proof we will need the following lemma.

LEMMA 5.2. *Let $1 < q < \infty$, and let $a = (a_n)_{n\ge 0} \subset L^{4q}(\mathcal{M})$ be a finite sequence. Then*

$$\begin{aligned} &\left\|\sum_{n\ge 0} \mathcal{E}_{n-1}|a_n|^4\right\|_q \\ &\quad \le \left\|\sum_{n\ge 0} \mathcal{E}_{n-1}|a_n|^2\right\|_{2q}^{2(q-1)/(2q-1)} \left(\sum_{n\ge 0} \|a_n\|_{4q}^{4q}\right)^{1/(2q-1)}. \end{aligned} \tag{5.1}$$

PROOF. We will show that for $\theta = \frac{1}{2q-1}$ we have

$$\begin{aligned} &\left\|\sum_{n\ge 0} \mathcal{E}_{n-1}[c_n b_n a_n]\right\|_q \\ &\quad \le \left(\sum_n \|b_n\|_{2q}^{2q}\right)^{1/2q} \left(\left(\sum_n \|a_n\|_{4q}^{4q}\right)\left(\sum_n \|c_n\|_{4q}^{4q}\right)\right)^{\theta/4q} \\ &\qquad \times \left(\left\|\sum_{n\ge 0} \mathcal{E}_{n-1}|c_n^*|^2\right\|_{2q} \left\|\sum_{n\ge 0} \mathcal{E}_{n-1}|a_n|^2\right\|_{2q}\right)^{(1-\theta)/2}. \end{aligned} \tag{5.2}$$

Clearly, this provides the assertion by setting $c_n = a_n^*$ and $b_n = a_n a_n^*$. For the proof of (5.2), we can assume by approximation that $a_n = D^{1/4q}A_n$, $b_n = D^{1/4q}B_nD^{1/4q}$ and $c_n = C_nD^{1/4q}$ such that $A_n$ and $C_n$ are analytic elements. By homogeneity, we also suppose $\sum_n \|b_n\|_{2q}^{2q} \le 1$. We define $1/r = 1 - 1/2q$. Using Kosaki's results [24], we may assume that there is an analytic function $B$ defined on the strip $S = \{z \in \mathbb{C} : 0 \le \mathrm{Re} z \le 1\}$ with values $N$ such that $B_n = B_n(\theta)$ and

$$\sup_t \max\left\{ \sup_n \|B_n(it)\|_\infty, \left( \sum_n \|D^{1/2r}B_n(1+it)D^{1/2r}\|_r^r \right)^{1/r} \right\} \le 1.$$

Let us consider $y = D^{1/2q'}YD^{1/2q'}$ and assume again by Kosaki's result that there is an analytic function $Y$ with values in $\mathcal{M}$ such that $Y(\theta) = Y$ and

$$\sup_t \max\{\|D^{1/2r}Y(it)D^{1/2r}\|_r, \|Y(1+it)\|_\infty\} \le 1.$$

Then, we may consider the analytic function

$$F(z) = \sum_n \mathrm{tr}\left(\mathcal{E}_{n-1}\left[\sigma_{z/2ir}(C_n)D^{1/4q}B_n(z)D^{1/4q}\sigma_{-z/2ir}(A_n)\right]D^{1/2r}Y(z)D^{1/2r}\right).$$

We deduce from the conditioned Hölder inequality in [19] that

$$\begin{aligned}
|F(it)| &\le \left\| \sum_{n\ge 0} \mathcal{E}_{n-1}\left[\sigma_{t/2r}(C_n)D^{1/4q}B_n(it)D^{1/4q}\sigma_{-t/2r}(A_n)\right] \right\|_{2q} \\
&\qquad \times \|D^{1/2r}Y(it)D^{1/2r}\|_r \\
&\le \left\| \sum_{n\ge 0} \mathcal{E}_{n-1}\left[\sigma_{t/2r}(C_n)D^{1/4q}B_n(it)B_n(it)^*D^{1/4q}\sigma_{t/2r}(C_n^*)\right] \right\|_{2q} \\
&\qquad \times \left\| \sum_{n\ge 0} \mathcal{E}_{n-1}\left[\sigma_{-t/2r}(A_n^*)D^{1/4q}D^{1/4q}\sigma_{-t/2r}(A_n)\right] \right\|_{2q} \\
&\le \left\| \sigma_{t/2r} \sum_{n\ge 0} \mathcal{E}_{n-1}[c_nc_n^*] \right\|_{2q}^{1/2} \left\| \sigma_{-t/2r} \sum_{n\ge 0} \mathcal{E}_{n-1}[a_n^*a_n] \right\|_{2q}^{1/2}.
\end{aligned}$$

Since $\sigma_t$ is an isometry on $L_{2q}$, we obtained the correct estimate for $z = it$. For $z = 1 + it$, we first observe

$$\begin{aligned}
F(1+it) &= \sum_n \mathrm{tr}\Big(\mathcal{E}_{n-1}\big[\sigma_{-i/2r}\big(\sigma_{t/2r}(C_n)\big)D^{1/4q}B_n(1+it) \\
&\qquad\qquad \times D^{1/4q}\sigma_{i/2r}\big(\sigma_{-t/2r}(A_n)\big)\big]D^{1/2r}Y(1+it)D^{1/2r}\Big) \\
&= \sum_n \mathrm{tr}\Big(\mathcal{E}_{n-1}\big[\sigma_{t/2r}(C_n)D^{1/4q+1/2r}B_n(1+it) \\
&\qquad\qquad \times D^{1/2r+1/4q}\big(\sigma_{-t/2r}(A_n)\big)\big]Y(1+it)\Big).
\end{aligned}$$

Hence, by Hölder's inequality we deduce

$$
\begin{aligned}
|F(1+it)| &\le \sum_n \big\|\mathcal{E}_{n-1}\big[\sigma_{t/2r}(C_n)D^{1/4q+1/2r}B_n(1+it)D^{1/2r+1/4q}\sigma_{-t/2r}(A_n)\big]\big\|_1 \\
&\quad \times \|Y(1+it)\|_\infty \\
&\le \sum_n \big\|\sigma_{t/2r}(C_n)D^{1/4q+1/2r}B_n(1+it)D^{1/2r+1/4q}\sigma_{-t/2r}(A_n)\big\|_1 \\
&\le \left(\sum_n \|\sigma_{t/2r}(C_n)D^{1/4q}\|_{4q}^{4q}\right)^{1/4q} \left(\sum_n \|D^{1/2r}B(1+it)D^{1/2r}\|_r^r\right)^{1/r} \\
&\quad \times \left(\sum_n \|D^{1/4q}\sigma_{-t/2r}(A_n)\|_{4q}^{4q}\right)^{1/4q} \\
&\le \left(\sum_n \|c_n\|_{4q}^{4q}\right)^{1/4q} \left(\sum_n \|a_n\|_{4q}^{4q}\right)^{1/4q}.
\end{aligned}
$$

Here we used again the isometric property of $\sigma_t$. With the three line lemma,

$$
\begin{aligned}
&\left|\sum_{n\ge 0} \operatorname{tr}(\mathcal{E}_{n-1}[c_n b_n a_n]y)\right| \\
&\quad = |F(\theta)| \\
&\quad \le \left(\sup_t |F(it)|\right)^{1-\theta}\left(\sup_t |F(1+it)|\right)^{\theta}.
\end{aligned}
$$

Therefore, we deduce (5.2) and thus the assertion. □

PROOF OF THEOREM 5.1. By Lemma 1.1, we only need to consider finite martingales in $\mathcal{M}_a D^{1/p}$. First we prove the second inequality of $(\mathrm{B}_p)$. In fact, we will show the following apparently stronger one: Let $1 \le p < \infty$. Then for any finite martingale $x = (x_n)_{n\ge 0}$ in $\mathcal{M}_a D^{1/p}$,

$$
(\mathrm{B}'_p) \qquad \|x\|_p \le \eta_p \max\big\{\|s_d(x)\|_p, \|s_c(x)\|_p, \|s_r(x)\|_p\big\}.
$$

Here $s_d(x)$, $s_c(x)$ and $s_r(x)$ are defined in the same way as in the beginning of this section. Note that since $dx_n \in \mathcal{M}_a D^{1/p}$, $\mathcal{E}_{n-1}(|dx_n|^2)$ and $\mathcal{E}_{n-1}(|dx_n^*|^2)$ are well defined. Note that in the case $1 \le p \le 2$, dualizing (3.1) yields

$$
(5.3) \qquad \|x\|_p \le 2^{2/p-1}\|s_d(x)\|_p
$$

for any finite $L^p$-martingale $x$. Therefore, we get $(\mathrm{B}'_p)$ for $1 \le p \le 2$. To prove $(\mathrm{B}'_p)$ for $2 < p < \infty$ we will proceed to prove the implication "$(\mathrm{B}'_p) \to (\mathrm{B}''_{2p})$."

This will show $(\mathrm{B}'_p)$ for all $2 < p < \infty$ by a standard iteration argument, starting from the case $1 \le p \le 2$ established in (5.3).

Assume now $(\mathrm{B}'_p)$ for some $1 \le p < \infty$. Let $x = (x_n)_{n\ge 0}$ be a finite martingale in $\mathcal{M}_a D^{1/2p}$. By homogeneity, we assume $s_{2p}(x) \le 1$. By Theorem 3.1,

$$\|x\|_{2p} \le \beta_{2p} \max\{\|S_c(x)\|_{2p}, \|S_r(x)\|_{2p}\}. \tag{5.4}$$

We first consider $\|S_c(x)\|_{2p}$. We have

$$\|S_c(x)\|_{2p}^2 = \|[S_c(x)]^2\|_p = \Big\|\sum_{n\ge 0} |dx_n|^2\Big\|_p.$$

Now, write

$$|dx_n|^2 = \mathcal{E}_{n-1}|dx_n|^2 + dy_n, \qquad n \ge 0,$$

where $dy_n = |dx_n|^2 - \mathcal{E}_{n-1}|dx_n|^2,\ n \ge 0$. Then

$$\begin{aligned} \|S_c(x)\|_{2p}^2 &\le \Big\|\sum_{n\ge 0} \mathcal{E}_{n-1}|dx_n|^2\Big\|_p + \|y\|_p \\ &= \|s_c(x)\|_{2p}^2 + \|y\|_p \le 1 + \|y\|_p. \end{aligned} \tag{5.5}$$

Note that $y$ is an $L^p$-martingale. If $1 \le p \le 2$, applying (5.3) to $y$ yields

$$\|y\|_p \le 2^{2/p-1}\|s_d(y)\|_p = 2^{2/p-1} \cdot 2\|s_d(x)\|_{2p}^2 \le 2^{2/p}.$$

Therefore, in this case,

$$\|S_c(x)\|_{2p}^2 \le 1 + 2^{2/p}. \tag{5.6}$$

Then suppose $2 < p < \infty$. This time, we apply $(\mathrm{B}'_p)$ to $y$ to infer

$$\|y\|_p \le \eta_p \max\{\|s_d(y)\|_p, \|s_c(y)\|_p\}. \tag{5.7}$$

Again,

$$\|s_d(y)\|_p \le 2\|s_d(x)\|_{2p}^2 \le 2. \tag{5.8}$$

To majorize $\|s_c(y)\|_p$, we observe that, by Proposition 2.3(ii),

$$\mathcal{E}_{n-1}|dy_n|^2 = \mathcal{E}_{n-1}|dx_n|^4 - (\mathcal{E}_{n-1}|dx_n|^2)^2 \le \mathcal{E}_{n-1}|dx_n|^4.$$

Thus

$$\|s_c(y)\|_p \le \Big\|\Big(\sum_{n\ge 0} \mathcal{E}_{n-1}|dx_n|^4\Big)^{1/2}\Big\|_p.$$

By Lemma 5.1 (applied to $q = p/2$), we have

$$\|s_c(y)\|_p \le \|s_d(x)\|_{2p}^{2p/(p-1)}\|s_c(x)\|_{2p}^{2(p-2)/(p-1)} \le 1.$$

Combining this with (5.5) and (5.7)–(5.8), we get, in the case $2 < p < \infty$,

$$\|S_c(x)\|_{2p}^2 \le 1 + 2\eta_p. \tag{5.9}$$

Therefore, (5.6) and (5.9) together imply that, for any $1 \le p < \infty$,

$$\|S_c(x)\|_{2p}^2 \le 1 + \max(2^{2/p}, 2\eta_p).$$

Passing to adjoints, we see that the same inequality holds for $S_r(x)$ instead of $S_c(x)$. Hence, by (5.4) we finally obtain

$$\|x\|_{2p} \le \beta_{2p}\big[1 + \max(2^{2/p}, 2\eta_p)\big]^{1/2}.$$

This yields $(\mathrm{B}'_{2p})$ with $\eta_{2p} = \beta_{2p}[1 + \max(2^{2/p}, 2\eta_p)]^{1/2}$. Thus the second inequality of $(\mathrm{B}_p)$ is proved.

Next we pass to the first inequality of $(\mathrm{B}_p)$. In fact, this inequality immediately follows from $(\mathrm{BG}_p)$ and the dual form of the noncommutative Doob inequality in [19]. However, we prefer to give a direct proof in the same spirit as the previous one. Let $x = (x_n)_{n\ge 0}$ be a finite martingale in $\mathcal{M}_a D^{1/p}$ such that $\|x\|_p \le 1$. By (3.1),

$$\|s_d(x)\|_p \le 2^{1-2/p}\|x\|_p \le 2^{1-2/p}.$$

Thus, it remains to majorize $\|s_c(x)\|_p$ and $\|s_r(x)\|_p$. Clearly, it suffices to do this for the former. We have

$$\|s_c(x)\|_p^2 \le \big\|[S_c(x)]^2\big\|_{p/2} + \|y\|_{p/2} = \|S_c(x)\|_p^2 + \|y\|_{p/2}, \tag{5.10}$$

where $y$ is the finite $L^{p/2}$-martingale defined by

$$dy_n = |dx_n|^2 - \mathcal{E}_{n-1}|dx_n|^2, \qquad n \ge 0.$$

By Theorem 3.1,

$$\|S_c(x)\|_p \le \alpha_p\|x\|_p \le \alpha_p. \tag{5.11}$$

If $2 \le p \le 4$, by (5.3) and (3.1),

$$\|y\|_{p/2} \le 2^{4/p-1}\|s_d(y)\|_{p/2} \le 2^{4/p}\|s_d(x)\|_p^2 \le 4.$$

Thus, by (5.10) and (5.11), for $2 \le p \le 4$, we obtain

$$\|S_c(x)\|_p^2 \le \alpha_p^2 + 4. \tag{5.12}$$

Now assume $4 < p < \infty$. Then, applying the second inequality of $(\mathrm{B}_p)$ (already proved above) to $y$, we get

$$\|y\|_{p/2} \le \eta_{p/2}\max\big\{\|s_d(y)\|_{p/2}, \|s_c(y)\|_{p/2}\big\}. \tag{5.13}$$

Clearly, by (3.1),

$$\|s_d(y)\|_{p/2} \le 2\|s_d(x)\|_p^2 \le 2^{3-4/p}. \tag{5.14}$$

Concerning $\|s_c(y)\|_{p/2}$, we deduce from Lemma 5.1 (applied to $q = p/4$) and (3.1),

$$\begin{aligned}\|s_c(y)\|_{p/2} &\leq \left\| \left( \sum_{n\geq 0} \mathcal{E}_{n-1} |dx_n|^4 \right)^{1/2} \right\|_{p/2} \\ &\leq \|s_d(x)\|_p^{p/(p-2)} \|s_c(x)\|_p^{(p-4)/(p-2)} \\ &\leq \|s_c(x)\|_p^{(p-4)/(p-2)}.\end{aligned}$$

Combining this with (5.10)–(5.11) and (5.13)–(5.14), we obtain, for $4 < p < \infty$,

$$\|s_c(x)\|_p^2 \leq \alpha_p^2 + \eta_{p/2} \max\left(2^{3-4/p}, \|s_c(x)\|_p^{(p-4)/(p-2)}\right).$$

Noting that $\frac{p-4}{p-2} < 1$, we then deduce

$$\|x\|_{h_c^p(\mathcal{M})} \leq \delta'_p,$$

where $\delta'_p$ is a constant depending only on $\alpha_p$ and $\eta_{p/2}$. This, together with (5.12), yields that for all $2 \leq p < \infty$,

$$\|s_c(x)\|_p \leq \delta_p.$$

Therefore, we have proved the first inequality of ($\mathrm{B}_p$), and thus Theorem 5.1. □

REMARK. Using the estimate $\beta_p \leq Cp$ for $p \geq 2$, we deduce from $\eta_{2p} \leq \beta_{2p}\sqrt{1+2\eta_p}$ that $\eta_p \leq C'p^2$ for some constant $C' = \frac{3}{4}C^2$. We do not know whether this order is optimal. For the lower estimate, we can use the dual version of Doob's inequality with the constant $Cp^2$ (see [22]) and deduce for $p \geq 2$ that

$$\frac{1}{C'p^2}\|x\|_p \leq s_p(x) \leq C'p^2\|x\|_p$$

for some universal constant $C'$. The estimate can be improved for sums of independent random variables of mean 0 (see [21] for details), then the Khintchine inequality applies and hence

$$\left\| \sum_k x_k \right\|_p \leq 2\mathbb{E} \left\| \sum_k r_k x_k \right\|_p \leq C\sqrt{p} \max\{\|S_c(x)\|, \|S_r(x)\|\}.$$

Using this improved estimate and denoting by $\eta_p^{\mathrm{Ros}}$ the noncommutative Rosenthal constant for the upper estimate, we obtain $\eta_{2p}^{\mathrm{Ros}} \leq C\sqrt{2p}\,[1+2\eta_p^{\mathrm{Ros}}]^{1/2}$ and hence

$$\frac{1}{C'p} s_p(x) \leq \left\| \sum_k x_k \right\|_p \leq C'p s_p(x)$$

for some universal constant $C' > 0$ and all $p \geq 2$. In view of the optimal order $p/\log(p)$ in the commutative case, this estimate seems not too bad.

**6. Noncommutative Burkholder inequalities: $1 < p < 2$.** We continue our study of the noncommutative Burkholder inequalities. The aim of this section is to extend ($\mathrm{B}_p$) in the last section to the case $p < 2$, and to all martingales (not only the finite ones). To this end we need to introduce some notation.

Let $1 \le p < \infty$. Let $a = (a_n)_{n\ge 0}$ be a finite sequence in $\mathcal{M}_a D^{1/p}$. We define (recalling $\mathcal{E}_{-1} = \mathcal{E}_0$)

$$\|a\|_{L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_c)} = \Big\| \Big( \sum_{n\ge 0} \mathcal{E}_{n-1}(a_n^* a_n) \Big)^{1/2} \Big\|_p .$$

It is shown in [19] that $\|\cdot\|_{L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_c)}$ is a norm on the vector space of all finite sequences in $\mathcal{M}_a D^{1/p}$. Then let $L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_c)$ be the corresponding completion. Note that $L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_c)$ is the conditioned version of $L^p(\mathcal{M};\ell^2_c)$. Similarly, we define the conditioned row space $L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_r)$. As the column and row spaces, $L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_c)$ and $L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_r)$ can be realized as spaces of matrices with operator entries. In fact, $L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_c)$ [resp. $L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_r)$] can be viewed as a closed subspace of the column (resp. row) subspace of $L^p(\mathcal{M} \otimes B(\ell^2(\mathbb{N}^2)))$. We refer to [19] for more details on this.

Let $x = (x_n)_{n\ge 0}$ be a finite martingale in $\mathcal{M}_a D^{1/p}$. We recall

$$s_c(x) = \Big( \sum_{n\ge 0} \mathcal{E}_{n-1}(|dx_n|^2) \Big)^{1/2} \quad \text{and} \quad s_r(x) = \Big( \sum_{n\ge 0} \mathcal{E}_{n-1}(|dx_n^*|^2) \Big)^{1/2} .$$

Then

$$\|s_c(x)\|_p = \|dx\|_{L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_c)} \quad \text{and} \quad \|s_r(x)\|_p = \|dx\|_{L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_r)} .$$

Let $h^p_c(\mathcal{M})$ [resp. $h^p_r(\mathcal{M})$] denote the closure in $L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_c)$ [resp. $L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_r)$] of all finite martingales in $\mathcal{M}_a D^{1/p}$. (As usual, we have identified a martingale with its difference sequence.) Then $h^p_c(\mathcal{M})$ and $h^p_r(\mathcal{M})$ are Banach spaces. We also need $\ell^p(L^p(\mathcal{M}))$, the space of all sequences $a = (a_n)_{n\ge 0}$ in $L^p(\mathcal{M})$ such that

$$\|a\|_{\ell^p(L^p(\mathcal{M}))} = \Big( \sum_{n\ge 0} \|a_n\|_p^p \Big)^{1/p} < \infty.$$

Recall

$$s_d(x) = \Big( \sum_{n\ge 0} |dx_n|^p \Big)^{1/p} .$$

Thus

$$\|s_d(x)\|_p = \|dx\|_{\ell^p(L^p(\mathcal{M}))} .$$

Let $h_d^p(\mathcal{M})$ be the subspace of $\ell^p(L^p(\mathcal{M}))$ consisting of martingale differences.

In virtue of the realization of $L^p_{\rm cond}(\mathcal{M};\ell^2_c)$ and $L^p_{\rm cond}(\mathcal{M};\ell^2_r)$ as subspaces of $L^p(\mathcal{M}\otimes B(\ell^2(\mathbb{N}^2)))$, mentioned above, all the previous spaces $h_c^p(\mathcal{M})$, $h_r^p(\mathcal{M})$ and $h_d^p(\mathcal{M})$ are compatible in the sense that they are all embedable into a large Hausdorff topological vector space. Now we define the conditioned version $h^p(\mathcal{M})$ of the Hardy spaces $\mathcal{H}^p(\mathcal{M})$ introduced in Section 3. For $2\le p<\infty$, we define

$$h^p(\mathcal{M}) = h_d^p(\mathcal{M})\cap h_c^p(\mathcal{M})\cap h_r^p(\mathcal{M})$$

equipped with

$$\|x\|_{h^p(\mathcal{M})} = \max\big\{\|x\|_{h_d^p(\mathcal{M})}, \|x\|_{h_c^p(\mathcal{M})}, \|x\|_{h_r^p(\mathcal{M})}\big\}.$$

Clearly, all finite martingales in $\mathcal{M}_a D^{1/p}$ are in $h_d^p(\mathcal{M})$ for $k\in\{d,c,r\}$, and thus $\|\cdot\|_{h^p(\mathcal{M})}$ defines a norm on the vector space of all such martingales. We will see in Lemma 6.3 below that $h^p(\mathcal{M})$ is exactly the corresponding completion.

On the other hand, for $1\le p<2$, we define

$$h^p(\mathcal{M}) = h_d^p(\mathcal{M})+h_c^p(\mathcal{M})+h_r^p(\mathcal{M})$$

equipped with

$$\|x\|_{h^p}(\mathcal{M}) = \inf\big\{\|x^d\|_{h_d^p(\mathcal{M})} + \|x^c\|_{h_c^p(\mathcal{M})} + \|x^r\|_{h_r^p(\mathcal{M})}\big\},$$

where the infimum runs over all triples $(x^d,x^c,x^r)$ such that $dx_n = dx_n^d + dx_n^c + dx_n^r$ holds for all $n\in\mathbb{N}$. Then $h^p(\mathcal{M})$ is a Banach space.

REMARK. For $2\le p<\infty$ all elements in $h^p(\mathcal{M})$ are $L^p$-martingales, since $h^p(\mathcal{M})\subset h_d^p(\mathcal{M})$.

Now we are ready to state Theorem 5.1 in the general case.

THEOREM 6.1. *Let $1<p<\infty$. Then an $L^p$-martingale $x$ is bounded in $L^p(\mathcal{M})$ iff $x$ belongs to $h^p(\mathcal{M})$; moreover, if this is the case, we have*

$$(\mathrm{B}_p)\qquad \delta_p^{-1}\|x\|_{h^p(\mathcal{M})}\le\|x\|_p\le\eta_p\|x\|_{h^p(\mathcal{M})}.$$

We can reformulate Theorem 6.1 simply as follows.

COROLLARY 6.2. *For any $1<p<\infty$ we have $L^p(\mathcal{M}) = h^p(\mathcal{M})$ with equivalent norms.*

In the case $p\ge 2$, Theorem 6.1 is just Theorem 5.1, modulo a density result (i.e., Lemma 6.3 below). We will reduce the case $p<2$ to the case $p>2$ by duality.

LEMMA 6.3. *Let $1\le p<\infty$. Then all finite martingales in $\mathcal{M}_a D^{1/p}$ are dense in $h^p(\mathcal{M})$.*

PROOF. This is evident if $1 \le p < 2$, because all finite martingales in $\mathcal{M}_a D^{1/p}$ are dense in each of the three spaces $h_d^p(\mathcal{M})$, $h_c^p(\mathcal{M})$ and $h_r^p(\mathcal{M})$. Now assume $2 \le p < \infty$. For any finite sequence $a = (a_n)_{n\ge 0}$ in $\mathcal{M}_a D^{1/p}$ define

$$P_n(a) = (a_0, a_1, \dots, a_n, 0, \dots).$$

Then clearly,

$$\|P_n(a)\|_{L^p_{\rm cond}(\mathcal{M};\ell^2_c)} \le \|a\|_{L^p_{\rm cond}(\mathcal{M};\ell^2_c)}.$$

Thus, $P_n$ extends to a contractive projection on $L^p_{\rm cond}(\mathcal{M};\ell^2_c)$. Since finite sequences in $\mathcal{M}_a D^{1/p}$ are dense in $L^p_{\rm cond}(\mathcal{M};\ell^2_c)$, we deduce that

$$\lim_{n\to\infty} P_n(a) = a \qquad \forall\, a \in L^p_{\rm cond}(\mathcal{M};\ell^2_c).$$

A similar statement holds for $L^p_{\rm cond}(\mathcal{M};\ell^2_r)$ and $\ell^p(L^p(\mathcal{M}))$ as well. Therefore, for any $x \in h^p(\mathcal{M})$,

$$\lim_{n\to\infty} P_n(dx) = dx \qquad \text{in } \ell^p\big(L^p(\mathcal{M})\big) \cap L^p_{\rm cond}(\mathcal{M};\ell^2_c) \cap L^p_{\rm cond}(\mathcal{M};\ell^2_r).$$

Thus, we may assume that $x$ is a finite martingale, say, $x_k = x_n$ for all $k \ge n$. Next observe that if $a = (a_0, \dots, a_n, 0, \dots)$ is a finite sequence in $\mathcal{M}_a D^{1/p}$,

$$\begin{aligned}\|a\|_{L^p_{\rm cond}(\mathcal{M};\ell^2_c)} &= \Big\| \sum_{k=0}^n \mathcal{E}_{k-1}|a_k|^2 \Big\|_{p/2}^{1/2} \\ &\le \Big( \sum_{k=0}^n \|a_k\|_p^2 \Big)^{1/2} \\ &\le (n+1)^{1/2-1/p} \|a\|_{\ell^p(L^p(\mathcal{M}))}.\end{aligned}$$

Similarly,

$$\|a\|_{L^p_{\rm cond}(\mathcal{M};\ell^2_r)} \le (n+1)^{1/2-1/p} \|a\|_{\ell^p(L^p(\mathcal{M}))}.$$

It follows that

$$P_n\big[\ell^p\big(L^p(\mathcal{M})\big) \cap L^p_{\rm cond}(\mathcal{M};\ell^2_c) \cap L^p_{\rm cond}(\mathcal{M};\ell^2_r)\big] = \ell^p_{n+1}\big(L^p(\mathcal{M})\big).$$

Then we easily see that the given finite martingale $x$ above can be approximated by finite martingales in $\mathcal{M}_a D^{1/p}$, in $\ell^p_{n+1}(L^p(\mathcal{M}))$, and hence in $h^p(\mathcal{M})$ as well. □

LEMMA 6.4. *Let $1 \le p < \infty$. For any finite sequence $a = (a_n)_{n\ge 0}$ in $\mathcal{M}_a D^{1/p}$ define*

$$R(a) = (\mathcal{E}_n a_n)_{n\ge 0} \quad \textit{and} \quad R'(a) = (\mathcal{E}_{n-1} a_n)_{n\ge 0}.$$

*Then $R$ and $R'$ extend to contractive projections on $L^p_{\rm cond}(\mathcal{M};\ell^2_c)$ and $L^p_{\rm cond}(\mathcal{M};\ell^2_r)$. Consequently, $h_c^p(\mathcal{M})$ and $h_r^p(\mathcal{M})$ are respectively one-complemented in $L^p_{\rm cond}(\mathcal{M};\ell^2_c)$ and $L^p_{\rm cond}(\mathcal{M};\ell^2_r)$.*

PROOF. Let $a = (a_n)_{n\geq 0}$ be a finite sequence in $\mathcal{M}_a D^{1/p}$. Then by Proposition 2.3(iii),

$$|\mathcal{E}_n a_n|^2 = (\mathcal{E}_n(a_n))^* \mathcal{E}_n(a_n) \leq \mathcal{E}_n |a_n|^2$$

whence

$$\mathcal{E}_{n-1}|\mathcal{E}_n a_n|^2 \leq \mathcal{E}_{n-1}|a_n|^2.$$

Therefore,

$$\|R(a)\|_{L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_c)} \leq \|a\|_{L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_c)}.$$

Thus, $R$ extends to a contractive projection on $L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_c)$. The same argument applies to $R'$. $\square$

We will also need the conditioned version of the duality for the column spaces contained in Corollary 1.3. This is the following lemma, which is taken from [19] and plays an essential role therein.

LEMMA 6.5. *Let $1 < p < \infty$. Then for any $b \in L^{p'}_{\mathrm{cond}}(\mathcal{M};\ell^2_c)$ the functional*

$$\xi_b : L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_c) \to \mathbb{C},$$
$$a \mapsto \sum_{n\geq 0} \mathrm{tr}(b_n^* a_n)$$

*is continuous and*

$$\|\xi_b\| \leq \|b\|_{L^{p'}_{\mathrm{cond}}(\mathcal{M};\ell^2_c)} \leq \gamma_{p'}\|\xi_b\|,$$

*where $\gamma_{p'}$ is the constant in* $(\mathrm{S}_{p'})$. *Conversely, any $\xi \in (L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_c))^*$ is given by some $b \in L^{p'}_{\mathrm{cond}}(\mathcal{M};\ell^2_c)$ in this way. Consequently,*

$$\big(L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_c)\big)^* = L^{p'}_{\mathrm{cond}}(\mathcal{M};\ell^2_c) \qquad \textit{with equivalent norms.}$$

*A similar statement holds for the conditioned row spaces, too.*

Now the proof of Theorem 6.1 is easy.

PROOF OF THEOREM 6.1. By Lemma 6.3, it suffices to prove $(\mathrm{B}_p)$ for finite martingales in $\mathcal{M}_a D^{1/p}$. Thus for $p \geq 2$ Theorem 6.1 reduces to Theorem 5.1. In the sequel, we suppose $1 < p < 2$. Let us show the first inequality of $(\mathrm{B}_p)$. Let $x = (x_n)_{n\geq 0}$ be a finite martingale in $\mathcal{M}_a D^{1/p}$ with $\|x\|_p = 1$ and $x_n = x_N$ for all $n \geq N$. Then $x$ defines a linear functional on $L^{p'}(\mathcal{M})$ of norm 1, denoted by $\omega$. By the second inequality of $(\mathrm{B}_{p'})$, $\omega$ can be also considered as a linear functional on $h^{p'}(\mathcal{M})$, and then $\omega$ is of norm $\leq \eta_{p'}$. By our definition of $h^{p'}(\mathcal{M})$ (noting that $2 < p' < \infty$), $h^{p'}(\mathcal{M})$ is the intersection of the three spaces $h_d^{p'}(\mathcal{M})$, $h_c^{p'}(\mathcal{M})$ and $h_r^{p'}(\mathcal{M})$. Recall that $h_d^{p'}(\mathcal{M})$ [resp. $h_c^{p'}(\mathcal{M})$, $h_r^{p'}(\mathcal{M})$] is a closed subspace of $\ell^{p'}(L^{p'}(\mathcal{M}))$ [resp. $L^{p'}_{\text{cond}}(\mathcal{M}; \ell^2_c)$, $L^{p'}_{\text{cond}}(\mathcal{M}; \ell^2_r)$]. Hence, by the Hahn–Banach theorem, $\omega$ extends to a continuous linear functional $\widetilde{\omega}$ of norm $\le \eta_{p'}$ on

$$\ell^{p'}\big(L^{p'}(\mathcal{M})\big) \bigoplus_{\infty} L^{p'}_{\text{cond}}(\mathcal{M}; \ell^2_c) \bigoplus_{\infty} L^{p'}_{\text{cond}}(\mathcal{M}; \ell^2_r).$$

In particular, $\widetilde{\omega}$ has three components $\widetilde{\omega} = (\omega^d, \omega^c, \omega^r)$ . It is evident that

$$\big[\ell^{p'}\big(L^{p'}(\mathcal{M})\big)\big]^* = \ell^p\big(L^p(\mathcal{M})\big) \qquad \text{isometrically.}$$

Here we use the anti-duality given by

$$\langle a, b\rangle = \operatorname{tr}\Bigg[\sum_{n\ge 0} b_n^* a_n\Bigg], \qquad a = (a_n) \in \ell^p\big(L^p(\mathcal{M})\big),\ b = (b_n) \in \ell^{p'}\big(L^{p'}(\mathcal{M})\big).$$

On the other hand, by Lemma 6.5, we have isomorphically (with the same duality as the previous one)

$$\big[L^{p'}_{\text{cond}}(\mathcal{M}; \ell^2_c)\big]^* = L^p_{\text{cond}}(\mathcal{M}; \ell^2_c), \qquad \big[L^{p'}_{\text{cond}}(\mathcal{M}; \ell^2_r)\big]^* = L^p_{\text{cond}}(\mathcal{M}; \ell^2_r).$$

Thus we deduce

$$\omega^d \in \ell^p\big(L^p(\mathcal{M})\big), \qquad \omega^c \in L^p_{\text{cond}}(\mathcal{M}; \ell^2_c), \qquad \omega^r \in L^p_{\text{cond}}(\mathcal{M}; \ell^2_r)$$

satisfying

$$\|\omega^d\| + \|\omega^c\| + \|\omega^r\| \le \eta_{p'}\gamma_p,$$

where $\gamma_p$ is the constant in $(\mathrm{S}_p)$ (see Lemma 6.5). Now let $y = (y_n)_{n\ge 0}$ be a finite martingale in $\mathcal{M}_a D^{1/p'}$. Then

$$\begin{aligned}\operatorname{tr}(x_N^* y_N) &= \operatorname{tr}\Bigg[\sum_{n\ge 0} (dx_n)^* dy_n\Bigg]\\ &= \omega(dy) = \langle dy, \omega^d\rangle + \langle dy, \omega^c\rangle + \langle dy, \omega^r\rangle.\end{aligned}$$

Using the projections $R$ and $R'$ in Lemma 6.4, we have

$$\langle dy, \omega^k\rangle = \langle (R - R')dy, \omega^k\rangle = \langle dy, (R - R')\omega^k\rangle, \qquad k \in \{d, c, r\}.$$

Set

$$dx^k = (R - R')\omega^k, \qquad k \in \{d, c, r\}.$$

Then

$$x^k \in h^p_k(\mathcal{M}) \quad \text{and} \quad \|x^k\| \le 2\|\omega^k\|, \qquad k \in \{d, c, r\}.$$

Thus

$$\mathrm{tr}(x_N^* y_N) = \langle dy, dx^d + dx^c + dx^r \rangle$$

for all finite martingales $y = (y_n)$ in $\mathcal{M}_a D^{1/p'}$. It then follows that $x = x^d + x^c + x^r$, and so $x \in h^p(\mathcal{M})$ and $\|x\|_{h^p(\mathcal{M})} \le 2\eta_{p'}\gamma_p$. This shows the first inequality of $(\mathrm{B}_p)$ (for $1 < p < 2$).

The second inequality of $(\mathrm{B}_p)$ for $1 < p < 2$ follows from $(\mathrm{BG}_p)$ in Section 3 and Theorem 7.1 below. However, we can give a proof similar to the previous one. Again, we use duality. Let $x = (x_n)_{n\ge 0}$ be a finite martingale in $\mathcal{M}_a D^{1/p}$, say, $x_n = x_N$ for all $n \ge N$. Let

$$x = x^d + x^c + x^r \qquad \text{with } x^k \in h_k^p(\mathcal{M}),\ k = d, c, r$$

be a decomposition of $x$. Let $y_N \in \mathcal{M}_N D^{1/p'}$ with $\|y_N\|_{p'} \le 1$. Then $y_N$ defines a finite martingale $y = (dy_n)_{n\ge 0}$ with $y_n = \mathcal{E}_n(y_N)$, $n \ge 0$. We have

$$\begin{aligned}\mathrm{tr}(y_N^* x_N) &= \mathrm{tr}\sum_{n\ge 0}(dy_n)^* dx_n = \langle dx, dy\rangle \\ &= \langle dx^d, dy\rangle + \langle dx^c, dy\rangle + \langle dx^r, dy\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}|\,\mathrm{tr}(y_N^* x_N)| &\le (\|x^d\| + \|x^c\| + \|x^r\|)\|y\|_{h^{p'}(\mathcal{M})} \\ &\le (\|x^d\| + \|x^c\| + \|x^r\|)\eta_{p'}\|y\|_{p'} \qquad [\text{by } (\mathrm{B}_{p'})] \\ &\le \eta_{p'}(\|x^d\| + \|x^c\| + \|x^r\|),\end{aligned}$$

whence

$$\|x\|_p \le \eta_{p'}\|x\|_{h^p(\mathcal{M})}.$$

Hence we have proved the first inequality of $(\mathrm{B}_p)$ for $1 < p < 2$, and so completed the proof of Theorem 6.1. □

REMARK. Using Corollary 4.3 and Theorem 7.1, we observe that the constant $\eta_p$ in $(\mathrm{B}_p)$ remains bounded for $1 < p \le 2$, and that the second inequality of $(\mathrm{B}_p)$ holds for $p = 1$ as well. However, we do not know whether this can be extended to $p < 1$.

**7. A norm inequality on conditional expectations.** One of the main results in [19] is the following dual form of the noncommutative Doob inequality. Let $1 \le p < \infty$. Then for every finite sequence $(a_n)$ of positive elements in $L^p(\mathcal{M})$,

$$\left\|\sum_{n\ge 0}\mathcal{E}_n a_n\right\|_p \le \lambda_p\left\|\sum_{n\ge 0} a_n\right\|_p, \tag{7.1}$$

where $\lambda_p$ is a positive constant depending on $p$. In the case $p < 1$ we have the following reverse inequality.

THEOREM 7.1. *Let $0 < p < 1$. Then for all finite sequences of positive elements in $\mathcal{M}_a D^{1/p}$,*

$$\Big\| \sum_{n\geq 0} a_n \Big\|_p \leq 2^{1/p} \Big\| \sum_{n\geq 0} \varepsilon_n a_n \Big\|_p . \tag{7.2}$$

The proof will be based on the following lemmas. The first one is a variant of [19], Lemma 4.1.

LEMMA 7.2. *Let $s, t$ be two real numbers such that $s < t$ and $0 \leq s \leq 1 \leq t \leq 2$. Let $x, y$ be two positive operators such that $x^{t-s}, y^{t-s} \in L^1(\mathcal{M})$ and such that the support of $y$ is equal to 1. Then*

$$\operatorname{tr}\big[y^{-s/2}(y^t - x^t)y^{-s/2}\big] \leq 2 \operatorname{tr}\big[y^{-(s+1-t)/2}(y - x)y^{-(s+1-t)/2}\big]. \tag{7.3}$$

PROOF. Let us first justify both operators in the brackets in (7.3) are in $L^1(\mathcal{M})$. For this we recall that $y \in L^0(\mathcal{M} \rtimes_\sigma \mathbb{R})$ (see Section 1). In particular, $y$ is a positive closed densely defined operator on $L^2(\mathbb{R}, H)$. In this case, the support of $y$ is equal to 1 iff $y$ is invertible with $y^{-1}$ being a closed densely defined operator on $L^2(\mathbb{R}, H)$, too. Moreover, $y^{-1}$ is also affiliated with $\mathcal{M} \rtimes_\sigma \mathbb{R}$. Now since $x^s \leq y^s$, there is a contraction $u \in \mathcal{M} \rtimes_\sigma \mathbb{R}$ such that $x^{s/2} = uy^{s/2}$. Since $x^{t-s}$ and $y^{t-s}$ are both in $L^1(\mathcal{M})$, $u$ is in fact in $\mathcal{M}$. By the invertibility of $y$, $u = x^{s/2}y^{-s/2}$. Consequently,

$$y^{-s/2}x^t y^{-s/2} = u^* x^{t-s} u \in L^1(\mathcal{M}).$$

This shows that the operator in the brackets on the left-hand side of (7.3) belongs to $L^1(\mathcal{M})$. Similar arguments apply to the right-hand side of (7.3).

Using $x^{2-t} \leq y^{2-t}$, we see that

$$v = x^{(2-t)/2} y^{-(2-t)/2} \in L^\infty(\mathcal{M}) \quad \text{and} \quad \|v\|_\infty \leq 1.$$

Now set

$$a = x^{(t-s)/2} u \quad \text{and} \quad b = v y^{(t-s)/2}.$$

Then $a, b \in L^2(\mathcal{M})$. Using the traciality of tr, we have

$$\begin{aligned}\operatorname{tr}(a^* b) &= \operatorname{tr}\big[a^*(v y^{(1-s)/2}) y^{(t-1)/2}\big] \\ &= \operatorname{tr}\big[y^{(t-1)/2} a^*(v y^{(1-s)/2})\big] \\ &= \operatorname{tr}\big[y^{-(s+1-t)/2} x y^{-(s+1-t)/2}\big].\end{aligned}$$

Consequently, $\operatorname{tr}(a^*b) \geq 0$, so $\operatorname{tr}(a^*b) = \operatorname{tr}(b^*a)$. On the other hand,

$$a^*a = y^{-s/2} x^t y^{-s/2} \quad \text{and} \quad b^*b = y^{(t-s)/2} v^* v y^{(t-s)/2} \leq y^{t-s}.$$

Combining the preceding inequalities we deduce

$$\begin{aligned} 2\operatorname{tr}\bigl[y^{-(s+1-t)/2}xy^{-(s+1-t)/2}\bigr] &= \operatorname{tr}(a^*b)+\operatorname{tr}(b^*a) \\ &\le \operatorname{tr}(a^*a)+\operatorname{tr}(b^*b) \\ &\le \operatorname{tr}(y^{-s/2}x^t y^{-s/2})+\operatorname{tr}(y^{t-s}) \end{aligned}$$

whence (7.3). □

LEMMA 7.3. *Let $1 \le q < \infty$. Let $0 \le a \le b$ such that the support of $b$ is equal to* 1 *and such that $a^{1/q}, b^{1/q} \in L^1(\mathcal{M})$. Then*

$$\operatorname{tr}\bigl[b^{-(q-1)/2q}(b-a)b^{-(q-1)/2q}\bigr] \le 2^q \operatorname{tr}[b^{1/q}-a^{1/q}]. \tag{7.4}$$

PROOF. Let $n$ be a positive integer such that $n \le q \le n+1$. Then $a^{n/q} \le b^{n/q}$. Let

$$s=\frac{q-1}{n},\qquad t=\frac{q}{n},\qquad x=a^{n/q},\qquad y=b^{n/q}.$$

Then $s, t, x, y$ satisfy the conditions in Lemma 7.2. Thus we get

$$\begin{aligned} \operatorname{tr}\bigl[b^{-(q-1)/2q}(b-a)b^{-(q-1)/2q}\bigr] &= \operatorname{tr}\bigl[y^{-s/2}(y^t-x^t)y^{-s/2}\bigr] \\ &\le 2\operatorname{tr}\bigl[y^{-(s+1-t)/2}(y-x)y^{-(s+1-t)/2}\bigr] \\ &= 2\operatorname{tr}\bigl[b^{-(n-1)/2q}(b^{n/q}-a^{n/q})b^{-(n-1)/2q}\bigr]. \end{aligned}$$

Applying again Lemma 7.2 $(n-1)$ times, we deduce

$$\operatorname{tr}\bigl[b^{-(q-1)/2q}(b-a)b^{-(q-1)/2q}\bigr] \le 2^n \operatorname{tr}[b^{1/q}-a^{1/q}]$$

whence (7.4). □

LEMMA 7.4. *Let $0 < p \le 1$, and let $a, b$ be two positive elements in $L^p(\mathcal{M})$. Suppose that the support of $b$ is equal to* 1 *and that $a \le Mb$ for some positive real number $M$. Then*

$$\operatorname{tr}(a^p) \le [\operatorname{tr}(b^p)]^{1-p}\bigl[\operatorname{tr}(b^{-(1-p)/2}ab^{-(1-p)/2})\bigr]^p. \tag{7.5}$$

PROOF. By approximation (with $a+\varepsilon b$), we may and will assume that also $a$ has full support. Then

$$a^{pit}xb^{-pit} \in \mathcal{M} \quad\text{and}\quad b^{-pit}xa^{ipt} \in \mathcal{M}$$

because they are $\tau$-measurable and $\theta_s$ invariant. Let $z = s+it$ be a complex number such that $0 \le s \le 1$. Note that

$$b^{-s/2}a^s b^{-s/2} \le M^s b^{-s/2}b^s b^{-s/2} \le M^s$$

and therefore

$$a^{z/2}b^{-z/2} = a^{it/2}a^{s/2}b^{-s/2}b^{-it/2} \in \mathcal{M}$$

and $\|a^z b^{-z/2}\| \le M^{s/2}$. Similarly, we have $b^{-z/2}a^{z/2} \in \mathcal{M}$. Let $f_n = 1_{[\frac{1}{n},n]}(b)$ and $e_n = 1_{[\frac{1}{n},n]}(a)$ be the spectral projection of $a$, $b$, respectively. Then

$$g_n(z) = e_n a^{z/2}b^{-z/2} f_n$$

is norm differentiable (since $e_n \log a$ and $f_n \log b$ are bounded) and therefore

$$g(z) = a^{z/2}b^{-z/2}b^{p/2} = \lim_n g_n(z)b^{p/2}$$

is a norm differentiable function in $L_{2,\infty}(\mathcal{M} \rtimes_{\sigma_t} \mathbb{R})$. Moreover, $g(z) \in L_2(\mathcal{M})$. Since the embedding of $L_2(\mathcal{M})$ into $L_{2,\infty}(\mathcal{M} \rtimes_{\sigma_t} \mathbb{R})$ is isometric (see [42]), we deduce that $g$ has values in $L_2(\mathcal{M})$ is continuous on the strip and analytic in the interior. The same applies for $h(z) = b^{p/2}b^{-z/2}a^{z/2}$. Therefore, we may define the analytic function

$$f(z) = \operatorname{tr}\big(g_2(z)h_2(z)\big) = \operatorname{tr}(b^{p/2}b^{-z/2}a^z b^{-z/2}b^{p/2}).$$

For $t \in \mathbb{R}$, we have

$$\begin{aligned}|f(it)| &= \big|\operatorname{tr}(b^{p/2}b^{-it/2}a^{it/2}a^{it/2}b^{-it/2}b^{p/2})\big| \\ &\le \|b^{p/2}\|_2\|b^{-it/2}a^{it/2}\|_\infty\|a^{it/2}b^{-it/2}\|_\infty\|b^{p/2}\|_2 \\ &= \operatorname{tr}(b^p).\end{aligned}$$

For $z = 1 + it$ we deduce from Hölder's inequality and [42], Lemma 20, that

$$\begin{aligned}|f(it)| &= \big|\operatorname{tr}\big(b^{p/2}(b^{-1/2-it/2}a^{1/2+it/2})(a^{1/2+it/2}b^{-1/2-it/2})b^{p/2}\big)\big| \\ &\le \big|\operatorname{tr}\big(b^{p/2}(b^{-1/2-it/2}a^{1/2+it/2})(b^{-1/2-it/2}a^{1/2+it/2})^*b^{p/2}\big)\big|^{1/2} \\ &\quad \times \big|\operatorname{tr}\big(b^{p/2}(a^{1/2+it/2}b^{-1/2-it/2})^*(a^{1/2+it/2}b^{-1/2-it/2})b^{p/2}\big)\big|^{1/2} \\ &= \big|\operatorname{tr}\big(b^{p/2}b^{-it/2}b^{-1/2}a^{1/2}a^{1/2}b^{-1/2}b^{+it/2}b^{p/2}\big)\big|^{1/2} \\ &\quad \times \big|\operatorname{tr}\big(b^{p/2}b^{it/2}b^{-1/2}a^{1/2}a^{1/2}b^{-1/2}b^{-it/2}b^{p/2}\big)\big|^{1/2} \\ &= \big|\operatorname{tr}\big(b^{p/2}b^{-1/2}a^{1/2}a^{1/2}b^{-1/2}b^{p/2}\big)\big|.\end{aligned}$$

Hence, by the three lines lemma,

$$|f(p)| \le [\operatorname{tr}(b^p)]^{1-p}\big[\operatorname{tr}(b^{-(1-p)/2}ab^{-(1-p)/2})\big]^p.$$

This is exactly (7.5). $\square$

Now we are ready to prove Theorem 7.1.

PROOF OF THEOREM 7.1. Let $(a_n) \subset \mathcal{M}_a D^{1/p}$ be a finite positive sequence. Then by Lemma 1.1, $a_n = D^{1/2p} b_n D^{1/2p}$ for some $b_n \in M_a$ with $b_n \geq 0$; so

$$\mathcal{E}_n a_n = D^{1/2p} \mathcal{E}_n(b_n) D^{1/2p}.$$

Let $\varepsilon > 0$. Set

$$A_n = \sum_{k=0}^{n} a_k, \qquad B_n = \varepsilon D^{1/p} + \sum_{k=0}^{n} \mathcal{E}_k a_k$$

and $A = A_\infty$, $B = B_\infty$. Note that $(A_n)$ and $(\mathrm{B}_n)$ are increasing sequences of positive operators, and the $B_n$'s are invertible. Consequently, $B_n^{(1-p)/2} B^{-(1-p)/2}$ is a contraction in $L^\infty(\mathcal{M})$ for every $n \geq 0$. Thus it follows that

$$\operatorname{tr}\big(B^{-(1-p)/2} a_n B^{-(1-p)/2}\big) \leq \operatorname{tr}\big(B_n^{-(1-p)/2} a_n B_n^{-(1-p)/2}\big).$$

On the other hand, since $B_n \geq \varepsilon D^{1/p}$, for the same reason we see that the operator $u_n = D^{(1-p)/2} B_n^{-(1-p)/2}$ belongs to $L^\infty(\mathcal{M}_n)$ for every $n$. Hence, by (2.4) and Proposition 2.3 [with $r$ determined by $1/r = 1/(2p) - (1-p)/2$],

$$\begin{aligned}\operatorname{tr}\big(B_n^{-(1-p)/2} a_n B_n^{-(1-p)/2}\big) &= \operatorname{tr}\big(u_n^* D^{1/r} b_n D^{1/r} u_n\big)\\ &= \operatorname{tr}\big(u_n^* D^{1/r} \mathcal{E}_n(b_n) D^{1/r} u_n\big)\\ &= \operatorname{tr}\big(B_n^{-(1-p)/2} \mathcal{E}_n(a_n) B_n^{-(1-p)/2}\big).\end{aligned}$$

Therefore, by Lemma 7.3 (applied with $q = 1/p$),

$$\begin{aligned}\operatorname{tr}\big(B^{-(1-p)/2} A B^{-(1-p)/2}\big) &\leq \sum_{n\geq 0} \operatorname{tr}\big(B_n^{-(1-p)/2} \mathcal{E}_n(a_n) B_n^{-(1-p)/2}\big)\\ &= \sum_{n\geq 0} \operatorname{tr}\big(B_n^{-(1-p)/2} (B_n - B_{n-1}) B_n^{-(1-p)/2}\big)\\ &\leq 2^{1/p} \sum_{n\geq 0} \operatorname{tr}(B_n^p - B_{n-1}^p)\\ &= 2^{1/p} \operatorname{tr}(B^p).\end{aligned}$$

By the form of the $a_n$'s, we see that $A$ and $B$ satisfy the assumptions of Lemma 7.4. Thus, applying this lemma, we conclude that $\operatorname{tr}(A^p) \leq 2 \operatorname{tr}(B^p)$; so letting $\varepsilon \to 0$, we finally get (7.2). □

In the same spirit as the proof of Theorem 7.1, we can give a very simple proof of (7.1) in the case $1 \leq p \leq 2$. The main point is the following elementary inequality [similar to (7.3)]: Let $1 \leq p \leq 2$, $x, y \in L^p(\mathcal{M})$ with $0 \leq x \leq y$. Then

$$\operatorname{tr}(y^p - x^p) \leq 2 \operatorname{tr}\big(y^{p-1}(y - x)\big). \tag{7.6}$$

In fact, letting $a = x^{p/2}$ and $b = x^{(2-p)/2}y^{p-1}$, we have

$$\begin{aligned} 2\,\mathrm{tr}(y^{p-1}x) &= \mathrm{tr}(a^*b) + \mathrm{tr}(b^*a) \\ &\le \mathrm{tr}(a^*a) + \mathrm{tr}(b^*b) \le \mathrm{tr}(x^p) + \mathrm{tr}(y^p). \end{aligned}$$

This yields (7.6). Then we can easily show (7.1) for $1 \le p \le 2$ as in the previous proof of Theorem 7.1. Again let $(a_n)$ be a finite positive sequence in $L^p(\mathcal{M})$. Define $A_n, B_n$ as in that proof (with $\varepsilon = 0$ now). Then by (7.5),

$$\begin{aligned} \mathrm{tr}(B^p) &= \sum_{n\ge 0} \mathrm{tr}(B_n^p - B_{n-1}^p) \le 2\sum_{n\ge 0} \mathrm{tr}\left(B_n^{p-1}(B_n - B_{n-1})\right) \\ &\le 2\sum_{n\ge 0} \mathrm{tr}(B_n^{p-1}a_n) \\ &\le 2\sum_{n\ge 0} \mathrm{tr}(B^{p-1}a_n) \\ &= 2\,\mathrm{tr}(B^{p-1}A) \le 2\|B\|_p^{p-1}\|A\|_p \end{aligned}$$

whence $\|B\|_p \le 2\|A\|_p$. This is (7.1) with $\lambda_p \le 2$ for $1 \le p \le 2$.

**8. The nonfaithful case.** So far, we have only considered faithful conditional expectation. However, in $B(\ell^2)$ nonfaithful conditional expectations which are invariant under the modular group of a given state are very natural. In general, the analysis of the $L^p$-spaces of hyperfinite von Neumann algebras, requires the investigation of nonfaithful states. The purpose of this section is to give a brief discussion on the modifications needed to derive the nonfaithful from the faithful case.

Let $\mathcal{M}, \varphi$ and $\sigma_t$ be as before in Sections 1 and 2. Now let $\mathcal{N} \subset \mathcal{M}$ be a $w^*$-closed $*$-subalgebra (not necessarily unital). Our basic assumption on $\mathcal{N}$ is still the invariance of $\mathcal{N}$ under the modular automorphism group $\{\sigma_t\}_{t\in\mathbb{R}}$, namely, we again suppose (2.1) for $\mathcal{N}$. Let $e \in \mathcal{N}$ be the unit of $\mathcal{N}$ ($e$ is then a projection of $\mathcal{M}$). Then $\mathcal{N}$ is a von Neumann subalgebra of $e\mathcal{M}e$. By (2.1), $\sigma_t(e) = e,\ t \in \mathbb{R}$. Hence, $e$ belongs to the centralizer of $\varphi$. Let $\varphi_e = \varphi|_{e\mathcal{M}e}$. Then $\varphi_e$ is a normal faithful state on $e\mathcal{M}e$ and

$$\sigma_t^{\varphi_e}(x) = e\sigma_t(x)e, \qquad t \in \mathbb{R},\ x \in e\mathcal{M}e.$$

Therefore, (2.1) implies that $\mathcal{N}$ is also invariant under $\{\sigma_t^{\varphi_e}\}_{t\in\mathbb{R}}$, and so we are again in the faithful situation considered in Section 2 with $(e\mathcal{M}e, \varphi_e)$ in place of $(\mathcal{M}, \varphi)$. Hence, there is a normal faithful conditional expectation $\mathcal{E}_e$ from $e\mathcal{M}e$ onto $\mathcal{N}$ such that $\varphi_e \circ \mathcal{E}_e = \varphi_e$. Define $P_e(x) = exe,\ x \in \mathcal{M}$. Note that $P_e$ is a normal conditional expectation from $\mathcal{M}$ onto $e\mathcal{M}e$. Finally, we define $\mathcal{E} = \mathcal{E}_e \circ P_e$. Then, $\mathcal{E}$ is a normal conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$. This $\mathcal{E}$ is faithful iff $e = 1$. Note that

$$\varphi \circ \mathcal{E} = \varphi \circ P_e \quad \text{and} \quad \mathcal{E} \circ \sigma_t \circ P_e = \sigma_t \circ \mathcal{E}, \qquad t \in \mathbb{R}.$$

Also, observe in the present setting, that the crossed product $e\mathcal{M}e \rtimes_{\sigma^{\varphi_e}} \mathbb{R}$ can be naturally identified with the subalgebra $(e \otimes 1)\mathcal{R}(e \otimes 1)$ of $\mathcal{R}$ (recalling that $\mathcal{R} = \mathcal{M} \rtimes_\sigma \mathbb{R}$), and that the canonical trace on $e\mathcal{M}e \rtimes_{\sigma^{\varphi_e}} \mathbb{R}$ is equal to the restriction of $\tau$ to $(e \otimes 1)\mathcal{R}(e \otimes 1)$; moreover, the Radon–Nikodym derivative of the dual weight of $\varphi_e$ with respect to the canonical trace of $e\mathcal{M}e \rtimes_{\sigma^{\varphi_e}} \mathbb{R}$ is equal to $eDe = eD$ (since $D$ and $e$ commute). Thus we can, and will do regard $e\mathcal{M}e \rtimes_{\sigma^{\varphi_e}} \mathbb{R}$, $\mathcal{N} \rtimes_{\sigma^{\varphi_e}} \mathbb{R}$ as subalgebras of $\mathcal{R}$, and use $\tau$ to denote the canonical trace on each of these three crossed products. Accordingly, $L^p(e\mathcal{M}e)$ and $L^p(\mathcal{N})$ (associated with $\varphi_e$) are considered as subspaces of $L^p(\mathcal{M})$. Again, $\mathcal{E}$ induces a contractive projection from $L^p(\mathcal{M})$ onto $L^p(\mathcal{N})$. All the properties established for the faithful case in Section 2 continue to hold in the present setting. In particular, for any $0 \le \theta \le 1$ and $0 < p < \infty$, we define

$$\mathcal{E}_{p,\theta}\big(D^{(1-\theta)/p} x D^{\theta/p}\big) = D_e^{(1-\theta)/p} \mathcal{E}(x) D_e^{\theta/p}, \qquad x \in \mathcal{M}_a,$$

where $D_e = eDe$. Then as in Lemma 2.1, $\mathcal{E}_{p,\theta}$ is a well-defined mapping on $\mathcal{M}_a D^{1/p}$ and independent of $\theta$. As before in Section 2, we will denote it simply by $\mathcal{E}$. Note that (2.4) now becomes

$$\mathrm{tr}(\mathcal{E}(x)) = \mathrm{tr}(ex), \qquad x \in L^1(\mathcal{M}).$$

Thus we still have $\mathrm{tr}(\mathcal{E}(x)) = \mathrm{tr}(x)$ for all $x \in L^1(\mathcal{M})$ whose left or right support is dominated by $e$. Proposition 2.3 is still valid in the present setting with almost the same proof. Alternatively, applying Lemma 2.1 and Proposition 2.3 to $\mathcal{E}_e$ above (which is faithful), we reduce the nonfaithful version of Proposition 2.3 to the special case $\mathcal{E} = P_e$. In this latter case, everything is almost obvious.

We turn to our main concern on noncommutative martingales. Thus let $\{\mathcal{M}_n\}_{n\ge 0}$ be an increasing filtration of $w^*$-closed $*$-subalgebras of $\mathcal{M}$ such that $\bigcup_{n\ge 0} \mathcal{M}_n$ is $w^*$-dense in $\mathcal{M}$. We assume that every $\mathcal{M}_n$ is invariant under $\{\sigma_t\}_{t\in\mathbb{R}}$. Then by the preceding discussions, for each $n \ge 0$, there is a normal conditional expectation $\mathcal{E}_n$ from $\mathcal{M}$ onto $\mathcal{M}_n$ such that

$$\varphi(\mathcal{E}_n(x)) = \varphi(e_n x e_n), \qquad x \in \mathcal{M},$$

where $e_n$ is the unit of $\mathcal{M}_n$. As in the faithful case, we have

$$\mathcal{E}_m \mathcal{E}_n = \mathcal{E}_n \mathcal{E}_m = \mathcal{E}_{\min(m,n)}, \qquad m, n \ge 0.$$

By Proposition 2.3 (used in the nonfaithful case), each $\mathcal{E}_n$ induces a contractive projection from $L^p(\mathcal{M})$ onto $L^p(\mathcal{M}_n)$, $1 \le p \le \infty$. Then all notions of noncommutative martingales can be transferred to the present setting word by word.

Now we consider the problem of the validity of the results in the previous sections for the nonfaithful conditional expectations. It is clear that Theorem 6.1 in the case $p < 2$ and Theorem 7.1 are no longer true. With these two exceptions, all other results remain valid. One way to justify this is to examine the proofs of these results given before. Then one can see that they still work in the nonfaithful case. However, there is a more interesting alternate way. It consists in reducing the nonfaithful case to the faithful one. The rest of this section is devoted to explain this reduction.

Let $\mathcal{M}, \mathcal{M}_n, e_n, \mathcal{E}_n$ be fixed as above. Then $(e_n)$ is an increasing sequence of projections converging to 1 in $\mathcal{M}$. As already observed, each $e_n$ is in the centralizer of $\varphi$. Let $\mathcal{D}$ denote the (commutative) von Neumann subalgebra of $\mathcal{M}$ generated by the $e_n$'s. Then $\mathcal{D}$ is invariant under $\sigma_t$. Thus there is a unique faithful normal conditional expectation from $\mathcal{M}$ onto $\mathcal{D}$, which we denote especially by $T$. It will be useful to represent $T$ as a sum of mutually orthogonal normal conditional expectations of rank 1. To this end, let $f_n = e_n - e_{n-1}$ (with $e_{-1} = 0$). Then $(f_n)$ is a sequence of orthogonal projections in $\mathcal{M}$ whose sum is 1. Set $\mathcal{D}_n = \mathbb{C} f_n$, and define $T_n : \mathcal{M} \to \mathcal{D}_n$ by $T_n(x) = \frac{\varphi(x f_n)}{\varphi(f_n)} f_n$. Note that $T_n$ is nothing but the normal conditional expectation from $\mathcal{M}$ onto $\mathcal{D}_n$. Then clearly

$$T(x) = \sum_{n\geq 0} T_n(x), \qquad x \in \mathcal{M}. \tag{8.1}$$

Now we introduce a new filtration $(\tilde{\mathcal{M}}_n)_n$ of von Neumann subalgebras by defining $\tilde{\mathcal{M}}_n$ as the von Neumann subalgebra generated by $\mathcal{M}_n$ and $\mathcal{D}$. Since both $\mathcal{M}_n$ and $\mathcal{D}$ are invariant under $\sigma_t$, so is $\tilde{\mathcal{M}}_n$. Therefore, there is a faithful normal conditional expectation $\tilde{\mathcal{E}}_n$ from $\mathcal{M}$ onto $\tilde{\mathcal{M}}_n$ preserving $\varphi$. It is easy to check that

$$\tilde{\mathcal{E}}_n(x) = \mathcal{E}_n(x) + \sum_{k>n} T_k(x), \qquad x \in \mathcal{M}. \tag{8.2}$$

As before, all these mappings extend to contractions on $L^p(\mathcal{M})$. Then (8.1) and (8.2) hold with $L^p(\mathcal{M})$ $(1 \leq p < \infty)$ instead of $\mathcal{M}$, and moreover the series there converge in $L^p(\mathcal{M})$. Let $d_0 = \mathcal{E}_0, \tilde{d}_0 = \tilde{\mathcal{E}}_0$ and $d_n = \mathcal{E}_n - \mathcal{E}_{n-1},\ \tilde{d}_n = \tilde{\mathcal{E}}_n - \tilde{\mathcal{E}}_{n-1}$ for $n \geq 1$. Note, that $dx_n = d_n x$ for all $n$. Then for any $x \in L^p(\mathcal{M})$, $(d_n x)_n$ [resp. $(\tilde{d}_n x)_n$] is the martingale difference sequence with respect to $(\mathcal{M}_n)$ [resp. $(\tilde{\mathcal{M}}_n)$]. We have the following easily checked relations between these martingale differences

$$\tilde{d}_0 = d_0 x + \sum_{k\geq 1} T_k x \quad \text{and} \quad \tilde{d}_n x = d_n x - T_n x, \qquad n \geq 1. \tag{8.3}$$

Also observe the following simple fact:

$$T_n(d_n x) = d_n(T x) = T_n x, \qquad n \geq 0. \tag{8.4}$$

Let us now explain the reduction. We will only explain this reduction for Theorems 3.1 and 6.1 and leave the rest to the interested reader. First, consider $(\mathrm{BG}_p)$. We claim that for any $x \in L^p(\mathcal{M})$ $(1 \leq p \leq \infty)$,

$$\|Tx\|_p \leq \min\left\{ \|x\|_p, \left\| \Big(\sum |d_n x|^2\Big)^{1/2} \right\|_p, \left\| \Big(\sum |d_n x^*|^2\Big)^{1/2} \right\|_p \right\}. \tag{8.5}$$

Indeed, since $T$ is contractive, $\|Tx\|_p \le \|x\|_p$. To see why $\|Tx\|_p$ is majorized by the second member in the brackets, we proceed as follows. Consider the tensor product $\mathcal{M} \otimes B(\ell^2)$ as in Section 1. Since $T \otimes \mathrm{id}$ is contractive on $L^p(\mathcal{M} \otimes B(\ell^2))$, we have

$$\Big\|\Big(\sum |d_n x|^2\Big)^{1/2}\Big\|_p \ge \left\| T \otimes \mathrm{id} \begin{pmatrix} d_0 x & 0 & \cdots \\ d_1 x & 0 & \cdots \\ \vdots & \vdots & \end{pmatrix} \right\|_p$$

$$= \left\| \begin{pmatrix} T(d_0 x) & 0 & \cdots \\ T(d_1 x) & 0 & \cdots \\ \vdots & \vdots & \end{pmatrix} \right\|_p$$

$$= \Big\|\Big(\sum |T(d_n x)|^2\Big)^{1/2}\Big\|_p.$$

By (8.1),

$$Tx = \sum_{n \ge 0} T_n(x) = \sum_{n \ge 0} T_n(d_n x).$$

On the other hand,

$$T(d_n x) = T_n(d_n x) + \sum_{k=0}^{n-1} T_k(d_n x).$$

Since $(T_k(d_n x))_{k \ge 0}$ are of disjoint support, $|T(d_n x)| \ge |T_n(d_n x)|$. Hence, we deduce

$$\Big\|\Big(\sum |d_n x|^2\Big)^{1/2}\Big\|_p \ge \|Tx\|.$$

Passing to adjoints, we get the last inequality in (8.3).

We have an inequality similar to (8.5) for $(\tilde{d}_n x)$, namely

$$\|Tx\|_p \le \min\left\{ \Big\|\Big(\sum |\tilde{d}_n x|^2\Big)^{1/2}\Big\|_p, \Big\|\Big(\sum |\tilde{d}_n x^*|^2\Big)^{1/2}\Big\|_p \right\}. \tag{8.6}$$

This is obvious for

$$\Big\|\Big(\sum |\tilde{d}_n x|^2\Big)^{1/2}\Big\|_p \ge \|\tilde{\mathcal{E}}_0 x\|_p \ge \|\tilde{T} x\|_p.$$

Combining (8.3), (8.5) and (8.6), we get

$$\tfrac{1}{3}\Big\|\Big(\sum |d_n x|^2\Big)^{1/2}\Big\|_p \le \Big\|\Big(\sum |\tilde{d}_n x|^2\Big)^{1/2}\Big\|_p \le 3\Big\|\Big(\sum |d_n x|^2\Big)^{1/2}\Big\|_p.$$

This clearly shows that the validity of $(\mathrm{BG}_p)$ for $x$ and $(\tilde{d}_n x)$ implies that for $x$ and $(d_n x)$. Therefore, we have reduced $(\mathrm{BG}_p)$ in the nonfaithful case to that in the faithful case.

Next we explain how to do the same for $(\mathrm{B}_p)$ for $2 \le p < \infty$. Fix $x \in L^p(\mathcal{M})$. By the definitions in (8.3), we obtain

$$\frac{1}{3}\Big(\sum \|d_n x\|_p^p\Big)^{1/p} \le \Big(\sum \|\tilde{d}_n x\|_p^p\Big)^{1/p} \le 3\Big(\sum \|d_n x\|_p^p\Big)^{1/p}. \tag{8.7}$$

On the other hand, again by (8.3) and also (8.4),

$$\tilde{\mathcal{E}}_0|\tilde{d}_0 x|^2 = \mathcal{E}_0|d_0 x|^2 + \sum_{k\ge 1} |T_k x|^2, \tag{8.8}$$

$$\tilde{\mathcal{E}}_{n-1}|\tilde{d}_n x|^2 = \mathcal{E}_{n-1}|d_n x|^2 + T_n(|d_n x|^2) - |T_n(d_n x)|^2, \qquad n \ge 1. \tag{8.9}$$

Since $|T_n(d_n x)|^2 \le T_n(|d_n x|^2)$, it follows that

$$\mathcal{E}_{n-1}|d_n x|^2 \le \tilde{\mathcal{E}}_{n-1}|\tilde{d}_n x|^2, \qquad n \ge 0.$$

Therefore, the validity of the first inequality of $(\mathrm{B}_p)$ for $x$ and $(\tilde{d}_n x)$ implies that for $x$ and $(d_n x)$. To show the same is true concerning the second inequality of $(\mathrm{B}_p)$, it suffices to note that (8.9) yields

$$\begin{aligned}\Big\| \sum \tilde{\mathcal{E}}_{n-1}|d_n y|^2\Big\|_{p/2} &\le \Big\| \sum \mathcal{E}_{n-1}|d_n x|^2\Big\|_{p/2} + \Big\| \sum T_n|d_n x|^2\Big\|_{p/2} \\ &= \Big\| \sum \mathcal{E}_{n-1}|d_n x|^2\Big\|_{p/2} + \Big(\sum \|T_n|d_n x|^2\|_{p/2}^{2/p}\Big)^{2/p} \\ &\le \Big\| \sum \mathcal{E}_{n-1}|d_n x|^2\Big\|_{p/2} + \Big(\sum \|d_n x\|_p^p\Big)^{2/p}.\end{aligned}$$

Thus we have obtained $(\mathrm{B}_p)$ for $2 \le p < \infty$ in the nonfaithful case.

REMARK. The previous arguments show that the first inequality in $(\mathrm{B}_p)$ still holds in the case $1 < p < 2$. Concerning the second one, it is true for all $x$ satisfying $Tx = 0$. Even in the commutative situation one can easily construct counterexamples if this condition is not satisfied.

## REFERENCES

[1] ACCARDI, L. and CECCHINI, C. (1982). Conditional expectations in von Neumann algebras and a theorem of Takesaki. *J. Funct. Anal.* **45** 245–273.

[2] ACCARDI, L. and VON WALDENFELS, M. (1990). *Quantum Probability and Applications. Lecture Notes in Math.* **1442**. Springer, Berlin.

[3] BIANE, P. and SPEICHER, R. (1998). Stochastic calculus with respect to free Brownian motion and analysis on Wigner space. *Probab. Theory Related Fields* **112** 373–409.

[4] BURKHOLDER, D. L. (1973). Distribution function inequalities for martingales. *Ann. Probab.* **1** 19–42.

[5] BURKHOLDER, D. L. (1979). Martingale theory and harmonic analysis in Euclidean spaces. In *Harmonic Analysis in Euclidean Spaces* (G. Weiss and S. Wainger, eds.) 283–301. Amer. Math. Soc., Providence, RI.

[6] BURKHOLDER, D. L. and GUNDY, R. F. (1970). Extrapolation and interpolation of quasi-linear operators on martingales. *Acta Math.* **124** 249–304.

[7] CARLEN, E. A. and KRÉE, P. (1998). On martingale inequalities in non-commutative stochastic analysis. *J. Funct. Anal.* **158** 475–508.

[8] CECCHINI, C. and PETZ, D. (1985). Norm convergence of generalized martingales in $L^p$-spaces over von Neumann algebras. *Acta Sci. Math. (Szeged)* **48** 55–63.

[9] CONNES, A. (1973). Une classification des facteurs de type III. *Ann. Sci. École Norm. Sup.* **6** 133–252.

[10] CUCULESCU, I. and OPREA, A. G. (1994). *Noncommutative Probability.* Kluwer, Dordrecht.

[11] EFFROS, E. and RUAN, Z.-J. (2000). *Operator Spaces*. Oxford Univ. Press.

[12] GARSIA, A. M. (1973). *Martingale Inequalities*. Benjamin, Reading, MA.

[13] FIDALEO, F. (2001). Weak and strong martingale convergence of generalized conditional expectations in non-commutative $L^p$-spaces. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **4** 195–213.

[14] HAAGERUP, U. (1979). $L^p$-spaces associated with an arbitrary von Neumann algebra. In *Algèbres d'opérateurs et leurs applications en physique mathématique* 175–184. Éditions du CNRS, Paris.

[15] HAAGERUP, U. (1980). Non-commutative integration theory. Lecture given at the Symposium in Pure Mathematics of the Amer. Math. Soc., Queens Univ., Kingston, Ontario.

[16] HIAI, F. and TSUKADA, M. (1987). Generalized conditional expectations and martingales in non-commutative $L^p$-spaces. *J. Operator Theory* **18** 265–288.

[17] JAJTE, R. (1985). *Strong Limit Theorems in Non-Commutative Probability. Lecture Notes in Math.* **1110**. Springer, Berlin.

[18] JAJTE, R. (1991). *Strong Limit Theorems in Non-Commutative $L_2$-Spaces. Lecture Notes in Math.* **1477**. Springer, Berlin.

[19] JUNGE, M. (2002). Doob's inequality for non-commutative martingales. *J. Reine Angew. Math.* **549** 149–190.

[20] JUNGE, M. and MUSAT, M. (2002). A noncommutative version of the John–Nirenberg theorem. Preprint.

[21] JUNGE, M. and XU, Q. (2002). Non-commutative Burkholder/Rosenthal inequalities: Applications. Preprint.

[22] JUNGE, M. and XU, Q. (2002). The optimal orders of growth of the best constants in some non-commutative martingale inequalities. Preprint.

[23] KADISON, R. V. and RINGROSE, J. R. (1986). *Fundamentals of the Theory of Operator Algebras II.* Academic Press, New York.

[24] KOSAKI, H. (1984). Applications of the complex interpolation method to a von Neumann algebra. *J. Funct. Anal.* **56** 29–78.

[25] LUST-PIQUARD, F. (1986). Inégalités de Khintchine dans $C_p$ $(1 < p < \infty)$. *C. R. Acad. Sci. Paris* **303** 289–292.

[26] LUST-PIQUARD, F. and PISIER, G. (1991). Non commutative Khintchine and Paley inequalities. *Arkiv Mat.* **29** 241–260.

[27] MEYER, P. A. (1995). *Quantum Probability for Probabilists. Lecture Notes in Math.* **1538**. Springer, Berlin.

[28] MITREA, M. (1994). *Clifford Wavelets, Singular Integrals, and Hardy Spaces. Lecture Notes in Math.* **1575**. Springer, Berlin.

[29] MUSAT, M. (2002). Interpolation between non-commutative BMO and noncommutative $L_p$-spaces. *J. Funct. Anal.* To appear.

[30] PARTHASARATHY, K. R. (1992). *An Introduction to Quantum Stochastic Calculus*. Birkhäuser, Boston.

[31] PEDERSEN, G. K. and TAKESAKI, M. (1973). The Radon–Nikodym theorem for von Neumann algebras. *Acta Math.* **130** 53–87.

[32] PISIER, G. (1998). Non-commutative vector valued $L_p$-spaces and completely $p$-summing maps. *Astérisque* **247**.

[33] PISIER, G. (2000). An inequality for $p$-orthogonal sums in non-commutative $L_p$. *Illinois J. Math.* **44** 901–923.

[34] PISIER, G. and XU, Q. (1997). Non-commutative martingale inequalities. *Comm. Math. Phys.* **189** 667–698.

[35] RANDRIANANTOANINA, N. (2002). Non-commutative martingale transforms. *J. Funct. Anal.* **194** 181–212.

[36] ROSENTHAL, H. P. (1970). On the subspaces of $L^p$ $(p > 2)$ spanned by sequences of independent random variables. *Israel J. Math.* **8** 273–303.

[37] STEIN, E. M. (1970). *Topics in Harmonic Analysis Related to the Littlewood–Paley Theory.* Princeton Univ. Press.

[38] STRATILA, S. (1981). *Modular Theory in Operator Algebras*. Abacus Press, Tunbridge, Wells, UK.

[39] TAKESAKI, M. (1970). *Tomita's Theory of Modular Hilbert Algebras and Its Applications. Lecture Notes in Math.* **128**. Springer, Berlin.

[40] TAKESAKI, M. (1972). Conditional expectations in von Neumann algebras. *J. Funct. Anal.* **9** 306–321.

[41] TAKESAKI, M. (1973). Duality for crossed products and the structure of von Neumann algebras of type III. *Acta Math.* **131** 249–310.

[42] TERP, M. (1981). $L^p$ spaces associated with von Neumann algebras. Notes, Math. Institute, Copenhagen Univ.

[43] TERP, M. (1982). Interpolation spaces between a von Neumann algebra and its predual. *J. Operator Theory* **8** 327–360.

[44] VOICULESCU, D. V. (1991). Limit laws from random matrices and free products. *Invent. Math.* **104** 201–220.

[45] VOICULESCU, D. V., DYKEMA, K. J. and NICA, A. (1992). *Free Random Variables*. Amer. Math. Soc., Providence, RI.

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