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NONCOMMUTATIVE DEFORMATIONS AND FLOPS

WILL DONOVAN AND MICHAEL WEMYSS

ABSTRACT. We prove that the functor of noncommutative deformations of every flipping or flopping irreducible rational curve in a 3-fold is representable, and hence associate to every such curve a noncommutative deformation algebra A_{con}. This new invariant extends and unifies known invariants for flopping curves in 3-folds, such as the width of Reid [R83], and the bidegree of the normal bundle. It also applies in the settings of flips and singular schemes. We show that the noncommutative deformation algebra A_{con} is finite dimensional, and give a new way of obtaining the commutative deformations of the curve, allowing us to make explicit calculations of these deformations for certain $(-3,1)$-curves.

We then show how our new invariant A_{con} also controls the homological algebra of flops. For any flopping curve in a projective 3-fold with only Gorenstein terminal singularities, we construct an autoequivalence of the derived category of the 3-fold by twisting around a universal family over the noncommutative deformation algebra A_{con}, and prove that this autoequivalence is an inverse of Bridgeland's flop–flop functor. This demonstrates that it is strictly necessary to consider noncommutative deformations of curves in order to understand the derived autoequivalences of a 3-fold, and thus the Bridgeland stability manifold.

CONTENTS

1. INTRODUCTION

To understand the birational geometry of algebraic varieties via the minimal model program, it is necessary to understand the geometry of certain codimension two modifications known as *flips* and *flops*. Even for modifications of irreducible rational curves in dimension three this geometry is extraordinarily rich, and has been discussed by many authors, for example [P83, R83, CKM, KaM, KoM, K94]. A central problem is to classify flips and flops in a satisfying manner, and to construct appropriate invariants.

In the first half of this paper we associate a new invariant to every flipping or flopping curve in a 3-fold, using noncommutative deformation theory. This generalises and unifies the classical invariants into one new object, the *noncommutative deformation algebra* A_{con} associated to the curve. Our new invariant is a finite dimensional algebra, and can be

associated to any contractible rational curve in any 3-fold, regardless of singularities. It recovers classical invariants in natural ways. Moreover, unlike these classical invariants, A_{con} is an algebra, and in the second half of the paper we exploit this to give the first intrinsic description of a derived autoequivalence associated to a general flopping curve.

1.1. **Background on 3-folds.** The simplest example of a 3-fold flop is the well-known Atiyah flop. In this case the flopping curve has normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and is rigid, and the flop may be factored as a blow-up of the curve followed by a blow-down. However, for more general flops the curve is no longer rigid, and the factorization of the birational map becomes more complicated. For a general flopping irreducible rational curve C in a smooth 3-fold X, the following classical invariants are associated to C.

(1) **Normal bundle.** Denoting $\mathcal{O}(a) \oplus \mathcal{O}(b)$ as simply (a, b), the normal bundle must be $(-1, -1)$, $(-2, 0)$ or $(-3, 1)$ [P83, Prop. 2].
(2) **Width** [R83, 5.3]**.** In the first two cases above, the width of C is defined to be

$$\sup\{n \mid \exists \text{ a scheme } C_n \cong C \times \operatorname{Spec} \mathbb{C}[x]/x^n \text{ with } C \subset C_n \subset X\}.$$

More generally, in the case $(-3, 1)$, it is not possible to define the width invariant in this way.
(3) **Dynkin type** [KaM, K94]**.** By taking a generic hyperplane section, Katz–Morrison and Kawamata assign to C the data of a Dynkin diagram with a marked vertex. Only some possibilities arise, as shown in Table 1 and (3.B).
(4) **Length** [CKM, p95–96]**.** This is defined to be the multiplicity of the curve C in the fundamental cycle of a generic hyperplane section.
(5) **Normal bundle sequence** [P83, Thm. 4]**.** The flop $f \colon X \dashrightarrow X'$ factors into a sequence of blow-ups in centres $C_1, \ldots, C_n$, followed by blow-downs. The normal bundles of these curves form the $\mathcal{N}$-sequence.

The following table summarises the relations between the invariants above, and shows that none of these invariants classify all analytic equivalence types of flopping curves.

(1) $\mathcal{N}_{C\mid X}$	(2) width	(3) Dynkin type	(4) length	(5) $\mathcal{N}$-sequence
$(-1, -1)$	1	A_1	1	$(-1, -1)$
$(-2, 0)$	$n > 1$			$\overbrace{(-2, 0) \ldots (-2, 0)}^{n}\, (-1, -1)$
$(-3, 1)$	undefined	D_4	2	$(-3, 1)\,(-2, -1)\,(-1, -1)$
		E_6	3	$(-3, 1)\,(-3, 0)\,(-2, -1)\,(-1, -1)$
		E_7	4	
		$E_8(6)$	6	
		$E_8(5)$	5	$(-3, 1)\,(-3, 0)\,(-3, 0)\,(-2, -1)\,(-1, -1)$

TABLE 1. Classical invariants of flopping rational curves in smooth 3-folds.

1.2. **Noncommutative deformations and invariants.** Our new invariant of flopping and flipping rational curves C in a 3-fold X is constructed by noncommutatively deforming the associated sheaf $E := \mathcal{O}_C(-1)$. Classically, infinitesimal deformations of C are controlled by $\operatorname{Ext}^1_X(E, E)$, the dimension of which is determined by the normal bundle $\mathcal{N}_{C\mid X}$. If $\dim_{\mathbb{C}} \operatorname{Ext}^1_X(E, E) \leq 1$, the curve C deforms over an artinian base $\mathbb{C}[x]/x^n$. However, when $\dim_{\mathbb{C}} \operatorname{Ext}^1_X(E, E) \geq 2$, deformations of C are less well understood, and for example in the setting of flops the width invariant of Reid is no longer sufficient to characterize them.

In this paper, we solve this problem by using noncommutative deformation theory. We give preliminary definitions here, leaving details to §2. Recall that the formal commutative deformations of the simple sheaf E associated to C are described by a functor from commutative local artinian $\mathbb{C}$-algebras to sets, given by

$$c\mathcal{D}ef_E \colon \mathsf{CArt} \to \mathsf{Sets}$$

$$R \mapsto \{\text{flat } R\text{-families of coherent sheaves deforming } E\}/\sim .$$

In the above we take the point of view that a family is an R-module in $\operatorname{coh} X$, as this turns out to generalise. Following [L02, E07], we may then define a *noncommutative deformation functor* $\mathcal{D}ef_E \colon \mathsf{Art}_1 \to \mathsf{Sets}$ from 1-pointed artinian not-necessarily-commutative $\mathbb{C}$-algebras to sets, which extends $c\mathcal{D}ef_E$ (for details see 2.4). Our first main theorem is the following.

Theorem 1.1 (=3.3(2)). *For a flopping or flipping curve C in a 3-fold X, the noncommutative deformation functor $\mathcal{D}ef_E$ is representable.*

As a consequence of 1.1, from the noncommutative deformation theory we obtain a universal family $\mathcal{E} \in \operatorname{coh} X$. At the same time, $c\mathcal{D}ef_E$ is also representable, giving another universal family $\mathcal{F} \in \operatorname{coh} X$. The following are our new invariants.

(6) **Noncommutative deformation algebra.** This is defined $A_{\mathrm{def}} := \operatorname{End}_X(\mathcal{E})$.

(7) **Commutative deformation algebra.** We define this to be $\operatorname{End}_X(\mathcal{F})$, and prove in 3.9 that it is isomorphic to the *abelianization* $A_{\mathrm{def}}^{\mathrm{ab}}$, given by quotienting A_{def} by the ideal generated by commutators.

The above objects are algebras, and not just vector spaces. Nevertheless, taking dimensions we obtain the following new numerical invariants.

(8) **Noncommutative width.** We take $\operatorname{wid}(C) := \dim_{\mathbb{C}} A_{\mathrm{def}}$.

(9) **Commutative width.** We set $\operatorname{cwid}(C) := \dim_{\mathbb{C}} A_{\mathrm{def}}^{\mathrm{ab}}$.

For flopping and flipping curves these numerical invariants are always finite by 1.2, regardless of singularities. We explain in 5.2 that the noncommutative width $\operatorname{wid}(C)$ is readily calculated, and is indeed easier to handle than the commutative width $\operatorname{cwid}(C)$. From the viewpoint of the homological algebra, we will see later in §1.7 that the noncommutative width is the more natural invariant.

For floppable curves of Type A, when Reid's width invariant is defined [R83], we prove in 3.15 that all three width invariants agree. For all other contractible curves, including flips and singular flops, our invariants are new. We will see in 1.3 below that the noncommutative width may be strictly larger than the commutative width. Remark 3.17 also gives lower bounds for the width according to Dynkin type. This makes it clear that the algebras A_{def} can be quite complex: for instance in type E_8 we find that $\operatorname{wid}(C) \geq 40$.

1.3. **Contraction algebras.** In addition to the deformation-theoretic viewpoint given above, we explain in §3 how A_{def} arises also as a *contraction algebra* associated to an algebra obtained by tilting. It is this alternative description of A_{def} that allows us to calculate it, and also control it homologically.

Briefly, for any contractible curve C in a 3-fold X, by passing to an open neighbourhood U of C, and then to the formal fibre $\widehat{U} \to \operatorname{Spec} \widehat{R}$, it is well known [V04] that $\widehat{U}$ is derived equivalent to an algebra $A := \operatorname{End}_{\widehat{R}}(\widehat{R} \oplus N_1)$. See §2.4 for more details. Let $[\widehat{R}]$ be the two-sided ideal of A consisting of homomorphisms $\widehat{R} \oplus N_1 \to \widehat{R} \oplus N_1$ that factor through $\operatorname{add} \widehat{R}$. We write $A_{\mathrm{con}} := A/[\widehat{R}]$ and call A_{con} the *contraction algebra*.

Theorem 1.2 (=3.9, 2.13). *For a flopping or flipping curve in a 3-fold, $A_{\mathrm{def}} \cong A_{\mathrm{con}}$, and furthermore this is a finite dimensional algebra.*

We refer the reader to 3.10 for the calculation of A_{con} in the case of the Pagoda flop, in which case $A_{\mathrm{con}} \cong \mathbb{C}[x]/x^n$, as expected. More generally, our approach allows us to make calculations beyond Dynkin type A. We illustrate this in the following example.

Example 1.3. The simplest non-type-A flopping contraction is obtained by setting

$$\widehat{R} := \frac{\mathbb{C}[[u,v,x,y]]}{u^2 + v^2 y = x(x^2 + y^3)},$$

and considering the morphism $X \to \operatorname{Spec} \widehat{R}$ obtained by blowing up a generator of the class group. There is a single floppable $(-3,1)$-curve above the origin, and this is a length two D_4 flop, known as the Laufer flop [R83, AM]. In this case, by [AM], $A := \operatorname{End}_{\widehat{R}}(\widehat{R} \oplus N_1)$ can be presented as the completion of the quiver with relations

$$\widehat{R} \underset{b}{\overset{a}{\rightleftarrows}} N_1 \overset{x}{\underset{y}{\circlearrowright}} \qquad \begin{array}{l} ay^2 = -aba \\ y^2 b = -bab \\ xy = -yx \\ x^2 + yba + bay = y^3. \end{array}$$

It follows immediately that

$$A_{\mathrm{con}} \cong \frac{\mathbb{C}\langle\langle x, y \rangle\rangle}{xy = -yx,\ x^2 = y^3} = \frac{\mathbb{C}\langle x, y \rangle}{xy = -yx,\ x^2 = y^3}.$$

We call this algebra a *quantum cusp*. From this presentation, it can be seen that $\operatorname{wid}(C) = 9$ (for details, see 3.14). Furthermore, factoring by commutators it follows that the commutative deformation algebra is given by

$$A_{\mathrm{con}}^{\mathrm{ab}} \cong \frac{\mathbb{C}[x, y]}{xy,\ x^2 = y^3}$$

and so $\operatorname{cwid}(C) = \dim_{\mathbb{C}} A_{\mathrm{con}}^{\mathrm{ab}} = 5$.

The deformations are sketched in Figure 1. In the commutative case on the left, the original curve moves infinitesimally, fibred over $\operatorname{Spec} A_{\mathrm{con}}^{\mathrm{ab}}$, with the dots in the base representing the vector space basis $\{1, x, x^2, y, y^2\}$ of $A_{\mathrm{con}}^{\mathrm{ab}}$. The noncommutative deformations on the right are harder to draw, but roughly speaking they thicken the universal sheaf $\mathcal{F}$ to a sheaf $\mathcal{E} \in \operatorname{coh} X$ over A_{con}, where the nine dots represent a basis of A_{con}.
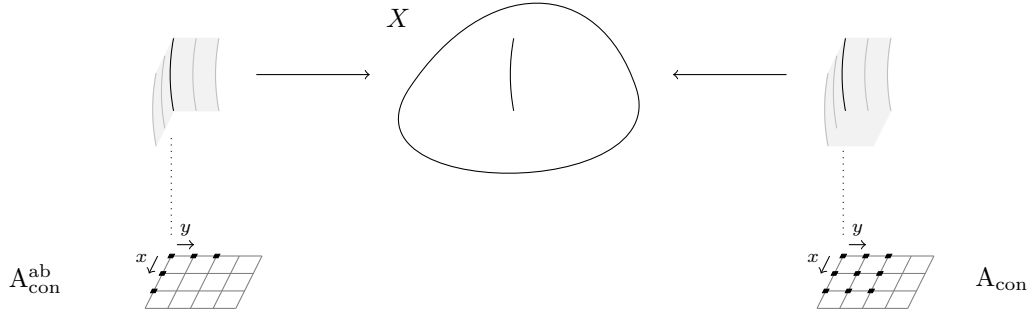
FIGURE 1. Comparing commutative and noncommutative deformations of the Laufer D_4 floppable curve in 1.3.

Using this example, and its generalization 3.14, we are able to extend the previous known invariants in Table 1. This is summarised in Table 2. We feel that the further study of the structure of these newly-arising algebras A_{con}, and of their representations, is of interest in itself. Furthermore, we conjecture that the contraction algebra distinguishes the analytic type of the flop:

Conjecture 1.4. Suppose that $Y \to Y_{\mathrm{con}}$ and $Z \to Z_{\mathrm{con}}$ are flopping contractions of an irreducible rational curve in smooth projective 3-folds, to points p and q respectively. To these, associate the contraction algebras A_{con} and B_{con}. Then the completions of the stalks at p and q are isomorphic if and only if $A_{\mathrm{con}} \cong B_{\mathrm{con}}$.

In the Type A and D cases for which we know the algebra A_{con} explicitly, the conjecture is true. We will return to the general case in future work.

Example	(3) Dynkin type	(6) A_{con}	(7) A_{con}^{ab}	(8) $\mathrm{wid}(C)$	(9) $\mathrm{cwid}(C)$
Atiyah	A_1	$\mathbb{C}$	$\mathbb{C}$	1	1
Pagoda (3.10)		$\frac{\mathbb{C}[x]}{x^n}$	$\frac{\mathbb{C}[x]}{x^n}$	n	n
Laufer (3.14)	D_4	$\frac{\mathbb{C}\langle x,y\rangle}{xy=-yx,\,x^2=y^{2n+1}}$ quantum cusp	$\frac{\mathbb{C}[x,y]}{xy,\,x^2=y^{2n+1}}$ cusp	$3(2n+1)$	$2n+3$

TABLE 2. Comparison of new invariants in known examples.

1.4. **Homological algebra and derived autoequivalences.** We now restrict our attention to flopping contractions, and use the contraction algebra A_{con} to give a unified construction of spherical-type twists related to flops. This works even in the mildly singular setting.

Given X, a projective 3-fold with at worst terminal Gorenstein singularities, and a flopping contraction $X \to X_{con}$ with flop $X' \to X_{con}$, then there is an autoequivalence of $D^b(\mathrm{coh}\,X)$ given as the composition of the flop equivalences

$$D^b(\mathrm{coh}\,X) \xrightarrow{\ \mathsf{F}\ } D^b(\mathrm{coh}\,X') \xrightarrow{\ \mathsf{F}'\ } D^b(\mathrm{coh}\,X)$$

where the Fourier–Mukai kernel $\mathcal{O}_{X \times_{X_{con}} X'}$ is used for both F and F' [B02, C02]. We denote this composition by FF.

Without assuming knowledge of the flop, we construct an autoequivalence of $D^b(\mathrm{coh}\,X)$ which we call the *noncommutative twist functor*, using the noncommutative deformation theory from the previous section. Then, using that FF is an autoequivalence, we show that it is an inverse of the noncommutative twist. This gives an intrinsic description of FF in terms of noncommutative deformation theory.

1.5. **Known autoequivalence results.** When X is smooth and the flopping contraction is of Type A, the functor FF has previously been described intrinsically as some form of spherical twist, which we now briefly review.

Spherical twists about spherical objects, in the sense of [ST01], are the simplest type of non-trivial derived autoequivalence. If C is a $(-1,-1)$-curve in a smooth 3-fold, the sheaf $E := \mathcal{O}_{\mathbb{P}^1}(-1) \in \mathrm{coh}\,X$ is a spherical object, and it was very well known to experts that the resulting twist autoequivalence was an inverse of FF.

For all other types of floppable curves, the structure sheaf E is not spherical, so the twist of [ST01] is not an autoequivalence. Toda proposed in [T07] a conceptual way to overcome this. The commutative deformation functor $c\mathcal{D}ef_E$ defined above gives us a universal sheaf $\mathcal{F} \in \mathrm{coh}\,X$. Using this sheaf, Toda constructed a Fourier–Mukai functor

$$T_{\mathcal{F}} \colon D^b(\mathrm{coh}\,X) \to D^b(\mathrm{coh}\,X).$$

Under the assumption that X is smooth, using explicit knowledge of the deformation base S, Toda proved the following.

- There is a functorial triangle $\mathbf{R}\mathrm{Hom}_X(\mathcal{F},-) \otimes_S^{\mathbf{L}} \mathcal{F} \to \mathrm{Id} \to T_{\mathcal{F}} \to$.
- For flopping contractions of Type A, $T_{\mathcal{F}}$ is an autoequivalence.
- For flopping contractions of Type A, $T_{\mathcal{F}}$ is an inverse of FF.

The techniques in [T07] have been very influential, and may be considered in the much more general framework of 'spherical functors', see [AL10, AL13]. However, it has remained an open question as to whether the above results hold for all flopping curves,

and whether they extend to mildly singular schemes. The commutative deformation base is extremely hard to calculate, and so it is difficult to determine whether the techniques in [T07] can be extended.

1.6. **New autoequivalence results.** We now consider the general setting where X is a projective 3-fold with at worst terminal Gorenstein singularities, and there is a flopping contraction $X \to X_{\mathrm{con}}$. By exploiting the universal sheaf $\mathcal{E}$ provided by the noncommutative deformation theory in §1.2, in 7.13 we construct a Fourier–Mukai functor

$$T_{\mathcal{E}} \colon \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X) \to \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X).$$

Toda's functor $T_{\mathcal{F}}$ was constructed using $\mathrm{Spec}\, \mathbb{C}[x]/x^n$ in a crucial way, but in this general setting we cannot take Spec of $\mathrm{A}_{\mathrm{con}}$ since it may not be commutative, and so our construction of $T_{\mathcal{E}}$ is necessarily more abstract.

The following is the main result of this paper.

Theorem 1.5 (=7.14, 7.18). *For any flopping contraction of an irreducible rational curve C in X, a projective 3-fold with only terminal Gorenstein singularities,*

- *There is a functorial triangle $\mathbf{R}\mathrm{Hom}_X(\mathcal{E}, -) \otimes^{\mathbf{L}}_{\mathrm{A}_{\mathrm{con}}} \mathcal{E} \to \mathrm{Id} \to T_{\mathcal{E}} \to$.*
- *The functor $T_{\mathcal{E}} \colon \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X) \to \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$ is an autoequivalence.*
- *The functor $T_{\mathcal{E}}$ is an inverse of FF.*

Thus in addition to extending the previous work to cover $(-3,1)$-curves, our noncommutative deformation technique also extends to the setting of mildly singular schemes. In the course of the proof, we establish the following.

Proposition 1.6 (=5.7, 7.1). *With the assumptions as in 1.5, and writing as above $E := \mathcal{O}_{\mathbb{P}^1}(-1) \in \mathrm{coh}\, X$,*

- *The contraction algebra $\mathrm{A}_{\mathrm{con}}$ is a self-injective finite dimensional algebra.*
- *The universal object $\mathcal{E}$ is a perfect complex (although E need not be).*
- *We have*

$$\mathrm{Ext}^t_X(\mathcal{E}, E) = \left\{ \begin{array}{ll} \mathbb{C} & \textit{if } t = 0, 3 \\ 0 & \textit{else.} \end{array} \right.$$

Remark 1.7. We will not discuss here whether a suitable notion of spherical functors could be applied in our noncommutative setting, but the above suggests that it is possible to use this axiomatic approach, potentially giving noncommutative twists in other triangulated and DG categories. However, we remark that the results in the proposition are not formal consequences of deformation theory, and so proving results in any axiomatic framework will necessarily need to use results obtained in this paper, or find another approach to show 1.6.

The noncommutative deformation base $\mathrm{A}_{\mathrm{con}}$ is artinian, and thus the noncommutative deformations of flopping curves are infinitesimal and may be analysed on the formal fibre. Consequently, to establish the Ext vanishing and other properties we pass to this formal fibre. There, we adapt the techniques of [IW10, §6] and prove 1.6 without an explicit presentation of $\mathrm{A}_{\mathrm{con}}$, or equivalently a filtration structure on $\mathcal{E}$.

In the course of proving that $T_{\mathcal{E}}$ is an autoequivalence, once we have shown that $T_{\mathcal{E}}$ is fully faithful, the usual Serre functor trick [BKR, 2.4] to establish equivalence does not work in our possibly singular setting. However we are able to bypass this by instead giving an elementary argument based on the 'difficult to check' criterion in [H06, 1.50]. This may be of independent interest.

Lemma 1.8 (=7.11). *Let $\mathcal{C}$ be a triangulated category, and $F \colon \mathcal{C} \to \mathcal{C}$ an exact fully faithful functor with right adjoint F^{RA}. Suppose that there exists an object $c \in \mathcal{C}$ such that $F(c) \cong c[i]$ for some i, and further $F(x) \cong x$ for all $x \in c^{\perp}$. Then F is an equivalence.*

1.7. **Necessity of noncommutative deformations.** Restricting to the case when X is smooth, by studying A_{con} we can also prove that Toda's functor $T_{\mathcal{F}}$ defined using commutative deformations is not an autoequivalence for $(-3,1)$-curves. This shows that our noncommutative deformation approach is strictly necessary. To achieve this, we investigate the conditions under which A_{con} is commutative.

Theorem 1.9 (=3.15). *Let $X \to X_{con}$ be the flopping contraction of an irreducible rational curve C, where X is smooth. Then the contraction algebra A_{con} is commutative if and only if C is a $(-1,-1)$-curve or a $(-2,0)$-curve.*

The main content in the above theorem is the 'only if' direction, and we prove it without any explicit case-by-case analysis. When A_{con} is not commutative, we prove in 3.19 that Toda's functor $T_{\mathcal{F}}$ is not an equivalence. Combining with 1.9, we obtain the following corollary.

Corollary 1.10 (=3.19). *With the setup as in 1.9, suppose that C is a $(-3,1)$-curve. Then Toda's functor $T_{\mathcal{F}}$ is not an autoequivalence of $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$.*

1.8. **Conventions.** Unqualified uses of the word 'module' refer to right modules, and $\mathrm{mod}\, A$ denotes the category of finitely generated right A-modules. Sometimes left modules will appear, in which case we will either emphasize the fact that they are left modules, or consider them as objects of $\mathrm{mod}\, A^{\mathrm{op}}$. If $M \in \mathrm{mod}\, A$, we let $\mathrm{add}\, M$ denote all possible summands of finite sums of M. We say that M is a generator if $R \in \mathrm{add}\, M$.

We use the functional convention for composing arrows, so $f \cdot g$ means g then f. With this convention, M is a $\mathrm{End}_R(M)^{\mathrm{op}}$-module. Furthermore, $\mathrm{Hom}_R(M, N)$ is a $\mathrm{End}_R(M)$-module and a $\mathrm{End}_R(N)^{\mathrm{op}}$-module, in fact a bimodule. Note also that $\mathrm{Hom}_R({}_S M_R, {}_T M_R)$ is an S–T bimodule and $\mathrm{Hom}_{R^{\mathrm{op}}}({}_R M_S, {}_R M_T)$ is a T–S bimodule.

Throughout, we work over the field of complex numbers $\mathbb{C}$, but many of our arguments work without this assumption.

1.9. **Notations.** In Appendix A we list our main notations, along with cross-references to their definitions.

2. Noncommutative Deformation Functors

Noncommutative deformations of modules over algebras were first introduced and studied by Laudal [L02]. Later this was extended to the setting of quasi-coherent sheaves over schemes by Eriksen [E07], and also studied by Segal [S08], Efimov–Lunts–Orlov [ELO1, ELO2, ELO3] and others. These works all establish prorepresentability of certain functors, whereas in the situation of flips and flops we will require the much stronger property of representability. The above approaches are not well suited to this problem, so in this section we will adopt a different strategy, based on tilting.

Classically, formal deformation functors are certain covariant functors $\mathsf{CArt} \to \mathsf{Sets}$ from local commutative artinian $\mathbb{C}$-algebras to sets. The idea behind noncommutative deformation theory is that the source category should be enlarged to include noncommutative algebras. It is well known that contractible irreducible rational curves have only infinitesimal commutative deformations, and that this can be detected by passing to a

suitable formal neighbourhood. In this section we will establish that this is also the case for noncommutative deformations, by relating deformations on the 3-fold X to deformations over a certain tilting algebra associated to a formal neighbourhood of the curve C. We prove this in a sequence of reduction steps, first to a Zariski local neighbourhood U in 2.6, then to a $\mathbb{C}$-algebra Λ in 2.8, then finally to the formal fibre in 2.19. From this, representability will be established in §3.

2.1. **Algebras and functors.** A key feature of the theory of noncommutative deformations, following [L02, E07, S08], is to consider simultaneous deformations of discrete families of n objects, which may be parametrised by the n-pointed algebras defined below. For the purposes of this paper, only the case $n = 1$ will be used, however we give the general definition as it highlights features that are hidden in the $n = 1$ case.

Definition 2.1. *An n-pointed $\mathbb{C}$-algebra Γ is an associative $\mathbb{C}$-algebra, together with $\mathbb{C}$-algebra morphisms $p \colon \Gamma \to \mathbb{C}^n$ and $i \colon \mathbb{C}^n \to \Gamma$ such that $p \circ i = \mathrm{Id}$. A morphism of n-pointed $\mathbb{C}$-algebras $\psi \colon (\Gamma, p, i) \to (\Gamma', p', i')$ is an algebra homomorphism $\psi \colon \Gamma \to \Gamma'$ such that*

$$
\begin{array}{ccc}
 & \Gamma & \\
{}^{i}\nearrow & \downarrow{\scriptstyle\psi} & \searrow^{p} \\
\mathbb{C}^n & & \mathbb{C}^n \\
{}_{i'}\searrow & \downarrow & \nearrow_{p'} \\
 & \Gamma' &
\end{array}
$$

commutes. We denote the category of n-pointed $\mathbb{C}$-algebras by Alg_n. We denote the full subcategory consisting of those objects that are commutative rings by CAlg_n.

We write Art_n for the full subcategory of Alg_n consisting of objects (Γ, p, i) for which $\dim_{\mathbb{C}} \Gamma < \infty$ and the augmentation ideal $\mathfrak{n} := \mathrm{Ker}(p)$ is nilpotent. We write CArt_n for the full subcategory of Art_n consisting of those objects that are commutative rings.

Remark 2.2. In the literature, an n-pointed $\mathbb{C}$-algebra is sometimes called an augmented $\mathbb{C}^n$-algebra. We also remark that when $n = 1$ the morphism i is the structure morphism, so can be omitted. Furthermore, CArt_1 is simply the category of commutative artinian local $\mathbb{C}$-algebras, with maximal ideal $\mathfrak{n}$. In our setting below, Art_1 will be the source category for our noncommutative deformation functors, since we will restrict to contractions of curves with 1 irreducible component. The category Art_n is required for contractions of curves with n irreducible components [DW2].

Throughout we will only consider abelian categories $\mathcal{A} = \mathrm{Mod}\,\Lambda$ or $\mathcal{A} = \mathrm{Qcoh}\,Y$ where Λ is a $\mathbb{C}$-algebra and Y is a quasi-projective $\mathbb{C}$-scheme, since considering a general $\mathbb{C}$-linear abelian category requires significantly more technology [LV, S08, ELO1, DW2]. With this restriction, it is possible to pick an object of $\mathcal{A}$ and study its noncommutative deformation functor in a naive way sufficient for our purposes.

Definition 2.3. *For $\mathcal{A}$ an abelian category as above, and an associative $\mathbb{C}$-algebra Γ, the category $\mathrm{Pairs}(\mathcal{A}, \Gamma)$ has as objects pairs (b, ϕ) where $b \in \mathcal{A}$ and ϕ is a $\mathbb{C}$-algebra homomorphism $\phi \colon \Gamma \to \mathrm{End}_{\mathcal{A}}(b)$. Morphisms in $\mathrm{Pairs}(\mathcal{A}, \Gamma)$ are defined to intertwine the action maps ϕ: explicitly, a morphism $f \colon (b, \phi) \to (b', \phi')$ is given by a morphism $f \colon b \to b'$ in $\mathcal{A}$ such that*

$$
\begin{array}{ccc}
b & \overset{f}{\longrightarrow} & b' \\
{\scriptstyle\phi(r)}\big\downarrow & & \big\downarrow{\scriptstyle\phi'(r)} \\
b & \underset{f}{\longrightarrow} & b'
\end{array}
$$

commutes for all $r \in \Gamma$.

Definition 2.4. *Let $\mathcal{A}$ be the abelian category $\mathcal{A} = \operatorname{Mod}\Lambda$ or $\mathcal{A} = \operatorname{Qcoh}Y$ where Λ is a $\mathbb{C}$-algebra and Y is a quasi-projective $\mathbb{C}$-scheme. Pick $a \in \mathcal{A}$, then the* noncommutative deformation functor *is defined*

$$\mathcal{D}ef_a^{\mathcal{A}} \colon \mathsf{Art}_1 \to \mathsf{Sets}$$

by sending

$$(\Gamma, \mathfrak{n}) \mapsto \left\{ ((b,\phi),\delta) \left| \begin{array}{l} (b,\phi) \in \operatorname{Pairs}(\mathcal{A},\Gamma) \\ - \otimes_\Gamma b \colon \operatorname{mod}\Gamma \to \mathcal{A} \text{ is exact} \\ \delta \colon (\Gamma/\mathfrak{n}) \otimes_\Gamma b \xrightarrow{\sim} a \end{array} \right. \right\} \middle/ \sim$$

where $((b,\phi),\delta) \sim ((b',\phi'),\delta')$ if there exists an isomorphism $\tau \colon (b,\phi) \to (b',\phi')$ in $\operatorname{Pairs}(\mathcal{A},\Gamma)$ such that

$$\begin{array}{ccc} (\Gamma/\mathfrak{n}) \otimes_\Gamma b & \xrightarrow{1 \otimes \tau} & (\Gamma/\mathfrak{n}) \otimes_\Gamma b' \\ & \underset{\delta}{\searrow} \quad \underset{a}{\swarrow} \delta' & \end{array}$$

commutes. The commutative deformation functor *is defined to be the restriction of the above functor to the category CArt_1, and is denoted by*

$$c\mathcal{D}ef_a^{\mathcal{A}} \colon \mathsf{CArt}_1 \to \mathsf{Sets}.$$

Remark 2.5. When $\mathcal{A} = \operatorname{Mod}\Lambda$, the functor $- \otimes_\Gamma b$ in 2.4 is the standard tensor functor of modules. When $\mathcal{A} = \operatorname{Qcoh}Y$, the functor $- \otimes_\Gamma b$ is defined in exactly the same way as in the case of tilting equivalences, and can be thought of as tensoring over the noncommutative sheaf of algebras $\mathcal{O}_Y \otimes_{\underline{\mathbb{C}}} \underline{\Gamma}$, where $\underline{\mathbb{C}}$ and $\underline{\Gamma}$ are the constant sheaves on Y associated to $\mathbb{C}$ and Γ (see e.g. §6.4).

The fact that $\mathcal{D}ef_a^{\mathcal{A}}$ is a functor is routine, and we refer the reader to [L02, E07].

2.2. **Contractions and deformation functors.** Our setup is the contraction of an irreducible rational curve C ($C^{\mathrm{red}} \cong \mathbb{P}^1$) in a quasi-projective normal 3-fold X with only Cohen–Macaulay canonical singularities. We remark that throughout the paper, there will be no extra assumptions on the singularities of X unless stated. By a contraction, we mean a projective birational morphism $f \colon X \to X_{\mathrm{con}}$ contracting C to a point p, satisfying $\mathbf{R}f_* \mathcal{O}_X = \mathcal{O}_{X_{\mathrm{con}}}$, such that $X \backslash C \cong X_{\mathrm{con}} \backslash p$. This incorporates both flips and flops.

We are interested in deformations of the sheaf $\mathcal{O}_{\mathbb{P}^1}(-1)$ viewed as a sheaf on X, so we choose an affine open neighbourhood $U_{\mathrm{con}} \cong \operatorname{Spec}R$ around p. Putting $U := f^{-1}(U_{\mathrm{con}})$, we have a commutative diagram

$$\begin{array}{ccccc} C & \overset{e}{\lhook\joinrel\longrightarrow} & U & \overset{i}{\lhook\joinrel\longrightarrow} & X \\ \Big\downarrow & & \Big\downarrow{\scriptstyle f|_U} & & \Big\downarrow{\scriptstyle f} \\ p & \in & U_{\mathrm{con}} & \lhook\joinrel\longrightarrow & X_{\mathrm{con}} \\ \| & & \|\wr & & \\ \mathfrak{m} & \in & \operatorname{Spec}R & & \end{array} \qquad (2.\mathrm{A})$$

where $C^{\mathrm{red}} \cong \mathbb{P}^1$, e is a closed embedding, and i is an open embedding. Since X_{con} is separated and U_{con} is affine, the morphism $U_{\mathrm{con}} \hookrightarrow X_{\mathrm{con}}$ is affine, and since affine morphisms are preserved under pullback it follows that i is an affine morphism. This, together with the fact that i is an open embedding, implies that there is a fully faithful embedding of derived categories

$$\mathbf{R}i_* = i_* \colon \operatorname{D}(\operatorname{Qcoh}U) \hookrightarrow \operatorname{D}(\operatorname{Qcoh}X) \qquad (2.\mathrm{B})$$

with adjoints $i^!$ and i^*.

Throughout we write $E := e_* \mathcal{O}_{\mathbb{P}^1}(-1) \in \operatorname{coh}U$, so that $i_* E \in \operatorname{coh}X$ is just the sheaf $\mathcal{O}_{\mathbb{P}^1}(-1)$ viewed as a sheaf on X. From 2.4, we thus have two deformation functors

$\mathcal{D}ef_E^{\mathrm{Qcoh}\,U}$ and $\mathcal{D}ef_{i_*E}^{\mathrm{Qcoh}\,X}$, both of which take $\mathsf{Art}_1 \to \mathsf{Sets}$. For brevity, we denote them by $\mathcal{D}ef_E^U$ and $\mathcal{D}ef_{i_*E}^X$ respectively. The isomorphism of these deformation functors below is not surprising: it says that noncommutative deformations can be detected Zariski locally. Although elementary, we give the proof in full, since we will need variations of it later.

Proposition 2.6. *There is a natural isomorphism $\mathcal{D}ef_E^U \xrightarrow{\sim} \mathcal{D}ef_{i_*E}^X$ induced by i_*, with inverse induced by i^*. By restriction, there is also a natural isomorphism $c\mathcal{D}ef_E^U \xrightarrow{\sim} c\mathcal{D}ef_{i_*E}^X$, with a corresponding inverse.*

Proof. Given $((\mathcal{F},\phi),\delta) \in \mathcal{D}ef_E^U(\Gamma)$, composing $\Gamma \xrightarrow{\phi} \mathrm{End}_U(\mathcal{F}) \xrightarrow{i_*} \mathrm{End}_X(i_*\mathcal{F})$ gives a pair $(i_*\mathcal{F}, i_*\phi) \in \mathrm{Pairs}(\mathrm{Qcoh}\,X, \Gamma)$. We check that this belongs to $\mathcal{D}ef_{i_*E}^X(\Gamma)$. Firstly, by the projection formula and (2.B), the functor $- \otimes_\Gamma i_*\mathcal{F} \colon \mathrm{mod}\,\Gamma \to \mathrm{Qcoh}\,X$ factorizes as

$$- \otimes_\Gamma i_*\mathcal{F} \colon \mathrm{mod}\,\Gamma \xrightarrow{-\otimes_\Gamma \mathcal{F}} \mathrm{Qcoh}\,U \xrightarrow{i_*} \mathrm{Qcoh}\,X. \tag{2.C}$$

The first functor is exact by definition, and the second functor is exact by (2.B), so the composition $- \otimes_\Gamma i_*\mathcal{F}$ is exact. Next, we have that

$$\Gamma/\mathfrak{n} \otimes_\Gamma i_*\mathcal{F} \xrightarrow{\sim} i_*(\Gamma/\mathfrak{n} \otimes_\Gamma \mathcal{F}) \xrightarrow{i_*(\delta)} i_*E$$

is an isomorphism, so denoting the first isomorphism by α, it follows that

$$\big((i_*\mathcal{F}, i_*\phi),\, i_*(\delta) \cdot \alpha\big) \in \mathcal{D}ef_{i_*E}^X(\Gamma).$$

It is easy to check that this preserves the equivalence relation $\sim$ and is functorial, so gives a natural transformation

$$i_* \colon \mathcal{D}ef_E^U \to \mathcal{D}ef_{i_*E}^X.$$

Conversely, given $((\mathcal{G},\psi),\gamma) \in \mathcal{D}ef_{i_*E}^X(\Gamma)$, composing $\Gamma \xrightarrow{\psi} \mathrm{End}_X(\mathcal{G}) \xrightarrow{i^*} \mathrm{End}_U(i^*\mathcal{G})$ gives a pair $(i^*\mathcal{G}, i^*\psi) \in \mathrm{Pairs}(\mathrm{Qcoh}\,U, \Gamma)$. The functor $- \otimes_\Gamma i^*\mathcal{G} \colon \mathrm{mod}\,\Gamma \to \mathrm{Qcoh}\,U$ factorizes as

$$- \otimes_\Gamma i^*\mathcal{G} \colon \mathrm{mod}\,\Gamma \xrightarrow{-\otimes_\Gamma \mathcal{G}} \mathrm{Qcoh}\,X \xrightarrow{i^*} \mathrm{Qcoh}\,U, \tag{2.D}$$

so as a composition of two exact functors it is exact. Further we have that

$$\Gamma/\mathfrak{n} \otimes_\Gamma i^*\mathcal{G} \xrightarrow{\sim} i^*(\Gamma/\mathfrak{n} \otimes_\Gamma \mathcal{G}) \xrightarrow{i^*(\gamma)} i^*i_*E \xrightarrow{\sim} E$$

is an isomorphism, so denoting the first isomorphism by β, it follows that

$$\big((i_*\mathcal{F}, i_*\phi),\, \varepsilon_E \cdot i^*(\gamma) \cdot \beta\big) \in \mathcal{D}ef_{i_*E}^X(\Gamma)$$

where ε_E is the counit morphism. Again it is easy to check that this preserves the equivalence relation $\sim$ and is functorial, so gives a natural transformation

$$i^* \colon \mathcal{D}ef_{i_*E}^X \to \mathcal{D}ef_E^U.$$

To show that these natural transformations are isomorphisms, we need to show that

$$((\mathcal{F},\phi),\delta) \sim \big((i^*i_*\mathcal{F}, i^*i_*\phi),\, \varepsilon_E \cdot i^*(i_*(\delta) \cdot \alpha) \cdot \beta\big)$$
$$((\mathcal{G},\psi),\gamma) \sim \big((i_*i^*\mathcal{G}, i_*i^*\psi),\, i_*(\varepsilon_E \cdot i^*(\gamma) \cdot \beta) \cdot \alpha\big).$$

We explain the second, the first being slightly easier. Since $\mathcal{G}$ is filtered by the sheaf i_*E, its support is contained in U and so the unit map $\eta_\mathcal{G} \colon \mathcal{G} \to i_*i^*\mathcal{G}$ is an isomorphism. By functoriality of the adjunction it is clear that

$$
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\ \eta_\mathcal{G}\ } & i_*i^*\mathcal{G} \\
{\scriptstyle \psi(r)}\big\downarrow & & \big\downarrow {\scriptstyle i_*i^*\psi(r)} \\
\mathcal{G} & \xrightarrow{\ \eta_\mathcal{G}\ } & i_*i^*\mathcal{G}
\end{array}
$$

commutes for all $r \in \Gamma$, and so $(\mathcal{G}, \psi) \cong (i_* i^* \mathcal{G}, i_* i^* \psi)$ in $\mathrm{Pairs}(\mathrm{Qcoh}\, X, \Gamma)$. Finally, we claim that

$$
\begin{array}{ccc}
(\Gamma/\mathfrak{n}) \otimes_\Gamma \mathcal{G} & \xrightarrow{1 \otimes \eta_\mathcal{G}} & (\Gamma/\mathfrak{n}) \otimes_\Gamma i_* i^* \mathcal{G} \\
& \searrow{\scriptstyle \gamma} \quad \swarrow{\scriptstyle (i_* \varepsilon_E) \cdot (i_* i^*(\gamma)) \cdot (i_* \beta) \cdot \alpha} & \\
& i_* E &
\end{array}
$$

commutes. We rewrite this diagram, adding in a unit morphism as the dotted arrow:

$$
\begin{array}{ccc}
(\Gamma/\mathfrak{n}) \otimes_\Gamma \mathcal{G} & \xrightarrow{\quad 1 \otimes \eta_\mathcal{G} \quad} & (\Gamma/\mathfrak{n}) \otimes_\Gamma i_* i^* \mathcal{G} \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \alpha} \\
{\scriptstyle \eta_{(\Gamma/\mathfrak{n} \otimes \mathcal{G})}} & i_*((\Gamma/\mathfrak{n}) \otimes_\Gamma i^* \mathcal{G}) & \\
{\scriptstyle \gamma}\Big\downarrow & & \Big\downarrow{\scriptstyle i_* \beta} \\
& i_* i^*((\Gamma/\mathfrak{n}) \otimes_\Gamma \mathcal{G}) & \\
& & \Big\downarrow{\scriptstyle i_* i^*(\gamma)} \\
i_* E & \xleftarrow{\quad i_* \varepsilon_E \quad} & i_* i^* i_* E
\end{array}
$$

The bottom half commutes by the functoriality of the unit, since $i_* \varepsilon_E = (\eta_{i_* E})^{-1}$ by the triangular identity. It remains to show that the top half commutes. This just follows by inspection, since locally the maps send

$$
\begin{array}{ccc}
1 \otimes a & \longmapsto & 1 \otimes i_* i^* a \\
& & \Big\uparrow \\
& i_*(1 \otimes i^* a) & \\
& & \Big\uparrow \\
& i_* i^*(1 \otimes a). &
\end{array}
$$

$\square$

2.3. **From geometry to algebra.** In 2.6 we reduced the problem of deforming $i_* E$ in X to deforming E in U. Now we further reduce the problem to deforming a simple module over a certain noncommutative algebra, using a derived equivalence of Van den Bergh [V04]. With respect to our applications later, this step is crucial, since it yields a different description of the universal object, one which is homologically much easier to control.

We keep the setup as in (2.A), so we have a projective birational map $f \colon U \to \mathrm{Spec}\, R$ of 3-folds, with at most one-dimensional fibres, such that $\mathbf{R}f_* \mathcal{O}_U = \mathcal{O}_R$, where R is a Cohen–Macaulay $\mathbb{C}$-algebra. Since the fibre is at most one-dimensional, in this setup it is known [V04, 3.2.8] that there is a bundle $\mathcal{V} := \mathcal{O}_U \oplus \mathcal{N}$ inducing a derived equivalence

$$
\begin{array}{ccc}
\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, U) & \xrightarrow[\sim]{\mathbf{R}\mathrm{Hom}_U(\mathcal{V}, -)} & \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, \mathrm{End}_U(\mathcal{V})) \\
\Big\updownarrow & & \Big\updownarrow \\
{}^0\mathrm{Per}\, U & \xrightarrow[\quad\sim\quad]{} & \mathrm{mod}\, \mathrm{End}_U(\mathcal{V})
\end{array}
\tag{2.E}
$$

Here ${}^0\mathrm{Per}\, U$ denotes the category of perverse sheaves, defined to be

$$
{}^0\mathrm{Per}\, U := \left\{ a \in \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, U) \;\middle|\; \begin{array}{c} H^i(a) = 0 \text{ if } i \neq 0, -1 \\ f_* H^{-1}(a) = 0,\ \mathbf{R}^1 f_* H^0(a) = 0 \\ \mathrm{Hom}(c, H^{-1}(a)) = 0 \text{ for all } c \in \mathcal{C} \end{array} \right\}
$$

where

$$
\mathcal{C} := \{ c \in \mathrm{coh}\, U \mid \mathbf{R}f_* c = 0 \}.
$$

By birationality [V04, 4.2.1], $\mathrm{End}_U(\mathcal{V}) \cong \mathrm{End}_R(f_*\mathcal{V})$, hence we write $N := f_*\mathcal{N}$ and throughout this section we set

$$\Lambda := \mathrm{End}_U(\mathcal{V}) = \mathrm{End}_U(\mathcal{O}_U \oplus \mathcal{N}) \cong \mathrm{End}_R(R \oplus N).$$

Since it is clear that $E = e_*\mathcal{O}_{\mathbb{P}^1}(-1)$, viewed as a complex in degree zero, belongs to $^0\mathrm{Per}\,U$, sending it across the derived equivalence (2.E) yields a module in degree zero.

Definition 2.7. *We write $T := \mathrm{Hom}_U(\mathcal{V}, E) \cong \mathbf{R}\mathrm{Hom}_U(\mathcal{V}, E)$.*

By 2.4, there is a deformation functor $\mathcal{D}ef_T^{\mathrm{Mod}\,\Lambda} \colon \mathsf{Art}_1 \to \mathsf{Sets}$, which to ease notation we denote by $\mathcal{D}ef_T^\Lambda$. The following is an analogue of 2.6, and is proved in a very similar way.

Proposition 2.8. *There is a natural isomorphism $\mathcal{D}ef_E^U \cong \mathcal{D}ef_T^\Lambda$ induced by $\mathrm{Hom}_U(\mathcal{V}, -)$ and $- \otimes_\Lambda \mathcal{V}$, and consequently by restriction a natural isomorphism $c\mathcal{D}ef_E^U \cong c\mathcal{D}ef_T^\Lambda$.*

Proof. The functors $\mathrm{Hom}_U(\mathcal{V}, -) \colon \mathrm{Qcoh}\,U \to \mathrm{Mod}\,\Lambda$ and $- \otimes_\Lambda \mathcal{V} \colon \mathrm{Mod}\,\Lambda \to \mathrm{Qcoh}\,U$ are adjoint. Since T is a module corresponding under the derived equivalence to a sheaf E, we know that $T \cong \mathrm{Hom}_U(\mathcal{V}, T \otimes_\Lambda \mathcal{V})$ and $E \cong \mathrm{Hom}_U(\mathcal{V}, E) \otimes_\Lambda \mathcal{V}$ via the unit and counit morphisms.

The proof is now identical to 2.6, where in the analogue of (2.C) and (2.D), the second functors are no longer exact, but they are exact out of the image of the first functor (using $\mathbf{R}\mathrm{Hom}_U(\mathcal{V}, E) = \mathrm{Hom}_U(\mathcal{V}, E)$ and $T \otimes_\Lambda^{\mathbf{L}} \mathcal{V} = T \otimes_\Lambda \mathcal{V}$ respectively), which is sufficient for the remainder of the proof. $\square$

Definition 2.9. *We define $[R]$ to be the two-sided ideal of $\Lambda = \mathrm{End}_R(R \oplus N)$ consisting of those morphisms $R \oplus N \to R \oplus N$ which factor*

$$R \oplus N \xrightarrow{\hspace{3cm}} R \oplus N$$
$$\searrow \qquad \nearrow$$
$$P$$

through an object $P \in \mathrm{add}\,R$. We set $I_{\mathrm{con}} := [R]$ and define the contraction algebra *associated to Λ to be $\Lambda_{\mathrm{con}} := \Lambda/I_{\mathrm{con}}$.*

Remark 2.10. The contraction algebra is the fundamental object in our paper, but we remark that the algebra Λ_{con} defined above in 2.9 depends on Λ, which in turn depends on the choice of derived equivalence in (2.E); different equivalences may be obtained due to the lack of Krull–Schmidt decompositions of tilting sheaves, see [V04, 3.2.7]. In 2.12 we will define a contraction algebra associated to C which is intrinsic to the geometry of X, and does not involve choices, as shown in 3.9. It will turn out in 5.4 that Λ_{con} will be morita equivalent, though not necessarily isomorphic, to the contraction algebra associated to C.

2.4. **Complete local geometric setting.** Having in 2.6 and 2.8 reduced the problem of deforming i_*E in X to the purely algebraic problem of deforming T in $\mathrm{mod}\,\Lambda$, we next search for a candidate representing object. This requires passing through a morita equivalence; indeed we would like to say that Λ_{con} belongs to Art_1, since then it is a natural candidate for the representing object. However, the algebras in Art_1 have only one simple module, and it is one-dimensional: this is not in general true for Λ_{con} for the reasons in 2.10. Thus to obtain a candidate representing object in Art_1, we need to pass to the basic algebra associated to Λ_{con}. The purpose of this subsection is to obtain this from the formal fibre, and construct the contraction algebra $\mathrm{A}_{\mathrm{con}}$. We will show that $\mathrm{A}_{\mathrm{con}}$ is indeed the representing object in §3.

For $X \in \mathrm{Mod}\,\Lambda$ and $\mathfrak{n} \in \mathrm{Max}\,R$, we write $X_\mathfrak{n} := X \otimes_R R_\mathfrak{n}$, and for $Y \in \mathrm{Mod}\,\Lambda$ we often write $\widehat{Y} := Y_\mathfrak{n} \otimes_{R_\mathfrak{n}} \widehat{R_\mathfrak{n}}$. We require a better description of $\widehat{\Lambda}$. Completing the base

with respect to $\mathfrak{m}$ in (2.A), since endomorphism rings of finitely generated modules behave well under completion [E95, 2.10], it is standard that

$$\widehat{\Lambda} := \Lambda \otimes_R \widehat{R} = \operatorname{End}_R(R \oplus N) \otimes_R \widehat{R} \cong \operatorname{End}_{\widehat{R}}(\widehat{R} \oplus \widehat{N}).$$

On the other hand, considering $\operatorname{Spec} \widehat{R}$ and the formal fibre $f \colon \widehat{U} \to \operatorname{Spec} \widehat{R}$, it is also standard that the tilting bundle $\mathcal{V} = \mathcal{O}_U \oplus \mathcal{N}$ on U restricts to a tilting bundle $\mathcal{O}_{\widehat{U}} \oplus \widehat{\mathcal{N}}$ on $\widehat{U}$. Then by birationality [V04, 3.2.10]

$$\operatorname{End}_{\widehat{U}}(\mathcal{O}_{\widehat{U}} \oplus \widehat{\mathcal{N}}) \cong \operatorname{End}_{\widehat{R}}(f_*(\mathcal{O}_{\widehat{U}} \oplus \widehat{\mathcal{N}})),$$

and this is clearly just $\operatorname{End}_{\widehat{R}}(\widehat{R} \oplus \widehat{N})$. Hence passing to the formal fibre induces a derived equivalence

$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \widehat{U}) \xrightarrow[\sim]{\mathbf{R}\operatorname{Hom}_{\widehat{U}}(\mathcal{O}_{\widehat{U}} \oplus \widehat{\mathcal{N}}, -)} \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \widehat{\Lambda}).$$

There is a more explicit tiling bundle on $\widehat{U}$, constructed in [V04], whose associated tilting algebra, defined in 2.12 below, is the basic algebra morita equivalent to $\widehat{\Lambda}$. For this, let $C = \pi^{-1}(\mathfrak{m})$ where $\mathfrak{m}$ is the unique closed point of $\operatorname{Spec} \widehat{R}$, then giving C the reduced scheme structure, we have $C^{\mathrm{red}} \cong \mathbb{P}^1$. Let $\mathcal{L}$ denote the line bundle on $\widehat{U}$ such that $\mathcal{L} \cdot C = 1$. If the multiplicity of C is equal to one, set $\mathcal{M} := \mathcal{L}$ and $\mathcal{N}_1 := \mathcal{M}^*$ [V04, 3.5.4], else define $\mathcal{M}$ to be given by the maximal extension

$$0 \to \mathcal{O}_{\widehat{U}}^{\oplus (r-1)} \to \mathcal{M} \to \mathcal{L} \to 0$$

associated to a minimal set of $r - 1$ generators of $H^1(\widehat{U}, \mathcal{L}^*)$ and set $\mathcal{N}_1 := \mathcal{M}^*$. Then $\mathcal{O}_{\widehat{U}} \oplus \mathcal{N}_1$ is a tilting bundle on $\widehat{U}$ [V04, 3.5.5]. Again by birationality

$$\operatorname{End}_{\widehat{U}}(\mathcal{O}_{\widehat{U}} \oplus \mathcal{N}_1) \cong \operatorname{End}_{\widehat{R}}(f_*(\mathcal{O}_{\widehat{U}} \oplus \mathcal{N}_1)),$$

and we have $f_*(\mathcal{O}_{\widehat{U}} \oplus \mathcal{N}_1) = \widehat{R} \oplus N_1$ where $N_1 := f_* \mathcal{N}_1$. We remark that $\operatorname{rank}_{\widehat{R}} N_1$ is equal to the scheme-theoretic multiplicity of the curve C [V04, 3.5.4].

Lemma 2.11. *We can write* $\widehat{R} \oplus \widehat{N} \cong \widehat{R}^{\oplus a_0} \oplus N_1^{\oplus a_1}$ *for some* $a_0, a_1 \in \mathbb{N}$.

Proof. Since $\mathcal{O}_U \oplus \mathcal{N} \in {}^0\operatorname{Per} U$ is a progenerator [V04, 3.2.7], $\mathcal{O}_{\widehat{U}} \oplus \widehat{\mathcal{N}} \in {}^0\operatorname{Per} \widehat{U}$ is a progenerator, so by [V04, 3.5.5], there exist positive integers such that

$$\mathcal{O}_{\widehat{U}} \oplus \widehat{\mathcal{N}} \cong \mathcal{O}_{\widehat{U}}^{\oplus a_0} \oplus \mathcal{N}_1^{\oplus a_1}.$$

Pushing down, it follows that $\widehat{R} \oplus \widehat{N} \cong \widehat{R}^{\oplus a_0} \oplus N_1^{\oplus a_1}$. $\qquad\square$

To ease notation with hats, we put $\mathfrak{R} := \widehat{R}$.

Definition 2.12. *We define* $\mathrm{A} := \operatorname{End}_{\mathfrak{R}}(\mathfrak{R} \oplus N_1)$, *the basic algebra morita equivalent to* $\widehat{\Lambda}$, *and define the* contraction algebra *associated to* C *to be* $\mathrm{A}_{\mathrm{con}} := \operatorname{End}_{\mathfrak{R}}(\mathfrak{R} \oplus N_1)/[\mathfrak{R}]$, *where* $[\mathfrak{R}]$ *is defined similarly as in 2.9.*

We show later in 3.9 that the contraction algebra $\mathrm{A}_{\mathrm{con}}$ is intrinsic to the geometry of X, and in particular does not depend on the choice of derived equivalence in (2.E). It is not hard to see, and we will prove it later in 5.4, that $\mathrm{A}_{\mathrm{con}}$ is morita equivalent to $\widehat{\Lambda}_{\mathrm{con}}$. However, the morita equivalence between A and $\widehat{\Lambda}$ is easy to describe, and we do this now. We write

$$Y := \mathfrak{R} \oplus N_1, \qquad Z := \mathfrak{R}^{\oplus a_0} \oplus N_1^{\oplus a_1},$$

so that $\mathrm{A} = \operatorname{End}_{\mathfrak{R}}(Y)$ and $\widehat{\Lambda} = \operatorname{End}_{\mathfrak{R}}(Z)$. Then we put

$$P := \operatorname{Hom}_{\mathfrak{R}}(Y, Z), \qquad Q := \operatorname{Hom}_{\mathfrak{R}}(Z, Y).$$

These have the structure of bimodules, namely ${}_{\widehat{\Lambda}}P_{\mathrm{A}}$ and ${}_{\mathrm{A}}Q_{\widehat{\Lambda}}$. It is clear that P is a progenerator, and we have a morita context

$$(\mathrm{A}, \widehat{\Lambda}, P, Q).$$

In particular this implies that

$$\operatorname{mod} A \xleftarrow[\operatorname{Hom}_{\widehat{\Lambda}}(Q,-)=-\otimes_{\widehat{\Lambda}} P]{\mathbb{F}:=\operatorname{Hom}_A(P,-)=-\otimes_A Q} \operatorname{mod} \widehat{\Lambda}$$

are equivalences.

We now show that, in our situation of flips and flops, the algebras Λ_{con} and A_{con} are finite dimensional. This will be used later to obtain numerical invariants associated to the curve C.

Proposition 2.13. *With the setup as in §2.2, and with notation as above,*

(1) Λ_{con} *and* A_{con} *are finite dimensional algebras, which as R-modules are supported only at $\mathfrak{m}$.*

(2) *As an R-module, $\mathbb{F}A_{\mathrm{con}}$ is supported only at $\mathfrak{m}$.*

(3) *T is a simple Λ-module, which as an R-module is supported only at $\mathfrak{m}$.*

Proof. (1) The contraction $U \to \operatorname{Spec} R$ is an isomorphism away from a single point $\mathfrak{m} \in \operatorname{Max} R$. By base change, it is clear that $\operatorname{add} N_{\mathfrak{p}} = \operatorname{add} R_{\mathfrak{p}}$ for all $\mathfrak{p} \neq \mathfrak{m}$, so $\Lambda_{\mathrm{con}\,\mathfrak{p}} = 0$ for all $\mathfrak{p} \neq \mathfrak{m}$. It follows that Λ_{con} is supported only on $\mathfrak{m}$, and so in particular is finite dimensional. The proof for A_{con} is identical, using the fact that $\widehat{U} \to \operatorname{Spec} \widehat{R}$ is an isomorphism away from the closed point.

(2) follows immediately from (1).

(3) The fact that E is simple in $^0\mathrm{Per}\,\widehat{U}$ is [V04, 3.5.8]. Thus $\widehat{T}$ is a simple $\widehat{\Lambda}$-module, since it corresponds to a simple module via a morita equivalence. Now let e denote the idempotent in Λ corresponding to R, then

$$\begin{array}{ccc} \mathrm{D}^{\mathrm{b}}(\operatorname{coh} U) & \xrightarrow[\sim]{\mathbf{R}\operatorname{Hom}_U(\mathcal{V},-)} & \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda) \\ \downarrow{\scriptstyle \mathbf{R}f_*} & & \downarrow{\scriptstyle (-)e} \\ \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \operatorname{Spec} R) & =\!=\!=\!=\!=\!= & \mathrm{D}^{\mathrm{b}}(\operatorname{mod} R) \end{array}$$

commutes. Since $\mathbf{R}f_*E = 0$, it follows that $Te = 0$, and so T is a finitely generated Λ_{con}-module. In particular, by (1) T is supported only at $\mathfrak{m}$, so $T \cong \widehat{T}$. It follows that T is a simple Λ-module. $\square$

Definition 2.14. *We denote by S the simple A-module corresponding across the morita equivalence to $\widehat{T}$.*

Note that since A is basic, $\dim_{\mathbb{C}} S = 1$. It is also clear that S can be viewed as a simple A_{con}-module, and it is the unique simple A_{con}-module. Under the running flips and flops setting, A_{con} is further finite dimensional by 2.13(1). Thus $A_{\mathrm{con}} \in \mathsf{Art}_1$, making it the candidate for the representing object.

2.5. **Reduction to complete local setting.** The proof that A_{con} represents the deformation functors in 2.6 and 2.8 requires one more reduction step, namely we must relate them to a similar deformation functor on the formal fibre. This step is largely routine, although in this subsection we do introduce notation and known results that will be used later.

The following is well known, and will be used throughout.

Lemma 2.15. *Let $X \in \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda)$, $Y \in \mathrm{D}(\operatorname{Mod} \Lambda)$, then*

(1) $\mathbf{R}\operatorname{Hom}_{\Lambda}(X,Y) \otimes_R R_{\mathfrak{m}} \cong \mathbf{R}\operatorname{Hom}_{\Lambda_{\mathfrak{m}}}(X_{\mathfrak{m}}, Y_{\mathfrak{m}})$.

(2) $\mathbf{R}\operatorname{Hom}_{\Lambda}(X,Y)_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} \widehat{R}_{\mathfrak{m}} \cong \mathbf{R}\operatorname{Hom}_{\widehat{\Lambda}}(\widehat{X}, \widehat{Y})$.

(3) *If $X \in \operatorname{mod} \Lambda$ with $\dim_{\mathbb{C}} X < \infty$, then $X \cong \bigoplus_{\mathfrak{m} \in \operatorname{Supp}_R X} X_{\mathfrak{m}} \cong \bigoplus_{\mathfrak{m} \in \operatorname{Supp}_R X} \widehat{X}$ as Λ-modules.*

Proof. (1) and (2) are [IR08, p1100].

(3) The natural map $\psi \colon X \to \bigoplus_{\mathfrak{m} \in \mathrm{Supp}_R X} X_{\mathfrak{m}}$ is clearly a Λ-module homomorphism. Viewing ψ as an R-module homomorphism, since X has finite length, ψ is bijective by [E95, 2.13b]. It follows that ψ is a Λ-module isomorphism. The last isomorphism follows since if $X_{\mathfrak{m}}$ has finite length, then $X_{\mathfrak{m}} \cong \widehat{X}$ as $\Lambda_{\mathfrak{m}}$-modules. $\square$

Now $\Lambda = \mathrm{End}_R(R \oplus N)$ and $\Lambda_{\mathrm{con}} = \Lambda/[R]$. Completing with respect to $\mathfrak{m}$, $\widehat{\Lambda} \cong \mathrm{End}_{\widehat{R}}(\widehat{R} \oplus \widehat{N})$ and we already know from §2.4 that $\widehat{N}$ may decompose into many more summands than N does. However, the following states that the localization (resp. completion) of the contraction algebra associated to Λ is the contraction algebra associated to the localization $\Lambda_{\mathfrak{m}}$ (resp. completion $\widehat{\Lambda}$). This will allow us to reduce many of our problems to the formal fibre.

Lemma 2.16. *With notation as above,*

(1) $[R] \otimes_R R_{\mathfrak{m}} \cong [R_{\mathfrak{m}}]$. *We denote this by* $I_{\mathfrak{m}\mathrm{con}}$.
(2) $[R_{\mathfrak{m}}] \otimes_R \widehat{R} \cong [\widehat{R}]$. *We denote this by* $\widehat{I}_{\mathrm{con}}$.
(3) $\Lambda_{\mathrm{con}} \otimes_R R_{\mathfrak{m}} \cong \Lambda_{\mathfrak{m}}/[R_{\mathfrak{m}}]$ *and* $\Lambda_{\mathrm{con}} \otimes_R \widehat{R} \cong \widehat{\Lambda}/[\widehat{R}]$.

Proof. Since localization and completion are exact, (3) follows from the first two. We prove (2), with (1) being similar. To ease notation, we temporarily write $M := R \oplus N$.

It is clear that $[R_{\mathfrak{m}}] \otimes_R \widehat{R} \subseteq [\widehat{R}]$, so let $\psi \in [\widehat{R}]$. Then $\psi \colon \widehat{M} \to \widehat{M}$ factors through add $\widehat{R}$, so by definition ψ factors through a summand of $\widehat{R}^a$ for some $a \in \mathbb{N}$. This implies that ψ factors through $\widehat{R}^a$, so there is a commutative diagram

$$\widehat{M} \xrightarrow{\ \ \psi\ \ } \widehat{M}$$
$$\widehat{M} \xrightarrow{\alpha} \widehat{R}^a \xrightarrow{\beta} \widehat{M}$$

As in 2.15(2) $\mathrm{Hom}_{\Lambda_{\mathfrak{m}}}(M_{\mathfrak{m}}, R_{\mathfrak{m}}^a) \otimes_{R_{\mathfrak{m}}} \widehat{R} \cong \mathrm{Hom}_{\widehat{\Lambda}}(\widehat{M}, \widehat{R}^a)$ etc, so we can find $\Lambda_{\mathfrak{m}}$-module homomorphisms

$$M_{\mathfrak{m}} \xrightarrow{\ \ f\ \ } M_{\mathfrak{m}}$$
$$M_{\mathfrak{m}} \xrightarrow{g} R_{\mathfrak{m}}^a \xrightarrow{h} M_{\mathfrak{m}} \tag{2.F}$$

such that $\widehat{f} = \psi$, $\widehat{g} = \alpha$ and $\widehat{h} = \beta$. It remains to show that (2.F) commutes, since then $f \in [R_{\mathfrak{m}}]$. But since the completion is exact

$$\mathrm{Im}(f - hg) \otimes \widehat{R} \cong \mathrm{Im}\,\widehat{f - hg} = 0,$$

so the completion of $\mathrm{Im}(f - hg)$ is zero, which implies that $\mathrm{Im}(f - hg) = 0$, i.e. $f = hg$. $\square$

Below, we write $\Lambda_{\mathfrak{m}\mathrm{con}}$ for the contraction algebra associated to $\Lambda_{\mathfrak{m}}$, and $\widehat{\Lambda}_{\mathrm{con}}$ for the contraction algebra associated to $\widehat{\Lambda}$. By the above, $\Lambda_{\mathfrak{m}\mathrm{con}} \cong (\Lambda_{\mathrm{con}})_{\mathfrak{m}}$ and $\widehat{\Lambda}_{\mathrm{con}} \cong \widehat{(\Lambda_{\mathrm{con}})}$, so there is no ambiguity.

Lemma 2.17. $\Lambda_{\mathrm{con}} \cong \widehat{\Lambda}_{\mathrm{con}}$ *both as Λ-modules and as algebras.*

Proof. Since Λ_{con} is supported only at $\mathfrak{m}$ by 2.13, it follows from 2.16 and 2.15(3) that the natural map $\Lambda_{\mathrm{con}} \to \widehat{\Lambda}_{\mathrm{con}}$ is bijective. Since it is both a Λ-module homomorphism and an algebra homomorphism, we have $\Lambda_{\mathrm{con}} \cong \widehat{\Lambda}_{\mathrm{con}}$ both as Λ-modules, and as algebras. $\square$

Warning 2.18. Often when studying the birational geometry of global 3-folds, we are forced to flip or flop together several irreducible curves that don't intersect. When this occurs, $\Lambda_{\mathrm{con}} \not\cong \widehat{\Lambda}_{\mathrm{con}}$, instead $\Lambda_{\mathrm{con}} \cong \bigoplus_{\mathfrak{m} \in \mathrm{Supp}\,\Lambda_{\mathrm{con}}} \widehat{\Lambda}_{\mathrm{con}}$.

Hence we have the following functors

$$
\mathrm{D^b(Mod\,\Lambda)} \underset{\mathrm{res}}{\overset{\otimes_R R_{\mathfrak{m}}}{\rightleftarrows}} \mathrm{D^b(Mod\,\Lambda_{\mathfrak{m}})} \underset{\mathrm{res}}{\overset{\otimes_{R_{\mathfrak{m}}} \widehat{R}}{\rightleftarrows}} \mathrm{D^b(Mod\,\widehat{\Lambda})} \overset{\mathbb{F}}{\longleftrightarrow} \mathrm{D^b(Mod\,A)}
$$

$$
\begin{array}{ccccccc}
T & \longmapsto & T_{\mathfrak{m}} & \longmapsto & \widehat{T} & \longleftrightarrow & S \\
\Lambda_{\mathrm{con}} & \longmapsto & (\Lambda_{\mathfrak{m}})_{\mathrm{con}} & \longmapsto & \widehat{\Lambda}_{\mathrm{con}} & & \\
\mathbb{F}\mathrm{A_{con}} & \longleftarrow\!\shortmid & \mathbb{F}\mathrm{A_{con}} & \longleftarrow\!\shortmid & \mathbb{F}\mathrm{A_{con}} & \longleftrightarrow & \mathrm{A_{con}}
\end{array}
$$

where res denotes restriction of scalars along the natural ring homomorphisms. The line for Λ_{con} follows by 2.16. Note that the first two functors are not equivalences, however, since T, Λ_{con} and $\mathrm{A_{con}}$ are supported only at $\mathfrak{m}$ we have $T \cong \mathrm{res}(T_{\mathfrak{m}})$ etc, so the arrows in the bottom three lines in the above diagram can also be drawn in the opposite direction.

Given this, the following is not surprising, and is a standard application of the proofs and techniques developed so far.

Proposition 2.19. *There are natural isomorphisms*

$$
\mathcal{D}ef_T^\Lambda \cong \mathcal{D}ef_{T_{\mathfrak{m}}}^{\Lambda_{\mathfrak{m}}} \cong \mathcal{D}ef_{\widehat{T}}^{\widehat{\Lambda}} \cong \mathcal{D}ef_S^{\mathrm{A}},
$$

and consequently by restriction natural isomorphisms

$$
c\mathcal{D}ef_T^\Lambda \cong c\mathcal{D}ef_{T_{\mathfrak{m}}}^{\Lambda_{\mathfrak{m}}} \cong c\mathcal{D}ef_{\widehat{T}}^{\widehat{\Lambda}} \cong c\mathcal{D}ef_S^{\mathrm{A}}.
$$

Proof. The functors $-\otimes_R R_{\mathfrak{m}} \colon \mathrm{Mod\,\Lambda} \to \mathrm{Mod\,\Lambda_{\mathfrak{m}}}$ and $\mathrm{res} \colon \mathrm{Mod\,\Lambda_{\mathfrak{m}}} \to \mathrm{Mod\,\Lambda}$ are adjoint, and are both exact. Since, by 2.13, T is a finite length module supported only at $\mathfrak{m}$, $T \cong \mathrm{res}(T \otimes_R R_{\mathfrak{m}})$ and $T_{\mathfrak{m}} \cong \mathrm{res}(T_{\mathfrak{m}}) \otimes_R R_{\mathfrak{m}}$ via the unit and counit morphisms.

The proof is now identical to 2.6, where in the analogue of (2.C) and (2.D), the second functors are replaced by $-\otimes_R R_{\mathfrak{m}}$ and res, which are still exact. This establishes the first isomorphism.

For the second, again restriction and extension of scalars give an exact adjoint pair, and since $T_{\mathfrak{m}}$ is a finite length module, the necessary unit and counit maps are isomorphisms, since restriction and extension of scalars are an equivalence on finite length modules. The proof is then identical to the above, establishing the second isomorphism. The third isomorphism also follows in an identical manner. $\square$

Thus, in conclusion, combining 2.6, 2.8 and 2.19 gives a natural isomorphism

$$
\mathcal{D}ef_{i_*E}^X \cong \mathcal{D}ef_S^{\mathrm{A}} \colon \mathsf{Art}_1 \to \mathsf{Sets}, \tag{2.G}
$$

and so to prove representability, we have reduced the problem to showing that $\mathcal{D}ef_S^{\mathrm{A}}$ is representable. Since A is basic (unlike Λ), by 2.13 $\mathrm{A_{con}} \in \mathsf{Art}_1$ and so this is the natural candidate for the representing object.

3. Representability and Contraction Algebras

We keep the setup in §2.2. In the previous section we established natural isomorphisms of functors

$$
\mathcal{D}ef_{i_*E}^X \cong \mathcal{D}ef_E^U \cong \mathcal{D}ef_S^{\mathrm{A}} \colon \mathsf{Art}_1 \to \mathsf{Sets} \tag{3.A}
$$

and showed that $\mathrm{A_{con}} \in \mathsf{Art}_1$. In this section we will show that $\mathrm{A_{con}}$ is a representing object for these functors, and describe the associated universal families. This then allows us to define numerical invariants associated to the curve C, and show that Toda's commutative deformation functor is not an equivalence in general.

3.1. **Representability on the basic algebra** A. In this subsection we show that $\mathcal{D}ef_S^A$ and $c\mathcal{D}ef_S^A$ are representable. This statement turns out to be an elementary consequence of the definitions, and is implicit in [E03, E07, S08], albeit in a slightly modified setting.

Proposition 3.1. $\mathcal{D}ef_S^A \cong \operatorname{Hom}_{\mathsf{Art}_1}(A_{\mathrm{con}}, -)\colon \mathsf{Art}_1 \to \mathsf{Sets}$.

Proof. Since $S = \mathbb{C}$ is an A-module, this induces a homomorphism $A \to \mathbb{C}$ which we denote by q. Since objects in $\operatorname{Pairs}(\operatorname{Mod} A, \Gamma)$ are bimodules $_\Gamma M_A$,

$$\mathcal{D}ef_S^A(\Gamma) = \left\{ (_\Gamma M_A, \delta) \,\middle|\, \begin{array}{l} - \otimes_\Gamma M\colon \operatorname{mod}\Gamma \to \operatorname{mod}A \text{ is exact} \\ \delta\colon (\Gamma/\mathfrak{n}) \otimes_\Gamma M \xrightarrow{\sim} S \text{ as A-modules} \end{array} \right\} \Big/ \sim$$

$$= \left\{ \begin{array}{l} \bullet \quad \text{a right A-module structure on } \Gamma \otimes_\mathbb{C} S \text{ such that} \\ \quad \Gamma \otimes_\mathbb{C} S \text{ becomes a } \Gamma\text{-A bimodule.} \\ \bullet \quad \delta\colon (\Gamma/\mathfrak{n}) \otimes_\Gamma (\Gamma \otimes_\mathbb{C} S) \xrightarrow{\sim} S \text{ as A-modules} \end{array} \right\} \Big/ \sim$$

$$= \left\{ \begin{array}{l} \bullet \quad \text{a } \mathbb{C}\text{-algebra homomorphism } A \to \Gamma \text{ such that} \\ \quad \text{the composition } A \to \Gamma \to \Gamma/\mathfrak{n} = \mathbb{C} \text{ is } q \end{array} \right\} \Big/ \sim$$

which is just $\operatorname{Hom}_{\mathsf{Alg}_1}(A_{\mathrm{con}}, \Gamma)$ since q factors through A_{con}. The result then follows since $A_{\mathrm{con}} \in \mathsf{Art}_1$ which is by definition a full subcategory of Alg_1. $\square$

Corollary 3.2. $c\mathcal{D}ef_S^A \cong \operatorname{Hom}_{\mathsf{CArt}_1}(A_{\mathrm{con}}^{\mathrm{ab}}, -)\colon \mathsf{CArt}_1 \to \mathsf{Sets}$.

Proof. Since any homomorphism from A_{con} to a commutative ring Γ must kill the two-sided ideal in A_{con} generated by the commutators, the result follows since

$$c\mathcal{D}ef_S^A(\Gamma) \overset{3.1}{=} \operatorname{Hom}_{\mathsf{Alg}_1}(A_{\mathrm{con}}, \Gamma) \cong \operatorname{Hom}_{\mathsf{Alg}_1}(A_{\mathrm{con}}^{\mathrm{ab}}, \Gamma) = \operatorname{Hom}_{\mathsf{CArt}_1}(A_{\mathrm{con}}^{\mathrm{ab}}, \Gamma). \quad \square$$

3.2. **Chasing through the representing couple.** It follows from 3.1 that all the functors in (3.A) are representable. In this subsection we chase the representing couples through the relevant isomorphisms, as we need the universal sheaves in geometric terms for our applications later.

If $F\colon \mathsf{Art}_1 \to \mathsf{Sets}$ is a deformation functor, recall that (Γ, ξ) is a *couple* for F if $\Gamma \in \mathsf{Art}_1$ and $\xi \in F(\Gamma)$. A couple (Γ, ξ) induces a natural transformation

$$\alpha_\xi\colon \operatorname{Hom}_{\mathsf{Art}_1}(\Gamma, -) \to F$$

which when applied to $\Gamma_2 \in \mathsf{Art}_1$ simply takes $\phi \in \operatorname{Hom}_{\mathsf{Art}_1}(\Gamma, \Gamma_2)$ to the element $F(\phi)(\xi)$ of $F(\Gamma_2)$. We say that a couple (Γ, ξ) *represents* F if α_ξ is a natural isomorphism of functors on Art_1. If a couple (Γ, ξ) represents F, then it is unique up to unique isomorphism of couples.

Now given the natural isomorphism $\beta\colon \operatorname{Hom}_{\mathsf{Art}_1}(A_{\mathrm{con}}, -) \xrightarrow{\sim} \mathcal{D}ef_S^A$ in 3.1, the couple $(A_{\mathrm{con}}, \beta(\operatorname{Id}_{A_{\mathrm{con}}})) = (A_{\mathrm{con}}, {}_{A_{\mathrm{con}}}(A_{\mathrm{con}})_A)$ represents $\mathcal{D}ef_S^A$. Simply tracking the element ${}_{A_{\mathrm{con}}}(A_{\mathrm{con}})_A \in \mathcal{D}ef_S^A(A_{\mathrm{con}})$ through the isomorphisms

$$\mathcal{D}ef_S^A \xrightarrow{\mathbb{F}} \mathcal{D}ef_{\widehat{T}}^{\widehat{\Lambda}} \xrightarrow{\operatorname{res}} \mathcal{D}ef_T^\Lambda \xrightarrow{-\otimes_\Lambda \mathcal{V}} \mathcal{D}ef_E^U \xrightarrow{i_*} \mathcal{D}ef_{i_*E}^X$$

we obtain the following:

Corollary 3.3. *In the setup of* §2.2,
  (1) *The couple* $(A_{\mathrm{con}}, \mathbb{F}A_{\mathrm{con}} \otimes_\Lambda \mathcal{V})$ *represents* $\mathcal{D}ef_E^U$.
  (2) *The couple* $(A_{\mathrm{con}}, i_*(\mathbb{F}A_{\mathrm{con}} \otimes_\Lambda \mathcal{V}))$ *represents* $\mathcal{D}ef_{i_*E}^X$.

We now fix notation.

Definition 3.4. *We define* $\mathcal{E}_U := \mathbb{F}A_{\mathrm{con}} \otimes_\Lambda \mathcal{V} \in \operatorname{coh}U$, *and* $\mathcal{E} := i_*(\mathcal{E}_U)$.

Thus $(A_{\mathrm{con}}, \mathcal{E})$ is the representing couple for $\mathcal{D}ef_{i_*E}^X$. Similarly to the above, we can deduce from 3.2 that all the commutative deformation functors have representing couples, with the left-hand term $A_{\mathrm{con}}^{\mathrm{ab}}$. Chasing through, it is obvious that the representing couple for $c\mathcal{D}ef_E^U$ is given by a coherent sheaf. We again fix notation.

Definition 3.5. *We write $\mathcal{F}_U \in \operatorname{coh} U$ for the coherent sheaf such that $(\mathrm{A}_{\mathrm{con}}^{\mathrm{ab}}, \mathcal{F}_U)$ represents $c\mathcal{D}ef_E^U$, and write $\mathcal{F} := i_*\mathcal{F}_U$ so that $(\mathrm{A}_{\mathrm{con}}^{\mathrm{ab}}, \mathcal{F})$ represents $c\mathcal{D}ef_{i_*E}^X$.*

Since i_* need not preserve coherence, we must prove that $\mathcal{E}$ and $\mathcal{F}$ are coherent.

Lemma 3.6. *With assumptions and notation as above,*

(1) $\mathcal{F}_U$ *and* $\mathcal{E}_U$ *are filtered by the sheaf* E.
(2) $\mathbf{R}i_*\mathcal{E}_U = i_*\mathcal{E}_U = \mathcal{E}$ *and* $\mathbf{R}i_*\mathcal{F}_U = i_*\mathcal{F}_U = \mathcal{F}$.
(3) $\mathcal{F}$ *and* $\mathcal{E}$ *are filtered by the sheaf* i_*E, *hence are coherent.*

Proof. Denote the radical of $\mathrm{A}_{\mathrm{con}}$ by J.
(1) Since $\mathcal{E}_U \in \mathcal{D}ef_E^U(\mathrm{A}_{\mathrm{con}})$, the functor $- \otimes_{\mathrm{A}_{\mathrm{con}}} \mathcal{E}_U \colon \operatorname{mod} \mathrm{A}_{\mathrm{con}} \to \operatorname{Qcoh} U$ is exact, and $(\mathrm{A}_{\mathrm{con}}/J) \otimes_{\mathrm{A}_{\mathrm{con}}} \mathcal{E}_U \cong E$. Since $\mathrm{A}_{\mathrm{con}}$ is a finite dimensional algebra, filtering $\mathrm{A}_{\mathrm{con}}$ by its simple module $(\mathrm{A}_{\mathrm{con}}/J)$ and applying $- \otimes_{\mathrm{A}_{\mathrm{con}}} \mathcal{E}_U$ yields the result. The case of $\mathcal{F}_U$ is identical.
(2) follows from (2.B), and (3) follows by combining (1) and (2). $\qquad\square$

In a similar vein, using the fact $\operatorname{Hom}_U(\mathcal{V}, E) = \mathbf{R}\operatorname{Hom}_U(\mathcal{V}, E)$ in 2.7 together with a filtration argument gives the following, which will be used later.

Lemma 3.7. *With assumptions and notation as above,*

(1) $\mathbf{R}\operatorname{Hom}_U(\mathcal{V}, \mathcal{E}_U) \cong \mathbb{F}\mathrm{A}_{\mathrm{con}}$ *and* $\mathbb{F}\mathrm{A}_{\mathrm{con}} \otimes_\Lambda^{\mathbf{L}} \mathcal{V} \cong \mathcal{E}_U$.
(2) $\mathbf{R}\operatorname{Hom}_U(\mathcal{V}, \mathcal{F}_U) \cong \mathbb{F}\mathrm{A}_{\mathrm{con}}^{\mathrm{ab}}$ *and* $\mathbb{F}\mathrm{A}_{\mathrm{con}}^{\mathrm{ab}} \otimes_\Lambda^{\mathbf{L}} \mathcal{V} \cong \mathcal{F}_U$.

3.3. **The contraction algebra of C and $\operatorname{wid}(C)$.**

Definition 3.8. *With the setup and assumptions as in §2.2, suppose that $X \to X_{\mathrm{con}}$ is the contraction of an irreducible rational curve C. We define the* noncommutative deformation algebra *of C to be* $\operatorname{End}_X(\mathcal{E})$.

The noncommutative deformation algebra is the fundamental object in our paper. The above definition is intrinsic to the geometry of X, since $\mathcal{E}$ arises from the representing couple of the noncommutative deformations of the sheaf $\mathcal{O}_{\mathbb{P}^1}(-1)$ in X. However, it is hard to calculate in the above form. It is the following description of $\operatorname{End}_X(\mathcal{E})$ as the contraction algebra $\mathrm{A}_{\mathrm{con}} = \operatorname{End}_{\widehat{R}}(\widehat{R} \oplus N_1)/[\widehat{R}]$ that both makes it possible to calculate $\operatorname{End}_X(\mathcal{E})$, and also to control it homologically.

Lemma 3.9. *There are isomorphisms*

$$\operatorname{End}_X(\mathcal{E}) \cong \operatorname{End}_U(\mathcal{E}_U) \cong \operatorname{End}_\Lambda(\mathbb{F}\mathrm{A}_{\mathrm{con}}) \cong \operatorname{End}_{\widehat{\Lambda}}(\mathbb{F}\mathrm{A}_{\mathrm{con}}) \cong \operatorname{End}_{\mathrm{A}}(\mathrm{A}_{\mathrm{con}}) \cong \mathrm{A}_{\mathrm{con}}.$$

Furthermore, $\operatorname{End}_X(\mathcal{F}) \cong \operatorname{End}_X(\mathcal{E})^{\mathrm{ab}}$.

Proof. The first isomorphism uses the fully faithful embedding of derived categories $\mathbf{R}i_*$, together with the fact that $\mathbf{R}i_*\mathcal{E}_U \cong \mathcal{E}$ by (2.B). The second isomorphism follows from the equivalence (2.E) together with 3.7. The third isomorphism follows since $\mathbb{F}\mathrm{A}_{\mathrm{con}}$ is supported only at $\mathfrak{m}$ (by 2.13), and the fact that restriction and extension of scalars are an equivalence on these finite length subcategories. The fourth isomorphism is just chasing across a morita equivalence, and the last is simply

$$\operatorname{End}_{\mathrm{A}}(\mathrm{A}_{\mathrm{con}}) = \operatorname{End}_{\mathrm{A}}(\mathrm{A}/\mathrm{A}e\mathrm{A}) \cong \mathrm{A}/\mathrm{A}e\mathrm{A} = \mathrm{A}_{\mathrm{con}}$$

where e is the idempotent in A corresponding to $\widehat{R}$.

For the final statement that $\operatorname{End}_X(\mathcal{F}) \cong \operatorname{End}_X(\mathcal{E})^{\mathrm{ab}}$, observe that

$$\mathrm{A}_{\mathrm{con}}^{\mathrm{ab}} \cong \operatorname{Hom}_{\mathrm{A}}(\mathrm{A}, \mathrm{A}_{\mathrm{con}}^{\mathrm{ab}}) \cong \operatorname{Hom}_{\mathrm{A}}(\mathrm{A}_{\mathrm{con}}, \mathrm{A}_{\mathrm{con}}^{\mathrm{ab}}) \cong \operatorname{End}_{\mathrm{A}}(\mathrm{A}_{\mathrm{con}}^{\mathrm{ab}}) \overset{3.7}{\cong} \operatorname{End}_U(\mathcal{F}_U) \overset{3.6}{\cong} \operatorname{End}_X(\mathcal{F})$$

and chasing through this establishes that $\mathrm{A}_{\mathrm{con}}^{\mathrm{ab}} \cong \operatorname{End}_X(\mathcal{F})$ as rings. $\qquad\square$
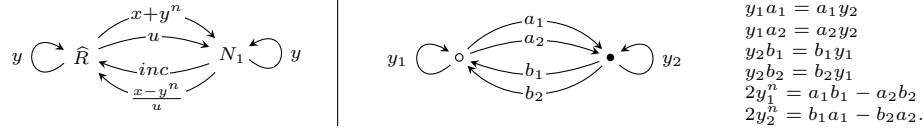
Thus throughout the remainder of the paper, we use the terms 'contraction algebra' and 'noncommutative deformation algebra' interchangeably.

Example 3.10. (Pagoda [R83]) Consider

$$\widehat{R} := \frac{\mathbb{C}[[u,v,x,y]]}{uv = (x-y^n)(x+y^n)}$$

for some $n \geq 1$. It is well known that $N_1 \cong (u, x+y^n)$.

We can present $A := \operatorname{End}_{\widehat{R}}(\widehat{R} \oplus N_1)$ as the completion of the quiver with relations

$$\begin{aligned}
y_1 a_1 &= a_1 y_2 \\
y_1 a_2 &= a_2 y_2 \\
y_2 b_1 &= b_1 y_1 \\
y_2 b_2 &= b_2 y_1 \\
2y_1^n &= a_1 b_1 - a_2 b_2 \\
2y_2^n &= b_1 a_1 - b_2 a_2.
\end{aligned}$$

To obtain $A_{\operatorname{con}} = \operatorname{End}_{\widehat{R}}(\widehat{R} \oplus N_1)/[\widehat{R}]$ we factor by all arrows that factor through the vertex $\widehat{R}$, and so $A_{\operatorname{con}} \cong \mathbb{C}[[y_2]]/2y_2^n \cong \mathbb{C}[y]/y^n$.

Definition 3.11. *Let $X \to X_{\operatorname{con}}$ be a contraction of an irreducible rational curve C and let $\mathcal{E} \in \operatorname{coh} X$ be as defined in 3.4. We define*

(1) *the* width *of C to be* $\operatorname{wid}(C) := \dim_{\mathbb{C}} \operatorname{End}_X(\mathcal{E})$.
(2) *the* commutative width *of C to be* $\operatorname{cwid}(C) := \dim_{\mathbb{C}} \operatorname{End}_X(\mathcal{E})^{\operatorname{ab}}$.

We remark that the width of a $(-3,1)$-curve is not defined [R83], and so in this case (as well as in the singular setting, and the flipping setting) the invariants in 3.11 are new. In contrast to determining an explicit presentation for $\operatorname{End}_X(\mathcal{E})$, which sometimes can be hard, to calculate $\operatorname{wid}(C)$ is much simpler, and can be achieved by using just commutative algebra on the base singularity $\widehat{R}$. We refer the reader to 5.2 later.

Example 3.12. (Pagoda) In 3.10, $A_{\operatorname{con}} \cong \mathbb{C}[y]/y^n$ which is commutative, so

$$\operatorname{wid}(C) = \operatorname{cwid}(C) = n.$$

Example 3.13. (Francia flip) Consider the ring $R := \mathbb{C}[[a,b,c_1,c_2,d]]/I$ where I is generated by the 2×2 minors of the following matrix:

$$\begin{pmatrix} a & b & c_1 \\ b & c_2 & d \end{pmatrix}$$

There is a flip

contracting a curve C in X and producing a nef curve C^{nef} in X^{nef}. The scheme X is obtained by blowing up the ideal $S_1 := (c_2, d^2)$ whereas X^{nef} is obtained by blowing up the ideal $S_2 := (c_1, d)$. In fact X is derived equivalent to $A := \operatorname{End}_R(R \oplus S_1)$, which can be presented as

At the vertex S_1, the loops c_1 and d commute since they belong to $\operatorname{End}_R(S_1) \cong R$, which is commutative. Also, at the vertex S_1, the path c_1^2 factors through the vertex R as the arrow $\frac{c_1^2}{d^2}$ followed by the arrow d^2. Similarly $c_1 d$ and d^2 factor through the vertex R, so

$$A_{\operatorname{con}} \cong \frac{\mathbb{C}[[c_1,d]]}{c_1^2,\ c_1 d,\ d^2} \cong \frac{\mathbb{C}[x,y]}{x^2,\ xy,\ y^2}.$$

In particular A_{con} is commutative, thus

$$\mathrm{wid}(C) = \mathrm{cwid}(C) = 3.$$

On the other hand, X^{nef} is derived equivalent to $B := \mathrm{End}_R(R \oplus S_2)$ which can be presented as

$$B \cong \qquad R \underset{\substack{\frac{a}{c_1} = \frac{b}{d} \\ \frac{b}{c_1} = \frac{c_2}{d} \\ \text{inc}}}{\overset{\substack{c_1 \\ d}}{\rightleftarrows}} S_2$$

Thus we see $B_{con} = \mathbb{C}$, which is commutative, and $\mathrm{wid}(C^{nef}) = \mathrm{cwid}(C^{nef}) = 1$.

Example 3.14. (The Laufer D_4 flop) We consider the more general case than in the introduction, namely the family of D_4 flops given by

$$R_n := \frac{\mathbb{C}[[u,v,x,y]]}{u^2 + v^2 y = x(x^2 + y^{2n+1})}$$

where $n \geq 1$. In this case $A := \mathrm{End}_R(R \oplus N_1)$ where N_1 is a rank 2 Cohen–Macaulay module, and by [AM], A can be written abstractly as the completion of the quiver with relations

$$x \circlearrowright R \underset{b}{\overset{a}{\rightleftarrows}} N_1 \quad \substack{x \\ \circlearrowright \\ y} \qquad \begin{array}{l} ay^2 = -xa \\ y^2 b = -bx \\ ab = -x^n \\ xy = -yx \\ x^2 + yba + bay = (-1)^{n+1} y^{2n+1}. \end{array}$$

Thus we see immediately that

$$A_{con} \cong \frac{\mathbb{C}\langle\langle x, y \rangle\rangle}{xy = -yx, \ x^2 = (-1)^{n+1} y^{2n+1}} \cong \frac{\mathbb{C}\langle\langle x, y \rangle\rangle}{xy = -yx, \ x^2 = y^{2n+1}}.$$

The contraction algebra A_{con} is finite dimensional by 2.13. To see this explicitly, post- and premultiplying the second equation by x we obtain $x^3 = y^{2n+1}x = xy^{2n+1}$. But using the first equation repeatedly gives $xy^{2n+1} = -y^{2n+1}x$, thus $x^3 = 0$. Consequently the algebra A_{con} is spanned by the monomials

$$\begin{array}{ccccc} 1 & y & y^2 & \dots & y^{2n} \\ x & xy & xy^2 & \dots & xy^{2n} \\ x^2 & x^2 y & x^2 y^2 & \dots & x^2 y^{2n} \end{array}$$

In fact this is a basis (which for example can be checked using the Diamond Lemma) and so $\mathrm{wid}(C) = \dim_{\mathbb{C}} A_{con} = 3(2n+1)$. Furthermore

$$A_{con}^{ab} \cong \frac{\mathbb{C}[[x,y]]}{xy = 0, \ x^2 = y^{2n+1}} \cong \frac{\mathbb{C}[x,y]}{xy = 0, \ x^2 = y^{2n+1}}$$

and so $\mathrm{cwid}(C) = \dim_{\mathbb{C}} A_{con}^{ab} = 2n + 3$.

3.4. **Commutativity of A_{con} and flops.** When C is an irreducible rational flopping curve and U is smooth, necessarily R is Gorenstein. Below in 3.15 we show that in this flopping setup, $\mathrm{wid}(C) = \mathrm{cwid}(C)$ if and only if C has normal bundle $(-1,-1)$ or $(-2,0)$, and furthermore in this case this invariant recovers Reid's notion of width from [R83].

Theorem 3.15. *Let $U \to \mathrm{Spec}\, R$ be the flopping contraction of an irreducible rational curve C, where U is smooth. Then $\mathrm{End}_U(\mathcal{E}_U)$ is commutative if and only if C is a $(-1,-1)$ or $(-2,0)$-curve.*

Proof. Since $\mathrm{End}_U(\mathcal{E}_U) \cong \mathrm{A}_{\mathrm{con}}$ by 3.9, we assume throughout that R is complete local. We check commutativity on $\mathrm{A}_{\mathrm{con}}$.

($\Leftarrow$) Suppose that C is not a $(-3, 1)$-curve, then it is well known that

$$R \cong \frac{\mathbb{C}[[u, v, x, y]]}{uv = (x - y^n)(x + y^n)}$$

[R83] for some $n \geq 1$, and that A is isomorphic to the completion of the quiver with relations in 3.10. As shown in 3.10, $\mathrm{A}_{\mathrm{con}} \cong \mathbb{C}[y]/y^n$, which is evidently commutative and has dimension n, which is the same as Reid's width.

($\Rightarrow$) By contraposition. Suppose that C is a $(-3, 1)$-curve, then since U is smooth,

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_U(E, E) = \dim_{\mathbb{C}} \mathrm{Ext}^2_U(E, E) = 2$$

by CY duality. Hence by the standard deformation theory argument [L02, 1.1]

$$\mathrm{A}_{\mathrm{con}} \cong \frac{\mathbb{C}\langle\langle x, y \rangle\rangle}{(f_1, f_2)}$$

where (f_1, f_2) is the closure of the ideal generated by f_1 and f_2, and further when we write both f_1 and f_2 as a sum of words, each word has degree two or higher. We know $\mathrm{A}_{\mathrm{con}}$ is finite dimensional by 2.13, hence it follows using the argument of [S13, 1.2] that $\mathrm{A}_{\mathrm{con}} \cong \mathrm{End}_U(\mathcal{E}_U)$ cannot be commutative. $\square$

The above is somewhat remarkable, since it does not use the classification of flops, and in fact it turns out later (3.19) that the above is the key step in showing that Toda's commutative deformation twist functor is not an equivalence. However, below we give a second proof of ($\Rightarrow$) in 3.15 that does use the classification of flops, since this proof gives us extra information regarding lower bounds of the possible widths of curves. We do not use this second proof (and thus the classification) anywhere else in this paper.

Proof. As in the previous proof, we can assume that R is complete local. Since R is a compound Du Val singularity and C is a $(-3, 1)$-curve, a generic hyperplane section $g \in R$ gives a Du Val surface singularity R/gR of type D_4, E_6, E_7 or E_8. In fact, taking the pullback

$$\begin{array}{ccc} X & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R/gR) & \longrightarrow & \mathrm{Spec}\,R \end{array}$$

then X is a partial resolution of the Du Val singularity where there is only one curve above the origin, and that curve must correspond to a marked curve in one of the following diagrams

$$\tag{3.B}$$

[KaM, K94]. Now set $Y := R \oplus N_1$ as in §2.4, so $\mathrm{A} = \mathrm{End}_R(Y)$, and let e be the idempotent corresponding to R, so $\mathrm{A}_{\mathrm{con}} = \mathrm{A}/\mathrm{A}e\mathrm{A}$. We denote the category of maximal Cohen–Macaulay R-modules by $\mathrm{CM}\,R$. Since flopping contractions are crepant, $\mathrm{A} \in \mathrm{CM}\,R$ [V04, 3.2.10], so since R is an isolated singularity, by the depth lemma $\mathrm{Ext}^1_R(Y, Y) = 0$. Also, since Y is a generator, necessarily $Y \in \mathrm{CM}\,R$.

Thus applying $\mathrm{Hom}_R(Y, -)$ to

$$0 \to Y \xrightarrow{g} Y \to Y/gY \to 0$$

gives

$$0 \to \operatorname{Hom}_R(Y,Y) \xrightarrow{g\cdot} \operatorname{Hom}_R(Y,Y) \to \operatorname{Hom}_R(Y,Y/gY) \to 0,$$

which implies that

$$A/gA \cong \operatorname{Hom}_R(Y,Y/gY) \cong \operatorname{End}_{R/gR}(Y/gY)$$

where the last isomorphism arises from the extension–restriction of scalars adjunction for the ring homomorphism $R \to R/gR$. Since $\operatorname{End}_{R/gR}(Y/gY) \in \operatorname{CM} R/gR$ with $\operatorname{rank}_R N_1 = \operatorname{rank}_{R/gR}(N_1/gN_1)$, in the McKay correspondence N_1/gN_1 is the CM module corresponding to one of the marked curves in (3.B).

Now let e' be the idempotent in $\Delta := \operatorname{End}_{R/gR}(Y/gY)$ corresponding to R/gR, so $\Delta_{\mathrm{con}} = \Delta/\Delta e'\Delta$. We claim that Δ_{con} is a factor of A_{con}. This follows since setting $I = AeA$ we have

$$\frac{A_{\mathrm{con}}}{gA_{\mathrm{con}}} \cong \frac{A/I}{(gA+I)/I} \cong \frac{A}{gA+I} \cong \frac{A/gA}{(gA+I)/gA} \tag{3.C}$$

and it is easy to show that as a Δ-ideal, $(gA+I)/gA$ is equal to $\Delta e'\Delta$. Hence (3.C) is isomorphic to Δ_{con}, and so indeed Δ_{con} is a factor of A_{con}.

It follows that if we can show Δ_{con} is not commutative, then A_{con} is not commutative. But $\Delta_{\mathrm{con}} \cong \underline{\operatorname{Hom}}_{R/gR}(N_1/gN_1, N_1/gN_1)$, and it is well known that we can calculate this using the AR quiver of $\underline{\operatorname{CM}} R/gR$ (see e.g. [IW08, Thm 4.5, Example 4.6]). Below in 3.16 we show by a case-by-case analysis of the five possible N_1 appearing in (3.B) that each Δ_{con} is not commutative, and so each A_{con} is not commutative. $\qquad\square$

Lemma 3.16. *With notation as in the proof of 3.15, the Δ_{con} corresponding to partial resolutions of the Du Val surfaces involving only the marked curves in (3.B) are never commutative.*

Proof. We give details for the E_7 flop, the remaining cases being very similar. We wish to calculate $\underline{\operatorname{Hom}}_{R/gR}(N_1/gN_1, N_1/gN_1)$, which is graded by path length in the AR quiver of $\underline{\operatorname{CM}} R/gR$. To calculate the dimension of each graded piece, we begin by placing a 1 (corresponding to the identity) in the place of N_1/gN_1, and proceed by knitting:

$$
\begin{array}{ccccccccc}
0 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 0 \\
& 1 & 2 & 2 & 3 & 3 & 2 & 2 & 1 & 0 \\
①\,1\,②\,1\,③\,2\,④\,2\,④\,2\,④\,2\,③\,1\,②\,1\,①\,0 \\
& 1 & 2 & 3 & 3 & 3 & 3 & 2 & 1 & 0 \\
0 & 1 & 2 & 2 & 2 & 2 & 2 & 1 & 0 \\
& 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0
\end{array}
$$

For details, see for example [IW08, §4]. The above shows that the degree zero morphism set is one-dimensional (spanned by the identity), the degree two morphism set is two-dimensional, the degree four morphism set is three-dimensional, etc. Summing up, we see that $\dim_\mathbb{C} \Delta_{\mathrm{con}} = 1 + 2 + 3 + 4 + 4 + 4 + 3 + 2 + 1 = 24$. Further, if we set

$$
\begin{array}{ccc}
0 & & 1 \\
& 1 \xrightarrow{x} & 2 \\
① \leftarrow 1 \rightarrow ② & 1 \\
& \underset{y}{1} & 2 \\
0 & 1 & \\
& 0 & 1
\end{array}
$$

then both x and y have degree two, and by the mesh relations $y^2 = 0$. Since the degree four morphism set is three-dimensional, necessarily $xy \neq yx$. $\qquad\square$

Remark 3.17. Although we do not know explicitly all the contraction algebras $\operatorname{End}_X(\mathcal{E}) \cong A_{\mathrm{con}}$ for all flopping contractions (except for some of type A and D_4), a more detailed

analysis of the proof of 3.16 gives a precise value for $\dim_{\mathbb{C}}(\Delta_{\mathrm{con}})$, and thus gives a lower bound for the possible $\mathrm{wid}(C)$:

Dynkin type	$\mathrm{wid}(C)$
A	≥ 1
D_4	≥ 4
E_6	≥ 12
E_7	≥ 24
$E_8(5)$	≥ 40
$E_8(6)$	≥ 60

3.5. **On Toda's commutative deformation functor.** In this subsection we show that Toda's functor defined in [T07] using commutative deformations is never an equivalence for floppable $(-3,1)$-curves. In fact, we work much more generally, without any assumptions on the singularities of X, then specialize down to establish the result. We first establish the results locally on U, then lift to X.

Theorem 3.18. *Let $U \to \operatorname{Spec} R$ be a contraction of an irreducible rational curve C. Then*

(1) *If $\operatorname{End}_U(\mathcal{E}_U)$ is not commutative, then $\operatorname{Ext}^1_U(\mathcal{F}_U, E) \neq 0$.*
(2) *If U is smooth and C is a flopping $(-3,1)$-curve, then Toda's functor does not give an autoequivalence of $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} U)$.*

Proof. (1) Since the Ext group is supported only on $\mathfrak{m}$,

$$\operatorname{Ext}^1_U(\mathcal{F}_U, E) \cong \operatorname{Ext}^1_{\mathrm{A}}(\mathrm{A}^{\mathrm{ab}}_{\mathrm{con}}, S).$$

We can present $\mathrm{A} = \operatorname{End}_{\widehat{R}}(\widehat{R} \oplus N_1)$ as the (completion of) a quiver with relations, where there are two vertices 0 and 1, corresponding to projectives as follows:

$$P_0 := \operatorname{Hom}_{\widehat{R}}(\widehat{R} \oplus N_1, \widehat{R}), \qquad P_1 := \operatorname{Hom}_{\widehat{R}}(\widehat{R} \oplus N_1, N_1).$$

We remark that P_1 is the projective cover of S, and that S is the vertex simple at 1.

We know that $\operatorname{End}_U(\mathcal{E}_U) \cong \mathrm{A}_{\mathrm{con}}$ by 3.9. Since $\mathrm{A}_{\mathrm{con}}$ is not commutative, $\mathrm{A}_{\mathrm{con}} \neq \mathrm{A}^{\mathrm{ab}}_{\mathrm{con}}$. Hence in the minimal projective resolution of $\mathrm{A}^{\mathrm{ab}}_{\mathrm{con}}$ the kernel K of the natural map $P_1 \to \mathrm{A}^{\mathrm{ab}}_{\mathrm{con}}$ contains some non-zero element x which is a composition of cycles at vertex 1, such that x is zero in $\mathrm{A}^{\mathrm{ab}}_{\mathrm{con}}$ but not in $\mathrm{A}_{\mathrm{con}}$. Since this element x does not factor through other vertices, there cannot be a surjective map $P_0^a \to K$, as we need some $P_1 \to K$ in order to surject onto the element x. Consequently the projective cover of K must contain P_1, and hence $\operatorname{Ext}^1_{\mathrm{A}}(\mathrm{A}^{\mathrm{ab}}_{\mathrm{con}}, S) \neq 0$.

(2) By 3.15, $\operatorname{End}_U(\mathcal{E}_U)$ is not commutative, so this follows by (1) [AL10, Theorem 3.1]. $\quad\square$

Corollary 3.19. *Suppose that $X \to X_{\mathrm{con}}$ is a contraction of an irreducible rational curve C.*

(1) *If $\operatorname{End}_X(\mathcal{E})$ is not commutative, then $\operatorname{Ext}^1_X(\mathcal{F}, i_*E) \neq 0$.*
(2) *If X is smooth and C is a flopping $(-3,1)$-curve, Toda's functor does not give an autoequivalence of $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$.*

Proof. (1) Since $\operatorname{End}_X(\mathcal{E}) \cong \operatorname{End}_U(\mathcal{E}_U)$ is not commutative, we know that $\operatorname{Ext}^1_U(\mathcal{F}_U, E) \neq 0$ by 3.18. The result then follows since

$$0 \neq \operatorname{Ext}^1_U(\mathcal{F}_U, E) \overset{(2.\mathrm{B})}{\cong} \operatorname{Ext}^1_X(\mathcal{F}, i_*E).$$

(2) By 3.15, $\operatorname{End}_X(\mathcal{E}) \cong \operatorname{End}_U(\mathcal{E}_U)$ is not commutative, so this follows by (1). $\quad\square$

4. Strategy for Noncommutative Twists

In the remainder of the paper we restrict to the setting of a flopping contraction $X \to X_{\mathrm{con}}$ of an irreducible rational curve C, where X is projective and has only Gorenstein terminal singularities. Our aim is to overcome 3.19 and use the universal sheaf $\mathcal{E}$ to produce a noncommutative twist functor, then prove that it is an autoequivalence which furthermore gives an intrinsic description of the flop–flop functor of Bridgeland and Chen.

4.1. **Notation.** As before, and so as to fix notation for the remainder of the paper, our setup is the contraction of a irreducible rational floppable curve C in a projective normal 3-fold X with at worst Gorenstein terminal singularities (e.g. X is smooth). We denote the contraction map $f \colon X \to X_{\mathrm{con}}$ and remark that necessarily $\mathbf{R}f_*\mathcal{O}_X = \mathcal{O}_{X_{\mathrm{con}}}$ and X_{con} has only Gorenstein terminal singularities. As before, there is a commutative diagram

$$
\begin{array}{ccccc}
C & \overset{e}{\hookrightarrow} & U & \overset{i}{\longrightarrow} & X \\
\big\updownarrow & & \Big\downarrow{\scriptstyle f|_U} & & \Big\downarrow{\scriptstyle f} \\
p & \in & U_{\mathrm{con}} & \hookrightarrow & X_{\mathrm{con}} \\
\| & & \|\wr & & \\
\mathfrak{m} & \in & \operatorname{Spec} R & &
\end{array}
\tag{4.A}
$$

where $C^{\mathrm{red}} \cong \mathbb{P}^1$, e is a closed embedding and i is an open embedding. As at the beginning of §3, U is derived equivalent to an algebra $\Lambda \cong \operatorname{End}_R(R \oplus N)$, and we set $\Lambda_{\mathrm{con}} := \Lambda/I_{\mathrm{con}}$ where $I_{\mathrm{con}} = [R]$. Since X_{con} has only terminal Gorenstein singularities, R has only isolated hypersurface singularities. Furthermore, since a single irreducible curve has been contracted to a point $\mathfrak{m}$, as an R-module Λ_{con} is supported only on $\mathfrak{m}$ and $\Lambda_{\mathrm{con}} \cong \Lambda_{\mathrm{con}} \otimes_R \widehat{R} \cong \widehat{\Lambda}_{\mathrm{con}}$. To ease notation, as before we set $\mathfrak{R} := \widehat{R}$.

4.2. **Mutation.** Our strategy is to first work on the formal fibre $\mathfrak{R}$, and here our main new tool is that of mutation, an algebraic operation developed in [IW10, §6] that extended other theories of mutation to the setting of quivers with loops, 2-cycles, and no superpotential, all of which occur for general flops.

One of the key properties of mutation exploited below is that it always gives rise to derived equivalences as follows (see §5.3 for more details). As input, we consider a *modifying* $\mathfrak{R}$-module Y, chosen as in §2.3 so that $\operatorname{End}_{\mathfrak{R}}(Y)$ is derived equivalent to the formal fibre. The output is another $\mathfrak{R}$-module, the *mutation* νY, together with a tilting bimodule inducing an equivalence

$$
\Phi \colon \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \operatorname{End}_{\mathfrak{R}}(Y)) \to \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \operatorname{End}_{\mathfrak{R}}(\nu Y)).
$$

It turns out that Φ is isomorphic to the inverse of the flop functor on the formal fibre [W, 4.2], but we will not need this fact. Instead, we compose with a similar functor in the opposite direction to obtain a *mutation–mutation* autoequivalence of $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} \operatorname{End}_{\mathfrak{R}}(Y))$, denoted by $\Phi \circ \Phi$. This will be easier to study, and forms our algebraic model for the inverse of the (analytic) flop–flop functor. Writing $\mathrm{A} = \operatorname{End}_{\mathfrak{R}}(Y)$,

(1) For the A-bimodule I_{A} defined by the natural exact sequence

$$
I_{\mathrm{A}} \to \mathrm{A} \to \mathrm{A}_{\mathrm{con}},
\tag{4.B}
$$

we will show in 5.10(1) that

$$
\Phi \circ \Phi = \mathbf{R}\operatorname{Hom}_{\mathrm{A}}(I_{\mathrm{A}}, -).
\tag{4.C}
$$

Although we do not give the details here, the functor $\mathbf{R}\operatorname{Hom}_{\mathrm{A}}(I_{\mathrm{A}}, -)$ may be interpreted as an (inverse) twist around the noncommutative deformation family of the simple A-module S, and so (4.C) gives a twist description of $\Phi \circ \Phi$. In the process of proving (1), we further obtain:

(2) A minimal projective resolution of $\mathrm{A}_{\mathrm{con}}$ by A-modules in 5.6(2).

(3) A description of the action of $\Phi \circ \Phi$ on $\mathrm{A}_{\mathrm{con}}$ in 5.9(2).

Point (3) is needed to produce a global spanning class later, and point (2) will establish that $\mathcal{E}$ is a perfect complex such that

$$\mathrm{Ext}^t_X(\mathcal{E}, i_*E) = \begin{cases} \mathbb{C} & \text{if } t = 0, 3 \\ 0 & \text{else,} \end{cases}$$

generalising [T07].

4.3. **Strategy.** The remainder of the paper involves lifting the above formal fibre results to algebraic flops. From the complete local algebraic model in §4.2, a twist autoequivalence around the universal sheaf $\mathcal{E}$ on X is produced using the following strategy:

Tilting algebra Λ: Using an algebraic idempotent trick, together with a morita equivalence, we are able to lift the tilting bimodule I_{A} giving rise to the complete local equivalence $\Phi \circ \Phi$ to a Λ-bimodule I_{con}. We prove in 6.2 that this bimodule gives rise to a derived autoequivalence of Λ, and we thus obtain an algebraic local model $\mathbf{R}\mathrm{Hom}_\Lambda(I_{\mathrm{con}}, -)$ for our twist functor.

Local geometry U: Since U is derived equivalent to Λ, the above algebraic Λ-bimodule I_{con} induces a Fourier–Mukai kernel on $U \times U$. Since I_{con} is a tilting module, this kernel gives a twist autoequivalence on U, in §6.4.

Global geometry X: A gluing construction then produces a twist functor $T_\mathcal{E}$ on X in §7. It remains to show that $T_\mathcal{E}$ is an autoequivalence. We are able to conclude that $\mathcal{E}$ is perfect from the minimal projective resolution of $\mathrm{A}_{\mathrm{con}}$ in point (2) above, then, by inferring the action of $T_\mathcal{E}$ on the universal sheaf $\mathcal{E}$ from the knowledge of $\Phi \circ \Phi$ in (3), we are able to construct (in §7.4) a spanning class on which the action of $T_\mathcal{E}$ is known. A spanning class argument establishes that $T_\mathcal{E}$ is fully faithful, and from there a straightforward lemma, 7.11, allows us to conclude that $T_\mathcal{E}$ is an autoequivalence in §7.5.

Furthermore, in 7.14(2) we show that $T_\mathcal{E}$ fits into a functorial triangle

$$\mathbf{R}\mathrm{Hom}_X(\mathcal{E}, -) \otimes^{\mathbf{L}}_{\mathrm{A}_{\mathrm{con}}} \mathcal{E} \to \mathrm{Id} \to T_\mathcal{E} \to$$

as a consequence of the short exact sequence (4.B) above. We finally prove in 7.18 that $T_\mathcal{E}$ is an inverse of the flop–flop functor FF, giving the global geometric version of (4.C).

5. COMPLETE LOCAL MUTATION AND EXT VANISHING

We keep the notation and assumptions as in §4. There, $U \to \mathrm{Spec}\, R$ is a crepant morphism, which in fact is equivalent to the condition $\Lambda = \mathrm{End}_R(R \oplus N) \in \mathrm{CM}\, R$ [IW11], where recall $\mathrm{CM}\, R$ denotes the category of maximal Cohen–Macaulay R-modules. Since $N \cong \mathrm{Hom}_R(R, N)$ is a summand of Λ, this in turn forces $N \in \mathrm{CM}\, R$. Writing $\mathrm{ref}\, R$ for the category of reflexive R-modules, modules $Y \in \mathrm{ref}\, R$ satisfying $\mathrm{End}_R(Y) \in \mathrm{CM}\, R$ are thus of interest, and are called *modifying* R-modules [IW10].

5.1. **Mutation.** Now $\Lambda = \mathrm{End}_R(R \oplus N)$ and completing with respect to the maximal ideal $\mathfrak{m}$ in (4.A)

$$\widehat{\Lambda} \cong \mathrm{End}_{\widehat{R}}(\widehat{R} \oplus \widehat{N}).$$

As in 2.11, we decompose

$$\widehat{R} \oplus \widehat{N} \cong \widehat{R}^{\oplus a_0} \oplus N_1^{\oplus a_1}.$$

Throughout the section, as in §2.4, to ease notation with hats we write $\mathfrak{R} := \widehat{R}$, and

$$Y := \mathfrak{R} \oplus N_1, \qquad Z := \mathfrak{R}^{\oplus a_0} \oplus N_1^{\oplus a_1},$$

so that $\mathrm{A} = \mathrm{End}_\mathfrak{R}(Y)$ and $\widehat{\Lambda} = \mathrm{End}_\mathfrak{R}(Z)$. We write $(-)^*$ for the duality functor

$$\mathrm{Hom}_\mathfrak{R}(-, \mathfrak{R})\colon \mathrm{ref}\,\mathfrak{R} \to \mathrm{ref}\,\mathfrak{R}.$$

Definition 5.1. *With the setup as above,*

(1) *We take a minimal right* $(\mathrm{add}\,\mathfrak{R})$*-approximation*

$$\mathfrak{R}_0 \xrightarrow{a} N_1$$

of N_1, *which by definition means that*
(a) $\mathfrak{R}_0 \in \mathrm{add}\,\mathfrak{R}$ *and* $(a\cdot)\colon \mathrm{Hom}_{\mathfrak{R}}(\mathfrak{R}, \mathfrak{R}_0) \to \mathrm{Hom}_{\mathfrak{R}}(\mathfrak{R}, N_1)$ *is surjective,*
(b) *if* $g \in \mathrm{End}_{\mathfrak{R}}(\mathfrak{R}_0)$ *satisfies* $a = ag$, *then* g *is an automorphism.*
In other words, a is a projective cover of N_1. Necessarily a is surjective. Since $\mathfrak{R}$ is complete, such an a exists and is unique up to isomorphism. We put $K_0 := \mathrm{Ker}\,a$, so we have an exact sequence

$$0 \to K_0 \xrightarrow{c} \mathfrak{R}_0 \xrightarrow{a} N_1 \to 0. \tag{5.A}$$

Trivially

$$0 \to \mathrm{Hom}_{\mathfrak{R}}(\mathfrak{R}, K_0) \xrightarrow{c\cdot} \mathrm{Hom}_{\mathfrak{R}}(\mathfrak{R}, \mathfrak{R}_0) \xrightarrow{a\cdot} \mathrm{Hom}_{\mathfrak{R}}(\mathfrak{R}, N_1) \to 0$$

is exact, thus since by definition $\mathrm{A}_{\mathrm{con}} = \mathrm{End}_{\mathfrak{R}}(Y)/[\mathfrak{R}] = \mathrm{End}_{\mathfrak{R}}(N_1)/[\mathfrak{R}]$, *applying* $\mathrm{Hom}_{\mathfrak{R}}(Y, -)$ *to* (5.A) *yields an exact sequence*

$$0 \to \mathrm{Hom}_{\mathfrak{R}}(Y, K_0) \xrightarrow{c\cdot} \mathrm{Hom}_{\mathfrak{R}}(Y, \mathfrak{R}_0) \xrightarrow{a\cdot} \mathrm{Hom}_{\mathfrak{R}}(Y, N_1) \to \mathrm{A}_{\mathrm{con}} \to 0 \tag{5.B}$$

of A*-modules.*
(2) *We define the* right mutation *of* Y *as*

$$\upmu Y := \mathfrak{R} \oplus K_0,$$

that is we remove the summand N_1 *and replace it with* K_0.
(3) *Dually, we consider a minimal right* $(\mathrm{add}\,\mathfrak{R}^*)$*-approximation*

$$\mathfrak{R}_1^* \xrightarrow{b} N_1^*$$

of N_1^*, *and we put* $K_1 := \mathrm{Ker}\,b$. *Thus again we have an exact sequence*

$$0 \to K_1 \xrightarrow{d} \mathfrak{R}_1^* \xrightarrow{b} N_1^* \to 0. \tag{5.C}$$

(4) *We define the* left mutation *of* Y *as*

$$\upnu Y := \mathfrak{R} \oplus K_1^*.$$

In fact $\upnu Y = (\upmu Y^*)^*$, so really we only need to define right mutation.

Remark 5.2. Applying $\mathrm{Hom}_{\mathfrak{R}}(Y, -)$ to (5.A) and observing the resulting long exact sequence in Ext gives

$$0 \to \mathrm{Hom}_{\mathfrak{R}}(Y, K_0) \xrightarrow{c\cdot} \mathrm{Hom}_{\mathfrak{R}}(Y, \mathfrak{R}_0) \xrightarrow{a\cdot} \mathrm{Hom}_{\mathfrak{R}}(Y, N_1) \to \mathrm{Ext}^1_{\mathfrak{R}}(Y, K_0) \to 0.$$

Combining this with (5.B) shows that, as $\mathfrak{R}$-modules

$$\mathrm{A}_{\mathrm{con}} \cong \mathrm{Ext}^1_{\mathfrak{R}}(Y, K_0) \cong \mathrm{Ext}^1_{\mathfrak{R}}(N_1, \Omega N_1) \cong D\,\mathrm{Ext}^1_{\mathfrak{R}}(\Omega N_1, N_1) \cong D\,\mathrm{Ext}^2_{\mathfrak{R}}(N_1, N_1)$$

where the third isomorphism is just AR duality, since $\mathfrak{R}$ is an isolated singularity. Taking dimensions of both sides shows that

$$\mathrm{wid}(C) = \dim_{\mathbb{C}} \mathrm{Ext}^2_{\widehat{R}}(N_1, N_1).$$

This may be calculated by using computer algebra on the base $\widehat{R}$, and gives a very easy way to calculate $\mathrm{wid}(C)$. In contrast, computing the algebra structure on $\mathrm{A}_{\mathrm{con}}$ is always a little harder, and this is needed to calculate $\mathrm{A}_{\mathrm{con}}^{\mathrm{ab}}$ and hence $\mathrm{cwid}(C)$. We remark that there does not seem to be any easy description of $\mathrm{cwid}(C)$ in terms of Ext groups.

Remark 5.3. Mutation is defined in [IW10, §6] for any modifying module, so in particular also for $Z = \mathfrak{R}^{\oplus a_0} \oplus N_1^{\oplus a_1}$. Summing the exact sequence (5.A) gives an exact sequence

$$0 \to K_0^{\oplus a_1} \xrightarrow{c^{\oplus a_1}} \mathfrak{R}_0^{\oplus a_1} \xrightarrow{a^{\oplus a_1}} N_1^{\oplus a_1} \to 0 \tag{5.D}$$

where $a^{\oplus a_1}$ is a minimal $(\mathrm{add}\,\mathfrak{R})$-approximation of $N_1^{\oplus a_1}$. Thus

$$\upmu Z = \mathfrak{R}_0^{\oplus a_0} \oplus K_0^{\oplus a_1}.$$

Similarly, summing (5.C) gives a minimal $(\operatorname{add}\mathfrak{R}^*)$-approximation of $(N_1^{\oplus a_1})^*$ and so

$$\nu Z = \mathfrak{R}_0^{\oplus a_0} \oplus (K_1^*)^{\oplus a_1}.$$

From this, it is clear that $\operatorname{End}_{\mathfrak{R}}(\nu Y)$ is morita equivalent to $\operatorname{End}_{\mathfrak{R}}(\nu Z)$.

Recall that $\mathbb{F} = \operatorname{Hom}_A(P, -)\colon \operatorname{mod} A \to \operatorname{mod} \widehat{\Lambda}$ is the morita equivalence from §2.4, where $P = \operatorname{Hom}_{\mathfrak{R}}(Y, Z)$.

Lemma 5.4. $\widehat{\Lambda}_{\mathrm{con}} \cong (\mathbb{F}A_{\mathrm{con}})^{\oplus a_1}$ *as $\widehat{\Lambda}$-modules. In particular*

$$\Lambda_{\mathrm{con}} \cong \widehat{\Lambda}_{\mathrm{con}} \cong M_{a_1}(A_{\mathrm{con}}),$$

where $M_{a_1}(A_{\mathrm{con}})$ denotes the ring of $a_1 \times a_1$ matrices over A_{con}. Consequently $\Lambda_{\mathrm{con}} \cong \widehat{\Lambda}_{\mathrm{con}}$ and A_{con} are morita equivalent.

Proof. Applying $\operatorname{Hom}_{\mathfrak{R}}(Z, -)$ to (5.D) yields the exact sequence

$$\operatorname{Hom}_{\mathfrak{R}}(Z, \mathfrak{R}_0^{\oplus a_1}) \xrightarrow{(a\cdot)^{\oplus a_1}} \operatorname{Hom}_{\mathfrak{R}}(Z, N_1^{\oplus a_1}) \to \widehat{\Lambda}_{\mathrm{con}} \to 0$$

of $\widehat{\Lambda}$-modules. On the other hand applying $\mathbb{F}$ to (5.B) shows that the top sequence in the following commutative diagram is exact

$$\operatorname{Hom}_A(\operatorname{Hom}_{\mathfrak{R}}(Y, Z), \operatorname{Hom}_{\mathfrak{R}}(Y, \mathfrak{R}_0)) \to \operatorname{Hom}_A(\operatorname{Hom}_{\mathfrak{R}}(Y, Z), \operatorname{Hom}_{\mathfrak{R}}(Y, N_1)) \to \mathbb{F}(A_{\mathrm{con}}) \to 0$$

$$\Big\uparrow\wr \qquad\qquad\qquad\qquad\qquad \Big\uparrow\wr$$

$$\operatorname{Hom}_{\mathfrak{R}}(Z, \mathfrak{R}_0) \xrightarrow{\quad a\cdot \quad} \operatorname{Hom}_{\mathfrak{R}}(Z, N_1)$$

where the vertical isomorphisms are just reflexive equivalence (see e.g. [IW10, 2.5]). It follows that $\mathbb{F}A_{\mathrm{con}} \cong \operatorname{Cok}(a\cdot)$ and thus

$$\mathbb{F}A_{\mathrm{con}}^{\oplus a_1} \cong \operatorname{Cok}(a^{\oplus a_1}\cdot) \cong \widehat{\Lambda}_{\mathrm{con}}.$$

The last statement follows by 2.17 since

$$\widehat{\Lambda}_{\mathrm{con}} \cong \operatorname{End}_{\widehat{\Lambda}}(\widehat{\Lambda}_{\mathrm{con}}) \cong \operatorname{End}_{\widehat{\Lambda}}(\mathbb{F}A_{\mathrm{con}}^{\oplus a_1}) \cong \operatorname{End}_A(A_{\mathrm{con}}^{\oplus a_1}) \cong M_{a_1}(\operatorname{End}_A(A_{\mathrm{con}})) \cong M_{a_1}(A_{\mathrm{con}}).$$

$\square$

The following is important, and is a consequence of the fact that $\mathfrak{R}$ is a hypersurface singularity.

Proposition 5.5. *With assumptions and notation as above,*

(1) $\nu(\nu Y) \cong Y$.

(2) $K_1^* \cong K_0$, *so that* $\nu Y \cong \mu Y$.

Proof. (1) By definition of mutation, $\nu = \Omega^{-1}$, where Ω^{-1} denotes the cosyzygy functor on $\operatorname{CM}\mathfrak{R}$. But $N_1 \in \operatorname{CM}\mathfrak{R}$ since $A \in \operatorname{CM}\mathfrak{R}$, and it is well known that on hypersurfaces $\Omega^{-2}(Z) \cong Z$ for all $Z \in \operatorname{CM}\mathfrak{R}$ [E80].

(2) To calculate $\nu(\nu Y)$, we take a minimal right $(\operatorname{add}(\frac{\nu Y}{K_1^*})^*)$-approximation, equivalently a minimal right $(\operatorname{add}\mathfrak{R}^*)$-approximation, of $(K_1^*)^* \cong K_1$ and by doing so obtain an exact sequence

$$0 \to K_3 \to \mathfrak{R}_2^* \to K_1 \to 0$$

with $\mathfrak{R}_2^* \in \operatorname{add}\mathfrak{R}^*$. Since we have $\nu(\nu Y) := \mathfrak{R} \oplus K_3^*$, by (1) and Krull–Schmidt it follows that $K_3^* \cong N_1$. Now dualizing the above sequence gives an exact sequence

$$0 \to K_1^* \to \mathfrak{R}_2 \to N_1 \to 0$$

where the last map is a minimal right $(\operatorname{add}\mathfrak{R})$-approximation. By uniqueness of minimal approximations, this implies that $\mathfrak{R}_2 \cong \mathfrak{R}_0$, $K_1^* \cong K_0$ and further $\nu Y = \mathfrak{R} \oplus K_1^* \cong \mu Y$. $\square$

Since all modules $\mathfrak{R}, N_1, K_0, K_1 \in \mathrm{CM}\,\mathfrak{R}$ and $(-)^*$ is an exact duality on $\mathrm{CM}\,\mathfrak{R}$, we can dualize (5.C) and obtain an exact sequence

$$0 \to N_1 \xrightarrow{b^*} \mathfrak{R}_1 \xrightarrow{d^*} K_1^* \to 0. \tag{5.E}$$

5.2. **Ext vanishing.** With regards to our applications, the point of this subsection is to prove 5.7. The following is an application of [IW10, §6].

Proposition 5.6. *With assumptions and notation as above,*

(1) *Applying* $\mathrm{Hom}_{\mathfrak{R}}(Y, -)$ *to the sequence (5.E) gives an exact sequence*

$$0 \to \mathrm{Hom}_{\mathfrak{R}}(Y, N_1) \xrightarrow{b^* \cdot} \mathrm{Hom}_{\mathfrak{R}}(Y, \mathfrak{R}_1) \xrightarrow{d^* \cdot} \mathrm{Hom}_{\mathfrak{R}}(Y, K_1^*) \to 0. \tag{5.F}$$

(2) *The minimal projective resolution of* $\mathrm{A}_{\mathrm{con}}$ *as an* A-*module has the form*

$$0 \to P \to Q_1 \to Q_0 \to P \to \mathrm{A}_{\mathrm{con}} \to 0$$

where $P := \mathrm{Hom}_{\mathfrak{R}}(Y, N_1)$, *and* $Q_i \in \mathrm{add}\,Q$ *for* $Q := \mathrm{Hom}_{\mathfrak{R}}(Y, \mathfrak{R})$.

(3) $\mathrm{pd}_{\widehat{\Lambda}}\,\widehat{\Lambda}_{\mathrm{con}} = 3$ *and* $\mathrm{pd}_{\widehat{\Lambda}}\,\widehat{I}_{\mathrm{con}} = 2$.

Proof. (1) Since $\mathrm{A}_{\mathrm{con}}$ is finite dimensional by 2.13, this follows from the argument exactly as in [IW10, (6.Q)].
(2) Since $K_1^* \cong K_0$ by 5.5, splicing (5.F) and (5.B) gives the minimal projective resolution

$$0 \to \mathrm{Hom}_{\mathfrak{R}}(Y, N_1) \to \mathrm{Hom}_{\mathfrak{R}}(Y, \mathfrak{R}_1) \to \mathrm{Hom}_{\mathfrak{R}}(Y, \mathfrak{R}_0) \to \mathrm{Hom}_{\mathfrak{R}}(Y, N_1) \to \mathrm{A}_{\mathrm{con}} \to 0.$$

(3) We have $\mathrm{pd}_{\mathrm{A}}\,\mathrm{A}_{\mathrm{con}} = 3$ by (2), and so chasing across the morita equivalence using 5.4, $\mathrm{pd}_{\widehat{\Lambda}}\,\widehat{\Lambda}_{\mathrm{con}} = 3$. The final statement follows. □

Corollary 5.7. *There are isomorphisms*

$$\mathrm{Ext}_{\Lambda}^t(\mathbb{F}\mathrm{A}_{\mathrm{con}}, T) \cong \mathrm{Ext}_{\widehat{\Lambda}}^t(\mathbb{F}\mathrm{A}_{\mathrm{con}}, \widehat{T}) \cong \mathrm{Ext}_{\mathrm{A}}^t(\mathrm{A}_{\mathrm{con}}, S) \cong \begin{cases} \mathbb{C} & \text{if } t = 0, 3 \\ 0 & \text{else,} \end{cases}$$

and further $\mathrm{A}_{\mathrm{con}}$ *is a self-injective algebra.*

Proof. The first two isomorphisms are consequences of the fact that the Ext groups are supported only at $\mathfrak{m}$. The third isomorphism is an immediate consequence of the minimal projective resolution in 5.6, since $\mathrm{Hom}_{\mathrm{A}}(Q_i, S) = 0$.

Now if the outer two terms in a short exact sequence are annihilated by an idempotent, so is the middle term. Hence $\mathrm{mod}\,\mathrm{A}_{\mathrm{con}}$ is extension-closed in $\mathrm{mod}\,\mathrm{A}$, and it follows that

$$\mathrm{Ext}_{\mathrm{A}_{\mathrm{con}}}^1(S, \mathrm{A}_{\mathrm{con}}) = \mathrm{Ext}_{\mathrm{A}}^1(S, \mathrm{A}_{\mathrm{con}}) \cong D\,\mathrm{Ext}_{\mathrm{A}}^2(\mathrm{A}_{\mathrm{con}}, S) \tag{5.G}$$

where the last isomorphism holds since A is 3-sCY [IW10, 2.22(2)], $\mathrm{pd}_{\mathrm{A}}\,\mathrm{A}_{\mathrm{con}} < \infty$ and S has finite length. Thus (5.G) shows that $\mathrm{Ext}_{\mathrm{A}_{\mathrm{con}}}^1(S, \mathrm{A}_{\mathrm{con}}) = 0$. Since $\mathrm{A}_{\mathrm{con}}$ is finite dimensional with unique simple S, it follows that $\mathrm{A}_{\mathrm{con}}$ is self-injective. □

5.3. **On the mutation functor Φ.** We retain the setup and notation from §5.1, namely $\mathrm{A} = \mathrm{End}_{\mathfrak{R}}(Y)$ and $\widehat{\Lambda} = \mathrm{End}_{\mathfrak{R}}(Z)$. For the case of left mutation νY, there is a derived equivalence between $\mathrm{End}_{\mathfrak{R}}(Y)$ and $\mathrm{End}_{\mathfrak{R}}(\nu Y)$ [IW10, §6] given by a tilting $\mathrm{End}_{\mathfrak{R}}(Y)$-module V constructed as follows. We consider the sequence (5.E)

$$0 \to N_1 \xrightarrow{b^*} \mathfrak{R}_1 \to K_1^* \to 0$$

obtained by dualizing (5.C). It is trivial that b^* is a minimal left $(\mathrm{add}\,\mathfrak{R})$-approximation of N_1. Applying $\mathrm{Hom}_{\mathfrak{R}}(Y, -)$, gives an induced map $(b^* \cdot) \colon \mathrm{Hom}_{\mathfrak{R}}(Y, N_1) \to \mathrm{Hom}_{\mathfrak{R}}(Y, \mathfrak{R}_1)$, and our tilting A-module is by definition given by

$$V := \mathrm{Hom}_{\mathfrak{R}}(Y, \mathfrak{R}) \oplus \mathrm{Cok}(b^* \cdot).$$

By (5.F) we already know that $\mathrm{Cok}(b^* \cdot) \cong \mathrm{Hom}_{\mathfrak{R}}(Y, K_1^*)$, so in fact

$$V \cong \mathrm{Hom}_{\mathfrak{R}}(Y, \mathfrak{R}) \oplus \mathrm{Hom}_{\mathfrak{R}}(Y, K_1^*) = \mathrm{Hom}_{\mathfrak{R}}(Y, \nu Y).$$

This gives rise to an equivalence [IW10, 6.8]

$$\Phi := \mathbf{R}\mathrm{Hom}_A(V, -) \colon \mathrm{D}^b(\mathrm{mod}\, A) \to \mathrm{D}^b(\mathrm{mod}\, \mathrm{End}_{\mathfrak{R}}(\nu Y)),$$

which we call the *mutation functor*. By 5.5 we can mutate $\mathrm{End}_{\mathfrak{R}}(\nu Y)$ back to obtain A, and in an identical way

$$W := \mathrm{Hom}_{\mathfrak{R}}(\nu Y, Y)$$

is a tilting module giving rise to an equivalence which by abuse of notation we also denote

$$\Phi := \mathbf{R}\mathrm{Hom}(W, -) \colon \mathrm{D}^b(\mathrm{mod}\, \mathrm{End}_{\mathfrak{R}}(\nu Y)) \to \mathrm{D}^b(\mathrm{mod}\, A).$$

Similarly

$$V' := \mathrm{Hom}_{\mathfrak{R}}(Z, \mathfrak{R}^{\oplus a_0}) \oplus \mathrm{Hom}_{\mathfrak{R}}(Z, (K_1^*)^{\oplus a_1}) = \mathrm{Hom}_{\mathfrak{R}}(Z, \nu Z)$$

is a tiling $\widehat{\Lambda}$-module, giving rise to an equivalence

$$\Phi' := \mathbf{R}\mathrm{Hom}_{\widehat{\Lambda}}(V', -) \colon \mathrm{D}^b(\mathrm{mod}\, \widehat{\Lambda}) \to \mathrm{D}^b(\mathrm{mod}\, \mathrm{End}_{\mathfrak{R}}(\nu Z))$$

and we mutate back via the tilting module $W' := \mathrm{Hom}_{\mathfrak{R}}(\nu Z, Z)$.

The following is easily seen, and is an elementary application of morita theory.

Lemma 5.8. *The following diagram commutes:*

$$
\begin{array}{ccc}
\mathrm{D}^b(\mathrm{mod}\, A) & \xrightarrow{\ \mathbb{F}\ } & \mathrm{D}^b(\mathrm{mod}\, \widehat{\Lambda}) \\
\Phi \downarrow & & \downarrow \Phi' \\
\mathrm{D}^b(\mathrm{mod}\, \mathrm{End}_{\mathfrak{R}}(\nu Y)) & \xrightarrow{morita} & \mathrm{D}^b(\mathrm{mod}\, \mathrm{End}_{\mathfrak{R}}(\nu Z)) \\
\Phi \downarrow & & \downarrow \Phi' \\
\mathrm{D}^b(\mathrm{mod}\, A) & \xrightarrow{\ \mathbb{F}\ } & \mathrm{D}^b(\mathrm{mod}\, \widehat{\Lambda})
\end{array}
$$

For our purposes later, we need to be able to track the object A_{con} through the derived equivalence Φ. This is taken care of in the following lemma.

Lemma 5.9. *Write* $B := \mathrm{End}_{\mathfrak{R}}(\nu Y)$ *and* $B_{\mathrm{con}} := B/[\mathfrak{R}]$, *then*

(1) $\Phi(A_{\mathrm{con}}) \cong B_{\mathrm{con}}[-1]$.
(2) $\Phi \circ \Phi(A_{\mathrm{con}}) \cong A_{\mathrm{con}}[-2]$.
(3) $\Phi' \circ \Phi'(\mathbb{F}A_{\mathrm{con}}) \cong \mathbb{F}A_{\mathrm{con}}[-2]$.

Proof. Note first that in fact $A_{\mathrm{con}} \cong B_{\mathrm{con}}$ as algebras [IW10, 6.20], so $\dim_{\mathbb{C}} B_{\mathrm{con}} < \infty$. (1) We already know that $V = \mathrm{Hom}_{\mathfrak{R}}(Y, \nu Y)$. Now since $K_1^* \cong K_0$ by 5.5, splicing (5.A) and (5.E) gives us an exact sequence of R-modules

$$0 \to K_1^* \xrightarrow{c} \mathfrak{R}_0 \xrightarrow{b^* \cdot a} \mathfrak{R}_1 \xrightarrow{d^*} K_1^* \to 0$$

to which applying $\mathrm{Hom}_{\mathfrak{R}}(Y, -)$ gives a complex of A-modules

$$0 \to \mathrm{Hom}_{\mathfrak{R}}(Y, K_1^*) \xrightarrow{c\cdot} \mathrm{Hom}_{\mathfrak{R}}(Y, \mathfrak{R}_0) \xrightarrow{(b^* \cdot a)\cdot} \mathrm{Hom}_{\mathfrak{R}}(Y, \mathfrak{R}_1) \xrightarrow{d^*\cdot} \mathrm{Hom}_{\mathfrak{R}}(Y, K_1^*) \to 0 \quad (5.\mathrm{H})$$

whereas applying $\mathrm{Hom}_{\mathfrak{R}}(\nu Y, -)$ gives a complex of B-modules

$$0 \to \mathrm{Hom}_{\mathfrak{R}}(\nu Y, K_1^*) \xrightarrow{c\cdot} \mathrm{Hom}_{\mathfrak{R}}(\nu Y, \mathfrak{R}_0) \xrightarrow{(b^* \cdot a)\cdot} \mathrm{Hom}_{\mathfrak{R}}(\nu Y, \mathfrak{R}_1) \xrightarrow{d^*\cdot} \mathrm{Hom}_{\mathfrak{R}}(\nu Y, K_1^*) \to 0 \quad (5.\mathrm{I})$$

In both (5.H) and (5.I), we consider the last Hom term to be in degree zero. Since every term in (5.H) is a summand of V, by reflexive equivalence it follows that under the derived equivalence Φ, (5.H) gets sent to (5.I). Hence the statement in the lemma follows if we can show that (5.H) is quasi-isomorphic to $A_{\mathrm{con}}[1]$ and that (5.I) is quasi-isomorphic to $B_{\mathrm{con}}[0]$.

Consider first (5.H). Combining the exact sequences (5.F) and (5.B)

$$0 \to \operatorname{Hom}_{\mathfrak{R}}(Y, N_1) \xrightarrow{b^*\cdot} \operatorname{Hom}_{\mathfrak{R}}(Y, \mathfrak{R}_1) \xrightarrow{d^*\cdot} \operatorname{Hom}_{\mathfrak{R}}(Y, K_1^*) \to 0$$

$$0 \to \operatorname{Hom}_{\mathfrak{R}}(Y, K_1^*) \xrightarrow{c\cdot} \operatorname{Hom}_{\mathfrak{R}}(Y, \mathfrak{R}_0) \xrightarrow{a\cdot} \operatorname{Hom}_{\mathfrak{R}}(Y, N_1) \to \mathrm{A}_{\mathrm{con}} \to 0$$

shows that (5.H) is quasi-isomorphic to $\mathrm{A}_{\mathrm{con}}[1]$.

Next, consider (5.I). Applying 5.6(1) with the module νY (instead of Y), it follows that

$$0 \to \operatorname{Hom}_{\mathfrak{R}}(\nu Y, K_1^*) \xrightarrow{c\cdot} \operatorname{Hom}_{\mathfrak{R}}(\nu Y, \mathfrak{R}_0) \xrightarrow{a\cdot} \operatorname{Hom}_{\mathfrak{R}}(\nu Y, N_1) \to 0$$

is exact. Further, since trivially d^* is a minimal $(\operatorname{add}\mathfrak{R})$-approximation, applying the functor $\operatorname{Hom}_{\mathfrak{R}}(\nu Y, -)$ to (5.E) gives an exact sequence

$$0 \to \operatorname{Hom}_{\mathfrak{R}}(\nu Y, N_1) \xrightarrow{b^*\cdot} \operatorname{Hom}_{\mathfrak{R}}(\nu Y, \mathfrak{R}_1) \xrightarrow{d^*\cdot} \operatorname{Hom}_{\mathfrak{R}}(\nu Y, K_1^*) \to \mathrm{B}_{\mathrm{con}} \to 0.$$

Splicing the last two exact sequences shows that (5.I) is quasi-isomorphic to $\mathrm{B}_{\mathrm{con}}[0]$.

(2) Since as above $\dim_{\mathbb{C}} \mathrm{B}_{\mathrm{con}} < \infty$, this follows since we can apply (1) twice.

(3) This follows from (2) using the commutativity of the diagram in 5.8. □

We write I_{A} for the ideal $[\mathfrak{R}]$ of $\mathrm{A} = \operatorname{End}_{\mathfrak{R}}(\mathfrak{R} \oplus N_1)$. The following gives a description of the mutation–mutation functors in terms of the bimodules I_{A} and $\widehat{I}_{\mathrm{con}}$.

Theorem 5.10. *There are functorial isomorphisms*
(1) $\Phi \circ \Phi \cong \mathbf{R}\operatorname{Hom}_{\mathrm{A}}(I_{\mathrm{A}}, -)$.
(2) $\Phi' \circ \Phi' \cong \mathbf{R}\operatorname{Hom}_{\widehat{\Lambda}}(\widehat{I}_{\mathrm{con}}, -)$.
Thus $\mathbf{R}\operatorname{Hom}_{\widehat{\Lambda}}(\widehat{I}_{\mathrm{con}}, -)$ *is an autoequivalence of* $\mathrm{D}^{\mathrm{b}}(\operatorname{mod}\widehat{\Lambda})$, *with inverse* $- \otimes_{\widehat{\Lambda}}^{\mathbf{L}} \widehat{I}_{\mathrm{con}}$, *and so* $\widehat{I}_{\mathrm{con}}$ *is a tilting* $\widehat{\Lambda}$-*module.*

Proof. (1) To avoid confusion we will write $\mathrm{B} = \operatorname{End}_{\mathfrak{R}}(\nu Y)$, and also denote

$$\mathrm{D}^{\mathrm{b}}(\operatorname{mod}\mathrm{A}) \xrightarrow{\Phi} \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\mathrm{B}) \text{ by } \Phi^1 = \mathbf{R}\operatorname{Hom}_{\mathrm{A}}(V, -),$$

and

$$\mathrm{D}^{\mathrm{b}}(\operatorname{mod}\mathrm{B}) \xrightarrow{\Phi} \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\mathrm{A}) \text{ by } \Phi^2 = \mathbf{R}\operatorname{Hom}_{\mathrm{B}}(W, -).$$

The composition $\Phi^2 \circ \Phi^1 \cong \mathbf{R}\operatorname{Hom}_{\mathrm{A}}(W \otimes_{\mathrm{B}}^{\mathbf{L}} V, -)$, so we show that $W \otimes_{\mathrm{B}}^{\mathbf{L}} V \cong I_{\mathrm{A}}$.

As above, $W := \operatorname{Hom}_{\mathfrak{R}}(\nu Y, Y)$. Hence W is quasi-isomorphic to its projective resolution

$$\ldots \to 0 \to \operatorname{Hom}_{\mathfrak{R}}(\nu Y, K_1^*) \xrightarrow{\binom{c\cdot}{0}} \operatorname{Hom}_{\mathfrak{R}}(\nu Y, \mathfrak{R}_0) \oplus \operatorname{Hom}_{\mathfrak{R}}(\nu Y, \mathfrak{R}) \to 0 \to \ldots \quad (5.\mathrm{J})$$

But Φ^1 is an equivalence that sends V to B. In fact, Φ^1 sends

$$\operatorname{Hom}_{\mathfrak{R}}(Y, \mathfrak{R}) \mapsto \operatorname{Hom}_{\mathfrak{R}}(\nu Y, \mathfrak{R})$$
$$\operatorname{Hom}_{\mathfrak{R}}(Y, K_1^*) \mapsto \operatorname{Hom}_{\mathfrak{R}}(\nu Y, K_1^*).$$

Hence applying the inverse of Φ^1, namely $- \otimes_{\mathrm{B}}^{\mathbf{L}} V$, to (5.J) gives the complex

$$\ldots \to 0 \to \operatorname{Hom}_{\mathfrak{R}}(Y, K_1^*) \xrightarrow{\binom{c\cdot}{0}} \operatorname{Hom}_{\mathfrak{R}}(Y, \mathfrak{R}_0) \oplus \operatorname{Hom}_{\mathfrak{R}}(Y, \mathfrak{R}) \to 0 \to \ldots \quad (5.\mathrm{K})$$

Then using $K_1^* \cong K_0$ and the fact that (5.B)

$$0 \to \operatorname{Hom}_{\mathfrak{R}}(Y, K_1^*) \xrightarrow{c\cdot} \operatorname{Hom}_{\mathfrak{R}}(Y, \mathfrak{R}_0) \to \operatorname{Hom}_{\mathfrak{R}}(Y, N_1) \to \mathrm{A}_{\mathrm{con}} \to 0$$

is exact, the complex (5.K) is quasi-isomorphic to

$$\ldots \to 0 \to \mathrm{A} = \operatorname{Hom}_{\mathfrak{R}}(Y, \mathfrak{R}) \oplus \operatorname{Hom}_{\mathfrak{R}}(Y, N_1) \to \mathrm{A}_{\mathrm{con}} \to \ldots$$

which is clearly quasi-isomorphic to I_{A}.

(2) is similar. The final statements follow since the mutation functors Φ' are always derived equivalences [IW10, 6.8]. □

Corollary 5.11. $\Phi \circ \Phi(S) \cong S[-2]$.

Proof. Let $J := \operatorname{Rad}(A_{\mathrm{con}})$ denote the Jacobson radical, then $A_{\mathrm{con}}/J \cong S$ and so

$$S \cong (A_{\mathrm{con}}/J) \otimes_{A_{\mathrm{con}}} A_{\mathrm{con}} \cong (A_{\mathrm{con}}/J) \otimes^{\mathbf{L}}_{A_{\mathrm{con}}} A_{\mathrm{con}}.$$

By 5.9(2) and 5.10 it follows that $S[-2] \cong (A_{\mathrm{con}}/J) \otimes^{\mathbf{L}}_{A_{\mathrm{con}}} \mathbf{R}\operatorname{Hom}_A(I_A, A_{\mathrm{con}})$, so the result follows from the fact that

$$(A_{\mathrm{con}}/J) \otimes^{\mathbf{L}}_{A_{\mathrm{con}}} \mathbf{R}\operatorname{Hom}_A(I_A, A_{\mathrm{con}}) \cong \mathbf{R}\operatorname{Hom}_A(I_A, (A_{\mathrm{con}}/J) \otimes^{\mathbf{L}}_{A_{\mathrm{con}}} A_{\mathrm{con}})$$
$$\cong \mathbf{R}\operatorname{Hom}_A(I_A, S),$$

where the first canonical isomorphism is standard (see e.g. [IR08, 2.10(2)]). $\qquad\square$

The equivalence $\mathbf{R}\operatorname{Hom}_{\widehat{\Lambda}}(\widehat{I}_{\mathrm{con}}, -)$ in 5.10 above is the complete local version of our noncommutative twist functor. The following is a corollary of the results in this section, and will be used later.

Corollary 5.12. *With the setup as above,*
(1) $\mathbf{R}\operatorname{Hom}_{\widehat{\Lambda}}(\widehat{I}_{\mathrm{con}}, \mathbb{F}A_{\mathrm{con}}) \cong \mathbb{F}A_{\mathrm{con}}[-2].$
(2) $\mathbf{R}\operatorname{Hom}_{\widehat{\Lambda}}(\widehat{I}_{\mathrm{con}}, \widehat{T}) \cong \widehat{T}[-2].$

Proof. (1) We have $\mathbf{R}\operatorname{Hom}_{\widehat{\Lambda}}(\widehat{I}_{\mathrm{con}}, \mathbb{F}A_{\mathrm{con}}) \stackrel{5.10}{\cong} \Phi' \circ \Phi'(\mathbb{F}A_{\mathrm{con}}) \stackrel{5.9}{\cong} \mathbb{F}A_{\mathrm{con}}[-2].$
(2) $\mathbf{R}\operatorname{Hom}_{\widehat{\Lambda}}(\widehat{I}_{\mathrm{con}}, \widehat{T}) \stackrel{5.10}{\cong} \Phi' \circ \Phi'(\widehat{T}) = \Phi' \circ \Phi'(\mathbb{F}S) \stackrel{5.8}{\cong} \mathbb{F} \circ \Phi \circ \Phi(S) \stackrel{5.11}{\cong} \mathbb{F}S[-2] = \widehat{T}[-2]. \quad\square$

6. Zariski Local Twists

We keep our running setup of a flopping contraction as in §4, with an irreducible rational curve contracting to a point $\mathfrak{m}$. Since R is Gorenstein, R is a canonical R-module, but in this non-local setting canonical modules are not unique. Throughout we set $\omega_R := g^!\mathbb{C}[-3]$, where $g\colon \operatorname{Spec} R \to \operatorname{Spec}\mathbb{C}$ is the structure morphism. This may or may not be isomorphic to R.

Throughout, as in §4, $\Lambda := \operatorname{End}_R(R \oplus N)$ and we already know that $\Lambda \in \operatorname{CM} R$. As before $I_{\mathrm{con}} := [R]$ and $\Lambda_{\mathrm{con}} := \Lambda/I_{\mathrm{con}}$. Since the category $\operatorname{CM} R$ is no longer Krull–Schmidt, we must be careful.

6.1. **Local tilting.** In this subsection, we show that the Λ-module I_{con} is still a tilting module, with endomorphism ring Λ. First, we need the following lemma.

Lemma 6.1. *Put $I := I_{\mathrm{con}}$, then*
(1) *Viewing I as a right Λ-module, we have $\Lambda \cong \operatorname{End}_\Lambda(I)$ under the map which sends $\lambda \in \Lambda$ to the map $(\lambda\cdot)\colon I \to I$ given by $i \mapsto \lambda i$.*
(2) *Under the isomorphism in (1), the bimodule $_{\operatorname{End}_\Lambda(I)}I_\Lambda$ coincides with the natural bimodule structure $_\Lambda I_\Lambda$.*

Proof. (1) First recall that $\Lambda \cong \operatorname{End}_\Lambda(\Lambda_\Lambda)$ via the ring homomorphism which sends $\lambda \in \Lambda$ to $(\lambda\cdot)\colon \Lambda \to \Lambda$ sending $x \mapsto \lambda x$, left multiplication by λ. Now consider the short exact sequence

$$0 \to I \to \Lambda \to \Lambda_{\mathrm{con}} \to 0 \tag{6.A}$$

of Λ-bimodules. Applying $\operatorname{Hom}_\Lambda(-, \Lambda)$ gives an exact sequence

$$0 \to \operatorname{Hom}_\Lambda(\Lambda_{\mathrm{con}}, \Lambda) \to \operatorname{Hom}_\Lambda(\Lambda, \Lambda) \to \operatorname{Hom}_\Lambda(I, \Lambda) \to \operatorname{Ext}^1_\Lambda(\Lambda_{\mathrm{con}}, \Lambda). \tag{6.B}$$

Since $\Lambda \in \operatorname{CM} R$, it follows that Λ is singular d-CY by [IW10, 2.22(2)], hence

$$\dim_R \Lambda_{\mathrm{con}} = d - \inf\{i \geq 0 \mid \operatorname{Ext}^i_\Lambda(\Lambda_{\mathrm{con}}, \Lambda) \neq 0\}$$

by [IR08, 3.4(5)(ii)]. But R is normal, which implies that $\dim_R \Lambda_{\mathrm{con}} \leq d - 2$ (see e.g. [IW10, 6.19]), thus combining we see that $\operatorname{Hom}_\Lambda(\Lambda_{\mathrm{con}}, \Lambda) \cong \operatorname{Ext}^1_\Lambda(\Lambda_{\mathrm{con}}, \Lambda) = 0$.

On the other hand, applying $\operatorname{Hom}_\Lambda(I, -)$ to (6.A) gives

$$0 \to \operatorname{Hom}_\Lambda(I, I) \to \operatorname{Hom}_\Lambda(I, \Lambda) \to \operatorname{Hom}_\Lambda(I, \Lambda_{\mathrm{con}}) = 0. \tag{6.C}$$

Combining (6.B) and (6.C) shows that $\mathrm{End}_\Lambda(\Lambda) \cong \mathrm{End}_\Lambda(I)$. Chasing through the sequences, this isomorphism is given by $(\lambda\cdot) \in \mathrm{End}_\Lambda(\Lambda) \mapsto (\lambda\cdot) \in \mathrm{End}_\Lambda(I)$, so the isomorphism is in fact a ring isomorphism.

(2) I is naturally a left $\mathrm{End}_\Lambda(I)$-module via $f \cdot i := f(i)$. By the isomorphism in (1), this is just left multiplication by λ. $\qquad\square$

We can now extend 5.10.

Theorem 6.2. I_{con} *is a tilting* Λ-*module with* $\mathrm{End}_\Lambda(I_{\mathrm{con}}) \cong \Lambda$. *Furthermore*

(1) $\mathrm{pd}_\Lambda I_{\mathrm{con}} = 2$, *so* $\mathrm{pd}_\Lambda \Lambda_{\mathrm{con}} = 3$.

(2) $\mathrm{pd}_{\Lambda^{\mathrm{op}}} \Lambda_{\mathrm{con}} = 3$.

Proof. The fact that $\mathrm{End}_\Lambda(I) \cong \Lambda$ is 6.1. Let $\mathfrak{n} \in \mathrm{Max}\, R$, then the short exact sequence $0 \to I_{\mathrm{con}} \to \Lambda \to \Lambda_{\mathrm{con}} \to 0$ shows that $(I_{\mathrm{con}})_\mathfrak{n} \cong \Lambda_\mathfrak{n}$ if $\mathfrak{n} \notin \mathrm{Supp}\,\Lambda_{\mathrm{con}} = \{\mathfrak{m}\}$. Further,

$$\mathrm{Ext}^i_\Lambda(I_{\mathrm{con}}, I_{\mathrm{con}})_\mathfrak{m} \otimes_{R_\mathfrak{m}} \widehat{R}_\mathfrak{m} \cong \mathrm{Ext}^i_{\widehat{\Lambda}}(\widehat{I}_{\mathrm{con}}, \widehat{I}_{\mathrm{con}}) \overset{5.10}{=} 0$$

for all $i > 0$. Thus, in either case, the completion of $\mathrm{Ext}^i_\Lambda(I_{\mathrm{con}}, I_{\mathrm{con}})$ at each maximal ideal is zero, hence $\mathrm{Ext}^i_\Lambda(I_{\mathrm{con}}, I_{\mathrm{con}}) = 0$ for all $i > 0$. Similarly

$$\mathrm{pd}_\Lambda I_{\mathrm{con}} = \sup\{\mathrm{pd}_{\widehat{\Lambda}} \widehat{\Lambda}_{\mathrm{con}} \mid \mathfrak{n} \in \mathrm{Supp}\,\Lambda_{\mathrm{con}} = \{\mathfrak{m}\}\} - 1$$

which is 2 by 5.6(3). For generation, suppose that $Y \in \mathrm{D}(\mathrm{Mod}\,\Lambda)$ with $\mathbf{R}\mathrm{Hom}_\Lambda(I_{\mathrm{con}}, Y) = 0$. Then

$$0 = \mathbf{R}\mathrm{Hom}_\Lambda(I_{\mathrm{con}}, Y)_\mathfrak{n} \otimes \widehat{R} \overset{2.15}{\cong} \mathbf{R}\mathrm{Hom}_{\widehat{\Lambda}}(\widehat{I}_{\mathrm{con}}, \widehat{Y})$$

and so since $\widehat{I}_{\mathrm{con}}$ is tilting by 5.10, $\widehat{Y} = 0$ for all $\mathfrak{n} \in \mathrm{Max}\,R$. Hence $Y = 0$, proving generation. Part (2) is shown similarly. $\qquad\square$

6.2. **Tracking the contraction algebra.** In this subsection we track the objects $\mathbb{F}\mathrm{A}_{\mathrm{con}}$ and T across the derived equivalence induced by I_{con} (extending 5.12), and also under the action of a certain Serre functor.

Proposition 6.3. *Under the setup above,*

(1) $\mathbf{R}\mathrm{Hom}_\Lambda(I_{\mathrm{con}}, \mathbb{F}\mathrm{A}_{\mathrm{con}}) \cong \mathbb{F}\mathrm{A}_{\mathrm{con}}[-2]$ *as* Λ-*modules.*

(2) $\mathbf{R}\mathrm{Hom}_\Lambda(I_{\mathrm{con}}, T) \cong T[-2]$ *as* Λ-*modules.*

Proof. (1) We have

$$\mathbf{R}\mathrm{Hom}_\Lambda(I_{\mathrm{con}}, \mathbb{F}\mathrm{A}_{\mathrm{con}})_\mathfrak{n} \otimes \widehat{R} \cong \mathbf{R}\mathrm{Hom}_{\widehat{\Lambda}}(\widehat{I}_{\mathrm{con}}, \widehat{\mathbb{F}\mathrm{A}_{\mathrm{con}}})$$

$$\overset{5.12}{\cong} \begin{cases} 0 & \text{if } \mathfrak{n} \notin \mathrm{Supp}\,\mathbb{F}\mathrm{A}_{\mathrm{con}} = \{\mathfrak{m}\} \\ \mathbb{F}\mathrm{A}_{\mathrm{con}}[-2] & \text{if } \mathfrak{n} \in \mathrm{Supp}\,\mathbb{F}\mathrm{A}_{\mathrm{con}} = \{\mathfrak{m}\}. \end{cases}$$

It follows that $C := \mathbf{R}\mathrm{Hom}(I_{\mathrm{con}}, \mathbb{F}\mathrm{A}_{\mathrm{con}})$ is a stalk complex in homological degree two, and further that the degree two piece is supported only on $\mathfrak{m}$. Thus

$$C \cong H^2(C)[-2] \overset{2.15}{\cong} \widehat{H^2(C)}[-2] \cong \mathbb{F}\mathrm{A}_{\mathrm{con}}[-2]$$

as required.

(2) is similar, using $\mathbf{R}\mathrm{Hom}_{\widehat{\Lambda}}(\widehat{I}_{\mathrm{con}}, \widehat{T}) \cong \widehat{T}[-2]$ by 5.12. $\qquad\square$

In our possibly singular setting, we need the following RHom version of Serre functors.

Definition 6.4. *We say that a functor* $\mathbb{S}\colon \mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda) \to \mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)$ *is a Serre functor relative to* ω_R *if there are functorial isomorphisms*

$$\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_\Lambda(a, b), \omega_R) \cong \mathbf{R}\mathrm{Hom}_\Lambda(b, \mathbb{S}(a)) \qquad (6.\mathrm{D})$$

in $\mathrm{D}(\mathrm{Mod}\,R)$ *for all* $a \in \mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)$ *and all* $b \in \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$.

Since under our assumptions in this section R is Gorenstein, the module ω_R is a dualizing module of R, i.e.

$$\mathbf{R}\mathrm{Hom}_R(-, \omega_R) \colon \mathrm{D}(\mathrm{mod}\,R) \to \mathrm{D}(\mathrm{mod}\,R)$$

gives a duality. Since $\mathrm{Hom}_R(-, \omega_R) \colon \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda^{\mathrm{op}}$, this induces a duality

$$\mathbf{R}\mathrm{Hom}_R(-, \omega_R) \colon \mathrm{D}^{\pm}(\mathrm{mod}\,\Lambda) \to \mathrm{D}^{\mp}(\mathrm{mod}\,\Lambda),$$

and we define the functor $\mathbb{S}_\Lambda$ to be the composition

$$\mathrm{D}^-(\mathrm{mod}\,\Lambda) \xrightarrow{\mathbf{R}\mathrm{Hom}_\Lambda(-, \Lambda)} \mathrm{D}^+(\mathrm{mod}\,\Lambda^{\mathrm{op}}) \xrightarrow{\mathbf{R}\mathrm{Hom}_R(-, \omega_R)} \mathrm{D}^-(\mathrm{mod}\,\Lambda).$$

As in the proof of [IR08, 3.5], functorially $\mathbb{S}_\Lambda(-) \cong \mathbf{R}\mathrm{Hom}_R(\Lambda, \omega_R) \otimes_\Lambda^{\mathbf{L}} -$. The following is known by [G06, 7.2.14] and [IR08, §3]. We include the proof for completeness.

Proposition 6.5. $\mathbb{S}_\Lambda$ *is a Serre functor relative to* ω_R.

Proof. First, since $\Lambda = \mathrm{End}_R(M) \in \mathrm{CM}\,R$, it follows that

$$\mathrm{inj.dim}_\Lambda \Lambda = \dim R = \mathrm{inj.dim}_{\Lambda^{\mathrm{op}}} \Lambda$$

by combining [IW10, 2.22] and [IR08, 3.1(6), 3.4(1)]. Given this, the fact that $\mathbb{S}_\Lambda$ maps $\mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)$ to $\mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)$ is then [G06, 7.2.14(i)], and the fact that (6.D) holds is [G06, 7.2.8] (see also [IR08, §3]). $\qquad\square$

The object $\mathbb{F}\mathrm{A}_{\mathrm{con}}$ does not change under the action of this Serre functor.

Proposition 6.6. $\mathbb{S}_\Lambda(\mathbb{F}\mathrm{A}_{\mathrm{con}}) \cong \mathbb{F}\mathrm{A}_{\mathrm{con}}$.

Proof. We have

$$\mathbb{S}_\Lambda(\mathbb{F}\mathrm{A}_{\mathrm{con}})_{\mathfrak{n}} = \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_\Lambda(\mathbb{F}\mathrm{A}_{\mathrm{con}}, \Lambda), \omega_R)_{\mathfrak{n}}$$
$$\overset{2.15}{\cong} \mathbf{R}\mathrm{Hom}_{R_{\mathfrak{n}}}(\mathbf{R}\mathrm{Hom}_{\Lambda_{\mathfrak{n}}}(\mathbb{F}\mathrm{A}_{\mathrm{con}\mathfrak{n}}, \Lambda_{\mathfrak{n}}), (\omega_R)_{\mathfrak{n}})$$

which as above is isomorphic to $\mathbf{R}\mathrm{Hom}_{R_{\mathfrak{n}}}(\Lambda_{\mathfrak{n}}, (\omega_R)_{\mathfrak{n}}) \otimes_{\Lambda_{\mathfrak{n}}}^{\mathbf{L}} \mathbb{F}\mathrm{A}_{\mathrm{con}\mathfrak{n}}$. Now since R is Gorenstein $(\omega_R)_{\mathfrak{n}} \cong R_{\mathfrak{n}}$, and further $\mathbf{R}\mathrm{Hom}_{R_{\mathfrak{n}}}(\Lambda_{\mathfrak{n}}, R_{\mathfrak{n}}) \cong \mathrm{Hom}_{R_{\mathfrak{n}}}(\Lambda_{\mathfrak{n}}, R_{\mathfrak{n}})$ since $\Lambda \in \mathrm{CM}\,R$. But since $\Lambda_{\mathfrak{n}}$ is a symmetric $R_{\mathfrak{n}}$-algebra [IR08, 2.4(3)], $\mathrm{Hom}_{R_{\mathfrak{n}}}(\Lambda_{\mathfrak{n}}, R_{\mathfrak{n}}) \cong \Lambda_{\mathfrak{n}}$ as $\Lambda_{\mathfrak{n}}$-bimodules. Thus

$$\mathbb{S}_\Lambda(\mathbb{F}\mathrm{A}_{\mathrm{con}})_{\mathfrak{n}} \cong \mathrm{Hom}_{R_{\mathfrak{n}}}(\Lambda_{\mathfrak{n}}, R_{\mathfrak{n}}) \otimes_{\Lambda_{\mathfrak{n}}}^{\mathbf{L}} \mathbb{F}\mathrm{A}_{\mathrm{con}\mathfrak{n}} \cong \left\{ \begin{array}{ll} 0 & \text{if } \mathfrak{n} \notin \mathrm{Supp}\,\mathbb{F}\mathrm{A}_{\mathrm{con}} = \{\mathfrak{m}\} \\ \mathbb{F}\mathrm{A}_{\mathrm{con}} & \text{if } \mathfrak{n} \in \mathrm{Supp}\,\mathbb{F}\mathrm{A}_{\mathrm{con}} = \{\mathfrak{m}\}. \end{array} \right.$$

It follows that $\mathbb{S}_\Lambda(\mathbb{F}\mathrm{A}_{\mathrm{con}})$ is a stalk complex in homological degree zero, and that degree zero piece has finite length, supported only on $\mathfrak{m}$. Thus

$$\mathbb{S}_\Lambda(\mathbb{F}\mathrm{A}_{\mathrm{con}}) \cong H^0(\mathbb{S}_\Lambda(\mathbb{F}\mathrm{A}_{\mathrm{con}})) \overset{2.15}{\cong} H^0(\mathbb{S}_\Lambda(\mathbb{F}\mathrm{A}_{\mathrm{con}}))_{\mathfrak{m}} \cong \mathbb{F}\mathrm{A}_{\mathrm{con}}. \qquad\square$$

6.3. **Algebraic noncommutative twist functors.** We can now give our definition of algebraic noncommutative twist functors. The ring homomorphism $\Lambda \to \Lambda_{\mathrm{con}}$ gives rise to the standard extension–restriction of scalars adjunction. To ease notation, we temporarily write $M := {}_\Lambda(\Lambda_{\mathrm{con}})_{\Lambda_{\mathrm{con}}}$ and $N := {}_{\Lambda_{\mathrm{con}}}(\Lambda_{\mathrm{con}})_\Lambda$. The adjunctions then read

$$\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda_{\mathrm{con}}) \quad \xleftrightarrow{\substack{F_{\mathrm{con}}^{\mathrm{LA}} := -\otimes_\Lambda^{\mathbf{L}} M \\ F_{\mathrm{con}} := \mathbf{R}\mathrm{Hom}_{\Lambda_{\mathrm{con}}}(M, -) = -\otimes_{\Lambda_{\mathrm{con}}}^{\mathbf{L}} N \\ F_{\mathrm{con}}^{\mathrm{RA}} := \mathbf{R}\mathrm{Hom}_\Lambda(N, -)}} \quad \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda) \qquad (6.\mathrm{E})$$

where we make use of the following:

Lemma 6.7. F_{con}, $F_{\mathrm{con}}^{\mathrm{RA}}$ *and* $F_{\mathrm{con}}^{\mathrm{LA}}$ *all preserve bounded complexes of finitely generated modules.*

Proof. F_{con} preserves boundedness since restriction of scalars is exact on the abelian level. We remark that if $Y \in \operatorname{Mod}\Lambda$ then $F_{\mathrm{con}}^{\mathrm{RA}}(Y)$ and $F_{\mathrm{con}}^{\mathrm{LA}}(Y)$ are both bounded since Λ_{con} has finite projective, hence finite flat, dimension as both a left and right Λ-module by 6.2. The fact that $F_{\mathrm{con}}^{\mathrm{RA}}$ and $F_{\mathrm{con}}^{\mathrm{LA}}$ both preserve boundedness then follows by induction on the length of the bounded complex. The preservation of finite generation is obvious. $\square$

Now $\Lambda_{\mathrm{con}} \cong \widehat{\Lambda}_{\mathrm{con}}$ as algebras by 2.17, and there is a morita equivalence

$$\operatorname{mod} A_{\mathrm{con}} \xleftarrow[\operatorname{Hom}_{\widehat{\Lambda}_{\mathrm{con}}}(\mathbb{F}A_{\mathrm{con}},-)]{-\otimes_{A_{\mathrm{con}}}\mathbb{F}A_{\mathrm{con}}} \operatorname{mod}\widehat{\Lambda}_{\mathrm{con}} \tag{6.F}$$

induced from 5.4.

Lemma 6.8. *Composing* (6.E) *with* (6.F) *gives adjunctions*

$$\begin{array}{c} F^{\mathrm{LA}} \\ \mathrm{D}^{\mathrm{b}}(\operatorname{mod} A_{\mathrm{con}}) \xrightarrow{\;\;F:=-\otimes_{A_{\mathrm{con}}}^{\mathbf{L}}\mathbb{F}A_{\mathrm{con}}\;\;} \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda) \\ F^{\mathrm{RA}}:=\mathbf{R}\operatorname{Hom}_{\Lambda}(\mathbb{F}A_{\mathrm{con}},-) \end{array} \tag{6.G}$$

Proof. This follows since $\mathbb{F}A_{\mathrm{con}} \otimes_{\Lambda_{\mathrm{con}}} \Lambda_{\mathrm{con}} \cong \mathbb{F}A_{\mathrm{con}}$ as A_{con}–Λ bimodules. $\square$

Now since $M \otimes_{\Lambda_{\mathrm{con}}}^{\mathbf{L}} N \cong M \otimes_{\Lambda_{\mathrm{con}}} N \cong {}_{\Lambda}(\Lambda_{\mathrm{con}})_{\Lambda}$ as Λ-bimodules, we have functorial isomorphisms

$$F \circ F^{\mathrm{RA}} \cong F_{\mathrm{con}} \circ F_{\mathrm{con}}^{\mathrm{RA}} \cong \mathbf{R}\operatorname{Hom}_{\Lambda}(M \otimes_{\Lambda_{\mathrm{con}}}^{\mathbf{L}} N, -) \cong \mathbf{R}\operatorname{Hom}_{\Lambda}({}_{\Lambda}(\Lambda_{\mathrm{con}})_{\Lambda}, -)$$

and

$$F \circ F^{\mathrm{LA}} \cong F_{\mathrm{con}} \circ F_{\mathrm{con}}^{\mathrm{LA}} \cong - \otimes_{\Lambda}^{\mathbf{L}} M \otimes_{\Lambda_{\mathrm{con}}}^{\mathbf{L}} N \cong - \otimes_{\Lambda}^{\mathbf{L}} {}_{\Lambda}(\Lambda_{\mathrm{con}})_{\Lambda}.$$

Definition 6.9. *Using the bimodule structure* ${}_{\Lambda}I_{\Lambda}$ *we define*

(1) *The twist functor* $T_{\mathrm{con}}\colon \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda) \to \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda)$ *by* $T_{\mathrm{con}} = \mathbf{R}\operatorname{Hom}_{\Lambda}(I_{\mathrm{con}}, -)$.
(2) *The dual twist functor* $T_{\mathrm{con}}^*\colon \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda) \to \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda)$ *by* $T_{\mathrm{con}}^* = - \otimes_{\Lambda}^{\mathbf{L}} I_{\mathrm{con}}$.

The twist is an equivalence, and its inverse is the dual twist, by 6.2. The terminology 'twist' is justified by the following lemma.

Lemma 6.10. *We have functorial triangles as follows*

(1) $F \circ F^{\mathrm{RA}} \to \operatorname{Id}_{\Lambda} \to T_{\mathrm{con}} \to$
(2) $T_{\mathrm{con}}^* \to \operatorname{Id}_{\Lambda} \to F \circ F^{\mathrm{LA}} \to$

Proof. (1) The short exact sequence (6.A) of Λ-bimodules gives rise to a triangle

$$I_{\mathrm{con}} \to \Lambda \to \Lambda_{\mathrm{con}} \to \tag{6.H}$$

in the derived category of Λ-bimodules. For $X \in \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda)$, applying $\mathbf{R}\operatorname{Hom}_{\Lambda}(-, X)$ to (6.H) gives a triangle

$$\mathbf{R}\operatorname{Hom}_{\Lambda}(\Lambda_{\mathrm{con}}, X) \to \mathbf{R}\operatorname{Hom}_{\Lambda}(\Lambda, X) \to \mathbf{R}\operatorname{Hom}_{\Lambda}(I_{\mathrm{con}}, X) \to$$

in $\mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda)$, which is simply

$$F \circ F^{\mathrm{RA}}(X) \to \operatorname{Id}_{\Lambda}(X) \to T_{\mathrm{con}}(X) \to .$$

(2) This follows by applying $X \otimes_{\Lambda}^{\mathbf{L}} -$ to (6.H). $\square$

6.4. **Geometric local noncommutative twists.** Again we keep the setup of §4. In this subsection we define the geometric noncommutative twist purely in terms of the geometry of U, and use it as the local model for our global twist functor later.

To fix notation, as in (2.E) we write $\mathcal{V} := \mathcal{O}_U \oplus \mathcal{N}$ and thus we have a derived equivalence

$$
D^b(\mathrm{mod}\,\Lambda) \underset{H^{\mathrm{RA}}:=\mathbf{R}\mathrm{Hom}_U(\mathcal{V},-)}{\overset{H^{\mathrm{LA}}:=\mathbf{R}\mathrm{Hom}_U(\mathcal{V},-)}{\xrightleftharpoons{\quad H:=-\otimes^{\mathbf{L}}_\Lambda \mathcal{V}\quad}}} D^b(\mathrm{coh}\,U) \tag{6.I}
$$

given by a tilting bundle $\mathcal{V}$ and $\Lambda \cong \mathrm{End}_Y(\mathcal{V})$. We know by 3.7 that $\mathcal{E}_U$ (in degree zero) corresponds to $\mathbb{F}A_{\mathrm{con}}$ across the derived equivalence.

Lemma 6.11. *With notation and setup as in §4, composing (6.G) with (6.I) gives*

$$
D^b(\mathrm{mod}\,A_{\mathrm{con}}) \underset{G^{\mathrm{RA}}_U:=\mathbf{R}\mathrm{Hom}_U(\mathcal{E}_U,-)}{\overset{G^{\mathrm{LA}}_U:=\mathbf{R}\mathrm{Hom}_{A_{\mathrm{con}}}(\mathbf{R}\mathrm{Hom}_U(-,\mathcal{E}_U),A_{\mathrm{con}})}{\xrightleftharpoons{\quad G_U:=-\otimes^{\mathbf{L}}_{A_{\mathrm{con}}} \mathcal{E}_U\quad}}} D^b(\mathrm{coh}\,U) \tag{6.J}
$$

Proof. The composition $H \circ F$ is clear since $\mathbb{F}A_{\mathrm{con}} \otimes^{\mathbf{L}}_\Lambda \mathcal{V} \cong \mathcal{E}_U$ by 3.7. Also by 3.7 $\mathbf{R}\mathrm{Hom}_U(\mathcal{V},\mathcal{E}_U) \cong \mathbb{F}A_{\mathrm{con}}$, so the composition $F^{\mathrm{RA}} \circ H^{\mathrm{RA}}$ is functorially isomorphic to

$$
\mathbf{R}\mathrm{Hom}_\Lambda(\mathbb{F}A_{\mathrm{con}}, \mathbf{R}\mathrm{Hom}_U(\mathcal{V},-)) \cong \mathbf{R}\mathrm{Hom}_\Lambda(\mathbf{R}\mathrm{Hom}_U(\mathcal{V},\mathcal{E}_U), \mathbf{R}\mathrm{Hom}_U(\mathcal{V},-))
$$
$$
\cong \mathbf{R}\mathrm{Hom}_U(\mathcal{E}_U,-).
$$

Lastly,

$$
\mathbf{R}\mathrm{Hom}_{A_{\mathrm{con}}}(F^{\mathrm{LA}} \circ H^{\mathrm{LA}}(-), A_{\mathrm{con}}) \cong \mathbf{R}\mathrm{Hom}_U(-, H \circ F(A_{\mathrm{con}}))
$$
$$
\cong \mathbf{R}\mathrm{Hom}_U(-, A_{\mathrm{con}} \otimes^{\mathbf{L}}_{A_{\mathrm{con}}} \mathcal{E}_U)
$$
$$
\cong \mathbf{R}\mathrm{Hom}_U(-, \mathcal{E}_U).
$$

But since A_{con} is self-injective $\mathbf{R}\mathrm{Hom}_{A_{\mathrm{con}}}(-, A_{\mathrm{con}}) = \mathrm{Hom}_{A_{\mathrm{con}}}(-, A_{\mathrm{con}})$ is a duality, so dualizing both sides gives

$$
F^{\mathrm{LA}} \circ H^{\mathrm{LA}}(-) \cong \mathbf{R}\mathrm{Hom}_{A_{\mathrm{con}}}(\mathbf{R}\mathrm{Hom}_U(-,\mathcal{E}_U), A_{\mathrm{con}}),
$$

as required. □

Remark 6.12. We remark that (6.J) is intrinsic to the geometry of U, and does not depend on the choice of Λ.

Definition 6.13.   (1) *We define $T_{\mathcal{E}_U}$ as the composition*

$$
D^b(\mathrm{coh}\,U) \xrightarrow{\mathbf{R}\mathrm{Hom}_U(\mathcal{V},-)} D^b(\mathrm{mod}\,\Lambda) \xrightarrow{T_{\mathrm{con}}} D^b(\mathrm{mod}\,\Lambda) \xrightarrow{-\otimes^{\mathbf{L}}_\Lambda \mathcal{V}} D^b(\mathrm{coh}\,U)
$$

and call it the geometric noncommutative twist functor.

(2) *We obtain an inverse twist $T^*_{\mathcal{E}_U}$ by composing similarly with T^*_{con}.*

Being a composition of equivalences, the geometric noncommutative twist is also an equivalence. We remark that the definition of the geometric noncommutative twist functor relies on the tilting equivalence, but below in 6.16 we show that it can be formulated as a Fourier–Mukai transform, by making use of the following setup.

Consider the commutative diagram

$$
\begin{array}{ccc}
U \times U & \xrightarrow{\ p_1\ } & U \\
{\scriptstyle p_2} \downarrow \quad {\scriptstyle \pi_{U \times U}} \searrow & & \downarrow {\scriptstyle \pi_U} \\
U & \xrightarrow{\ \pi_U\ } & \mathrm{pt}
\end{array}
\tag{6.K}
$$

and set $\mathcal{V}^\vee \boxtimes \mathcal{V} = p_1^* \mathcal{V}^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_{U \times U}} p_2^* \mathcal{V}$. Then there is an induced derived equivalence

$$
\mathrm{D^b}(\operatorname{coh} U \times U) \xrightleftharpoons[{-\otimes^{\mathbf{L}}_{\Lambda^{\mathrm{e}}}(\mathcal{V}^\vee \boxtimes \mathcal{V})}]{\mathbf{R}\operatorname{Hom}_{U \times U}(\mathcal{V}^\vee \boxtimes \mathcal{V}, -)} \mathrm{D^b}(\operatorname{mod} \Lambda \otimes_{\mathbb{C}} \Lambda^{\mathrm{op}})
\tag{6.L}
$$

as in [BH], where we denote the enveloping algebra by $\Lambda^{\mathrm{e}} := \Lambda \otimes_{\mathbb{C}} \Lambda^{\mathrm{op}}$.

Notation 6.14. If Y is a $\mathbb{C}$-scheme and Γ is a $\mathbb{C}$-algebra, we define

$$
\mathcal{O}_Y^\Gamma := \mathcal{O}_Y \otimes_{\underline{\mathbb{C}}} \underline{\Gamma},
$$

where $\underline{\mathbb{C}}$ and $\underline{\Gamma}$ are the constant sheaves associated to $\mathbb{C}$ and Γ.

To be slightly more pedantic, and armed with the notation above, we think of $\mathcal{V}^\vee \boxtimes \mathcal{V}$ as a left $\mathcal{O}_{U \times U}^{\Lambda^{\mathrm{e}}}$-module, and the lower functor in (6.L) is the derived version of the composition

$$
\operatorname{mod} \Lambda^{\mathrm{e}} \xrightarrow{\ \pi_{U \times U}^*(-)\ } \operatorname{mod} \mathcal{O}_{U \times U}^{\Lambda^{\mathrm{e}}} \xrightarrow{\ -\otimes_{\mathcal{O}_{U \times U}^{\Lambda^{\mathrm{e}}}}(\mathcal{V}^\vee \boxtimes \mathcal{V})\ } \operatorname{mod} \mathcal{O}_{U \times U}.
$$

Now, we may view the tilting equivalences in 6.13(1) as Fourier–Mukai functors between $\mathrm{D^b}(\operatorname{coh} U)$ and $\mathrm{D^b}(\operatorname{mod} \Lambda)$, given by $\mathcal{V}^\vee$ and $\mathcal{V}$ respectively, considered as objects in $\mathrm{D^b}(\operatorname{mod} \mathcal{O}_U^{\Lambda^{(\mathrm{op})}})$. The lower functor in (6.L) should then be seen as taking a Λ-bimodule, and composing it with these Fourier–Mukai functors as in 6.13(1). Hence, taking $\mathcal{W}'$ to be the object in $\mathrm{D^b}(\operatorname{coh} U \times U)$ corresponding to the Λ-bimodule $\mathbf{R}\operatorname{Hom}_\Lambda(I_{\mathrm{con}}, \Lambda)$, we will find in 6.16(1) below that our twist $T_{\mathcal{E}_U}$ is itself a Fourier–Mukai functor with kernel $\mathcal{W}'$. This motivates the following definition and lemma.

Definition 6.15. (1) *We define a* geometric twist kernel $\mathcal{W}'$ *as follows*

$$
\mathcal{W}' := \pi_{U \times U}^*(\mathbf{R}\operatorname{Hom}_\Lambda(I_{\mathrm{con}}, \Lambda)) \otimes^{\mathbf{L}}_{\mathcal{O}_{U \times U}^{\Lambda^{\mathrm{e}}}} (p_1^* \mathcal{V}^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_{U \times U}} p_2^* \mathcal{V}).
$$

(2) *We obtain an* inverse twist kernel $\mathcal{W}$ *as follows*

$$
\mathcal{W} := \pi_{U \times U}^* I_{\mathrm{con}} \otimes^{\mathbf{L}}_{\mathcal{O}_{U \times U}^{\Lambda^{\mathrm{e}}}} (p_1^* \mathcal{V}^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_{U \times U}} p_2^* \mathcal{V}).
\tag{6.M}
$$

The following lemma describes the twist and inverse twist.

Lemma 6.16. *There are functorial isomorphisms*

(1) $T_{\mathcal{E}_U} \cong \mathsf{FM}(\mathcal{W}') := \mathbf{R}p_{2*}\big(p_1^*(-) \otimes^{\mathbf{L}}_{\mathcal{O}_{U \times U}} \mathcal{W}'\big)$,
(2) $T_{\mathcal{E}_U}^* \cong \mathsf{FM}(\mathcal{W})$.

Proof. Put $I := I_{\text{con}}$. We prove (2) first. We have

$$T^*_{\mathcal{E}_U} = \pi^*_U(\mathbf{R}\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{V}, -) \otimes^{\mathbf{L}}_{\Lambda} I) \otimes^{\mathbf{L}}_{\mathcal{O}^{\Lambda}_U} \mathcal{V} \qquad (6.13(2))$$

$$\cong \pi^*_U(\mathbf{R}\Gamma(- \otimes^{\mathbf{L}}_{\mathcal{O}_U} \mathcal{V}^{\vee}) \otimes^{\mathbf{L}}_{\Lambda} I) \otimes^{\mathbf{L}}_{\mathcal{O}^{\Lambda}_U} \mathcal{V} \qquad (\text{definition of } \mathbf{R}\mathrm{Hom})$$

$$\cong \mathbf{R}p_{2*}p^*_1(- \otimes^{\mathbf{L}}_{\mathcal{O}_U} \mathcal{V}^{\vee}) \otimes^{\mathbf{L}}_{\mathcal{O}^{\Lambda}_U} \pi^*_U I \otimes^{\mathbf{L}}_{\mathcal{O}^{\Lambda}_U} \mathcal{V} \qquad (\text{base change for } (6.\mathrm{K}))$$

$$\cong \mathbf{R}p_{2*}\left(p^*_1(- \otimes^{\mathbf{L}}_{\mathcal{O}_U} \mathcal{V}^{\vee}) \otimes^{\mathbf{L}}_{\mathcal{O}^{\Lambda}_{U \times U}} p^*_2(\pi^*_U I \otimes^{\mathbf{L}}_{\mathcal{O}^{\Lambda}_U} \mathcal{V})\right) \qquad (\text{projection formula})$$

$$\cong \mathbf{R}p_{2*}\left(p^*_1(-) \otimes^{\mathbf{L}}_{\mathcal{O}_{U \times U}} \left(p^*_1\mathcal{V}^{\vee} \otimes^{\mathbf{L}}_{\mathcal{O}^{\Lambda}_{U \times U}} p^*_2\pi^*_U I \otimes^{\mathbf{L}}_{\mathcal{O}^{\Lambda}_{U \times U}} p^*_2\mathcal{V}\right)\right)$$

$$\cong \mathbf{R}p_{2*}\left(p^*_1(-) \otimes^{\mathbf{L}}_{\mathcal{O}_{U \times U}} \left(\pi^*_{U \times U} I \otimes^{\mathbf{L}}_{\mathcal{O}^{\Lambda^e}_{U \times U}} (p^*_1\mathcal{V}^{\vee} \otimes^{\mathbf{L}}_{\mathcal{O}_{U \times U}} p^*_2\mathcal{V})\right)\right)$$

$$\cong \mathbf{R}p_{2*}\left(p^*_1(-) \otimes^{\mathbf{L}}_{\mathcal{O}_{U \times U}} \mathcal{W}\right). \qquad (\text{by } (6.\mathrm{M}))$$

Since $T_{\text{con}} = \mathbf{R}\mathrm{Hom}_{\Lambda}(I, -) = (-) \otimes^{\mathbf{L}}_{\Lambda} \mathbf{R}\mathrm{Hom}_{\Lambda}(I, \Lambda)$, a similar calculation using 6.13(1) gives (1). $\qquad \square$

For our purposes later we must track the object $\mathcal{E}_U$ under a certain Serre functor. Again, since U is possibly singular, this involves the RHom version. Throughout, if Y is a $\mathbb{C}$-scheme of dimension d, although canonical modules are not unique, we write $\mathcal{D}_Y := g^!\mathbb{C}$ for the dualizing complex of Y, where $g \colon Y \to \mathrm{Spec}\,\mathbb{C}$ is the structure morphism. If further Y is normal CM of dimension d, we will always write ω_Y for $\mathcal{D}_Y[-d]$, and refer to ω_Y as the *geometric canonical*.

Definition 6.17. *Suppose that $f \colon Y \to \mathrm{Spec}\,T$ is a morphism where T is a normal CM $\mathbb{C}$-algebra with canonical module ω_T. Typically*

- $T = \mathbb{C}$ *(in the case $Y = X$ is projective), or*
- $T = R$ *with $\mathrm{Spec}\,R$ a 3-fold and f is also birational (in the case $Y = U$ of the local model).*

We say that a functor $\mathbb{S} \colon \mathrm{per}(Y) \to \mathrm{per}(Y)$ is a Serre functor relative to ω_T if there are functorial isomorphisms

$$\mathbf{R}\mathrm{Hom}_T(\mathbf{R}\mathrm{Hom}_Y(\mathcal{F}, \mathcal{G}), \omega_T) \cong \mathbf{R}\mathrm{Hom}_Y(\mathcal{G}, \mathbb{S}(\mathcal{F}))$$

in $\mathrm{D}(\mathrm{Mod}\,T)$ for all $\mathcal{F} \in \mathrm{per}(Y)$, $\mathcal{G} \in \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,Y)$.

Lemma 6.18. *Suppose that $f \colon Y \to \mathrm{Spec}\,T$ is a projective morphism, where Y and $\mathrm{Spec}\,T$ are both Gorenstein varieties. Then*

$$\mathbb{S}_Y := - \otimes_Y \omega_Y[\dim Y - \dim T] \colon \mathrm{per}(Y) \to \mathrm{per}(Y)$$

is a Serre functor relative to ω_T.

Proof. First, since Y is Gorenstein, ω_Y is a line bundle, so tensoring does preserve perfect complexes. The remainder is just Grothendieck duality: since $f^!\omega_T = \omega_Y[\dim Y - \dim T]$ we have

$$\mathbf{R}\mathrm{Hom}_Y(\mathcal{G}, \mathbb{S}_Y(\mathcal{F})) \cong \mathbf{R}\mathrm{Hom}_Y(\mathbf{R}\mathcal{H}om_Y(\mathcal{F}, \mathcal{G}), f^!\omega_T) \cong \mathbf{R}\mathrm{Hom}_T(\mathbf{R}\mathrm{Hom}_Y(\mathcal{F}, \mathcal{G}), \omega_T)$$

in $\mathrm{D}(\mathrm{Mod}\,T)$ for all $\mathcal{F} \in \mathrm{per}(Y)$, $\mathcal{G} \in \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,Y)$. $\qquad \square$

We now revert back to the assumptions and setup of §4. The following are geometric versions of 6.6 and 6.3.

Lemma 6.19. $\mathbb{S}_U(\mathcal{E}_U) \cong \mathcal{E}_U$, *i.e. $\mathcal{E}_U \otimes \omega_U \cong \mathcal{E}_U$.*

Proof. If we temporarily denote the derived equivalence by $\Psi := \mathbf{R}\mathrm{Hom}_U(\mathcal{V}, -)$ with inverse $\Phi := - \otimes^{\mathbf{L}}_{\Lambda} \mathcal{V}$, then on Λ, $\Psi \circ \mathbb{S}_U \circ \Phi$ is a Serre functor relative to ω_R, so since Serre functors are unique with respect to a fixed canonical, $\mathbb{S}_{\Lambda} \cong \Psi \circ \mathbb{S}_U \circ \Phi$. Hence

$$\mathbb{S}_U(\mathcal{E}_U) \overset{3.7}{\cong} \mathbb{S}_U(\Phi\mathbb{F}\mathrm{A}_{\text{con}}) \cong \Phi\Psi\mathbb{S}_U(\Phi\mathbb{F}\mathrm{A}_{\text{con}}) \cong \Phi\mathbb{S}_{\Lambda}(\mathbb{F}\mathrm{A}_{\text{con}}) \overset{6.6}{\cong} \Phi\mathbb{F}\mathrm{A}_{\text{con}} \overset{3.7}{\cong} \mathcal{E}_U. \qquad \square$$

Proposition 6.20. *With notation as above,*

(1) $T_{\mathcal{E}_U}(\mathcal{E}_U) \cong \mathcal{E}_U[-2]$ *and* $T_{\mathcal{E}_U}(E) \cong E[-2]$.
(2) $T_{\mathcal{E}_U}^*(\mathcal{E}_U) \cong \mathcal{E}_U[2]$ *and* $T_{\mathcal{E}_U}^*(E) \cong E[2]$.

Proof. (1) By definition $T_{\mathcal{E}_U}$ is the composition

$$\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,U) \xrightarrow{\mathbf{R}\mathrm{Hom}_U(\mathcal{V},-)} \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda) \xrightarrow{T_{\mathrm{con}}} \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda) \xrightarrow{-\otimes_{\Lambda}^{\mathbf{L}}\mathcal{V}} \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,U).$$

We know that the first functor takes $\mathcal{E}_U$ to $\mathbb{F}\mathrm{A}_{\mathrm{con}}$ by 3.7(1), the second functor takes $\mathbb{F}\mathrm{A}_{\mathrm{con}}$ to $\mathbb{F}\mathrm{A}_{\mathrm{con}}[-2]$ by 6.3(1), and the third functor takes $\mathbb{F}\mathrm{A}_{\mathrm{con}}[-2]$ to $\mathcal{E}_U[-2]$, again by 3.7(1). Tracking E is similar.
(2) This follows immediately from (1), since as we have already observed, $T_{\mathcal{E}_U}^*$ is the inverse of $T_{\mathcal{E}_U}$. $\qquad\square$

7. Noncommutative Twist Functors: Global Case

In this section we globalize the previous sections to obtain noncommutative twists of projective varieties. We keep the assumptions and setup of §4.

7.1. **Global** Ext **vanishing.** Using 3.6, we can deduce Ext vanishing on X by reducing the problem to Ext vanishing on U, which we have already solved in 5.7. Also, we remark that the following shows that although i_*E need not be perfect, its noncommutative thickening $\mathcal{E}$ automatically is.

Theorem 7.1. *We keep the assumptions as above, in particular X is a projective normal 3-fold X with at worst Gorenstein terminal singularities. Then $\mathcal{E}$ is a perfect complex and further*

$$\mathrm{Ext}_X^t(\mathcal{E}, i_*E) = \begin{cases} \mathbb{C} & \text{if } t = 0, 3 \\ 0 & \text{else.} \end{cases}$$

Proof. Now the first assertion in the statement is local, so it suffices to check that $\mathcal{E}_x$ has finite projective dimension for all closed points $x \in X$. Restricting $\mathcal{E}$ to U, by 3.6 it is clear that $i^*\mathcal{E} = \mathcal{E}_U$, which across the derived equivalence (2.E) corresponds to $\mathbb{F}\mathrm{A}_{\mathrm{con}}$ by 3.7. Since $\mathbb{F}\mathrm{A}_{\mathrm{con}}$ is supported only on the maximal ideal $\mathfrak{m} \in \mathrm{Max}\,R$ by 2.13, it follows that $\mathrm{pd}_\Lambda \mathbb{F}\mathrm{A}_{\mathrm{con}} = \mathrm{pd}_{\widehat{\Lambda}} \mathbb{F}\mathrm{A}_{\mathrm{con}} = \mathrm{pd}_{\mathrm{A}} \mathrm{A}_{\mathrm{con}}$ which by 5.6 is finite. Hence back across the equivalence $i^*\mathcal{E}$ is a perfect $\mathcal{O}_U$-module. Thus $\mathcal{E}_u$ has finite projective dimension for all $u \in U$. Since $\mathcal{E}_x = 0$ for all $x \notin U$, this implies that $\mathcal{E}$ is perfect.

For the second assertion, we have

$$\mathrm{Ext}_X^t(\mathcal{E}, i_*E) \stackrel{(2.B)}{\cong} \mathrm{Ext}_U^t(\mathcal{E}_U, E) \cong \mathrm{Ext}_\Lambda^t(\mathbb{F}\mathrm{A}_{\mathrm{con}}, T)$$

and so the result follows from 5.7. $\qquad\square$

The following result will be needed later. Again 3.6 reduces the proof to the local model, allowing us to use 6.19.

Proposition 7.2. $\mathcal{E} \otimes_X \omega_X \cong \mathcal{E}$.

Proof. Since X is Gorenstein, ω_X is a line bundle and so

$$\mathcal{E} \otimes_X \omega_X \stackrel{3.6}{\cong} \mathbf{R}i_*\mathcal{E}_U \otimes_X^{\mathbf{L}} \omega_X \cong \mathbf{R}i_*(\mathcal{E}_U \otimes_U^{\mathbf{L}} i^*\omega_X) \cong \mathbf{R}i_*(\mathcal{E}_U \otimes_U^{\mathbf{L}} \omega_U) \stackrel{6.19}{\cong} \mathbf{R}i_*(\mathcal{E}_U) \stackrel{3.6}{\cong} \mathcal{E}.$$

$\qquad\square$

7.2. **Global inverse twist.** We now extend the definition of the local inverse twist $T^*_{\mathcal{E}_U}$ on the open neighbourhood U (from 6.13) to a global inverse twist $T^*_{\mathcal{E}}$ on X. It turns out to be technically much easier to extend $T^*_{\mathcal{E}_U}$ rather than to extend $T_{\mathcal{E}_U}$, so we do this first. We return to the problem of defining the global noncommutative twist in §7.5.

Throughout, we keep the setting and assumptions of §4. We begin with a slight refinement of the results of §6.4. Consider the triangle of Λ-bimodules (6.H)

$$I_{\mathrm{con}} \to \Lambda \to \Lambda_{\mathrm{con}} \to,$$

then applying the functor $- \otimes^{\mathbf{L}}_{\Lambda^e} (\mathcal{V}^\vee \boxtimes \mathcal{V})$ we obtain an exact triangle

$$\mathcal{W}_U \to \mathcal{O}_{\Delta,U} \xrightarrow{\phi_U} \mathcal{Q}_U \to \tag{7.A}$$

of Fourier–Mukai kernels on $U \times U$.

Definition 7.3. *The* inverse twist $T^*_{\mathcal{E}}$ *is defined to be the Fourier–Mukai functor* $\mathsf{FM}(\mathcal{W}_X)$ *given by the kernel* $\mathcal{W}_X := \mathrm{Cone}(\phi_X)[-1]$, *where* ϕ_X *is the composition*

$$\mathcal{O}_{\Delta,X} \xrightarrow{\eta_\Delta} \mathbf{R}(i \times i)_* \mathcal{O}_{\Delta,U} \xrightarrow{\mathbf{R}(i \times i)_* \phi_U} \mathbf{R}(i \times i)_* \mathcal{Q}_U =: \mathcal{Q}_X. \tag{7.B}$$

Here η_Δ *is defined to be*

$$\mathbf{R}\Delta_{X*}\mathcal{O}_X \xrightarrow{\mathbf{R}\Delta_{X*}\eta} \mathbf{R}\Delta_{X*}\mathbf{R}i_*i^*\mathcal{O}_X \cong \mathbf{R}(i \times i)_*\mathbf{R}\Delta_{U*}\mathcal{O}_U, \tag{7.C}$$

with $\eta\colon \mathcal{O}_X \to \mathbf{R}i_*i^*\mathcal{O}_X$ *the unit morphism.*

Since $\mathbf{R}(i \times i)_*$ does not in general preserve coherence, we must work hard to establish that $\mathcal{W}_X, \mathcal{Q}_X \in \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X \times X)$. To do this, we first establish the triangle representation for $T^*_{\mathcal{E}}$ in (7.F) below, and thence show that the Fourier–Mukai functors for $\mathcal{W}_X$ and $\mathcal{Q}_X$ preserve $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$.

Consider the adjunctions in (6.J), which were used to describe the local inverse twist $T^*_{\mathcal{E}_U}$. These adjunctions are induced from adjunctions on the unbounded level, and composing them with (2.B) yields the diagram of adjoints

$$
\begin{array}{c}
G_X^{\mathrm{LA}}:=G_U^{\mathrm{LA}}\circ i^* \\[2pt]
\mathrm{D}(\mathrm{Mod}\, \mathrm{A}_{\mathrm{con}}) \xleftarrow{\quad} \overset{\displaystyle G_X:=\mathbf{R}i_*(-\otimes^{\mathbf{L}}_{\mathrm{A}_{\mathrm{con}}}\mathcal{E}_U)}{\xrightarrow{\hspace{4cm}}} \mathrm{D}(\mathrm{Qcoh}\, X) \\[2pt]
G_X^{\mathrm{RA}}:=G_U^{\mathrm{RA}}\circ i^!
\end{array}
\tag{7.D}
$$

By the projection formula we have $G_X \cong - \otimes^{\mathbf{L}}_{\mathrm{A}_{\mathrm{con}}} \mathbf{R}i_*\mathcal{E}_U$, so by 3.6 and adjunction, the above yields

$$
\begin{array}{c}
G_X^{\mathrm{LA}} \\[2pt]
\mathrm{D}(\mathrm{Mod}\, \mathrm{A}_{\mathrm{con}}) \xleftarrow{\quad} \overset{\displaystyle G_X\cong-\otimes^{\mathbf{L}}_{\mathrm{A}_{\mathrm{con}}}\mathcal{E}}{\xrightarrow{\hspace{3cm}}} \mathrm{D}(\mathrm{Qcoh}\, X) \\[2pt]
G_X^{\mathrm{RA}}\cong\mathbf{R}\mathrm{Hom}_X(\mathcal{E},-)
\end{array}
\tag{7.E}
$$

We remark that the above diagram is intrinsic to the geometry of X. The following lemma establishes, amongst other things, that the inverse twist preserves the bounded derived category.

Lemma 7.4. *Under the setup above,*

(1) $\mathsf{FM}(\mathcal{Q}_X) \cong G_X \circ G_X^{\mathrm{LA}}$, *and so for all* $x \in \mathrm{D}(\mathrm{Qcoh}\, X)$, *there is a triangle*

$$T^*_{\mathcal{E}}(x) \to x \to G_X \circ G_X^{\mathrm{LA}}(x) \to . \tag{7.F}$$

(2) *Both* $\mathsf{FM}(\mathcal{Q}_X)$ *and the inverse twist* $T^*_{\mathcal{E}}$ *preserve* $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$.

Proof. (1) From the definition 7.3, there is a functorial triangle

$$T_{\mathcal{E}}^* \to \mathrm{Id}_X \to \mathsf{FM}(\mathcal{Q}_X) \to .$$

Now since $\mathcal{Q}_X = \mathbf{R}(i \times i)_* \mathcal{Q}_U$ by definition (7.B), necessarily

$$\mathsf{FM}(\mathcal{Q}_X) \cong \mathbf{R}i_* \circ \mathsf{FM}(\mathcal{Q}_U) \circ i^*$$

by [H06, 5.12]. By the argument of 6.16, we have that $\mathsf{FM}(\mathcal{Q}_U) \cong G_U \circ G_U^{\mathrm{LA}}$, hence

$$\mathsf{FM}(\mathcal{Q}_X) \cong \mathbf{R}i_* \circ \mathsf{FM}(\mathcal{Q}_U) \circ i^* \cong \mathbf{R}i_* \circ G_U \circ G_U^{\mathrm{LA}} \circ i^* = G_X \circ G_X^{\mathrm{LA}} \qquad (7.\mathrm{G})$$

and so the statement follows.

(2) Since $\mathrm{A_{con}}$ is a finite dimensional algebra, its derived category $\mathrm{D^b(mod\,A_{con})}$ is generated by its simple module S. Note that G_X sends S to $i_*E \in \mathrm{D^b(coh\,X)}$, thus it follows that G_X sends $\mathrm{D^b(mod\,A_{con})}$ to $\mathrm{D^b(coh\,X)}$. Since G_X^{LA} is the composition $G_U^{\mathrm{LA}} \circ i^*$, and by (6.J) G_U^{LA} takes $\mathrm{D^b(coh\,X)}$ to $\mathrm{D^b(mod\,A_{con})}$, it follows that G_X^{LA} sends $\mathrm{D^b(coh\,X)}$ to $\mathrm{D^b(mod\,A_{con})}$. Combining, we see that $G_X \circ G_X^{\mathrm{LA}}$ takes $\mathrm{D^b(coh\,X)}$ to $\mathrm{D^b(coh\,X)}$, so the claim that the inverse twist preserves $\mathrm{D^b(coh\,X)}$ follows immediately from the triangle in (1), using the two-out-of-three property. $\qquad\square$

7.3. **Existence of adjoints.**

The existence of left and right adjoints of our inverse twist follows directly from the following known theorem [HMS1, HMS2]. Recall that for projections $p_1, p_2 \colon Y \times Y \to Y$, an object $\mathcal{G} \in \mathrm{D}(\mathrm{Qcoh}\,Y \times Y)$ is said to have *finite projective dimension* with respect to p_i, if $\mathbf{R}\mathcal{H}om_{\mathcal{O}_{Y \times Y}}(\mathcal{G}, p_i^! \mathcal{F}) \in \mathrm{D^b}(\mathrm{Qcoh}\,Y \times Y)$ for all $\mathcal{F} \in \mathrm{D^b}(\mathrm{Qcoh}\,Y)$.

Theorem 7.5. *Suppose that Y is a projective variety with only Gorenstein singularities, and $\mathcal{G} \in \mathrm{D^b}(\mathrm{Qcoh}\,Y \times Y)$. If $\mathsf{FM}(\mathcal{G}) := \mathbf{R}p_{2*}(\mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_{Y \times Y}} p_1^*(-))$ preserves $\mathrm{D^b}(\mathrm{coh}\,Y)$, then*

1. *$\mathcal{G} \in \mathrm{D^b}(\mathrm{coh}\,Y \times Y)$.*
2. *$\mathcal{G}$ has finite projective dimension with respect to p_1.*

If furthermore $\mathbf{R}p_{1*}(\mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_{Y \times Y}} p_2^*(-))$ also preserves $\mathrm{D^b}(\mathrm{coh}\,Y)$, then

3. *$\mathcal{G}$ has finite projective dimension with respect to both p_1 and p_2, and $\mathsf{FM}(\mathcal{G})$ admits left and right adjoints which also preserve $\mathrm{D^b}(\mathrm{coh}\,Y)$. The right adjoint of $\mathsf{FM}(\mathcal{G})$ is*

$$\mathsf{FM}(\mathcal{G}^\vee \otimes p_1^* \omega_Y[\dim Y]).$$

Proof. (1) Take a very ample bundle $\mathcal{O}(1)$ on Y, then $p_1^* \mathcal{O}(1)$ is very ample with respect to the morphism p_2. Since by assumption $\mathbf{R}p_{2*}(\mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_{Y \times Y}} p_1^* \mathcal{O}(r))$ has coherent cohomology for all r, it follows that $\mathcal{G} \in \mathrm{D^b}(\mathrm{coh}\,Y \times Y)$ by [HMS2, remark below 2.5].

(2) By (1) $\mathcal{G} \in \mathrm{D^b}(\mathrm{coh}\,Y \times Y)$, so since by assumption its Fourier–Mukai functor $\mathsf{FM}(\mathcal{G})$ takes $\mathrm{D^b}(\mathrm{coh}\,Y)$ to $\mathrm{D^b}(\mathrm{coh}\,Y)$, by [HMS2, 2.7] $\mathcal{G}$ must have finite homological dimension with respect to p_1 (for the definition of this, see [HMS1, 1.3]). Hence by [HMS1, 1.6] $\mathcal{G}$ has finite projective dimension with respect to p_1.

(3) With the additional assumption, applying (2) with the roles of p_1 and p_2 exchanged we see that $\mathcal{G}$ also has finite projective dimension with respect to p_2. Thus by [HMS1, 1.17] $\mathsf{FM}(\mathcal{G})$ now has both left and right adjoints, which also preserve $\mathrm{D^b}(\mathrm{coh}\,Y)$, and the right adjoint is of the form above. $\qquad\square$

The following is a consequence of 7.4 and 7.5.

Corollary 7.6. *We keep the assumptions as in §4, and consider the triangle*

$$\mathcal{W}_X \to \mathcal{O}_{\Delta, X} \to \mathcal{Q}_X \to$$

from 7.3. Then

1. *$\mathbf{R}p_{2*}(\mathcal{Q}_X \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times X}} p_1^*(-))$ and $\mathbf{R}p_{1*}(\mathcal{Q}_X \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times X}} p_2^*(-))$ both preserve $\mathrm{D^b}(\mathrm{coh}\,X)$.*
2. *$\mathcal{W}_X$, $\mathcal{O}_{\Delta, X}$ and $\mathcal{Q}_X$ all belong to $\mathrm{D^b}(\mathrm{coh}\,X \times X)$ and have finite projective dimension over both factors. Hence the corresponding Fourier–Mukai functors have right and left adjoints, which preserve $\mathrm{D^b}(\mathrm{coh}\,X)$.*

Proof. (1) We have $\mathsf{FM}(\mathcal{Q}_X) = \mathbf{R}p_{2*}(\mathcal{Q}_X \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times X}} p_1^*(-))$, which we established preserves $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,X)$ in 7.4(2). For the other, we note that by [H06, 5.12]

$$\mathbf{R}p_{1*}(\mathcal{Q}_X \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times X}} p_2^*(-)) \cong \mathbf{R}i_* \circ \mathbf{R}p_{1*}(\mathcal{Q}_U \otimes^{\mathbf{L}}_{\mathcal{O}_{U \times U}} p_2^*(-)) \circ i^* \qquad (7.\mathrm{H})$$

where by abuse of notation p_i also denote the projections $U \times U \to U$. But again using the argument in 6.16 it follows immediately that $\mathbf{R}p_{1*}(\mathcal{Q}_U \otimes^{\mathbf{L}}_{\mathcal{O}_{U \times U}} p_2^*(-))$ is the composition

$$\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,U) \xrightarrow{\mathbf{R}\mathrm{Hom}_U(\mathcal{V}^\vee, -)} \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda^{\mathrm{op}}) \xrightarrow{\Lambda_{\mathrm{con}} \otimes^{\mathbf{L}}_\Lambda -} \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda^{\mathrm{op}}) \xrightarrow{\mathcal{V}^\vee \otimes^{\mathbf{L}}_\Lambda -} \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,U)$$

so every object in the image is a bounded complex of coherent sheaves filtered by E. Since $\mathbf{R}i_*E = i_*E$ by (2.B) and is certainly bounded coherent, it follows that (7.H) preserves $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,X)$.

(2) The fact that $\mathcal{W}_X, \mathcal{Q}_X \in \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,X \times X)$ is a consequence of 7.4(2) and 7.5(1). Thus by (1) and 7.5(3), $\mathcal{Q}_X$ has finite projective dimension over both factors. Since $\mathcal{O}_{\Delta, X}$ also has this property, by two-out-of-three so does $\mathcal{W}_X$. The remaining statements now follow, again using 7.5(3). $\qquad\square$

Remark 7.7. If one could establish that the inverse twist $T^*_{\mathcal{E}}$ has both left and right adjoints that preserve $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,X)$ under the weaker assumption that X is only quasi-projective, then the proof that $T^*_{\mathcal{E}}$ is an autoequivalence in the next subsection goes through in that level of generality.

7.4. **Proof of equivalence for global flops.** In this subsection, keeping the assumptions and setting of §4, we give the proof that the inverse twist $T^*_{\mathcal{E}}$ is a derived equivalence in the situation of a flopping contraction. As above, our strategy is to reduce as much as possible to the local case and apply the results of §6. With this in mind, the following is useful.

Proposition 7.8. *The following diagram is naturally commutative:*

$$
\begin{array}{ccc}
\mathrm{D}(\mathrm{Qcoh}\,U) & \xrightarrow{\mathbf{R}i_*} & \mathrm{D}(\mathrm{Qcoh}\,X) \\
{\scriptstyle T^*_{\mathcal{E}_U}}\Big\downarrow & & \Big\downarrow{\scriptstyle T^*_{\mathcal{E}}} \\
\mathrm{D}(\mathrm{Qcoh}\,U) & \xrightarrow{\mathbf{R}i_*} & \mathrm{D}(\mathrm{Qcoh}\,X)
\end{array}
$$

Proof. Translating the question into Fourier–Mukai kernels, by [H06, 5.12] we require

$$(i \times 1)^* \mathcal{W}_X \cong \mathbf{R}(1 \times i)_* \mathcal{W}_U. \qquad (7.\mathrm{I})$$

We evaluate the left-hand side of (7.I). By definition (7.3), $\mathcal{W}_X = \mathrm{Cone}(\phi_X)[-1]$, where ϕ_X factors through the morphism

$$\mathcal{O}_{\Delta, X} \xrightarrow{\eta_\Delta} \mathbf{R}(i \times i)_* \mathcal{O}_{\Delta, U}.$$

By definition (7.C), the morphism η_Δ is given (up to isomorphism) as a pushforward $\mathbf{R}\Delta_{X*}\eta$ of a unit morphism η associated to the adjunction $i^* \dashv \mathbf{R}i_*$. A triangular identity for this adjunction yields that $i^*\eta$ is an isomorphism, and so using flat base change around the following Cartesian square

$$
\begin{array}{ccc}
U & \xrightarrow{1 \times i} & U \times X \\
{\scriptstyle i}\Big\downarrow & & \Big\downarrow{\scriptstyle i \times 1} \\
X & \xrightarrow{\Delta_X} & X \times X
\end{array}
$$

we find that $(i \times 1)^*\eta_\Delta$ is also an isomorphism.

It then follows, using the definition of ϕ_X (7.B), that

$$
\begin{aligned}
(i \times 1)^* \mathcal{W}_X &\cong (i \times 1)^* \operatorname{Cone}(\mathbf{R}(i \times i)_* \phi_U)[-1] \\
&\cong (i \times 1)^* \mathbf{R}(i \times i)_* \operatorname{Cone}(\phi_U)[-1] \\
&\cong (i \times 1)^* \mathbf{R}(i \times 1)_* \mathbf{R}(1 \times i)_* \operatorname{Cone}(\phi_U)[-1].
\end{aligned}
$$

Now $i \times 1$ is an open embedding and thence $(i \times 1)^* \mathbf{R}(i \times 1)_* = \operatorname{Id}$, and $\mathcal{W}_U \cong \operatorname{Cone}(\phi_U)[-1]$ by (7.A), so we have shown (7.I), and the statement is proved. $\qquad\square$

We now build up our next main result in 7.12 via a series of lemmas. Recall that $\Omega \subseteq \mathrm{D}^\mathrm{b}(\operatorname{coh} X)$ is called a *spanning class* if

   (1) $\mathbf{R}\operatorname{Hom}_X(a, c) = 0$ for all $c \in \Omega$ implies that $a = 0$.
   (2) $\mathbf{R}\operatorname{Hom}_X(c, a) = 0$ for all $c \in \Omega$ implies that $a = 0$.

To obtain a spanning class, usually one uses a Serre functor, which for us is $\mathbb{S}_X$ from 6.18. However in our setting, since E need not be perfect, obtaining a spanning class is a little delicate since the usual candidate $E \cup E^\perp$ will not do. The solution is to use $\mathcal{E}$, which we already know is perfect.

Lemma 7.9. *With the assumptions as in §4, define*

$$
\begin{aligned}
\mathcal{E}^\perp &:= \{a \in \mathrm{D}^\mathrm{b}(\operatorname{coh} X) \mid \mathbf{R}\operatorname{Hom}(\mathcal{E}, a) = 0\}, \\
{}^\perp \mathcal{E} &:= \{a \in \mathrm{D}^\mathrm{b}(\operatorname{coh} X) \mid \mathbf{R}\operatorname{Hom}(a, \mathcal{E}) = 0\}.
\end{aligned}
$$

Then $\mathcal{E}^\perp = {}^\perp \mathcal{E}$, and $\Omega := \mathcal{E} \cup \mathcal{E}^\perp$ is a spanning class in $\mathrm{D}^\mathrm{b}(\operatorname{coh} X)$.

Proof. For the first statement, note that

$$
\mathbf{R}\operatorname{Hom}_X(a, \mathcal{E}) = 0 \stackrel{7.2}{\iff} \mathbf{R}\operatorname{Hom}_X(a, \mathcal{E} \otimes \omega_X) = 0 \stackrel{6.18}{\iff} \mathbf{R}\operatorname{Hom}_X(a, \mathbb{S}\mathcal{E}) = 0.
$$

On the other hand

$$
\mathbf{R}\operatorname{Hom}_X(\mathcal{E}, a) = 0 \iff \mathbf{R}\operatorname{Hom}_T(\mathbf{R}\operatorname{Hom}_X(\mathcal{E}, a), \omega_T) = 0 \stackrel{6.17}{\iff} \mathbf{R}\operatorname{Hom}_X(a, \mathbb{S}\mathcal{E}) = 0
$$

where $T = \mathbb{C}$. Combining gives $\mathcal{E}^\perp = {}^\perp \mathcal{E}$.

Now we check that $\Omega := \mathcal{E} \cup \mathcal{E}^\perp$ is a spanning class. If $\mathbf{R}\operatorname{Hom}_X(a, c) = 0$ for all $c \in \Omega$, then $a \in {}^\perp \mathcal{E}$ and so by the above $a \in \mathcal{E}^\perp$. Thus taking $c := a$ gives $\mathbf{R}\operatorname{Hom}_X(a, a) = 0$ and hence $a = 0$. Also, if $\mathbf{R}\operatorname{Hom}_X(c, a) = 0$ for all $c \in \Omega$, in particular $\mathbf{R}\operatorname{Hom}_X(\mathcal{E}, a) = 0$ and so $a \in \mathcal{E}^\perp$. As above, this gives $a = 0$. $\qquad\square$

The following is expected from the standard theory of Seidel–Thomas twists. We note here that in our setting the proof follows immediately from the local model, bypassing subtleties with the octahedral axiom in the 'usual' proof [T07, p124].

Lemma 7.10. $T_\mathcal{E}^*(\mathcal{E}) \cong \mathcal{E}[2]$, $T_\mathcal{E}^*(i_* E) \cong i_* E[2]$ *and functorially* $\operatorname{Id}|_{\mathcal{E}^\perp} \cong T_\mathcal{E}^*|_{\mathcal{E}^\perp}$.

Proof. The first and second assertions follow from our local mutation calculation:

$$
T_\mathcal{E}^*(\mathcal{E}) \stackrel{(2.B)}{\cong} T_\mathcal{E}^*(\mathbf{R}i_* \mathcal{E}_U) \stackrel{7.8}{\cong} \mathbf{R}i_* T_{\mathcal{E}_U}^*(\mathcal{E}_U) \stackrel{6.20}{\cong} \mathbf{R}i_* \mathcal{E}_U[2] \stackrel{(2.B)}{\cong} \mathcal{E}[2].
$$

$$
T_\mathcal{E}^*(i_* E) \stackrel{(2.B)}{\cong} T_\mathcal{E}^*(\mathbf{R}i_* E) \stackrel{7.8}{\cong} \mathbf{R}i_* T_{\mathcal{E}_U}^*(E) \stackrel{6.20}{\cong} \mathbf{R}i_* E[2] \stackrel{(2.B)}{\cong} i_* E[2].
$$

For the last assertion, since any morphism between kernels induces a natural transformation between the resulting Fourier–Mukai functors, the morphism $\mathcal{W}_X \to \mathcal{O}_{\Delta, X}$ in 7.3 induces a natural transformation $\beta \colon T_\mathcal{E}^* \to \operatorname{Id}$. Since $\mathsf{FM}(\mathcal{Q}_X) = \mathbf{R}i_* \circ G_U \circ G_U^{\mathrm{LA}} \circ i^*$ by (7.G), and we have

$$
G_U^{\mathrm{LA}}(i^* a) \stackrel{6.11}{\cong} \mathbf{R}\operatorname{Hom}_{\mathrm{A}_{\mathrm{con}}}(\mathbf{R}\operatorname{Hom}_U(i^* a, \mathcal{E}_U), \mathrm{A}_{\mathrm{con}}) \stackrel{(2.B)}{\cong} \mathbf{R}\operatorname{Hom}_{\mathrm{A}_{\mathrm{con}}}(\mathbf{R}\operatorname{Hom}_X(a, \mathcal{E}), \mathrm{A}_{\mathrm{con}})
$$

which is zero for all $a \in {}^\perp \mathcal{E}$, we see that $\beta_a \colon T_\mathcal{E}^*(a) \to a$ is an isomorphism for all $a \in {}^\perp \mathcal{E}$. Thus functorially $\operatorname{Id}|_{{}^\perp \mathcal{E}} \cong T_\mathcal{E}^*|_{{}^\perp \mathcal{E}}$. Since $\mathcal{E}^\perp = {}^\perp \mathcal{E}$ by 7.9, the result follows. $\qquad\square$

We are almost ready to prove our main result. The key point below is that we check that the functor is full and faithful using the spanning class from 7.9, where the hard case (case 1 in the proof of 7.12 below) is already taken care of by the known derived equivalence on the local model (§6). Once we know our functor is fully faithful, the problem then is that the usual Serre functor trick to establish equivalence does not work in our possibly singular setting. However we are able to bypass this with the following.

Lemma 7.11. *Let $\mathcal{C}$ be a triangulated category, and $F\colon \mathcal{C} \to \mathcal{C}$ an exact fully faithful functor with right adjoint F^{RA}. Suppose that there exists an object $c \in \mathcal{C}$ such that $F(c) \cong c[i]$ for some i, and further $F(x) \cong x$ for all $x \in c^{\perp}$. Then F is an equivalence.*

Proof. Since F is fully faithful, by [H06, 1.24] we have

$$\mathrm{Id} \xrightarrow{\sim} F^{\mathrm{RA}} \circ F. \tag{7.J}$$

Now we know that $F(a) \cong a$ for all $a \in c^{\perp}$, hence it follows from (7.J) that $F^{\mathrm{RA}}(a) \cong a$ for all $a \in c^{\perp}$.

Now by [H06, 1.50], F is an equivalence provided that $F^{\mathrm{RA}}(x) = 0$ implies that $x = 0$. Hence suppose that $F^{\mathrm{RA}}(x) = 0$, then trivially $\mathrm{Hom}_{\mathcal{C}}(c, F^{\mathrm{RA}}(x)[j]) = 0$ for all $j \in \mathbb{Z}$. By adjunction $\mathrm{Hom}_{\mathcal{C}}(F(c), x[j]) = 0$ for all $j \in \mathbb{Z}$, so since $F(c) \cong c[i]$, we have $\mathrm{Hom}_{\mathcal{C}}(c, x[j]) = 0$ for all $j \in \mathbb{Z}$, i.e. $x \in c^{\perp}$. But then $F^{\mathrm{RA}}(x) = x$, so $x = 0$, and we are done. $\square$

Theorem 7.12. *With the assumptions as in §4, the inverse twist*

$$T_{\mathcal{E}}^{*}\colon \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X) \to \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$$

is an equivalence.

Proof. We know that $T_{\mathcal{E}}^{*}\colon \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X) \to \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$ by 7.4, and has left and right adjoints that preserve $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$ by 7.6. Combining 7.10 with 7.11, it suffices to show that $T_{\mathcal{E}}^{*}$ is fully faithful.

We know from 7.9 that $\Omega = \mathcal{E} \cup \mathcal{E}^{\perp}$ is a spanning class, so by [H06, 1.49] we just need to check that

$$T_{\mathcal{E}}^{*}\colon \mathrm{Hom}_{\mathrm{D}(X)}(\mathcal{G}_1, \mathcal{G}_2[i]) \to \mathrm{Hom}_{\mathrm{D}(X)}(T_{\mathcal{E}}^{*}(\mathcal{G}_1), T_{\mathcal{E}}^{*}(\mathcal{G}_2)[i]) \tag{7.K}$$

is a bijection for all $\mathcal{G}_1, \mathcal{G}_2 \in \Omega$ and all $i \in \mathbb{Z}$. This is taken care of in the following four cases.

Case 1: $\mathcal{G}_1 = \mathcal{G}_2 = \mathcal{E}$. Since $\mathbf{R}i_{*}\mathcal{E}_U = \mathcal{E}$, by 7.8 we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{D}(U)}(\mathcal{E}_U, \mathcal{E}_U[i]) & \xrightarrow[\sim]{\mathbf{R}i_{*}} & \mathrm{Hom}_{\mathrm{D}(X)}(\mathcal{E}, \mathcal{E}[i]) \\
\Big\downarrow{T_{\mathcal{E}_U}^{*}} & & \Big\downarrow{T_{\mathcal{E}}^{*}} \\
\mathrm{Hom}_{\mathrm{D}(U)}(T_{\mathcal{E}_U}^{*}(\mathcal{E}_U), T_{\mathcal{E}_U}^{*}(\mathcal{E}_U)[i]) & \xrightarrow[\sim]{\mathbf{R}i_{*}} & \mathrm{Hom}_{\mathrm{D}(X)}(T_{\mathcal{E}}^{*}(\mathcal{E}), T_{\mathcal{E}}^{*}(\mathcal{E})[i])
\end{array}
$$

We already know that the left-hand map is a bijection since $T_{\mathcal{E}_U}^{*}$ is an equivalence, hence the right-hand map is also a bijection.

Case 2: $\mathcal{G}_1 = \mathcal{E}$, $\mathcal{G}_2 \in \mathcal{E}^{\perp}$. The left-hand side of (7.K) is $H^i\mathbf{R}\mathrm{Hom}_X(\mathcal{E}, \mathcal{G}_2)$, which is zero. By 7.10, the right-hand side of (7.K) is $\mathrm{Hom}(\mathcal{E}[2], \mathcal{G}_2[i]) = H^{i-2}\mathbf{R}\mathrm{Hom}_X(\mathcal{E}, \mathcal{G}_2)$, which also equals zero.

Case 3: $\mathcal{G}_1 \in \mathcal{E}^{\perp}$, $\mathcal{G}_2 = \mathcal{E}$. Since $\mathcal{E}^{\perp} = {}^{\perp}\mathcal{E}$ by 7.9, we obtain $\mathbf{R}\mathrm{Hom}_X(\mathcal{G}_1, \mathcal{E}) = 0$. As in Case 2, this implies that both sides of (7.K) are zero.

Case 4: $\mathcal{G}_1, \mathcal{G}_2 \in \mathcal{E}^{\perp}$. Since functorially $\mathrm{Id}|_{\mathcal{E}^{\perp}} \cong T_{\mathcal{E}}^{*}|_{\mathcal{E}^{\perp}}$ by 7.10, (7.K) is bijective for all $\mathcal{G}_1, \mathcal{G}_2 \in \mathcal{E}^{\perp}$. $\square$

7.5. **The global noncommutative twist.** In this subsection we define the noncommutative twist functor, and show it sits in the expected functorial triangle. We consider the triangle of Fourier–Mukai kernels

$$\mathcal{W}_X \to \mathcal{O}_{\Delta,X} \xrightarrow{\phi_X} \mathcal{Q}_X \to \tag{7.L}$$

from 7.3, which by 7.4 give the Fourier–Mukai functors $T_{\mathcal{E}}^*$, Id and $G_X \circ G_X^{\mathrm{LA}}$ respectively, which all preserve $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$. Set $D := (-)^\vee \otimes p_1^* \omega_X[\dim X]$. By 7.5(3) and 7.6(2), applying D to (7.L) induces the right adjoints, which also preserve $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$.

Definition 7.13. *We define the* noncommutative twist $T_{\mathcal{E}} \colon \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) \to \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$ *to be* $\mathsf{FM}(D(\mathcal{W}_X))$.

Proposition 7.14. *With the assumptions as in §4,*

    (1) $T_{\mathcal{E}}$ *is the inverse functor to* $T_{\mathcal{E}}^*$*. In particular,* $T_{\mathcal{E}}$ *is an autoequivalence.*
    (2) *There is a functorial triangle*

$$G_X \circ G_X^{\mathrm{RA}} \to \mathrm{Id} \to T_{\mathcal{E}} \to$$

on $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$*, where* $G_X \circ G_X^{\mathrm{RA}} \cong \mathbf{R}\mathrm{Hom}_X(\mathcal{E}, -) \otimes_{\mathrm{A}_{\mathrm{con}}}^{\mathbf{L}} \mathcal{E}$*.*

Proof. (1) Since $T_{\mathcal{E}}^*$ is an equivalence, its right adjoint $T_{\mathcal{E}}$ is its inverse.
(2) By 7.5, the triangle

$$D(\mathcal{Q}_X) \to D(\mathcal{O}_{\Delta,X}) \to D(\mathcal{W}_X) \to$$

gives the right adjoints. The Fourier–Mukai functor of $D(\mathcal{Q}_X)$ gives the right adjoint to $G_X \circ G_X^{\mathrm{LA}}$, namely $G_X \circ G_X^{\mathrm{RA}}$, which by (7.E) is isomorphic to $\mathbf{R}\mathrm{Hom}_X(\mathcal{E}, -) \otimes_{\mathrm{A}_{\mathrm{con}}}^{\mathbf{L}} \mathcal{E}$. The Fourier–Mukai functor of $D(\mathcal{O}_{\Delta,X})$ gives the right adjoint to the identity, which is the identity. By definition, the Fourier–Mukai functor of $D(\mathcal{W}_X)$ gives $T_{\mathcal{E}}$. The result follows. $\square$

7.6. **Relation with flop–flop functors.** Finally, in this subsection we show that the noncommutative twist $T_{\mathcal{E}}$ is inverse to an autoequivalence naturally associated with the flop, namely the flop–flop functor FF. We thence obtain an intrinsic description of FF in terms of the noncommutative deformation theory of the curve, without assuming that the flop exists.

    Recall that the category ${}^0\mathrm{Per}\, X$ of perverse sheaves is defined as

$$ {}^0\mathrm{Per}\, X := \left\{ a \in \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) \;\middle|\; \begin{array}{c} H^i(a) = 0 \text{ if } i \neq 0, -1 \\ f_* H^{-1}(a) = 0,\ \mathbf{R}^1 f_* H^0(a) = 0 \\ \mathrm{Hom}(c, H^{-1}(a)) = 0 \text{ for all } c \in \mathcal{C} \end{array} \right\}, \tag{7.M}$$

with $\mathcal{C} := \{ c \in \operatorname{coh} X \mid \mathbf{R} f_* c = 0 \}$.

    The following two straightforward lemmas, 7.15 and 7.16, record the properties of $T_{\mathcal{E}}$ and FF that we will need. In 7.17, we deduce that the functor $\Psi := T_{\mathcal{E}} \circ \mathsf{FF}$ preserves the category ${}^0\mathrm{Per}\, X$ above, and preserves the numerical equivalence class of skyscraper sheaves. The proof that $T_{\mathcal{E}}$ is inverse to FF then concludes in 7.18, following an argument given by Toda.

Lemma 7.15. *The twist functor $T_{\mathcal{E}}$ has the following properties:*

    (1) $\mathbf{R}f_* \circ T_{\mathcal{E}} \cong \mathbf{R}f_*$.
    (2) $T_{\mathcal{E}}(\mathcal{O}_X) \cong \mathcal{O}_X$.
    (3) $T_{\mathcal{E}}(E) \cong E[-2]$.

Proof. Property (1) may be deduced by applying $\mathbf{R}f_*$ to the triangle representation of $T_{\mathcal{E}}$ in 7.14(2), using the fact that $\mathbf{R}f_* \circ G_X \cong 0$, which we prove now. The commutative

diagram

gives

$$\mathbf{R}f_* \circ G_X = \mathbf{R}f_* \mathbf{R}i_*(- \otimes^{\mathbf{L}}_{A_{con}} \mathcal{E}_U)$$
$$\cong \mathbf{R}i_{con*} \mathbf{R}f_*(- \otimes^{\mathbf{L}}_{A_{con}} \mathcal{E}_U)$$
$$\cong \mathbf{R}i_{con*}(- \otimes^{\mathbf{L}}_{A_{con}} \mathbf{R}f_* \mathcal{E}_U).$$

As $\mathcal{E}_U$ is filtered by E, it follows that $\mathbf{R}f_* \mathcal{E}_U \cong 0$, and the property follows.

To obtain (2), we note that $f^! \mathcal{O}_{X_{con}} = \mathcal{O}_X$ since f is crepant, so

$$\mathbf{R}\mathrm{Hom}_X(\mathcal{E}, \mathcal{O}_X) = \mathbf{R}\mathrm{Hom}_X(\mathcal{E}, f^! \mathcal{O}_{X_{con}}) \cong \mathbf{R}\mathrm{Hom}_{X_{con}}(\mathbf{R}f_* \mathcal{E}, \mathcal{O}_{X_{con}}) = 0,$$

again using $\mathbf{R}f_* \mathcal{E} \cong 0$. Hence $T_{\mathcal{E}}(\mathcal{O}_X) \cong \mathcal{O}_X$ by 7.14(2).

The last property (3) follows from 7.10. □

Lemma 7.16. *The flop–flop functor* FF *has the following properties:*

(1) $\mathbf{R}f_* \circ \mathsf{FF} \cong \mathbf{R}f_*$.
(2) $\mathsf{FF}(\mathcal{O}_X) \cong \mathcal{O}_X$.
(3) $\mathsf{FF}(E) \cong E[2]$.

Proof. We denote the flop by X^+. Property (1) follows immediately since the individual flop functors commute with the pushdown by [B02, (4.4)]. Further, by [B02, (4.8)] noting the correction to the sign in [T14, (27)], the flop functor induces an equivalence

$$^0\mathrm{Per}\, X \to {}^{-1}\mathrm{Per}\, X^+.$$

By morita theory, under this equivalence projectives must go to projectives, and simples must go to simples. By Van den Bergh's description of these objects [V04, 3.2.7, 3.5.7, 3.5.8], and the fact that the flop functor commutes with pushdown, it follows that the flop functor must take $\mathcal{O}_X$ to $\mathcal{O}_{X^+}$, and E to $E[1]$. Applying the flop functor again, properties (2) and (3) follow. □

The following observations are used in the proof of 7.18, where we will show that the functor Ψ defined below is in fact isomorphic to the identity.

Proposition 7.17. *The functor* $\Psi := T_{\mathcal{E}} \circ \mathsf{FF}$ *is an autoequivalence of* $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$*, preserving the category* $^0\mathrm{Per}\, X$ *of perverse sheaves given in* (7.M)*. It also preserves the numerical equivalence class of skyscraper sheaves, that is for points* $p \in X$ *we have*

$$\chi(\mathcal{L}, \mathcal{O}_p) = \chi(\mathcal{L}, \Psi(\mathcal{O}_p))$$

for all locally free sheaves $\mathcal{L}$ *on* X*, where* χ *is the Euler characteristic. For definitions, see for instance* [BBH, 7.61].

Proof. Note that Ψ is an equivalence by construction, using 7.14(1), and also that combining 7.15 and 7.16 gives

(1) $\mathbf{R}f_* \circ \Psi \cong \mathbf{R}f_*$,
(2) $\Psi(E) \cong E$.

We first argue that Ψ preserves the category $^0\mathrm{Per}\, X$ of perverse sheaves. As explained in [B02, Lemma 3.2], this category is obtained as the intersection, for perversity $p = 0$, of the subcategories

$$^p\mathcal{A}^{\leq 0} := \{a \in \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X) \mid \mathbf{R}f_* a \in \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X_{con})^{\leq 0}, \, a \in {}^\perp(\mathcal{C}^{>p})\},$$
$$^p\mathcal{A}^{\geq 0} := \{a \in \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X) \mid \mathbf{R}f_* a \in \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X_{con})^{\geq 0}, \, a \in (\mathcal{C}^{<p})^\perp\},$$

where $\mathcal{C}^{>p} := \{c \in \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) \mid \mathbf{R}f_*c = 0,\ H^i(c) = 0 \text{ for } i \leq p\}$ and $\mathcal{C}^{<p}$ is defined similarly. Now the property (1) implies that Ψ^{-1} preserves $\{c \in \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) \mid \mathbf{R}f_*c = 0\}$, and using property (2) it is easy to see that Ψ^{-1} preserves the induced standard t-structure on $\{c \in \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) \mid \mathbf{R}f_*c = 0\}$, and hence preserves $\mathcal{C}^{>p}$ and $\mathcal{C}^{<p}$. From this, it follows that Ψ^{-1} preserves ${}^p\mathcal{A}^{\leq 0}$ and ${}^p\mathcal{A}^{\geq 0}$, and hence ${}^0\mathrm{Per}\,X$, as claimed.

Next we argue that Ψ preserves the numerical equivalence class of skyscraper sheaves. Taking a sheaf $\mathcal{O}_p$ for a point $p \in X$, we consider two cases. If $p \notin C$, then f is an isomorphism at p. Property (1) above then gives that $\Psi(\mathcal{O}_p) \cong \mathcal{O}_p$, and so there is nothing to prove in this case. If $p \in C$, then $\mathcal{O}_p$ is no longer simple in ${}^0\mathrm{Per}\,X$ but it is filtered by the simples $\{\omega_C[1], E\}$. Thus we have a series of exact triangles

$$
\begin{array}{ccccccccc}
b_0 & \longrightarrow & a_1 & \longrightarrow & a_2 & \longrightarrow & \cdots & \longrightarrow & a_{n-1} & \longrightarrow & \mathcal{O}_p \\
& \searrow & \nearrow & \searrow & \nearrow & & & & & \searrow & \nearrow \\
& b_1 & & b_2 & & & & & & b_n
\end{array}
\tag{7.N}
$$

where all $b_i \in \{\omega_C[1], E\}$. But Ψ is an equivalence preserving ${}^0\mathrm{Per}\,X$, so it must permute the simples. Since we already know that Ψ fixes E and $\mathcal{O}_x$ for $x \in X\backslash C$, it follows that Ψ also fixes $\omega_C[1]$ and hence fixes all the simples.

Because of this, we also have a series of exact triangles

$$
\begin{array}{ccccccccc}
b_0 & \longrightarrow & \Psi a_1 & \longrightarrow & \Psi a_2 & \longrightarrow & \cdots & \longrightarrow & \Psi a_{n-1} & \longrightarrow & \Psi\mathcal{O}_p \\
& \searrow & \nearrow & \searrow & \nearrow & & & & & \searrow & \nearrow \\
& b_1 & & b_2 & & & & & & b_n
\end{array}
\tag{7.O}
$$

Using additivity of $\chi(\mathcal{L}, -)$ on triangles, the left-hand triangles in (7.N) and (7.O) imply that $\chi(\mathcal{L}, a_1) = \chi(\mathcal{L}, \Psi a_1)$. Inducting to the right on (7.N) and (7.O) gives $\chi(\mathcal{L}, \mathcal{O}_p) = \chi(\mathcal{L}, \Psi\mathcal{O}_p)$, as required. $\qquad\square$

The following is the main result of this subsection.

Proposition 7.18. *$T_\mathcal{E}$ is an inverse of FF, the Bridgeland–Chen flop–flop functor of §1.4.*

Proof. We use the argument indicated in [T07, end of proof of 3.1], but write the proof in full to show that there is nothing particular about commutative deformations or smoothness used in *loc. cit.* that affects the proof. As above, consider the autoequivalence

$$\Psi := T_\mathcal{E} \circ \mathsf{FF}.$$

Using 7.15 and 7.16 we know that $\Psi(\mathcal{O}_X) \cong \mathcal{O}_X$, and by 7.17 Ψ preserves the category ${}^0\mathrm{Per}\,X$ of perverse sheaves, and preserves the numerical equivalence class of skyscraper sheaves.

We now follow the argument in [T06, end of proof of 6.1] to deduce that in fact $\Psi \cong \mathrm{Id}$. We first show that for points $p \in X$, $\Psi(\mathcal{O}_p)$ is a skyscraper sheaf. Most of the work here is to show that $\Psi(\mathcal{O}_p)$ is a sheaf, as the result will then follow from the above remark on numerical equivalence classes. We take the short exact sequence of sheaves

$$0 \to \mathcal{I}_p \to \mathcal{O}_X \to \mathcal{O}_p \to 0,$$

and observe that it is also a short exact sequence in ${}^0\mathrm{Per}\,X$ (for this, we need only check that $\mathbf{R}^1 f_* = 0$ for each sheaf). Since Ψ preserves ${}^0\mathrm{Per}\,X$, and $\Psi(\mathcal{O}_X) \cong \mathcal{O}_X$,

$$0 \to \Psi(\mathcal{I}_p) \to \mathcal{O}_X \to \Psi(\mathcal{O}_p) \to 0$$

is also an exact sequence in ${}^0\mathrm{Per}\,X$. The perversity gives that $H^1\Psi(\mathcal{I}_p) = 0$, and so the associated long exact sequence yields a surjection

$$\mathcal{O}_X \to H^0\Psi(\mathcal{O}_p). \tag{7.P}$$

We claim that the target of this surjection must be non-trivial. Otherwise, the perversity implies that $\Psi(\mathcal{O}_p) = H^{-1}\Psi(\mathcal{O}_p)[1]$, which then implies that for $\mathcal{L}$ suitably anti-ample

$$1 = \chi(\mathcal{L}, \mathcal{O}_p) \stackrel{7.17}{=} \chi(\mathcal{L}, \Psi(\mathcal{O}_p)) = -\chi(\mathcal{L}, H^{-1}\Psi(\mathcal{O}_p)) \leq 0,$$

which is a contradiction. Hence the map (7.P) above is non-trivial. Furthermore, it is adjoint to a map $\mathcal{O}_{X_{\mathrm{con}}} \to f_* H^0 \Psi(\mathcal{O}_p)$, which must consequently also be non-trivial, and so we find that

$$f_* H^0 \Psi(\mathcal{O}_p) \neq 0. \tag{7.Q}$$

Now the fibres of f have dimension at most 1, so $\mathbf{R}^{>1} f_* = 0$ and therefore the Grothendieck spectral sequence

$$E_2^{p,q} = \mathbf{R}^p f_*(H^q \Psi(-)) \to \mathbf{R}^{p+q}(f_* \circ \Psi)(-) \cong \mathbf{R}^{p+q} f_*(-)$$

collapses. Hence the exact sequence of low-degree terms applied to $\mathcal{O}_p$ gives a short exact sequence

$$0 \to \mathbf{R}^1 f_* H^{-1} \Psi(\mathcal{O}_p) \to f_*(\mathcal{O}_p) \to f_* H^0 \Psi(\mathcal{O}_p) \to 0.$$

The middle term is just $\mathcal{O}_{f(p)}$. This surjects onto the last term, which we know by (7.Q) to be non-zero. We deduce that $\mathbf{R}^1 f_* H^{-1}\Psi(\mathcal{O}_p) = 0$. Recall that $\Psi(\mathcal{O}_p) \in {}^0\mathrm{Per}\, X$, and hence $f_* H^{-1}\Psi(\mathcal{O}_p) = 0$ also, so we conclude that

$$\mathbf{R} f_* H^{-1}\Psi(\mathcal{O}_p) = 0,$$

so that $H^{-1}\Psi(\mathcal{O}_p) \in \mathcal{C}$, the category used in the definition of ${}^0\mathrm{Per}\, X$. Using once more the fact that $\Psi(\mathcal{O}_p) \in {}^0\mathrm{Per}\, X$, we see from the definition that

$$\mathrm{Hom}_X(H^{-1}\Psi(\mathcal{O}_p), H^{-1}\Psi(\mathcal{O}_p)) = 0,$$

and thence that $H^{-1}\Psi(\mathcal{O}_p) = 0$, and that $\Psi(\mathcal{O}_p)$ is a sheaf. This sheaf is numerically equal to a skyscraper sheaf by 7.17, and thence by [BBH, 7.62(2)] is itself a skyscraper sheaf.

Finally, we therefore have that $\Psi(\mathcal{O}_p) \cong \mathcal{O}_{\phi(p)}$ for all p and some function ϕ. We deduce that ϕ is an automorphism because Ψ is an equivalence. Therefore

$$\Psi \cong \mathcal{L} \otimes \phi_*(-)$$

for some line bundle $\mathcal{L}$, which must be trivial as $\Psi(\mathcal{O}_X) \cong \mathcal{O}_X$. We must have $f \circ \phi = f$ because Ψ commutes with the contraction, and so $\phi = \mathrm{Id}$. This completes the proof. $\square$

Appendix A. List of Notations

η,ε	unit and counit morphisms	
$(-)^{\mathrm{ab}}$	abelianization	
$(-)^{\mathrm{e}}$	enveloping algebra	
pd	projective dimension	
$\operatorname{Mod}A$ $(\operatorname{mod}A)$	(finitely generated) right A-modules	
$\operatorname{add}M$	summands of finite sums of module M	
$\operatorname{CM}R$	category of maximal Cohen–Macaulay R-modules	
CArt	category of local commutative artinian $\mathbb{C}$-algebras	
D $(\mathrm{D}^{\mathrm{b}})$	derived category (bounded)	
$\mathrm{D}^-,\mathrm{D}^+$	derived category bounded above, and below	
$\mathsf{FM}(\mathcal{W})$	Fourier–Mukai functor with kernel $\mathcal{W}$	
$F^{\mathrm{RA}},F^{\mathrm{LA}}$	right and left adjoints, for a functor F	
F, F$'$	flop functors	§1.4
FF	flop–flop functor	§1.4
Alg_1 (CAlg_1)	category of 1-pointed (commutative) $\mathbb{C}$-algebras	2.1
Art_1 (CArt_1)	category of 1-pointed artinian (commutative) $\mathbb{C}$-algebras	2.1
Γ	1-pointed artinian $\mathbb{C}$-algebra	2.1
$a,b\in\mathcal{A}$	objects of abelian category $\mathcal{A}$	§2
$\operatorname{Pairs}(\mathcal{A},\Gamma)$	Γ-objects in $\mathcal{A}$	2.3
$\mathcal{D}ef_a^{\mathcal{A}}$ $(c\mathcal{D}ef_a^{\mathcal{A}})$	noncommutative (commutative) deformation functor	2.4
$C\subset X$	curve C in a 3-fold X	§2.2
C^{red}	reduced subscheme associated to C $(\cong\mathbb{P}^1)$	§2.2
E	sheaf $\mathcal{O}_{\mathbb{P}^1}(-1)$ supported on C^{red}	§2.2
$p\in X_{\mathrm{con}}$	singular point p in contracted 3-fold X_{con}	§2.2
f	contraction morphism	§2.2
U_{con}	affine open neighbourhood of singular point p	§2.2
R	$\mathbb{C}$-algebra, with $U_{\mathrm{con}}\cong\operatorname{Spec}R$	§2.2
U	$:=f^{-1}(U_{\mathrm{con}})$, open neighbourhood of curve C	§2.2
i	open embedding of U in X	§2.2
$\mathcal{V}$	$:=\mathcal{O}_U\oplus\mathcal{N}$, tilting bundle on U	§2.3
N	$:=f_*\mathcal{N}$	§2.3
T	$:=\operatorname{Hom}_U(\mathcal{V},E)$	§2.3
$^0\mathrm{Per}\,U$	perverse sheaves on U	§2.3
Λ	endomorphism algebra $\operatorname{End}_U(\mathcal{V})$	§2.3
Λ_{con}	contraction algebra associated to Λ	2.9
I_{con}	ideal associated to Λ_{con}	2.9
$[R]$	ideal of morphisms of Λ which factor through $\operatorname{add}R$	2.9
$\widehat{U}$	formal fibre of f over singular point p	§2.4
$\widehat{R},\mathfrak{R}$	completion of R at p	§2.4
$\widehat{\Lambda}$	completion of Λ	§2.4
$\mathcal{O}_{\widehat{U}}\oplus\mathcal{N}_1$	tilting bundle on $\widehat{U}$	§2.4
N_1	$:=f_*\mathcal{N}_1$	§2.4
$\mathbb{F}$	morita equivalence	§2.4
A	basic algebra, morita equivalent to $\widehat{\Lambda}$	2.12
$\mathrm{A}_{\mathrm{con}}$	contraction algebra	2.12
$\mathcal{E},\mathcal{E}_U$	noncommutative universal sheaf on X, and on U	§3.2
$\mathcal{F},\mathcal{F}_U$	commutative universal sheaf on X, and on U	§3.2
$\mathrm{A}_{\mathrm{def}}$	$:=\operatorname{End}_X(\mathcal{E})$, noncommutative deformation algebra	3.8
$\operatorname{wid}(C),\operatorname{cwid}(C)$	width, and commutative width, of C	3.11
$\upmu,\upnu$	right and left mutation	5.1
Φ	mutation functor	§5.3

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