PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 115, Number 4, AUGUST 1992

NONCOMPACT CHAIN RECURRENCE AND ATTRACTION

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(Communicated by Charles C. Pugh)

ABSTRACT. Both this paper and *Chain recurrence and attraction in noncompact spaces*, [Ergodic Theory Dynamical Systems (to appear)] are concerned with the question of extending certain results obtained by C. Conley for dynamical systems on compact spaces to systems on arbitrary metric spaces. The basic result is the analogue of Conley's theorem that characterizes the *chain recurrent set* of f in terms of the *attractors* of f and their *basins of attraction.* The point of view taken in the above-mentioned paper was that the given metric was of primary importance rather than the topology that it generated. The purpose of this note is to give results that depend on the topology induced by a metric rather than on the particular choice of the metric.

The goal of this paper is to extend a theorem of C. Conley from the setting of dynamical systems on compact metric spaces to metric spaces that are only locally compact. Conley's result connects the *chain recurrent set* of f with the collection of *attractors* and *basins of attraction* of f, as follows:

Theorem (Conley). *If X is a compact metric space and $f: X \to X$ is continuous, then the chain recurrent set of f is the complement of the union of the sets $B(A) - A$, as A varies over the collection of attractors of f; here $B(A)$ denotes the basin of attraction of A (the set of points whose omega-limit sets lie in A).*

Definitions are given in the next section. An earlier paper [7] described one extension of Conley's theorem to the noncompact case. In [7] it was assumed that the distances defined by the given metric on X were themselves important; as a consequence some of the dynamical structures (the analogues of the chain recurrent set and of attractors and their basins) could change if the metric was changed—even if the new metric induced the same topology as the old. There are circumstances where this point of view is appropriate (see [8]), but in general it is preferable to have a theory in which the dynamical structures are invariant under changes of metric (or equivalently, under topological conjugacies). Describing such a theory is the main goal of this paper. There are two main results. The first is the counterpart of Conley's theorem, and the second is the existence of global Lyapunov functions in the case that X is second countable (which is also a generalization of a theorem of Conley).

The approach taken in this paper was suggested by John Franks and the author benefitted from conversations with him. Part of the motivation for

Received by the editors January 26, 1991.
1991 *Mathematics Subject Classification.* Primary 58F12.

considering the connection between chain recurrence and attractors came from several recent papers dealing with dynamics on noncompact spaces [3, 5, 6, 10].

1. Definitions and background

Suppose that (X, d) is a metric space and that $f: X \to X$ is continuous. If $\varepsilon > 0$ then a nonempty open subset U of X is called *ε-absorbing* if U contains the ε-ball about $f(x)$ for each x in U. U is *absorbing* if it is ε-absorbing for some $\varepsilon > 0$; in other words, U is absorbing if f maps U a uniform distance into its interior. The closed set $A = \bigcap_{n\geq 0} \overline{f^n(U)}$ is called the *attractor-like set determined by* U. The open set $\mathscr{O}^-(U) = \bigcup_{n\geq 0} f^{-n}(U)$ is the set of all points whose omega-limit sets are contained in A, and it is called the *basin of A relative to U, $B(A; U)$*.

When X is compact these definitions can be simplified. Compactness implies that: (1) U is absorbing if and only if U contains the closure of $f(U)$; (2) A is nonempty and invariant $(f(A) = A)$; and (3) $B(A; U)$ is independent of U in the sense that if W is a second absorbing set that also determines the attractor-like set A, then $B(A; U) = B(A; W)$. Because of (3), in the compact case we can abbreviate $B(A; U)$ to $B(A)$; in the compact case A is usually referred to simply as the *attractor* determined by U.

Without compactness the situation is less straightforward [7]: A can be empty (even though we require that U be nonempty); A is forward-invariant $(f(A) \subset A)$, but may fail to be invariant; and different absorbing sets may define different basins even when they determine the same attractor-like set A. In order to deal with this last difficulty the *extended basin* of A is defined to be the union of the sets $B(A; U)$ as U varies over all the absorbing sets that determine A. The extended basin of A will be denoted as $B(A)$; in an attempt to avoid confusion we will usually write $\mathscr{O}^-(U)$ instead of $B(A; U)$ for the basin of A relative to U. It is worthwhile to note that the extended basins of two attractors might overlap without the attractors intersecting. This is obvious if one of the attractors is empty; more generally all that can be said about a point $x \in B(A_1) \cap B(A_2)$ is that either its omega-limit set is empty or else its omega-limit set is in $A_1 \cap A_2$.

An *ε-chain* (or an *ε-pseudo-orbit*) for f is a sequence $x_0, x_1, \ldots, x_n$ with the property that $d(f(x_j), x_{j+1}) < \varepsilon$ for $0 \leq j < n-1$. This ε-chain is said to *go from x_0 to x_n* and have *length n*. In this paper the length will always be finite and at least 1. The *chain recurrent set* of f is the set

$$\mathscr{C}R(f) = \{p | \text{for each } \varepsilon > 0 \text{ there is an } \varepsilon\text{-chain from } p \text{ back to } p\}.$$

The chain recurrent set is always closed and forward invariant; when X is compact it is invariant and nonempty. Note that if there is an ε-chain of length n from p back to p, then by concatenating this chain with itself k times we obtain an ε-chain of length kn from p back to p; thus for any chain recurrent point p there is an arbitrarily long ε-chain that begins and ends at p.

It is not hard to see a connection between the chain recurrent set of f and the collection of absorbing sets for f. The basic observation is that if x is a point of an ε-absorbing set U, then the ε-ball about $f(x)$ is contained in U so that any ε-chain of length 1 beginning in U must also end in U. The obvious induction now shows that any ε-chain beginning in U must also end in U. In fact slightly more is true: if δ is any positive constant less than ε, then any δ-chain beginning in U must end at a point within δ of $f(U)$. If $x \in U$ is chain recurrent then for any δ there is a δ-chain beginning and ending at x, and so (by letting $\delta \to 0$) we conclude that any chain recurrent point in U must actually be in $\overline{f(U)}$. This proves

Lemma 1.1. *If U is an absorbing set for f, then $\mathscr{C}R(f) \cap U$ is contained in $\overline{f(U)}$.*

Lemma 1.1 is a first step towards proving Conley's theorem, both in the compact case [1] and in the noncompact [7]. Unfortunately the most general result in [7] is not as nice as Conley's theorem, unless additional assumptions are made concerning f (for instance the assumption that $f^{-1}(K)$ is compact for every compact $K \subset X$). In the next section we will modify the definitions of absorbing set and of chain recurrence, replacing the positive constants ε in those definitions by continuous positive functions on X. With these modified definitions we are able to obtain precisely the same conclusion as in Conley's theorem without any additional assumptions on f; see Theorem 1 below.

2. Variable epsilons

When one takes the view that the only intrinsically important feature of the metric d on X is the topology that it defines, then the use of a constant ε in the definitions of absorbing sets and chain recurrence is unnatural. One alternative is to replace the fixed ε's of the last section by arbitrary positive functions. Let $\mathscr{P}$ denote the set $\{\varepsilon : X \to (0, \infty), \text{continuous}\}$.

Definitions. A nonempty open subset U of X is *weakly absorbing* for f if there is a function $\varepsilon \in \mathscr{P}$ with the property that $B_{\varepsilon(f(x))}(f(x)) \subset U$ for each $x \in U$ ($B_\delta(p)$ denotes the ball of radius δ centered at p). U is weakly absorbing if and only if $\overline{f(U)} \subset U$: the inclusion is a clear consequence of U being weakly absorbing, and the converse follows by setting $\varepsilon(x) = [d(x, \overline{f(U)}) + d(x, X - U)]/2$.

When U is weakly absorbing the set $A = \bigcap_{n \geq 0} \overline{f^n(U)}$ is the *weak attractor determined by* U. As before we will set $B(A; U) = \mathscr{O}^-(U)$ and $B(A)$ will be the union of the sets $B(A; U)$ as U ranges over the collection of weakly absorbing sets that determine A.

If $\varepsilon \in \mathscr{P}$ then $x_0, x_1, \dots, x_n$ is an $\varepsilon(x)$*-chain* if $d(f(x_j)), x_{j+1}) < \varepsilon(f(x_j))$ for $0 \leq j < n - 1$. A point p is called *strongly chain recurrent* for f if for each $\varepsilon \in \mathscr{P}$ there is an $\varepsilon(x)$-chain of length at least 1 that begins and ends at p. The set of all strongly chain recurrent points of f will be denoted as $\mathscr{C}R^+(f)$.

Clearly any absorbing set is weakly absorbing and $\mathscr{C}R^+(f) \subset \mathscr{C}R(f)$. When X is compact the new definitions are equivalent to the earlier ones where ε was constant. In the noncompact case it is not hard to give examples where the new definitions are indeed different from the old.

Example 1. Let X be the subset of the plane consisting of all points of either of the two forms $(n, 0)$ or $(n, 1/n)$ where $n \geq 1$ is an integer. Define $f: X \to X$ by $f(n, 0) = (n+1, 0)$ and $f(n, 1/n) = (1, 0)$. If d is the metric inherited from the usual metric on the plane, then the requirements that an absorbing set be nonempty, forward invariant, and mapped uniformly inside itself show that any absorbing set must contain each point of the form $(n, 0)$. However if $\varepsilon(n, y) = 1/2n$ then any set of the form $\{(n, 0) | n \geq k\}$ is weakly absorbing. Similarly, with this choice of $\varepsilon(x)$ it is apparent that no point of X is strongly chain recurrent; this contrasts with the fact that every point of X on the x-axis is chain recurrent in the weaker sense. Further features of this example are described in [7, Example 2.4].

The proof of Lemma 1.1 can be repeated to show

Lemma 2.1. *If U is weakly absorbing for f then any strongly chain recurrent point in U is actually in $\overline{f(U)}$.*

In fact, slightly more than this is true. If $p \in \mathscr{O}^-(U)$ then there is a positive integer n with $f^n(p) \in U$. An argument based on the continuity of f and the fact that U is open shows that if a constant $\delta_0 > 0$ is sufficiently small then any δ_0-chain of length n beginning at p must end in U. If we define $\delta(x) = \min\{\delta_0, \varepsilon(x)\}$ then the proof of Lemma 1.1 shows that any $\delta(x)$-chain of length at least $n+1$ that begins at p will end within δ_0 of $f(U)$. If p is strongly chain recurrent then there is such a chain that also ends at p. Letting δ_0 go to 0 yields the following.

Lemma 2.2. *If U is weakly absorbing for f then any strongly chain recurrent point in $\mathscr{O}^-(U)$ is actually in $\overline{f(U)}$.*

Proposition 2.3. *Suppose that X is a locally compact metric space. If U is weakly absorbing for f then any strongly chain recurrent point in $\mathscr{O}^-(U)$ is actually in A, the attractor-like set determined by U.*

Proof. Since U is weakly absorbing for f, it is certainly weakly absorbing for f^n for every $n \geq 1$. Thus the proposition follows from the last lemma provided that we can show that every strongly chain recurrent point for f is also strongly chain recurrent for each f^n,

$$\mathscr{C}R^+(f) \subset \mathscr{C}R^+(f^n). \tag{1}$$

(The opposite inclusion is trivial.) The proof of (1) follows from the following lemma, whose proof we defer to an appendix.

Lemma 2.4. *Suppose that X is a locally compact metric space, that $f: X \to X$ is continuous, and that $\varepsilon: X \to (0, \infty)$ is also continuous. Then there is a continuous map $\delta: X \to (0, \infty)$ with the property $d(x, y) < \delta(x)$ that implies $d(f(x), f(y)) < \varepsilon(f(x))$ for all $x \in X$.*

Now suppose that $p \in \mathscr{C}R^+(f)$ and that $n \geq 1$ and $\varepsilon \in \mathscr{P}$ are given. Using induction and the last lemma one sees that there is a function $\delta \in \mathscr{P}$ with the property that any $\delta(x)$-chain for f of length n beginning at a point y will end at a point that is within $\varepsilon(f^n(y))$ of $f^n(y)$. It follows that if $x_0, x_1, x_2, \ldots, x_{nk}$ is a $\delta(x)$-chain for f of length kn, then $x_0, x_n, x_{2n}, \ldots, x_{nk}$ is an $\varepsilon(x)$-chain for f^n of length k. The concatenation argument given after the definition of chain recurrence shows that there is a $\delta(x)$-chain for f that begins and ends at p and whose length is a multiple of n. Thus there is an $\varepsilon(x)$-chain for f^n that begins and ends at p. This establishes (1) and so completes the proof of the proposition. □

Theorem 1. *If X is a locally compact metric space and $f: X \to X$ is continuous, then $\mathscr{C}R^+(f)$ is the complement of the union of the sets $B(A) - A$, as A varies over the collection of attractor-like sets of f,*

$$X - \mathscr{C}R^+(f) = \bigcup B(A) - A. \tag{2}$$

Proof. The proposition shows that the set on the right in (2) is contained in the set on the left. The opposite inclusion is obtained via the same argument as in [1] or [7], which goes as follows. Suppose that p is a point that is not strongly chain recurrent, and pick $\varepsilon \in \mathscr{P}$ such that there is no $\varepsilon(x)$-chain from p back to p. Then p is not an element of the open set $U = \{y|$ there is an $\varepsilon(x)$-chain from p to $y\}$. Note, however, that $f(p)$ is certainly in U, so that $p \in \mathscr{O}^-(U)$. In addition U is weakly absorbing: if $y \in U$ then there is an $\varepsilon(x)$-chain from p to y, and, therefore, there is an $\varepsilon(x)$-chain from p to any point that is within $\varepsilon(f(y))$ of $f(y)$. In other words, U contains the ball of radius $\varepsilon(f(y))$ centered at $f(y)$, which shows that U is weakly absorbing. To finish, let A be the attractor-like set determined by U; then $p \in B(A; U) - U \subset B(A) - A$. □

3. Remarks

This section contains several remarks and examples to fill out the basic exposition of the previous section. The first question to be addressed is whether using arbitrary positive continuous functions in the definition of strong chain recurrence is too strong a restriction; for instance, does it force the conclusion that the closure of the forward orbit of a strongly chain recurrent point is compact? The following example shows that this is not the case; a related example is presented in more detail on page 343 of [11]. Let X be the horizontal strip in the plane $X = (-\infty, \infty) \times [-1, 1]$. The map f on X is the time 1 map of a flow ϕ. On the upper boundary of X, ϕ moves points to the right with unit speed, and on the lower boundary it moves points to the left with unit speed. ϕ has a repelling fixed point at the origin. All other orbits spiral clockwise away from the origin and have their omega-limit sets equal to the union of the two boundary lines. As long as we have some control over the time parametrization of the flow, say, if the speed is everywhere bounded by 1, then it is evident that we can arrange that any point on the boundary of X is strongly chain recurrent for f.

A second question that arises concerns the definition of the weakly attracting set determined by a weakly absorbing set U as $\bigcap \overline{f^n(U)}$. When X is compact it is not necessary to take the closures of the sets in the intersection, but the following example shows that without compactness it is necessary to take the closure.

Example 2 [7, §2.6]. Here X is a subset of the plane, consisting of a countable number of bounded horizontal line segments:

- the bottom segment: $B = \{(x, 0) | 0 < x < 1\}$,
- the top segment: $T = \{(x, 2) | 0 \leq x < 1/2\}$, and
- the intermediate segments: $I_n = \{(x, 1/n) | 0 < x < 1\}$.

Note that the top segment includes its left end point $(0, 2)$ while none of the other segments include either of their end points. The map f takes B to itself, sending $(x, 0)$ to $(x^2, 0)$, and it sends T into B, $f(x, 2) = (x + 0.5, 0)$.

Each of the remaining segments is mapped to the segment directly above, the map being $f(x, 1/n) = (x^2, 1/(n-1))$ if $n \geq 2$ and $f(x, 1) = (x/2, 2)$. It is not hard to see that the point $(0, 2)$ is strongly chain recurrent even though it is not in the image of f. Since X itself is trivially absorbing, we have a strongly chain recurrent point that is in an absorbing set but not in its image. Thus if the closure operations are omitted in the definition of a weakly attracting set the conclusion of Theorem 1 would be false.

Example 2 also shows that $\mathscr{C}R^+(f)$ is not necessarily invariant: $f(\mathscr{C}R^+(f))$ (which is closed in X) does not contain the strongly chain recurrent point $(0, 2)$. It also illustrates an important difference between the compact case and the noncompact case: in the example there is no ε-chain from $(0, 2)$ back to itself that is contained in $\mathscr{C}R^+(f)$. When X is compact there is always such an ε-chain; see [1, p. 38] or [12].

The fact that a set U is weakly absorbing if and only if $\overline{f(U)} \subset U$ makes it clear that the property of being weakly absorbing is maintained by topological conjugacies. It now follows from Theorem 1 that strong chain recurrence is also maintained by a topological conjugacy. In particular, these properties are unaffected if the given metric is changed to a different metric, provided only that the two metrics define the same topology on X. The failure of this result is perhaps the main defect in the (constant ε) definition of absorbing sets and $\mathscr{C}R$ used in [7]. This failure is illustrated by Example 1 of §2. In that example X was a discrete subset of the plane, and the given metric was the one inherited from the plane. Using that metric $\mathscr{C}R(f)$ was nonempty; if instead we use the equivalent metric $\rho(x, y) = 1$ if $x \neq y$ then $\mathscr{C}R(f)$ is easily seen to be empty.

4. Lyapunov functions

There is a natural decomposition of $\mathscr{C}R^+(f)$ into equivalence classes under the relation: $p \sim q$ if and only if for each $\varepsilon \in \mathscr{P}$ there are ε-chains from p to q and from q to p. Each equivalence class is called a *chain transitive component* of f. The arguments of §2 make it clear that if a chain transitive component C intersects a weakly absorbing set U then C must be contained in the weak attractor determined by U.

A *complete Lyapunov function* for $f: X \to X$ is a continuous map $L: X \to \mathbb{R}$ with the following properties:

(i) $L(f(x)) \leq x$ for all x, with equality if and only if $x \in \mathscr{C}R^+(f)$.

(ii) L is constant on each chain component and takes on different values on different chain components.

(iii) If C and C' are distinct chain components with the property that for each $\varepsilon \in \mathscr{P}$ there is an ε-chain from C to C' then $L(C) > L(C')$.

(iv) $L(\mathscr{C}R^+(f))$ is nowhere dense.

Theorem 2. *Suppose that X is a locally compact, second countable metric space, and that $f: X \to X$ is continuous. Then there is a complete Lyapunov function for f.*

The proof outlined below is adapted from the proof of the corresponding theorem in [7], which in turn was adapted from proofs in [1, 4]. There are two parts to the proof. In the first we construct a "Lyapunov-like" function for a given weak attractor A. This is a continuous function $h: X \to [0, 2]$ with the properties: h is 0 on A, h is 2 on the complement of $B(A)$, and $0 < h(f(x)) < h(x) < 2$ for all other x. This part of the argument is not much different than the corresponding arguments in [1, 4, 7]; consequently, we will only give a rough outline of this first part of the proof. The Lyapunov function L will be defined as an infinite linear combination of these Lyapunov-like functions; the second part of the proof is to show that for a given f there is a countable collection of Lyapunov-like functions that capture enough information about the dynamics for L to satisfy properties (i)–(iv). The reduction afforded by the second part of the proof is necessary, as it is possible for f to have an uncountable number of distinct weak attractors; see [7] for an example.

Throughout this section we will assume that X is a locally compact, second countable metric space. These assumptions ensure that we can write $X = \bigcup_{n\geq 0} K_n$ where each K_n is compact and is contained in the interior of K_{n+1}.

First we describe the construction of a "Lyapunov-like" function $h: X \to [0, 2]$ for a single weak attractor A. Write $X = \bigcup K_n$ as above and let $\psi: X \to [1, \infty)$ be a continuous function with the property that $\psi(x) \geq n$ for x in the complement of K_n. Let $M(x)$ be the minimum of 1 and the distance from x to A (set $M(x) = 1$ if $A = \varnothing$). Define

$$\phi(x) = \frac{M(x)}{M(x) + \psi(x) \cdot \operatorname{dist}(x, X - B(A))}$$

(again, take $\operatorname{dist}(x, X - B(A)) = 1$ if $X - B(A) = \varnothing$). ϕ is continuous, takes on values in $[0, 1]$, is 0 only on A, and is 1 only on $X - B(A)$. It can be verified that each $x \in B(A)$ has a compact neighborhood on which the functions $\phi \circ f^j$ converge uniformly to 0 as $j \to \infty$. Because of this last property, if we let $g(x) = \sup_{j\geq 0}\{\phi(f^j(x))\}$ then g is continuous and satisfies $0 \leq g(f(x)) \leq g(x) \leq 1$ for all x. Define

$$h(x) = \sum_{i=0}^{\infty} g(f^i(x))/2^i .$$

If $x \in A$ then $h(f(x)) = h(x) = 0$; if $x \in X - B(A)$ then $h(f(x)) = h(x) = 2$; and if $x \in B(A) - A$ then $0 \leq h(f(x)) < h(x) < 2$.

We will define L as

$$L(x) = \sum_{n=1}^{\infty} h_n(x)/3^n \tag{3}$$

where $\{h_n\}$ is a countable collection of Lyapunov-like functions. It is clear that L is continuous, nonnegative and that $L(f(x)) \leq L(x)$ for all x. To verify the rest of the properties (i)–(iv) of a Lyapunov function we need to show that if x is not strongly chain recurrent then $h_n(f(x)) < h_n(x)$ for some n, and that if C and C' are distinct chain components then there is an h_n that is 0 on one of C, C' and is 2 on the other.

Lemma 4.1. *For any given f there is a countable subset $\mathscr{E}_1$ of $\mathscr{P}$ with the property that if $y \in Y = X - \mathscr{C}R^+(f)$ then there is an $\varepsilon \in \mathscr{E}_1$ such that no $\varepsilon(x)$-chain both begins and ends at y.*

Proof. For brevity we shall say that a point p is *$\varepsilon(x)$-recurrent* if there is an $\varepsilon(x)$-chain that begins and ends at p. The proof is based on the observation that if y is not strongly chain recurrent there is a neighborhood N of y and an $\varepsilon \in \mathscr{P}$ with the property that no point of N is $\varepsilon(x)$-recurrent. To establish this fact choose $\delta \in \mathscr{P}$ such that y is not $\delta(x)$-recurrent. Let K and K' be compact neighborhoods of y, $f(y)$, respectively, and let δ_0 denote the minimum value of $\delta(x)$ on $K \cup K'$. Choose $\alpha > 0$ small enough that $d(y, z) < \alpha$ implies: (1) $z \in K$; (2) $f(z) \in K'$; and (3) $d(f(y), f(z)) < \delta_0/2$. Let β be the smaller of α and δ_0, define $\varepsilon(x) = \frac{1}{2}\min\{\beta, \delta(x)\}$, and let $N = B_{\beta/2}(y)$. Suppose that $z \in N$ and that $z = z_0, z_1, \ldots, z_n, z_{n+1} = z$ is an $\varepsilon(x)$-chain. We will show that $y, z_1, \ldots, z_n, y$ is a $\delta(x)$-chain, which contradicts the way that $\delta(x)$ was chosen. Since we know that $\varepsilon(x) < \delta(x)$, all that must be checked is the pair of inequalities: (a) $d(f(y), z_1) < \delta(f(y))$ and (b) $d(f(z_n), y) < \delta(f(z_n))$. Inequality (a) is true because (2) and (3) show that each of $f(y)$ and z_1 is within $\delta_0/2$ of $f(z)$ and $\delta_0 \leq \delta(f(y))$. To verify (b) note first that both y and $f(z_n)$ are within $\alpha/2$ of z, which shows that $f(z_n)$ is in K. Consequently $\varepsilon(f(z_n)) < \delta_0/2$, showing that $f(z_n)$ is within $\delta_0/2$ of z. Since y is also within $\delta_0/2$ of z, we conclude that $d(f(z_n), y) < \delta_0$; as $f(z_n) \in K$ this establishes (b).

Thus there is an open cover $\mathscr{U}$ of Y such that for each $U \in \mathscr{U}$ there is an $\varepsilon_U(x) \in \mathscr{P}$ such that no point of U is $\varepsilon_U(x)$-recurrent. Since Y is second countable, $\mathscr{U}$ has a countable subcover $\mathscr{W} = \{W_1, W_2, \ldots\}$. Let $\mathscr{E}_1$ denote the functions $\varepsilon_U(x)$ for the U's in $\mathscr{U}$ that are elements of $\mathscr{W}$. □

In the following we will say that two distinct chain transitive components C and C' are *distinguished* by $\varepsilon \in \mathscr{P}$ if either there is no $\varepsilon(x)$-chain going from any point of C to C' or else there is none going from any point of C' to C. Our goal is to show that there is a countable subset $\mathscr{E}_2$ of $\mathscr{P}$ with the property that any pair of distinct chain transitive components of f are distinguished by some element of $\mathscr{E}_2$.

Lemma 4.2. *Let $\beta > 0$ and a compact subset K of X be given. Suppose that $\mathscr{M}$ is a collection of chain transitive components of f satisfying*

(1) *$f(C) \cap K \neq \varnothing$ for each $C \in \mathscr{M}$.*
(2) *For each distinct pair C, C' in $\mathscr{M}$ there is an $\varepsilon \in \mathscr{P}$ that distinguishes between them and satisfies $\varepsilon(x) \geq \beta$ for all $x \in K$.*

Then $\mathscr{M}$ is finite.

Proof. The idea is to show that the intersections of K with distinct elements of $\mathscr{M}$ are uniformly bounded apart from each other. Suppose that C, C' are in $\mathscr{M}$, containing the points p, p', respectively, and that $f(p)$, $f(p')$ are both in K and within β of each other. Let $\varepsilon(x)$ be the element of $\mathscr{P}$ distinguishing C from C', as given by condition (2). Since $\beta \leq \varepsilon(f(p))$ we see that $\{p, f(p')\}$ is an $\varepsilon(x)$-chain going from C to C'. Similarly, $\{p', f(p)\}$ is an $\varepsilon(x)$-chain going from C' to C, and we have a contradiction to the assumption that C and C' were distinguished by $\varepsilon(x)$. □

Corollary 4.3. *There is a single element ε of $\mathscr{P}$ that satisfies $\varepsilon(x) = \beta$ for all $x \in K$ and with the property that $\varepsilon(x)$ distinguishes between any pair of distinct chain transitive components in $\mathscr{M}$.*

Proof. Let $\mu(x)$ be the minimum of the functions given by condition (2) of the lemma for each of the finitely many pairs of distinct elements of $\mathscr{M}$, and let $\varepsilon(x) = \min\{\beta, \mu(x)\}$. □

As before, write $X = \bigcup_{n\geq 0} K_n$ where each K_n is compact and is contained in the interior of K_{n+1}. Let $\mathscr{E}_2$ be the countable subset of $\mathscr{P}$ that is obtained by applying the corollary with $\beta = 1/m$ and $K = K_n$ as m and n range over the positive integers.

Lemma 4.4. *Any pair of distinct chain transitive components can be distinguished by an element of* $\mathscr{E}_2$.

Proof. Given a pair of distinct chain transitive components C and C', choose n large enough that both $f(C)$ and $f(C')$ meet K_n. Since they are distinct there is an $\varepsilon \in \mathscr{P}$ that distinguishes them. Pick m large enough that $1/m$ is less than the minimum value of $\varepsilon(x)$ on K_n. Then the element of $\mathscr{E}_2$ corresponding to this choice of m and n distinguishes C from C'. □

Now that we have Lemmas 4.1 and 4.4 we can finish the construction of the complete Lyapunov function L. Let $Z \subset X$ be a countable dense subset of X, and let $\mathscr{E}$ denote the union of the two countable subsets of $\mathscr{P}$ given by Lemmas 4.1 and 4.4. For each $z \in Z$ and each $\varepsilon \in \mathscr{E}$ let $U(z, \varepsilon)$ be the set of all possible end points of $\varepsilon(x)$-chains that begin at z; as in the proof of Theorem 1, each of these countably many sets is open and absorbing. List them in some order, $U_1, U_2, \ldots$; for each n let A_n be the weak attractor determined by U_n and let h_n be the associated Lyapunov-like function. Define L by

$$L(x) = \sum_{n=1}^{\infty} h_n(x)/3^n .$$

The properties (i)–(iv) defining a Lyapunov function are now verifiable: the density of Z combined with the properties of the collection $\mathscr{E}_1$ of Lemma 4.1 imply property (i), while properties (ii) and (iii) follow from Lemma 4.4. (iv) is a consequence of the fact that each h_n is either 0 or 2 on any given chain component, which shows that $L|\mathscr{C}R^+(f)$ is a subset of the Cantor middle-third set. For details of the argument consult [7].

5. Appendix: proof of Lemma 2.4

The proof of Lemma 2.4 uses the fact that any metric space is paracompact so that there is a continuous partition of unity that is subordinate to any given open cover of the space [2, 9]. Recall that X is locally compact and the two functions $f: X \to X$ and $\varepsilon: X \to (0, \infty)$ are continuous. Define a function $\kappa: X \to (0, \infty)$ by $\kappa(x) = \sup\{0 < \alpha \leq 1 | \overline{B_\alpha(x)}$ is compact$\}$. Note that if $\alpha < \kappa(x)$ then $\alpha - d(x, y) < \kappa(y)$, from which it follows that $\kappa(x) - \kappa(y) \leq d(x, y)$, so κ is continuous.

Now define $\eta: X \to (0, \infty)$ by

$$\eta(x) = \sup\{0 < \alpha \leq \kappa(x)/2 | \overline{f(B_\alpha(x))} \subset B_{\varepsilon(f(x))}(f(x))\} .$$

In general η is not continuous, but it is lower semicontinuous. To verify the semicontinuity of η at p choose constants α, β satisfying $0 < \alpha < \beta < \eta(p)$. By the definition of η we know that $\overline{B_\beta(p)}$ is compact and that $f(\overline{B_\beta(p)}) \subset B_{\varepsilon(f(x))}(f(x))$. Since $f(\overline{B_\beta(p)})$ is compact, there is a constant $\lambda < \varepsilon(f(x))$ such that $f(\overline{B_\beta(p)}) \subset B_\lambda(f(x))$. Now if y is close enough to p we can conclude all of the following: (i) $B_\alpha(y) \subset B_\beta(p)$; (ii) $B_\lambda(f(p)) \subset B_{\varepsilon(f(y))}(f(y))$; and (iii)

$\alpha < \kappa(y)/2$. It follows from (i)–(iii) that $\alpha \le \eta(y)$ as long as y is sufficiently close to p. Hence $\liminf_{y\to p} \eta(y) \ge \eta(p)$, and the lower semicontinuity of η is established.

Let $\mathcal{G}$ be the open cover of X by the balls $B_{\kappa(x)/2}(x)$. Note that each of these balls has compact closure. By paracompactness there is a locally finite open cover $\mathcal{F} = \{F_i\}$ refining $\mathcal{G}$ and a continuous partition of unity $\{\psi_i\}$ subordinate to $\mathcal{F}$. Since each F_i has compact closure, η is bounded away from 0 on F_i, say $\eta(x) \ge 2\gamma_i > 0$ on F_i. Now define $\delta(x) = \sum \gamma_i \psi_i(x)$. Since $\psi_i(x) = 0$ unless $x \in F_i$ we see that $\psi_i(x) \ne 0$ implies that $\gamma_i \le \eta(x)/2$, so that $\delta(x) \le \sum \eta(x)\psi_i(x)/2 = \eta(x)/2$, which by the definition of η implies that $f(B_{\delta(x)}(x)) \subset B_{\varepsilon(f(x))}(f(x))$, as desired.

References

1. C. Conley, *Isolated invariant sets and the Morse index*, CBMS Publ., vol. 38, Amer. Math. Soc., Providence, RI, 1978.
2. J. Dugundji, *Topology*, Allyn and Bacon, Boston, MA, 1966.
3. J. Franks, *Recurrence and fixed points of surface homeomorphisms*, Ergodic Theory and Dynamical Systems **8** (1988), 99–107.
4. ——, *A variation on the Poincaré-Birkhoff theorem*, Hamiltonian Dynamical Systems, Contemp. Math., vol. 81, Amer. Math. Soc., Providence, RI, 1988, pp. 111–117.
5. ——, *A new proof of the Brouwer plane translation theorem*, preprint.
6. B. M. Garay, *Uniform persistence and chain recurrence*, J. Math. Anal. Appl. **130** (1989), 372–381.
7. M. Hurley, *Chain recurrence and attraction in noncompact spaces*, Ergodic Theory and Dynamical Systems (to appear).
8. ——, *Attractors in restricted cellular automata*, Proc. Amer. Math. Soc. **115** (1992), 563–572.
9. J. L. Kelley, *General topology*, Van Nostrand, Princeton, NJ, 1967.
10. R. McGehee, *Some metric proprties of attractors with applications to computer simulations of dynamical systems*, Univ. of Minnesota, preprint, 1988.
11. V. V. Nemytskiĭ and V. V. Stepanov, *Qualitative theory of differential equations*, Princeton Univ. Press, Princeton, NJ, 1960.
12. C. Robinson, *Stability theorems and hyperbolicity in dynamical systems*, Rocky Mountain J. Math. **7** (1977), 425–437.

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