# NONCONFORMING FINITE ELEMENT STOKES COMPLEXES IN THREE DIMENSIONS 

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#### Abstract

Two nonconforming finite element Stokes complexes starting from the conforming Lagrange element and ending with the nonconforming $P_{1}-P_{0}$ element for the Stokes equation in three dimensions are studied. And commutative diagrams are also shown by combining nonconforming finite element Stokes complexes and interpolation operators. The lower order $\boldsymbol{H}$ (gradcurl)nonconforming finite element only has 14 degrees of freedom, whose basis functions are explicitly given in terms of the barycentric coordinates. The $\boldsymbol{H}$ (gradcurl)-nonconforming elements are applied to solve the quad-curl problem, and optimal convergence is derived. By the nonconforming finite element Stokes complexes, the mixed finite element methods of the quad-curl problem are decoupled into two mixed methods of the Maxwell equation and the nonconforming $P_{1}-P_{0}$ element method for the Stokes equation, based on which a fast solver is discussed. Numerical results are provided to verify the theoretical convergence rates.


## 1. Introduction

In this paper we shall consider nonconforming finite element discretization of the following Stokes complex in three dimensions

$$
\begin{equation*}
\mathbb{R} \xrightarrow{C} H^{1}(\Omega) \xrightarrow{\nabla} \boldsymbol{H}(\operatorname{grad} \operatorname{curl}, \Omega) \xrightarrow{\text { curl }} \boldsymbol{H}^{1}\left(\Omega ; \mathbb{R}^{3}\right) \xrightarrow{\text { div }} L^{2}(\Omega) \rightarrow 0, \tag{1.1}
\end{equation*}
$$

where $\boldsymbol{H}(\operatorname{grad} \operatorname{curl}, \Omega):=\left\{\boldsymbol{v} \in \boldsymbol{H}(\operatorname{curl}, \Omega): \operatorname{curl} \boldsymbol{v} \in \boldsymbol{H}^{1}\left(\Omega ; \mathbb{R}^{3}\right)\right\}$. Conforming finite element Stokes complexes on triangles and rectangles in two dimensions are devised in [27, 40]. And conforming finite element Stokes complexes on Alfeld split meshes in three dimensions are advanced in [28]. We refer to [5] for a conforming virtual element discretization of the Stokes complex (1.1). To the best of our knowledge, there is no finite element discretization of the Stokes complex (1.1) in three dimensions using pure polynomials as shape functions in literature. Recently $\boldsymbol{H}(\mathrm{grad}$ curl $)$-conforming finite elements in three dimensions are constructed with $k \geq 6$ in [41], whose space of shape functions includes all the polynomials with degree no more than $k$. The number of the degrees of freedom for the lowest-order element in [41] is 315 , which is reduced to 18 by enriching the shape function space with macro-element bubble functions in [28]. Nonconforming elements to discretize $\boldsymbol{H}(\operatorname{grad} \operatorname{curl}, \Omega)$ are another choices to reduce the large dimension of the conforming

[^0]element spaces. The $\boldsymbol{H}$ (grad curl)-nonconforming Zheng-Hu-Xu element in [44] has only 20 degrees of freedom, which is the first $\boldsymbol{H}$ (gradcurl)-nonconforming finite element.

We will construct an $\boldsymbol{H}(\operatorname{grad}$ curl)-nonconforming finite element possessing fewer degrees of freedom than the Zheng-Hu-Xu element, but preserving the same approximation error in energy norm. The finite element discretization of $\boldsymbol{H}^{1}\left(\Omega ; \mathbb{R}^{3}\right) \times$ $L^{2}(\Omega)$ in the Stokes complex (1.1) should be a stable divergence-free pair for the Stokes equation, which suggests us to use the nonconforming linear element and piecewise constant to discretize $\boldsymbol{H}^{1}\left(\Omega ; \mathbb{R}^{3}\right)$ and $L^{2}(\Omega)$ respectively. On the other hand, the direct sum decomposition $\mathbb{P}_{k}\left(K ; \mathbb{R}^{3}\right)=\nabla \mathbb{P}_{k+1}(K) \oplus\left(\left(\boldsymbol{x}-\boldsymbol{x}_{K}\right) \times\right.$ $\left.\mathbb{P}_{k-1}\left(K ; \mathbb{R}^{3}\right)\right)[2,3]$ implies the curl operator curl : $\left(\boldsymbol{x}-\boldsymbol{x}_{K}\right) \times \mathbb{P}_{1}\left(K ; \mathbb{R}^{3}\right) \rightarrow$ $\mathbb{P}_{1}\left(K ; \mathbb{R}^{3}\right)$ is injective. This motivates us to take the space of shape functions $\boldsymbol{W}_{k}(K)=\nabla \mathbb{P}_{k+1}(K) \oplus\left(\left(\boldsymbol{x}-\boldsymbol{x}_{K}\right) \times \mathbb{P}_{1}\left(K ; \mathbb{R}^{3}\right)\right)$ with $k=0,1$. Note that $\boldsymbol{W}_{1}(K)$ is exactly the space of shape functions of the Zheng-Hu-Xu element, hence we give a new understanding of the Zheng-Hu-Xu element by the space decomposition. The dimension of $\boldsymbol{W}_{0}(K)$ is 14 , which is six fewer than the dimension of $\boldsymbol{W}_{1}(K)$. The degrees of freedom $\mathcal{N}_{0}(K)$ for $\boldsymbol{W}_{0}(K)$ are given by

$$
\begin{array}{cl}
\int_{e} \boldsymbol{v} \cdot \boldsymbol{t}_{e} \mathrm{~d} s & \text { on each } e \in \mathcal{E}(K) \\
\int_{F}(\operatorname{curl} \boldsymbol{v}) \times \boldsymbol{n} \mathrm{d} s & \text { on each } F \in \mathcal{F}(K) .
\end{array}
$$

By comparing the degrees of freedom, the lower order nonconforming element $\left(K, \boldsymbol{W}_{0}(K), \mathcal{N}_{0}(K)\right)$ for $\boldsymbol{H}(\operatorname{grad} \operatorname{curl}, \Omega)$ is very similar as the Morley-Wang-Xu element [39] for $H^{2}(\Omega)$. The explicit expressions of the basis functions of $\boldsymbol{W}_{0}(K)$ are shown in terms of the barycentric coordinates.

Then we combine the conforming $(k+1)$ th order Lagrange element space $V_{h 0}^{g}$, the $\boldsymbol{H}$ (grad curl)-nonconforming finite element space $\boldsymbol{W}_{h 0}$ including the Zheng-HuXu element and the lower order one constructed in this paper, the nonconforming linear element space $\boldsymbol{V}_{h 0}^{s}$, and the piecewise constant space $\mathcal{Q}_{h 0}$ to build up the nonconforming finite element Stokes complexes

$$
\begin{equation*}
0 \xrightarrow{\subset} V_{h 0}^{g} \xrightarrow{\nabla} \boldsymbol{W}_{h 0} \xrightarrow{\text { curl }_{h}} \boldsymbol{V}_{h 0}^{s} \xrightarrow{\operatorname{div}_{h}} \mathcal{Q}_{h 0} \rightarrow 0 \tag{1.2}
\end{equation*}
$$

The divergence-free subspace of the nonconforming linear element space $\boldsymbol{V}_{h 0}^{s}$ is explicitly characterized due to this nonconforming finite element Stokes complex, which essentially extends the result of Falk and Morley [18] to three dimensions. Recently this nonconforming finite element Stokes complex is applied to prove the quasi-orthogonality of the adaptive finite element method for the quad-curl problem in [9]. Furthermore, we develop the commutative diagram for Stokes complex (1.1)

where $I_{h}^{S Z}$ is the Scott-Zhang interpolation operator [37], $\boldsymbol{\Pi}_{h}^{g c}$ is a quasi-interpolation operator, and both $\boldsymbol{I}_{h}^{s}$ and $I_{h}^{L^{2}}$ are the standard interpolation operators based on the degrees of freedom.

The $\boldsymbol{H}$ (grad curl)-nonconforming element together with the Lagrange element is then applied to solve the quad-curl problem. The discrete Poincaré inequality is established for the $\boldsymbol{H}\left(\operatorname{grad}\right.$ curl)-nonconforming element space $\boldsymbol{W}_{h 0}$, as a result the coercivity on the weak divergence-free space follows. Then we acquire the discrete stability of the bilinear form from the evident discrete inf-sup condition, and derive the optimal convergence of the nonconforming mixed finite methods. Since the interpolation operator $\boldsymbol{I}_{h}^{g c}$ is not well-defined on $\boldsymbol{H}_{0}(\operatorname{grad} \operatorname{curl}, \Omega)$, in the error analysis we exploit a quasi-interpolation operator $\boldsymbol{\Pi}_{h}^{g c}$ defined on $\boldsymbol{H}_{0}(\operatorname{grad} \operatorname{curl}, \Omega)$, which is constructed by combining a regular decomposition for the space $\boldsymbol{H}_{0}(\operatorname{grad} \operatorname{curl}, \Omega)$, the interpolation operator $\boldsymbol{I}_{h}^{\text {gc }}$ and the Scott-Zhang interpolation operator [37].

By the nonconforming finite element Stokes complex (1.2), we equivalently decouple the mixed finite element methods of the quad-curl problem into two mixed methods of the Maxwell equation and the nonconforming $P_{1}-P_{0}$ element method for the Stokes equation, as the decoupling of the quad-curl problem in the continuous level [11, 43]. A fast solver based on this equivalent decoupling is discussed for the mixed finite element methods of the quad-curl problem.

In addition to the Stokes complex (1.1), another kind of Stokes complex [30] is

$$
\begin{equation*}
\mathbb{R} \xrightarrow{C} H^{2}(\Omega) \xrightarrow{\text { curl }} \boldsymbol{H}^{1}\left(\Omega ; \mathbb{R}^{2}\right) \xrightarrow{\text { div }} L^{2}(\Omega) \rightarrow 0 \tag{1.3}
\end{equation*}
$$

in two dimensions, and

$$
\begin{equation*}
\mathbb{R} \xrightarrow{\subset} H^{2}(\Omega) \xrightarrow{\nabla} \boldsymbol{H}^{1}(\operatorname{curl}, \Omega) \xrightarrow{\text { curl }} \boldsymbol{H}^{1}\left(\Omega ; \mathbb{R}^{3}\right) \xrightarrow{\text { div }} L^{2}(\Omega) \rightarrow 0 \tag{1.4}
\end{equation*}
$$

in three dimensions, where $\boldsymbol{H}^{1}(\operatorname{curl}, \Omega):=\left\{\boldsymbol{v} \in \boldsymbol{H}^{1}\left(\Omega ; \mathbb{R}^{3}\right): \operatorname{curl} \boldsymbol{v} \in \boldsymbol{H}^{1}\left(\Omega ; \mathbb{R}^{3}\right)\right\}$. We refer to $[18,32,4,31,19,23,24,42,22]$ for some finite element discretizations of the Stokes complex (1.3) in two dimensions, and [38, 23, 35] for some finite element discretizations of the Stokes complex (1.4) in three dimensions. While the finite elements corresponding to the Stokes complexes (1.3)-(1.4) are not suitable to discretize the quad-curl problem, since $\nabla H^{1}(\Omega) \subset \boldsymbol{H}(\operatorname{grad} \operatorname{curl}, \Omega)$ is not a subspace of $\boldsymbol{H}^{1}(\operatorname{curl}, \Omega)$.

The rest of this paper is organized as follows. In Section 2, we devise a lower order $\boldsymbol{H}(\mathrm{grad}$ curl $)$-nonconforming finite element. Nonconforming finite element Stokes complexes are developed in Section 3. In Section 4, we propose the nonconforming mixed finite element methods for the quad-curl problem. And the decoupling of the mixed finite element methods and a fast solver are discussed in Section 5. Finally numerical results are presented in Section 6.

## 2. The $\boldsymbol{H}$ (grad curl)-NONCONFORMING FINITE ELEMENTS

In this section we will present $\boldsymbol{H}$ (grad curl)-nonconforming finite elements.
2.1. Notation. Given a bounded domain $G \subset \mathbb{R}^{3}$ and a nonnegative integer $m$, let $H^{m}(G)$ be the usual Sobolev space of functions on $G$, and $\boldsymbol{H}^{m}\left(G ; \mathbb{R}^{3}\right)$ the vector version of $H^{m}(G)$. The corresponding norm and semi-norm are denoted, respectively, by $\|\cdot\|_{m, G}$ and $|\cdot|_{m, G}$. Let $(\cdot, \cdot)_{G}$ be the standard inner product on $L^{2}(G)$ or $L^{2}\left(G ; \mathbb{R}^{3}\right)$. If $G$ is $\Omega$, we abbreviate $\|\cdot\|_{m, G},|\cdot|_{m, G}$ and $(\cdot, \cdot)_{G}$ by $\|\cdot\|_{m},|\cdot|_{m}$ and $(\cdot, \cdot)$, respectively. Denote by $H_{0}^{m}(G)\left(\boldsymbol{H}_{0}^{m}\left(G ; \mathbb{R}^{3}\right)\right)$ the closure of $C_{0}^{\infty}(G)\left(\boldsymbol{C}_{0}^{\infty}\left(G ; \mathbb{R}^{3}\right)\right)$ with respect to the norm $\|\cdot\|_{m, G}$. Let $\mathbb{P}_{m}(G)$ stand for the set of all polynomials in $G$ with the total degree no more than $m$, and $\mathbb{P}_{m}\left(G ; \mathbb{R}^{3}\right)$ be the vector version of $\mathbb{P}_{m}(G)$. Let $Q_{G}^{m}: L^{2}(G) \rightarrow \mathbb{P}_{m}(G)$ be the $L^{2}$-orthogonal projector, and its vector version is denoted by $\boldsymbol{Q}_{G}^{m}$. Set $Q_{G}:=Q_{G}^{0}$. The gradient operator,
curl operator and divergence operator are denoted by $\nabla$, curl and div respectively. And define Sobolev spaces $\boldsymbol{H}(\operatorname{curl}, G), \boldsymbol{H}_{0}(\operatorname{curl}, G), \boldsymbol{H}(\operatorname{div}, G), \boldsymbol{H}_{0}(\operatorname{div}, G)$ and $L_{0}^{2}(G)$ in the standard way.

Assume $\Omega \subset \mathbb{R}^{3}$ is a contractible polyhedron. Let $\left\{\mathcal{T}_{h}\right\}_{h>0}$ be a regular family of tetrahedral meshes of $\Omega$. For each element $K \in \mathcal{T}_{h}$, denote by $\boldsymbol{n}_{K}$ the unit outward normal vector to $\partial K$, which will be abbreviated as $\boldsymbol{n}$ for simplicity. Let $\mathcal{F}_{h}, \mathcal{F}_{h}^{i}, \mathcal{E}_{h}$ and $\mathcal{V}_{h}$ be the union of all faces, interior faces, edges and vertices of the partition $\mathcal{T}_{h}$, respectively. We fix a unit normal vector $\boldsymbol{n}_{F}$ for each face $F \in \mathcal{F}_{h}$, and a unit tangent vector $\boldsymbol{t}_{e}$ for each edge $e \in \mathcal{E}_{h}$. For any $K \in \mathcal{T}_{h}$, denote by $\mathcal{F}(K), \mathcal{E}(K)$ and $\mathcal{V}(K)$ the set of all faces, edges and vertices of $K$, respectively. For any $F \in \mathcal{F}_{h}$, let $\mathcal{E}(F)$ be the set of all edges of $F$. And for each $e \in \mathcal{E}(F)$, denote by $\boldsymbol{n}_{F, e}$ the unit vector being parallel to $F$ and outward normal to $\partial F$. Set $\boldsymbol{t}_{F, e}:=\boldsymbol{n}_{F} \times \boldsymbol{n}_{F, e}$, where $\times$ is the exterior product. For elementwise smooth function $\boldsymbol{v}$, define

$$
\|\boldsymbol{v}\|_{1, h}^{2}:=\sum_{K \in \mathcal{T}_{h}}\|\boldsymbol{v}\|_{1, K}^{2}, \quad|\boldsymbol{v}|_{1, h}^{2}:=\sum_{K \in \mathcal{T}_{h}}|\boldsymbol{v}|_{1, K}^{2} .
$$

Let $\nabla_{h}$, curl ${ }_{h}$ and $\operatorname{div}_{h}$ be the elementwise version of $\nabla$, curl and div with respect to $\mathcal{T}_{h}$.
2.2. Nonconforming finite elements. We focus on constructing nonconforming finite elements for the space $\boldsymbol{H}(\operatorname{grad} \operatorname{curl}, \Omega)$ in this subsection. To this end, recall the direct sum of the polynomial space $[2,3]$

$$
\begin{equation*}
\mathbb{P}_{k}\left(K ; \mathbb{R}^{3}\right)=\nabla \mathbb{P}_{k+1}(K) \oplus\left(\left(\boldsymbol{x}-\boldsymbol{x}_{K}\right) \times \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{3}\right)\right) \quad \forall K \in \mathcal{T}_{h}, \tag{2.1}
\end{equation*}
$$

where $\boldsymbol{x}_{K}$ is the barycenter of $K$. The decomposition (2.1) implies that curl : $\left(\boldsymbol{x}-\boldsymbol{x}_{K}\right) \times \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{3}\right) \rightarrow \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{3}\right)$ is injective. We intend to use the nonconforming linear element to discretize $\boldsymbol{H}^{1}\left(\Omega ; \mathbb{R}^{3}\right)$, then the decomposition (2.1) and the complex (1.1) motivate us that the space of shape functions to discrete $\boldsymbol{H}(\operatorname{grad} \operatorname{curl}, \Omega)$ should include $\left(\boldsymbol{x}-\boldsymbol{x}_{K}\right) \times \mathbb{P}_{1}\left(K ; \mathbb{R}^{3}\right)$. The direct sum in (2.1) also suggests to enrich $\left(\boldsymbol{x}-\boldsymbol{x}_{K}\right) \times \mathbb{P}_{1}\left(K ; \mathbb{R}^{3}\right)$ with $\nabla \mathbb{P}_{l}(K)$ for some positive integer $l$ to get the space of shape functions. Hence for each $K \in \mathcal{T}_{h}$, define the space of shape functions as

$$
\boldsymbol{W}_{k}(K):=\nabla \mathbb{P}_{k+1}(K) \oplus\left(\left(\boldsymbol{x}-\boldsymbol{x}_{K}\right) \times \mathbb{P}_{1}\left(K ; \mathbb{R}^{3}\right)\right) \quad \text { for } k=0,1
$$

By the decomposition (2.1), we have $\mathbb{P}_{k}\left(K ; \mathbb{R}^{3}\right) \subset \boldsymbol{W}_{k}(K) \subset \mathbb{P}_{2}\left(K ; \mathbb{R}^{3}\right)$, and

$$
\operatorname{dim} \boldsymbol{W}_{k}(K)= \begin{cases}14, & k=0 \\ 20, & k=1\end{cases}
$$

Then choose the following local degrees of freedom $\mathcal{N}_{k}(K)$

$$
\begin{array}{cl}
\int_{e} \boldsymbol{v} \cdot \boldsymbol{t}_{e} q \mathrm{~d} s & \forall q \in \mathbb{P}_{k}(e) \text { on each } e \in \mathcal{E}(K), \\
\int_{F}(\operatorname{curl} \boldsymbol{v}) \times \boldsymbol{n} \mathrm{d} s & \text { on each } F \in \mathcal{F}(K) . \tag{2.3}
\end{array}
$$

The degrees of freedom (2.2)-(2.3) are inspired by the degrees of freedom of nonconforming linear element and the Nédélec element $[33,34]$. Note that the triple $\left(K, \boldsymbol{W}_{1}(K), \mathcal{N}_{1}(K)\right)$ is exactly the nonconforming finite element in [44]. In this paper we will embed this nonconforming finite element into the discrete Stokes complex. And we also construct the lowest order triple $\left(K, \boldsymbol{W}_{0}(K), \mathcal{N}_{0}(K)\right)$.

Lemma 2.1. The degrees of freedom (2.2)-(2.3) are unisolvent for the shape function space $\boldsymbol{W}_{k}(K)$.

Proof. Notice that the number of the degrees of freedom (2.2)-(2.3) is same as the dimension of $\boldsymbol{W}_{k}(K)$. It is sufficient to show that $\boldsymbol{v}=\mathbf{0}$ for any $\boldsymbol{v} \in \boldsymbol{W}_{k}(K)$ with vanishing degrees of freedom (2.2)-(2.3).

For each $F \in \mathcal{F}(K)$, apply the integration by parts on face $F$ to obtain

$$
\begin{aligned}
\int_{F}(\operatorname{curl} \boldsymbol{v}) \cdot \boldsymbol{n}_{F} \mathrm{~d} s & =\int_{F} \operatorname{div}\left(\boldsymbol{v} \times \boldsymbol{n}_{F}\right) \mathrm{d} s=\left.\sum_{e \in \mathcal{E}(F)} \int_{e}\left(\boldsymbol{v} \times \boldsymbol{n}_{F}\right)\right|_{F} \cdot \boldsymbol{n}_{F, e} \mathrm{~d} s \\
& =\sum_{e \in \mathcal{E}(F)} \int_{e} \boldsymbol{v} \cdot\left(\boldsymbol{n}_{F} \times \boldsymbol{n}_{F, e}\right) \mathrm{d} s=\sum_{e \in \mathcal{E}(F)} \int_{e} \boldsymbol{v} \cdot \boldsymbol{t}_{F, e} \mathrm{~d} s
\end{aligned}
$$

We get from the vanishing degrees of freedom (2.2) that $\int_{F}(\operatorname{curl} \boldsymbol{v}) \cdot \boldsymbol{n}_{F} \mathrm{~d} s=0$, which together with the vanishing degrees of freedom (2.3) implies

$$
\int_{F} \operatorname{curl} \boldsymbol{v} \mathrm{~d} s=\mathbf{0} .
$$

Since curl $\boldsymbol{v} \subseteq \mathbb{P}_{1}\left(K ; \mathbb{R}^{3}\right)$, we acquire from the unisolvence of the nonconforming linear element that curl $\boldsymbol{v}=\mathbf{0}$. Employing the fact that curl : $\left(\boldsymbol{x}-\boldsymbol{x}_{K}\right) \times \mathbb{P}_{1}\left(K ; \mathbb{R}^{3}\right) \rightarrow$ $\mathbb{P}_{1}\left(K ; \mathbb{R}^{3}\right)$ is injective, there exists $q \in \mathbb{P}_{k+1}(K)$ such that $\boldsymbol{v}=\nabla q$. By the vanishing degrees of freedom (2.2), it holds $\partial_{t_{e}} q=0$, which implies that we can choose $q \in \mathbb{P}_{k+1}(K)$ such that $\left.q\right|_{e}=0$ for each $e \in \mathcal{E}(K)$. Noting that $k=0,1$, we acquire $q=0$ and $\boldsymbol{v}=\mathbf{0}$.

By comparing the degrees of freedom, the lower order nonconforming element $\left(K, \boldsymbol{W}_{0}(K), \mathcal{N}_{0}(K)\right)$ for $\boldsymbol{H}(\operatorname{grad} \operatorname{curl}, \Omega)$ is very similar as the Morley-Wang-Xu element [39] for $H^{2}(\Omega)$.

Next we give a norm equivalence of space $\boldsymbol{W}_{k}(K)$. To this end, recall the Poincaré operator $\mathcal{K}_{K}: \mathbb{P}_{1}\left(K ; \mathbb{R}^{3}\right) \rightarrow\left(\boldsymbol{x}-\boldsymbol{x}_{K}\right) \times \mathbb{P}_{1}\left(K ; \mathbb{R}^{3}\right)$ in [21, 25]

$$
\mathcal{K}_{K} \boldsymbol{q}:=-\left(\boldsymbol{x}-\boldsymbol{x}_{K}\right) \times \int_{0}^{1} t \boldsymbol{q}\left(t\left(\boldsymbol{x}-\boldsymbol{x}_{K}\right)+\boldsymbol{x}_{K}\right) \mathrm{d} t
$$

It holds the identity [21, Theorem 2.1]

$$
\begin{equation*}
\operatorname{curl} \mathcal{K}_{K}(\operatorname{curl} \boldsymbol{v})=\operatorname{curl} \boldsymbol{v} \quad \forall \boldsymbol{v} \in \boldsymbol{W}_{k}(K) . \tag{2.5}
\end{equation*}
$$

By the inverse inequality, we have

$$
\begin{equation*}
\left\|\mathcal{K}_{K} \boldsymbol{q}\right\|_{0, K} \lesssim h_{K}^{5 / 2}\|\boldsymbol{q}\|_{L^{\infty}(K)} \lesssim h_{K}\|\boldsymbol{q}\|_{0, K} \quad \forall \boldsymbol{q} \in \mathbb{P}_{1}\left(K ; \mathbb{R}^{3}\right) \tag{2.6}
\end{equation*}
$$

Lemma 2.2. For $\boldsymbol{v} \in \boldsymbol{W}_{k}(K)$, there exists $q \in \mathbb{P}_{k+1}(K)$ such that

$$
\begin{gather*}
\boldsymbol{v}=\nabla q+\mathcal{K}_{K}(\operatorname{curl} \boldsymbol{v})  \tag{2.7}\\
\|q\|_{0, K}^{2} \lesssim h_{K}^{4}\|\operatorname{curl} \boldsymbol{v}\|_{0, K}^{2}+h_{K}^{4} \sum_{e \in \mathcal{E}(K)}\left\|Q_{e}^{k}\left(\boldsymbol{v} \cdot \boldsymbol{t}_{e}\right)\right\|_{0, e}^{2} . \tag{2.8}
\end{gather*}
$$

Proof. Take a vertex $\delta \in \mathcal{V}(K)$. Due to (2.5), $\boldsymbol{v}-\mathcal{K}_{K}(\operatorname{curl} \boldsymbol{v}) \in \boldsymbol{W}_{k}(K) \cap \operatorname{ker}(\operatorname{curl})$, which means $\boldsymbol{v}-\mathcal{K}_{K}(\operatorname{curl} \boldsymbol{v}) \in \nabla \mathbb{P}_{k+1}(K)$. Choose $q \in \mathbb{P}_{k+1}(K)$ such that $\boldsymbol{v}$ -
$\mathcal{K}_{K}(\operatorname{curl} \boldsymbol{v})=\nabla q$ and $q(\delta)=0$. By the fact $q \in \mathbb{P}_{2}(K)$, the norm equivalence of Lagrange element and the inverse inequality,

$$
\begin{aligned}
\|q\|_{0, K}^{2} & \lesssim h_{K}^{2} \sum_{e \in \mathcal{E}(K)}\|q\|_{0, e}^{2} \lesssim h_{K}^{3} \sum_{e \in \mathcal{E}(K)}\|q\|_{L^{\infty}(e)}^{2}=h_{K}^{3} \sum_{e \in \mathcal{E}(K)}\|q(\boldsymbol{x})-q(\delta)\|_{L^{\infty}(e)}^{2} \\
& \lesssim h_{K}^{3} \sum_{e \in \mathcal{E}(K)} h_{e}^{2}\left\|\partial_{t} q\right\|_{L^{\infty}(e)}^{2} \lesssim h_{K}^{4} \sum_{e \in \mathcal{E}(K)}\left\|\partial_{t} q\right\|_{0, e}^{2} .
\end{aligned}
$$

Since $\partial_{t} q=Q_{e}^{k}\left(\partial_{t} q\right)=Q_{e}^{k}\left(\boldsymbol{v} \cdot \boldsymbol{t}_{e}\right)+Q_{e}^{k}\left(\mathcal{K}_{K}(\operatorname{curl} \boldsymbol{v}) \cdot \boldsymbol{t}_{e}\right)$ on edge $e$, we get from the inverse inequality that

$$
\begin{aligned}
\|q\|_{0, K}^{2} & \lesssim h_{K}^{4} \sum_{e \in \mathcal{E}(K)}\left(\left\|Q_{e}^{k}\left(\boldsymbol{v} \cdot \boldsymbol{t}_{e}\right)\right\|_{0, e}^{2}+\left\|\mathcal{K}_{K}(\operatorname{curl} \boldsymbol{v})\right\|_{0, e}^{2}\right) \\
& \lesssim h_{K}^{2}\left\|\mathcal{K}_{K}(\operatorname{curl} \boldsymbol{v})\right\|_{0, K}^{2}+h_{K}^{4} \sum_{e \in \mathcal{E}(K)}\left\|Q_{e}^{k}\left(\boldsymbol{v} \cdot \boldsymbol{t}_{e}\right)\right\|_{0, e}^{2} .
\end{aligned}
$$

Finally we conclude (2.8) from (2.6).
Lemma 2.3. For $\boldsymbol{v} \in \boldsymbol{W}_{k}(K)$, it holds the norm equivalence

$$
\begin{equation*}
\|\boldsymbol{v}\|_{0, K}^{2} \bar{\sim} h_{K}^{2} \sum_{e \in \mathcal{E}(K)}\left\|Q_{e}^{k}\left(\boldsymbol{v} \cdot \boldsymbol{t}_{e}\right)\right\|_{0, e}^{2}+h_{K}^{3} \sum_{F \in \mathcal{F}(K)}\left\|\boldsymbol{Q}_{F}^{0}((\operatorname{curl} \boldsymbol{v}) \times \boldsymbol{n})\right\|_{0, F}^{2} \tag{2.9}
\end{equation*}
$$

Proof. Since curl $\boldsymbol{v} \in \mathbb{P}_{1}\left(K ; \mathbb{R}^{3}\right)$, by the norm equivalence of the nonconforming $P_{1}$ element,

$$
\begin{aligned}
\|\operatorname{curl} \boldsymbol{v}\|_{0, K}^{2} & \lesssim h_{K} \sum_{F \in \mathcal{F}(K)}\left\|\boldsymbol{Q}_{F}^{0}(\operatorname{curl} \boldsymbol{v})\right\|_{0, F}^{2} \\
& \lesssim h_{K} \sum_{F \in \mathcal{F}(K)}\left(\left\|\boldsymbol{Q}_{F}^{0}((\operatorname{curl} \boldsymbol{v}) \times \boldsymbol{n})\right\|_{0, F}^{2}+\left\|Q_{F}^{0}((\operatorname{curl} \boldsymbol{v}) \cdot \boldsymbol{n})\right\|_{0, F}^{2}\right) .
\end{aligned}
$$

From (2.4) we get

$$
\begin{aligned}
& h_{K}\left\|Q_{F}^{0}((\operatorname{curl} \boldsymbol{v}) \cdot \boldsymbol{n})\right\|_{0, F}^{2} \lesssim h_{K}^{3}\left|Q_{F}^{0}((\operatorname{curl} \boldsymbol{v}) \cdot \boldsymbol{n})\right|^{2} \\
& \lesssim h_{K} \sum_{e \in \mathcal{E}(F)}\left|Q_{e}^{0}\left(\boldsymbol{v} \cdot \boldsymbol{t}_{e}\right)\right|^{2} \\
& \lesssim \sum_{e \in \mathcal{E}(F)}\left\|Q_{e}^{0}\left(\boldsymbol{v} \cdot \boldsymbol{t}_{e}\right)\right\|_{0, e}^{2}
\end{aligned}
$$

Combining the last two inequalities yields

$$
\begin{equation*}
\|\operatorname{curl} \boldsymbol{v}\|_{0, K}^{2} \lesssim \sum_{e \in \mathcal{E}(K)}\left\|Q_{e}^{k}\left(\boldsymbol{v} \cdot \boldsymbol{t}_{e}\right)\right\|_{0, e}^{2}+h_{K} \sum_{F \in \mathcal{F}(K)}\left\|\boldsymbol{Q}_{F}^{0}((\operatorname{curl} \boldsymbol{v}) \times \boldsymbol{n})\right\|_{0, F}^{2} . \tag{2.10}
\end{equation*}
$$

Applying Lemma 2.2 to $\boldsymbol{v}$, we derive from (2.7), the inverse inequality, (2.6) and (2.8) that

$$
\begin{aligned}
\|\boldsymbol{v}\|_{0, K}^{2} & \leq 2\|\nabla q\|_{0, K}^{2}+2\left\|\mathcal{K}_{K}(\operatorname{curl} \boldsymbol{v})\right\|_{0, K}^{2} \lesssim h_{K}^{-2}\|q\|_{0, K}^{2}+h_{K}^{2}\|\operatorname{curl} \boldsymbol{v}\|_{0, K}^{2} \\
& \lesssim h_{K}^{2}\|\operatorname{curl} \boldsymbol{v}\|_{0, K}^{2}+h_{K}^{2} \sum_{e \in \mathcal{E}(K)}\left\|Q_{e}^{k}\left(\boldsymbol{v} \cdot \boldsymbol{t}_{e}\right)\right\|_{0, e}^{2} .
\end{aligned}
$$

Then we acquire from (2.10) that

$$
\|\boldsymbol{v}\|_{0, K}^{2} \lesssim h_{K}^{2} \sum_{e \in \mathcal{E}(K)}\left\|Q_{e}^{k}\left(\boldsymbol{v} \cdot \boldsymbol{t}_{e}\right)\right\|_{0, e}^{2}+h_{K}^{3} \sum_{F \in \mathcal{F}(K)}\left\|\boldsymbol{Q}_{F}^{0}((\operatorname{curl} \boldsymbol{v}) \times \boldsymbol{n})\right\|_{0, F}^{2} .
$$

The another side of (2.9) follows from the inverse inequality.
2.3. Basis functions. We will figure out the basis functions of $\boldsymbol{W}_{0}(K)$ in this subsection. We refer to [44] for the basis functions of $\boldsymbol{W}_{1}(K)$. Let $\lambda_{1}, \lambda_{2}, \lambda_{3}$ and $\lambda_{4}$ be the barycentric coordinates of point $\boldsymbol{x}$ with respect to the vertices $\boldsymbol{x}_{1}, \boldsymbol{x}_{2}, \boldsymbol{x}_{3}$ and $\boldsymbol{x}_{4}$ of the tetrahedron $K$ respectively. Let $F_{l}$ be the face of $K$ opposite to $\boldsymbol{x}_{l}$. And the vertices of $F_{l}$ denoted by $\boldsymbol{x}_{l_{1}}, \boldsymbol{x}_{l_{2}}$ and $\boldsymbol{x}_{l_{3}}$ with $l_{1}<l_{2}<l_{3}$. Set $\boldsymbol{t}_{i j}:=\boldsymbol{x}_{j}-\boldsymbol{x}_{i}$, which is a tangential vector to the edge $e_{i j}$ with vertices $\boldsymbol{x}_{i}$ and $\boldsymbol{x}_{j}$, and similarly define other tangential vectors with different subscripts. For ease of presentation, let

$$
\begin{gathered}
M_{e_{i j}}(\boldsymbol{v}):=\frac{1}{\left|e_{i j}\right|} \int_{e_{i j}} \boldsymbol{v} \cdot \boldsymbol{t}_{i j} \mathrm{~d} s, \quad \boldsymbol{M}_{F_{l}}(\boldsymbol{v}):=\int_{F_{l}}(\operatorname{curl} \boldsymbol{v}) \times \boldsymbol{n}_{l} \mathrm{~d} s, \\
M_{F_{l}, 1}(\boldsymbol{v}):=\frac{1}{2\left|F_{l}\right|\left(\nabla \lambda_{l_{1}} \times \nabla \lambda_{l_{2}}\right) \cdot \boldsymbol{n}_{l}} \int_{F_{l}}(\operatorname{curl} \boldsymbol{v}) \cdot\left(\left(\boldsymbol{n}_{l} \times \nabla \lambda_{l_{2}}\right) \times \boldsymbol{n}_{l}\right) \mathrm{d} s, \\
M_{F_{l}, 2}(\boldsymbol{v}):=\frac{1}{2\left|F_{l}\right|\left(\nabla \lambda_{l_{2}} \times \nabla \lambda_{l_{1}}\right) \cdot \boldsymbol{n}_{l}} \int_{F_{l}}(\operatorname{curl} \boldsymbol{v}) \cdot\left(\left(\boldsymbol{n}_{l} \times \nabla \lambda_{l_{1}}\right) \times \boldsymbol{n}_{l}\right) \mathrm{d} s .
\end{gathered}
$$

The degrees of freedom $M_{F_{l}, 1}(\boldsymbol{v})$ and $M_{F_{l}, 2}(\boldsymbol{v})$ are equivalent to $\boldsymbol{M}_{F_{l}}(\boldsymbol{v})$, i.e. (2.3).
2.3.1. Basis functions corresponding to the face degrees of freedom. Define

$$
\begin{aligned}
\boldsymbol{\varphi}_{F_{l}, i}:= & \frac{1}{4}\left(8 \lambda_{l}-3\right)\left(\boldsymbol{x}-\boldsymbol{x}_{K}\right) \times\left(\boldsymbol{n}_{l} \times \nabla \lambda_{l_{i}}\right)+\frac{1}{4}\left(\boldsymbol{x}_{l}-\boldsymbol{x}_{K}\right) \cdot \boldsymbol{n}_{l} \nabla \lambda_{l_{i}}+\frac{1}{16} \boldsymbol{n}_{l} \\
= & \frac{1}{16}\left(8 \lambda_{l}-3\right)\left[\left(4 \lambda_{l_{i}}-1\right) \boldsymbol{n}_{l}-4\left(\boldsymbol{x}-\boldsymbol{x}_{K}\right) \cdot \boldsymbol{n}_{l} \nabla \lambda_{l_{i}}\right] \\
& +\frac{1}{4}\left(\boldsymbol{x}_{l}-\boldsymbol{x}_{K}\right) \cdot \boldsymbol{n}_{l} \nabla \lambda_{l_{i}}+\frac{1}{16} \boldsymbol{n}_{l}
\end{aligned}
$$

for $i=1,2$. We will show that $\boldsymbol{\varphi}_{F_{l}, 1}$ and $\boldsymbol{\varphi}_{F_{l}, 2}$ are the basis functions being dual to $M_{F_{l}, 1}(\boldsymbol{v})$ and $M_{F_{l}, 2}(\boldsymbol{v})$.

Lemma 2.4. Functions $\boldsymbol{\varphi}_{F_{l}, 1}$ and $\boldsymbol{\varphi}_{F_{l}, 2}$ are the basis functions of $\boldsymbol{W}_{0}(K)$ being dual to $M_{F_{l}, 1}(\boldsymbol{v})$ and $M_{F_{l}, 2}(\boldsymbol{v})$, respecrtively. That is

$$
\begin{gather*}
M_{e}\left(\boldsymbol{\varphi}_{F_{l}, 1}\right)=M_{e}\left(\boldsymbol{\varphi}_{F_{l}, 2}\right)=0 \quad \forall e \in \mathcal{E}(K)  \tag{2.11}\\
\boldsymbol{M}_{F}\left(\boldsymbol{\varphi}_{F_{l}, 1}\right)=\boldsymbol{M}_{F}\left(\boldsymbol{\varphi}_{F_{l}, 2}\right)=\mathbf{0} \quad \forall F \in \mathcal{F}(K) \backslash\left\{F_{l}\right\}  \tag{2.12}\\
M_{F_{l}, 2}\left(\boldsymbol{\varphi}_{F_{l}, 1}\right)=M_{F_{l}, 1}\left(\boldsymbol{\varphi}_{F_{l}, 2}\right)=0, \quad M_{F_{l}, 1}\left(\boldsymbol{\varphi}_{F_{l}, 1}\right)=M_{F_{l}, 2}\left(\boldsymbol{\varphi}_{F_{l}, 2}\right)=1 \tag{2.13}
\end{gather*}
$$

Proof. Apparently $\boldsymbol{\varphi}_{F_{l}, 1} \cdot \boldsymbol{t}_{l_{2} l_{3}}=0$. By $\boldsymbol{n}_{l} \cdot \boldsymbol{t}_{l_{1} l_{2}}=0, \nabla \lambda_{l_{1}} \cdot \boldsymbol{t}_{l_{1} l_{2}}=-1$ and $\left.\lambda_{l}\right|_{e_{l_{1} l_{2}}}=0$, we get

$$
\begin{aligned}
M_{e_{l_{1} l_{2}}}\left(\boldsymbol{\varphi}_{F_{l}, 1}\right) & =\frac{1}{4\left|e_{l_{1} l_{2}}\right|} \int_{e_{l_{1} l_{2}}}\left(\left(8 \lambda_{l}-3\right)\left(\boldsymbol{x}-\boldsymbol{x}_{K}\right) \cdot \boldsymbol{n}_{l}-\left(\boldsymbol{x}_{l}-\boldsymbol{x}_{K}\right) \cdot \boldsymbol{n}_{l}\right) \mathrm{d} s \\
& =-\frac{3}{4}\left(\boldsymbol{x}_{l_{1}}-\boldsymbol{x}_{K}\right) \cdot \boldsymbol{n}_{l}-\frac{1}{4}\left(\boldsymbol{x}_{l}-\boldsymbol{x}_{K}\right) \cdot \boldsymbol{n}_{l}
\end{aligned}
$$

Noting that $\boldsymbol{x}_{l}-\boldsymbol{x}_{K}+3\left(\boldsymbol{x}_{l_{1}}-\boldsymbol{x}_{K}\right)=\boldsymbol{x}_{l}+3 \boldsymbol{x}_{l_{1}}-4 \boldsymbol{x}_{K}=2 \boldsymbol{x}_{l_{1}}-\boldsymbol{x}_{l_{2}}-\boldsymbol{x}_{l_{3}}$ is parallel to face $F_{l}$, we have $\left(\boldsymbol{x}_{l}-\boldsymbol{x}_{K}+3\left(\boldsymbol{x}_{l_{1}}-\boldsymbol{x}_{K}\right)\right) \cdot \boldsymbol{n}_{l}=0$. Hence

$$
M_{e_{l_{1} l_{2}}}\left(\varphi_{F_{l}, 1}\right)=0
$$

Since $\boldsymbol{n}_{l}=\frac{\boldsymbol{n}_{l} \cdot \nabla \lambda_{l}}{\left|\nabla \lambda_{l}\right|^{2}} \nabla \lambda_{l}, 4\left(\boldsymbol{x}-\boldsymbol{x}_{K}\right) \cdot \nabla \lambda_{l}=4 \lambda_{l}-1$ and $\left.\left(\lambda_{l_{1}}+\lambda_{l}\right)\right|_{e_{l_{1} l}}=1$, it follows

$$
\begin{aligned}
M_{e_{l_{1} l}}\left(\boldsymbol{\varphi}_{F_{l}, 1}\right) & =\frac{\boldsymbol{n}_{l} \cdot \nabla \lambda_{l}}{8\left|e_{l_{1} l}\right|\left|\nabla \lambda_{l}\right|^{2}} \int_{e_{l_{1} l}}\left(8 \lambda_{l}-3\right) \mathrm{d} s+\frac{\boldsymbol{n}_{l} \cdot \nabla \lambda_{l}}{16\left|\nabla \lambda_{l}\right|^{2}}\left(1-4\left(\boldsymbol{x}_{l}-\boldsymbol{x}_{K}\right) \cdot \nabla \lambda_{l}\right) \\
& =\frac{\boldsymbol{n}_{l} \cdot \nabla \lambda_{l}}{16\left|\nabla \lambda_{l}\right|^{2}}\left(3-4\left(\boldsymbol{x}_{l}-\boldsymbol{x}_{K}\right) \cdot \nabla \lambda_{l}\right)=0
\end{aligned}
$$

Similarly we can show that $M_{e}\left(\boldsymbol{\varphi}_{F_{l}, 1}\right)=0$ for other edges and $M_{e}\left(\boldsymbol{\varphi}_{F_{l}, 2}\right)=0$. Hence (2.11) holds.

On the other hand, by the identity $\operatorname{curl}\left(\left(\boldsymbol{x}-\boldsymbol{x}_{K}\right) \times \boldsymbol{q}\right)=\left(\boldsymbol{x}-\boldsymbol{x}_{K}\right) \operatorname{div} \boldsymbol{q}-((\boldsymbol{x}-$ $\left.\left.\boldsymbol{x}_{K}\right) \cdot \nabla\right) \boldsymbol{q}-2 \boldsymbol{q}$, we have for $i=1,2$,

$$
\operatorname{curl} \boldsymbol{\varphi}_{F_{l}, i}=\frac{1}{4} \operatorname{curl}\left(\left(8 \lambda_{l}-3\right)\left(\boldsymbol{x}-\boldsymbol{x}_{K}\right) \times\left(\boldsymbol{n}_{l} \times \nabla \lambda_{l_{i}}\right)\right)=2\left(1-3 \lambda_{l}\right) \boldsymbol{n}_{l} \times \nabla \lambda_{l_{i}} .
$$

We conclude (2.12)-(2.13) by the fact that $1-3 \lambda_{l}$ is the basis function of the nonconforming $P_{1}$ element.
2.3.2. Basis functions corresponding to the edge degrees of freedom. Next we construct the basis function corresponding to the degree of freedom $M_{e_{i j}}(\boldsymbol{v})$. Recall the basis function of the lowest order Nédélec element of the first kind $\lambda_{i} \nabla \lambda_{j}-\lambda_{j} \nabla \lambda_{i}$. Thanks to (2.11)-(2.12), function $\lambda_{i} \nabla \lambda_{j}-\lambda_{j} \nabla \lambda_{i}$ can be modified by $\varphi_{F_{l}, 1}$ and $\boldsymbol{\varphi}_{F_{l}, 2}$ to derive the basis function of $\boldsymbol{W}_{0}(K)$ corresponding to the degree of free$\operatorname{dom} M_{e_{i j}}(\boldsymbol{v})$.

Lemma 2.5. Let

$$
\boldsymbol{\varphi}_{e_{i j}}:=\lambda_{i} \nabla \lambda_{j}-\lambda_{j} \nabla \lambda_{i}+\sum_{l=1}^{4}\left(c_{l, 1}^{i j} \boldsymbol{\varphi}_{F_{l}, 1}+c_{l, 2}^{i j} \boldsymbol{\varphi}_{F_{l}, 2}\right)
$$

with constants

$$
\begin{aligned}
c_{l, 1}^{i j} & :=\frac{1}{\left(\nabla \lambda_{l_{1}} \times \nabla \lambda_{l_{2}}\right) \cdot \boldsymbol{n}_{l}}\left(\nabla \lambda_{j} \times \nabla \lambda_{i}\right) \cdot\left(\left(\boldsymbol{n}_{l} \times \nabla \lambda_{l_{2}}\right) \times \boldsymbol{n}_{l}\right), \\
c_{l, 2}^{i j} & :=\frac{1}{\left(\nabla \lambda_{l_{2}} \times \nabla \lambda_{l_{1}}\right) \cdot \boldsymbol{n}_{l}}\left(\nabla \lambda_{j} \times \nabla \lambda_{i}\right) \cdot\left(\left(\boldsymbol{n}_{l} \times \nabla \lambda_{l_{1}}\right) \times \boldsymbol{n}_{l}\right) .
\end{aligned}
$$

Then

$$
M_{e_{i j}}\left(\boldsymbol{\varphi}_{e_{i j}}\right)=1, \quad M_{e}\left(\boldsymbol{\varphi}_{e_{i j}}\right)=0, \quad \boldsymbol{M}_{F}\left(\boldsymbol{\varphi}_{e_{i j}}\right)=\mathbf{0} .
$$

for each $e \in \mathcal{E}(K) \backslash\left\{e_{i j}\right\}$ and $F \in \mathcal{F}(K)$.
Proof. The identities $M_{e_{i j}}\left(\boldsymbol{\varphi}_{e_{i j}}\right)=1$ and $M_{e}\left(\boldsymbol{\varphi}_{e_{i j}}\right)=0$ follow from (2.11) and the fact

$$
M_{e_{i j}}\left(\lambda_{i} \nabla \lambda_{j}-\lambda_{j} \nabla \lambda_{i}\right)=1, \quad M_{e}\left(\lambda_{i} \nabla \lambda_{j}-\lambda_{j} \nabla \lambda_{i}\right)=0 \quad \forall e \in \mathcal{E}(K) \backslash\left\{e_{i j}\right\} .
$$

On the other hand, we get from (2.12)-(2.13) and $\operatorname{curl}\left(\lambda_{i} \nabla \lambda_{j}-\lambda_{j} \nabla \lambda_{i}\right)=2 \nabla \lambda_{i} \times \nabla \lambda_{j}$ that

$$
M_{F_{l}, r}\left(\varphi_{e_{i j}}\right)=M_{F_{l}, r}\left(\lambda_{i} \nabla \lambda_{j}-\lambda_{j} \nabla \lambda_{i}\right)+c_{l, r}^{i j}=0
$$

for $r=1,2$.
In summary, we arrive at the basis functions being dual to the degrees of freedom $M_{F_{l}, 1}(\boldsymbol{v}), M_{F_{l}, 2}(\boldsymbol{v})$ and $M_{e_{i j}}(\boldsymbol{v}):$
(1) Two basis functions on each face $F_{l}(1 \leq l \leq 4)$

$$
\boldsymbol{\varphi}_{F_{l}, i}=\frac{1}{4}\left(8 \lambda_{l}-3\right)\left(\boldsymbol{x}-\boldsymbol{x}_{K}\right) \times\left(\boldsymbol{n}_{l} \times \nabla \lambda_{l_{i}}\right)+\frac{1}{4}\left(\boldsymbol{x}_{l}-\boldsymbol{x}_{K}\right) \cdot \boldsymbol{n}_{l} \nabla \lambda_{l_{i}}+\frac{1}{16} \boldsymbol{n}_{l}
$$

for $i=1,2$, where $\boldsymbol{x}_{K}$ is the barycenter of $K$.
(2) One basis function on each edge $e_{i j}(1 \leq i<j \leq 4)$

$$
\boldsymbol{\varphi}_{e_{i j}}=\lambda_{i} \nabla \lambda_{j}-\lambda_{j} \nabla \lambda_{i}+\sum_{l=1}^{4}\left(c_{l, 1}^{i j} \boldsymbol{\varphi}_{F_{l}, 1}+c_{l, 2}^{i j} \boldsymbol{\varphi}_{F_{l}, 2}\right)
$$

with constants

$$
\begin{aligned}
& c_{l, 1}^{i j}:=\frac{1}{\left(\nabla \lambda_{l_{1}} \times \nabla \lambda_{l_{2}}\right) \cdot \boldsymbol{n}_{l}}\left(\nabla \lambda_{j} \times \nabla \lambda_{i}\right) \cdot\left(\left(\boldsymbol{n}_{l} \times \nabla \lambda_{l_{2}}\right) \times \boldsymbol{n}_{l}\right), \\
& c_{l, 2}^{i j}:=\frac{1}{\left(\nabla \lambda_{l_{2}} \times \nabla \lambda_{l_{1}}\right) \cdot \boldsymbol{n}_{l}}\left(\nabla \lambda_{j} \times \nabla \lambda_{i}\right) \cdot\left(\left(\boldsymbol{n}_{l} \times \nabla \lambda_{l_{1}}\right) \times \boldsymbol{n}_{l}\right) .
\end{aligned}
$$

## 3. Nonconforming finite element Stokes complexes

We will consider the nonconforming finite element discretization of the Stokes complex (1.1) in this section. The homogeneous version of the Stokes complex (1.1) is

$$
0 \xrightarrow{\subset} H_{0}^{1}(\Omega) \xrightarrow{\nabla} \boldsymbol{H}_{0}(\operatorname{grad} \operatorname{curl}, \Omega) \xrightarrow{\text { curl }} \boldsymbol{H}_{0}^{1}\left(\Omega ; \mathbb{R}^{3}\right) \xrightarrow{\text { div }} L_{0}^{2}(\Omega) \rightarrow 0,
$$

where $\boldsymbol{H}_{0}(\operatorname{grad} \operatorname{curl}, \Omega):=\left\{\boldsymbol{v} \in \boldsymbol{H}_{0}(\operatorname{curl}, \Omega): \operatorname{curl} \boldsymbol{v} \in \boldsymbol{H}_{0}^{1}\left(\Omega ; \mathbb{R}^{3}\right)\right\}$. Since $\boldsymbol{H}_{0}(\operatorname{curl}, \Omega) \cap$ $\boldsymbol{H}_{0}(\operatorname{div}, \Omega)=\boldsymbol{H}_{0}^{1}\left(\Omega ; \mathbb{R}^{3}\right)$, it holds $\boldsymbol{H}_{0}(\operatorname{grad} \operatorname{curl}, \Omega)=\boldsymbol{H}_{0}\left(\operatorname{curl}^{2}, \Omega\right)$, where

$$
\boldsymbol{H}_{0}\left(\operatorname{curl}^{2}, \Omega\right):=\left\{\boldsymbol{v} \in \boldsymbol{H}_{0}(\operatorname{curl}, \Omega): \operatorname{curl} \boldsymbol{v} \in \boldsymbol{H}_{0}(\operatorname{curl}, \Omega)\right\} .
$$

We can use the Lagrange element, the nonconforming linear element and piecewise constant to discretize $H^{1}(\Omega), \boldsymbol{H}^{1}\left(\Omega ; \mathbb{R}^{3}\right)$ and $L^{2}(\Omega)$ in the Stokes complex (1.1), respectively. Take the Lagrange element space

$$
V_{h}^{g}:=\left\{v_{h} \in H^{1}(\Omega):\left.v_{h}\right|_{K} \in \mathbb{P}_{k+1}(K) \text { for each } K \in \mathcal{T}_{h}\right\}
$$

with $k=0,1$, the nonconforming linear element space

$$
\begin{aligned}
& \boldsymbol{V}_{h}^{s}:=\left\{\boldsymbol{v}_{h} \in \boldsymbol{L}^{2}\left(\Omega ; \mathbb{R}^{3}\right):\left.\boldsymbol{v}_{h}\right|_{K} \in \mathbb{P}_{1}\left(K ; \mathbb{R}^{3}\right) \text { for each } K \in \mathcal{T}_{h}\right. \\
&\text { and } \left.\int_{F} \llbracket \boldsymbol{v}_{h} \rrbracket \mathrm{~d} s=\mathbf{0} \text { for each } F \in \mathcal{F}_{h}^{i}\right\}
\end{aligned}
$$

and the piecewise constant space

$$
\mathcal{Q}_{h}:=\left\{q_{h} \in L^{2}(\Omega):\left.q_{h}\right|_{K} \in \mathbb{P}_{0}(K) \text { for each } K \in \mathcal{T}_{h}\right\}
$$

Here $\llbracket \boldsymbol{v}_{h} \rrbracket$ is the jump of $\boldsymbol{v}_{h}$ across $F$. Define the global $\boldsymbol{H}$ (grad curl)-nonconforming element space
$\boldsymbol{W}_{h}:=\left\{\boldsymbol{v}_{h} \in \boldsymbol{L}^{2}\left(\Omega ; \mathbb{R}^{3}\right):\left.\boldsymbol{v}_{h}\right|_{K} \in \boldsymbol{W}_{k}(K)\right.$ for each $K \in \mathcal{T}_{h}$, and all the degrees of freedom (2.2)-(2.3) are single-valued $\}$.
According to the proof of Lemma 2.1, it holds

$$
\begin{equation*}
\int_{F} \llbracket \operatorname{curl} \boldsymbol{v}_{h} \rrbracket \mathrm{~d} s=\mathbf{0} \quad \forall \boldsymbol{v}_{h} \in \boldsymbol{W}_{h}, F \in \mathcal{F}_{h}^{i} \tag{3.1}
\end{equation*}
$$

To prove the exactness of the nonconforming discrete Stokes complexes, we need the help of the Nédélec element spaces [33, 34]

$$
\boldsymbol{V}_{h}^{c}:=\left\{\boldsymbol{v}_{h} \in \boldsymbol{H}(\operatorname{curl}, \Omega):\left.\boldsymbol{v}_{h}\right|_{K} \in \boldsymbol{V}_{k}^{c}(K) \text { for each } K \in \mathcal{T}_{h}\right\}
$$

where $\boldsymbol{V}_{k}^{c}(K):=\mathbb{P}_{k}\left(K ; \mathbb{R}^{3}\right)+\left(\boldsymbol{x}-\boldsymbol{x}_{K}\right) \times \mathbb{P}_{0}\left(K ; \mathbb{R}^{3}\right)$ with $k=0,1$. Apparently $\boldsymbol{V}_{k}^{c}(K) \subset \boldsymbol{W}_{k}(K)$. The degrees of freedom for $\boldsymbol{V}_{k}^{c}(K)$ are

$$
\begin{equation*}
\int_{e} \boldsymbol{v} \cdot \boldsymbol{t}_{e} q \mathrm{~d} s \quad \forall q \in \mathbb{P}_{k}(e) \text { on each } e \in \mathcal{E}(K) \tag{3.2}
\end{equation*}
$$

It is observed that the degrees of freedom (3.2) are exactly same as (2.2). By the finite element de Rham complexes [2, 3], we have

$$
\begin{equation*}
\boldsymbol{V}_{h}^{c} \cap \operatorname{ker}(\text { curl })=\nabla V_{h}^{g} \tag{3.3}
\end{equation*}
$$

The notation $\operatorname{ker}(\mathcal{A})$ means the kernel space of the operator $\mathcal{A}$.
Lemma 3.1. It holds

$$
\boldsymbol{W}_{h} \cap \operatorname{ker}\left(\operatorname{curl}_{h}\right)=\nabla V_{h}^{g} .
$$

Proof. Since curl : $\left(\boldsymbol{x}-\boldsymbol{x}_{K}\right) \times \mathbb{P}_{1}\left(K ; \mathbb{R}^{3}\right) \rightarrow \mathbb{P}_{1}\left(K ; \mathbb{R}^{3}\right)$ is injective $[2,3]$, we have

$$
\begin{aligned}
\boldsymbol{W}_{h} \cap \operatorname{ker}\left(\operatorname{curl}_{h}\right) & =\left\{\boldsymbol{v}_{h} \in \boldsymbol{W}_{h}:\left.\boldsymbol{v}_{h}\right|_{K} \in \nabla \mathbb{P}_{k+1}(K) \text { for each } K \in \mathcal{T}_{h}\right\}, \\
\boldsymbol{V}_{h}^{c} \cap \operatorname{ker}\left(\operatorname{curl}_{h}\right) & =\left\{\boldsymbol{v}_{h} \in \boldsymbol{V}_{h}^{c}:\left.\boldsymbol{v}_{h}\right|_{K} \in \nabla \mathbb{P}_{k+1}(K) \text { for each } K \in \mathcal{T}_{h}\right\} .
\end{aligned}
$$

Noting that the degrees of freedom (2.2) and (3.2) are the same, it follows $\boldsymbol{W}_{h} \cap$ $\operatorname{ker}\left(\operatorname{curl}_{h}\right)=\boldsymbol{V}_{h}^{c} \cap \operatorname{ker}\left(\operatorname{curl}_{h}\right)$. Thus we finish the proof from (3.3).

Lemma 3.2. The nonconforming discrete Stokes complex

$$
\begin{equation*}
\mathbb{R} \xrightarrow{\subset} V_{h}^{g} \xrightarrow{\nabla} \boldsymbol{W}_{h} \xrightarrow{\operatorname{curl}_{h}} \boldsymbol{V}_{h}^{s} \xrightarrow{\operatorname{div}_{h}} \mathcal{Q}_{h} \rightarrow 0 \tag{3.4}
\end{equation*}
$$

is exact.
Proof. We refer to $[16,6]$ for $\operatorname{div}_{h} \boldsymbol{V}_{h}^{s}=\mathcal{Q}_{h}$ and Lemma 3.1 for $\boldsymbol{W}_{h} \cap \operatorname{ker}\left(\operatorname{curl}_{h}\right)=$ $\nabla V_{h}^{g}$. By the definition of $\boldsymbol{W}_{h}$, apparently we have from (3.1) that

$$
\operatorname{curl}_{h} \boldsymbol{W}_{h} \subseteq \boldsymbol{V}_{h}^{s} \cap \operatorname{ker}\left(\operatorname{div}_{h}\right)
$$

Then we prove

$$
\operatorname{curl}_{h} \boldsymbol{W}_{h}=\boldsymbol{V}_{h}^{s} \cap \operatorname{ker}\left(\operatorname{div}_{h}\right)
$$

by counting the dimensions of these spaces. Indeed, we have

$$
\begin{aligned}
\operatorname{dim} \operatorname{curl}_{h} \boldsymbol{W}_{h} & =\operatorname{dim} \boldsymbol{W}_{h}-\operatorname{dim} V_{h}^{g}+1 \\
& =(k+1) \# \mathcal{E}_{h}+2 \# \mathcal{F}_{h}-\# \mathcal{V}_{h}-k \# \mathcal{E}_{h}+1 \\
& =\# \mathcal{E}_{h}+2 \# \mathcal{F}_{h}-\# \mathcal{V}_{h}+1
\end{aligned}
$$

$$
\operatorname{dim} \boldsymbol{V}_{h}^{s} \cap \operatorname{ker}\left(\operatorname{div}_{h}\right)=\operatorname{dim} \boldsymbol{V}_{h}^{s}-\operatorname{dim} \mathcal{Q}_{h}=3 \# \mathcal{F}_{h}-\# \mathcal{T}_{h}
$$

Finally apply the Euler's formula $\# \mathcal{V}_{h}-\# \mathcal{E}_{h}+\# \mathcal{F}_{h}-\# \mathcal{T}_{h}=1$ to end the proof.
Corollary 3.3. The nonconforming discrete Stokes complex with homogeneous boundary condition

$$
\begin{equation*}
0 \xrightarrow{\subset} V_{h 0}^{g} \xrightarrow{\nabla} \boldsymbol{W}_{h 0} \xrightarrow{\text { curl }_{h}} \boldsymbol{V}_{h 0}^{s} \xrightarrow{\operatorname{div}_{h}} \mathcal{Q}_{h 0} \rightarrow 0 \tag{3.5}
\end{equation*}
$$

is exact, where $V_{h 0}^{g}:=V_{h}^{g} \cap H_{0}^{1}(\Omega), \mathcal{Q}_{h 0}:=\mathcal{Q}_{h} \cap L_{0}^{2}(\Omega)$, and

$$
\begin{aligned}
\boldsymbol{W}_{h 0}:= & \left\{\boldsymbol{v}_{h} \in \boldsymbol{W}_{h}: \text { all the degrees of freedom (2.2)-(2.3) on } \partial \Omega \text { vanish }\right\}, \\
& \boldsymbol{V}_{h 0}^{s}:=\left\{\boldsymbol{v}_{h} \in \boldsymbol{V}_{h}^{s}: \int_{F} \boldsymbol{v}_{h} \mathrm{~d} s=\mathbf{0} \text { for each } F \in \mathcal{F}_{h} \backslash \mathcal{F}_{h}^{i}\right\} .
\end{aligned}
$$

The space $\boldsymbol{V}_{h 0}^{s}$ possesses the norm equivalence [7, Section 10.6]

$$
\begin{equation*}
\left\|\boldsymbol{v}_{h}\right\|_{1, h} \bar{\sim}\left|\boldsymbol{v}_{h}\right|_{1, h} \quad \forall \boldsymbol{v}_{h} \in \boldsymbol{V}_{h 0}^{s} \tag{3.6}
\end{equation*}
$$

Equip $\boldsymbol{W}_{h 0}$ with the discrete squared norm

$$
\left\|\boldsymbol{v}_{h}\right\|_{H_{h}(\operatorname{grad} \operatorname{curl})}^{2}:=\left\|\boldsymbol{v}_{h}\right\|_{0}^{2}+\left\|\operatorname{curl}_{h} \boldsymbol{v}_{h}\right\|_{0}^{2}+\left|\operatorname{curl}_{h} \boldsymbol{v}_{h}\right|_{1, h}^{2}
$$

Since $\operatorname{curl}_{h} \boldsymbol{v}_{h} \in \boldsymbol{V}_{h 0}^{s}$ for any $\boldsymbol{v}_{h} \in \boldsymbol{W}_{h 0}$, applying (3.6) to $\operatorname{curl}_{h} \boldsymbol{v}_{h}$ gives

$$
\left\|\boldsymbol{v}_{h}\right\|_{H_{h}(\operatorname{grad} \operatorname{curl})} \bar{\sim}\left\|\boldsymbol{v}_{h}\right\|_{0}+\left|\operatorname{curl}_{h} \boldsymbol{v}_{h}\right|_{1, h} \quad \forall \boldsymbol{v}_{h} \in \boldsymbol{W}_{h 0}
$$

Next we focus on the commutative diagrams for the Stokes complexes (3.4) and (3.5). For this, we introduce some interpolation operators. For each $K \in \mathcal{T}_{h}$, let $I_{K}^{g}: H^{2}(K) \rightarrow \mathbb{P}_{k+1}(K)$ be the nodal interpolation operator of the Lagrange element [14], and $\boldsymbol{I}_{K}^{s}: \boldsymbol{H}^{1}\left(K ; \mathbb{R}^{3}\right) \rightarrow \mathbb{P}_{1}\left(K ; \mathbb{R}^{3}\right)$ be the nodal interpolation operator of the nonconforming linear element [7]. We have [6]

$$
\begin{equation*}
\operatorname{div}\left(\boldsymbol{I}_{K}^{s} \boldsymbol{v}\right)=Q_{K} \operatorname{div} \boldsymbol{v} \quad \forall \boldsymbol{v} \in \boldsymbol{H}^{1}\left(K ; \mathbb{R}^{3}\right) \tag{3.7}
\end{equation*}
$$

Define $\boldsymbol{I}_{K}^{g c}: \boldsymbol{H}^{1}(\operatorname{curl}, K) \rightarrow \boldsymbol{W}_{k}(K)$ as the nodal interpolation operator based on the degrees of freedom (2.2)-(2.3). By Lemma 2.1, we get

$$
\begin{equation*}
\boldsymbol{I}_{K}^{g c} \boldsymbol{q}=\boldsymbol{q} \quad \forall \boldsymbol{q} \in \boldsymbol{W}_{k}(K) . \tag{3.9}
\end{equation*}
$$

Lemma 3.4. It holds

$$
\begin{equation*}
\left\|\boldsymbol{v}-\boldsymbol{I}_{K}^{g c} \boldsymbol{v}\right\|_{0, K} \lesssim h_{K}^{k+1}|\boldsymbol{v}|_{k+1, K}+h_{K}^{2}|\boldsymbol{v}|_{2, K} \quad \forall \boldsymbol{v} \in \boldsymbol{H}^{2}\left(K ; \mathbb{R}^{3}\right) \tag{3.10}
\end{equation*}
$$

Proof. Set $\boldsymbol{w}=\boldsymbol{v}-\boldsymbol{Q}_{K}^{k} \boldsymbol{v}$ for ease of presentation. By the norm equivalence (2.9) and the definition of $\boldsymbol{I}_{K}^{g c}$,

$$
\begin{aligned}
\left\|\boldsymbol{I}_{K}^{g c} \boldsymbol{w}\right\|_{0, K}^{2} & \lesssim h_{K}^{2} \sum_{e \in \mathcal{E}(K)}\left\|Q_{e}^{k}\left(\left(\boldsymbol{I}_{K}^{g c} \boldsymbol{w}\right) \cdot \boldsymbol{t}_{e}\right)\right\|_{0, e}^{2}+h_{K}^{3} \sum_{F \in \mathcal{F}(K)}\left\|\boldsymbol{Q}_{F}^{0}\left(\operatorname{curl}\left(\boldsymbol{I}_{K}^{g c} \boldsymbol{w}\right) \times \boldsymbol{n}\right)\right\|_{0, F}^{2} \\
& =h_{K}^{2} \sum_{e \in \mathcal{E}(K)}\left\|Q_{e}^{k}\left(\boldsymbol{w} \cdot \boldsymbol{t}_{e}\right)\right\|_{0, e}^{2}+h_{K}^{3} \sum_{F \in \mathcal{F}(K)}\left\|\boldsymbol{Q}_{F}^{0}((\operatorname{curl} \boldsymbol{w}) \times \boldsymbol{n})\right\|_{0, F}^{2} \\
& \leq h_{K}^{2} \sum_{e \in \mathcal{E}(K)}\|\boldsymbol{w}\|_{0, e}^{2}+h_{K}^{3} \sum_{F \in \mathcal{F}(K)}\|\operatorname{curl} \boldsymbol{w}\|_{0, F}^{2} .
\end{aligned}
$$

Then we obtain from (3.9), $\mathbb{P}_{k}\left(K ; \mathbb{R}^{3}\right) \subset \boldsymbol{W}_{k}(K)$ and the trace inequality that

$$
\begin{aligned}
\left\|\boldsymbol{v}-\boldsymbol{I}_{K}^{g c} \boldsymbol{v}\right\|_{0, K}^{2} & =\left\|\boldsymbol{w}-\boldsymbol{I}_{K}^{g c} \boldsymbol{w}\right\|_{0, K}^{2} \leq 2\|\boldsymbol{w}\|_{0, K}^{2}+2\left\|\boldsymbol{I}_{K}^{g c} \boldsymbol{w}\right\|_{0, K}^{2} \\
& \lesssim\|\boldsymbol{w}\|_{0, K}^{2}+h_{K}^{2} \sum_{e \in \mathcal{E}(K)}\|\boldsymbol{w}\|_{0, e}^{2}+h_{K}^{3} \sum_{F \in \mathcal{F}(K)}\|\operatorname{curl} \boldsymbol{w}\|_{0, F}^{2} \\
& \lesssim\|\boldsymbol{w}\|_{0, K}^{2}+h_{K} \sum_{F \in \mathcal{F}(K)}\left(\|\boldsymbol{w}\|_{0, F}^{2}+h_{K}^{2}|\boldsymbol{w}|_{1, F}^{2}+h_{K}^{2}\|\operatorname{curl} \boldsymbol{w}\|_{0, F}^{2}\right) \\
& \lesssim\|\boldsymbol{w}\|_{0, K}^{2}+h_{K}^{2}|\boldsymbol{w}|_{1, K}^{2}+h_{K}^{4}|\boldsymbol{w}|_{2, K}^{2} .
\end{aligned}
$$

Therefore the inequality (3.10) holds from the error estimate of $\boldsymbol{Q}_{K}^{k}$.
Lemma 3.5. The operators $I_{K}^{g}, \boldsymbol{I}_{K}^{g c}$ and $\boldsymbol{I}_{K}^{s}$ satisfy the following commuting properties

$$
\begin{equation*}
\nabla\left(I_{K}^{g} v\right)=\boldsymbol{I}_{K}^{g c}(\nabla v) \quad \forall v \in H^{2}(K) \tag{3.11}
\end{equation*}
$$

$$
\begin{equation*}
\operatorname{curl}\left(\boldsymbol{I}_{K}^{g c} \boldsymbol{v}\right)=\boldsymbol{I}_{K}^{s}(\operatorname{curl} \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{H}^{1}(\operatorname{curl}, K) . \tag{3.12}
\end{equation*}
$$

Proof. On each edge $e \in \mathcal{E}(K)$, it follows from the definitions of $I_{K}^{g}$ and $\boldsymbol{I}_{K}^{g c}$ that

$$
\int_{e}\left(\nabla\left(I_{K}^{g} v\right)-\boldsymbol{I}_{K}^{g c}(\nabla v)\right) \cdot \boldsymbol{t}_{e} q \mathrm{~d} s=\int_{e} \partial_{t_{e}}\left(I_{K}^{g} v-v\right) q \mathrm{~d} s=0 \quad \forall q \in \mathbb{P}_{k}(e)
$$

On each face $F \in \mathcal{F}(K)$, we have

$$
\int_{F} \operatorname{curl}\left(\nabla\left(I_{K}^{g} v\right)-\boldsymbol{I}_{K}^{g c}(\nabla v)\right) \times \boldsymbol{n} \mathrm{d} s=\int_{F} \operatorname{curl}\left(\nabla\left(I_{K}^{g} v-v\right)\right) \times \boldsymbol{n} \mathrm{d} s=\mathbf{0} .
$$

Hence (3.11) holds from $\nabla\left(I_{K}^{g} v\right)-\boldsymbol{I}_{K}^{g c}(\nabla v) \in \boldsymbol{W}_{k}(K)$.
On the other hand, we get from the Stokes formula that

$$
\begin{aligned}
& \int_{F}\left(\operatorname{curl}\left(\boldsymbol{I}_{K}^{g c} \boldsymbol{v}\right)-\boldsymbol{I}_{K}^{s}(\operatorname{curl} \boldsymbol{v})\right) \cdot \boldsymbol{n} \mathrm{d} s=\int_{F} \operatorname{curl}\left(\boldsymbol{I}_{K}^{g c} \boldsymbol{v}-\boldsymbol{v}\right) \cdot \boldsymbol{n} \mathrm{d} s \\
= & \int_{F}(\boldsymbol{n} \times \nabla) \cdot\left(\boldsymbol{I}_{K}^{g c} \boldsymbol{v}-\boldsymbol{v}\right) \mathrm{d} s=\int_{F} \boldsymbol{t}_{F, e} \cdot\left(\boldsymbol{I}_{K}^{g c} \boldsymbol{v}-\boldsymbol{v}\right) \mathrm{d} s=0 .
\end{aligned}
$$

And by the definitions of $\boldsymbol{I}_{K}^{g c}$ and $\boldsymbol{I}_{K}^{s}$,

$$
\int_{F}\left(\operatorname{curl}\left(\boldsymbol{I}_{K}^{g c} \boldsymbol{v}\right)-\boldsymbol{I}_{K}^{s}(\operatorname{curl} \boldsymbol{v})\right) \times \boldsymbol{n} \mathrm{d} s=\int_{F} \operatorname{curl}\left(\boldsymbol{I}_{K}^{g c} \boldsymbol{v}-\boldsymbol{v}\right) \times \boldsymbol{n} \mathrm{d} s=\mathbf{0} .
$$

Therefore (3.12) follows from the last two identities.
Now introduce the global version of $I_{K}^{g}, \boldsymbol{I}_{K}^{g c}, \boldsymbol{I}_{K}^{s}$ and $Q_{K}$. Let $I_{h}^{g}: H^{2}(\Omega) \rightarrow \boldsymbol{V}_{h}^{g}$, $\boldsymbol{I}_{h}^{g c}: \boldsymbol{H}^{1}(\operatorname{curl}, \Omega) \rightarrow \boldsymbol{W}_{h}, \boldsymbol{I}_{h}^{s}: \boldsymbol{H}^{1}\left(\Omega ; \mathbb{R}^{3}\right) \rightarrow \boldsymbol{V}_{h}^{s}$ and $I_{h}^{L^{2}}: L^{2}(\Omega) \rightarrow \mathcal{Q}_{h}$ be defined by $\left.\left(I_{h}^{g} v\right)\right|_{K}:=I_{K}^{g}\left(\left.v\right|_{K}\right),\left.\left(\boldsymbol{I}_{h}^{g c} \boldsymbol{v}\right)\right|_{K}:=\boldsymbol{I}_{K}^{g c}\left(\left.\boldsymbol{v}\right|_{K}\right),\left.\left(\boldsymbol{I}_{h}^{s} \boldsymbol{v}\right)\right|_{K}:=\boldsymbol{I}_{K}^{s}\left(\left.\boldsymbol{v}\right|_{K}\right)$ and $\left.\left(I_{h}^{L^{2}} v\right)\right|_{K}:=Q_{K}\left(\left.v\right|_{K}\right)$ for each $K \in \mathcal{T}_{h}$, respectively. As the direct result of (3.7), (3.11) and (3.12), we have

$$
\begin{array}{cl}
\nabla\left(I_{h}^{g} v\right)=\boldsymbol{I}_{h}^{g c}(\nabla v) & \forall v \in H^{2}(\Omega), \\
\operatorname{curl}_{h}\left(\boldsymbol{I}_{h}^{g c} \boldsymbol{v}\right)=\boldsymbol{I}_{h}^{s}(\operatorname{curl} \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{H}^{1}(\operatorname{curl}, \Omega), \\
\operatorname{div}_{h}\left(\boldsymbol{I}_{h}^{s} \boldsymbol{v}\right)=I_{h}^{L^{2}} \operatorname{div} \boldsymbol{v} \quad \forall \boldsymbol{v} \in \boldsymbol{H}^{1}\left(\Omega ; \mathbb{R}^{3}\right) \tag{3.15}
\end{array}
$$

Combining (3.13)-(3.15) and the complex (3.4) yields the commutative diagram

and the commutative diagram with homogeneous boundary conditions


## 4. Mixed finite element methods for the quad-curl problem

In this section, we will advance mixed finite element methods for the quad-curl problem

$$
\left\{\begin{align*}
(\operatorname{curl})^{4} \boldsymbol{u} & =\boldsymbol{f} & & \text { in } \Omega  \tag{4.1}\\
\operatorname{div} \boldsymbol{u} & =0 & & \text { in } \Omega \\
\boldsymbol{u} \times \boldsymbol{n}=(\operatorname{curl} \boldsymbol{u}) \times \boldsymbol{n} & =\mathbf{0} & & \text { on } \partial \Omega
\end{align*}\right.
$$

where $\boldsymbol{f} \in \boldsymbol{H}(\operatorname{div}, \Omega)$ with $\operatorname{div} \boldsymbol{f}=0$. The quad-curl problem arises in inverse electromagnetic scattering theory [8] and magnetohydrodynamics [44].

Due to the identity $\operatorname{curl}^{2} \boldsymbol{v}=-\Delta \boldsymbol{v}+\nabla(\operatorname{div} \boldsymbol{v})$ and the fact

$$
(\operatorname{curl} \boldsymbol{u}) \cdot \boldsymbol{n}=(\boldsymbol{n} \times \nabla) \cdot \boldsymbol{u}=(\boldsymbol{n} \times \nabla) \cdot(\boldsymbol{n} \times \boldsymbol{u} \times \boldsymbol{n})=0 \quad \text { on } \partial \Omega,
$$

the quad-curl problem (4.1) is equivalent to

$$
\left\{\begin{align*}
-\operatorname{curl} \Delta \operatorname{curl} \boldsymbol{u}=\boldsymbol{f} & \text { in } \Omega  \tag{4.2}\\
\operatorname{div} \boldsymbol{u}=0 & \text { in } \Omega \\
\boldsymbol{u} \times \boldsymbol{n}=\operatorname{curl} \boldsymbol{u}=\mathbf{0} & \text { on } \partial \Omega
\end{align*}\right.
$$

Then a mixed formulation of the quad-curl problem (4.1) is to find $\boldsymbol{u} \in \boldsymbol{H}_{0}(\operatorname{grad} \operatorname{curl}, \Omega)$ and $\lambda \in H_{0}^{1}(\Omega)$ such that

$$
\begin{align*}
(\nabla \operatorname{curl} \boldsymbol{u}, \nabla \operatorname{curl} \boldsymbol{v})+(\boldsymbol{v}, \nabla \lambda) & =(\boldsymbol{f}, \boldsymbol{v}) & & \forall \boldsymbol{v} \in \boldsymbol{H}_{0}(\operatorname{grad} \operatorname{curl}, \Omega)  \tag{4.3}\\
(\boldsymbol{u}, \nabla \mu) & =0 & & \forall \mu \in H_{0}^{1}(\Omega) \tag{4.4}
\end{align*}
$$

Replacing $\boldsymbol{v}$ in (4.3) with $\nabla \mu$ for any $\mu \in H_{0}^{1}(\Omega)$, we get $\lambda=0$ from the fact $\operatorname{div} \boldsymbol{f}=0$. Thus it follows from (4.3) that

$$
(\nabla \operatorname{curl} \boldsymbol{u}, \nabla \operatorname{curl} \boldsymbol{v})=(\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{H}_{0}(\operatorname{grad} \operatorname{curl}, \Omega)
$$

4.1. Mixed finite element methods. Based on the mixed formulation (4.3)(4.4), we propose the mixed finite element methods for the quad-curl problem (4.1) as follows: find $\boldsymbol{u}_{h} \in \boldsymbol{W}_{h 0}$ and $\lambda_{h} \in V_{h 0}^{g}$ such that

$$
\begin{align*}
\left(\nabla_{h} \operatorname{curl}_{h} \boldsymbol{u}_{h}, \nabla_{h} \operatorname{curl}_{h} \boldsymbol{v}_{h}\right)+\left(\boldsymbol{v}_{h}, \nabla \lambda_{h}\right) & =\left(\boldsymbol{f}, \boldsymbol{v}_{h}\right) & & \forall \boldsymbol{v}_{h} \in \boldsymbol{W}_{h 0},  \tag{4.5}\\
\left(\boldsymbol{u}_{h}, \nabla \mu_{h}\right) & =0 & & \forall \mu_{h} \in V_{h 0}^{g} . \tag{4.6}
\end{align*}
$$

Now we show the well-posedness of the mixed finite element methods (4.5)-(4.6) and the stability. To this end, we recall the discrete de Rham complex and the corresponding interpolation operators [2]. Based on the degrees of freedom (3.2), define $\boldsymbol{I}_{K}^{c}: \boldsymbol{H}^{2}\left(K ; \mathbb{R}^{3}\right) \rightarrow \boldsymbol{V}_{k}^{c}(K)$ for each $K \in \mathcal{T}_{h}$ by

$$
\int_{e} \boldsymbol{I}_{K}^{c} \boldsymbol{v} \cdot \boldsymbol{t}_{e} q \mathrm{~d} s=\int_{e} \boldsymbol{v} \cdot \boldsymbol{t}_{e} q \mathrm{~d} s \quad \forall \boldsymbol{v} \in \boldsymbol{H}^{2}\left(K ; \mathbb{R}^{3}\right), q \in \mathbb{P}_{k}(e), e \in \mathcal{E}(K)
$$

Then we have

$$
\begin{align*}
\boldsymbol{I}_{K}^{c} \boldsymbol{v} & =\boldsymbol{v} \quad \forall \boldsymbol{v} \in \boldsymbol{V}_{k}^{c}(K)  \tag{4.7}\\
\left\|\operatorname{curl}\left(\boldsymbol{v}-\boldsymbol{I}_{K}^{c} \boldsymbol{v}\right)\right\|_{0, K} & \lesssim h_{K}|\operatorname{curl} \boldsymbol{v}|_{1, K} \quad \forall \boldsymbol{v} \in \boldsymbol{H}^{2}\left(K ; \mathbb{R}^{3}\right)  \tag{4.8}\\
\left\|\boldsymbol{I}_{K}^{c} \boldsymbol{v}\right\|_{0, K} & \lesssim\|\boldsymbol{v}\|_{0, K} \quad \forall \boldsymbol{v} \in \boldsymbol{W}_{k}(K) \tag{4.9}
\end{align*}
$$

Let $\boldsymbol{I}_{h}^{c}: \boldsymbol{H}^{2}\left(\Omega ; \mathbb{R}^{3}\right)+\boldsymbol{W}_{h} \rightarrow \boldsymbol{V}_{h}^{c}$ be determined by

$$
\left.\left(\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right)\right|_{K}:=\boldsymbol{I}_{K}^{c}\left(\left.\boldsymbol{v}_{h}\right|_{K}\right) \quad \forall K \in \mathcal{T}_{h}
$$

The operator $\boldsymbol{I}_{h}^{c}$ is well-defined, since the degrees of freedom (2.2) for $\boldsymbol{W}_{k}(K)$ and (3.2) for $\boldsymbol{V}_{k}^{c}(K)$ are same. And we have $\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h} \in \boldsymbol{V}_{h 0}^{c}:=\boldsymbol{V}_{h}^{c} \cap \boldsymbol{H}_{0}(\operatorname{curl}, \Omega)$ when $\boldsymbol{v}_{h} \in \boldsymbol{W}_{h 0}^{c}$.

Let the lowest order Raviart-Thomas element space [33, 36]

$$
\boldsymbol{V}_{h 0}^{d}:=\left\{\boldsymbol{v}_{h} \in \boldsymbol{H}_{0}(\operatorname{div}, \Omega):\left.\boldsymbol{v}_{h}\right|_{K} \in \mathbb{P}_{0}\left(K ; \mathbb{R}^{3}\right)+\boldsymbol{x} \mathbb{P}_{0}(K) \text { for each } K \in \mathcal{T}_{h}\right\} .
$$

We have the discrete de Rham complex [2]

$$
0 \xrightarrow{\subset} V_{h 0}^{g} \xrightarrow{\nabla} \boldsymbol{V}_{h 0}^{c} \xrightarrow{\text { curl }} \boldsymbol{V}_{h 0}^{d} \xrightarrow{\text { div }} \mathcal{Q}_{h 0} \rightarrow 0 .
$$

Denote by $\boldsymbol{I}_{h}^{d}: \boldsymbol{H}_{0}^{1}\left(\Omega ; \mathbb{R}^{3}\right)+\boldsymbol{V}_{h 0}^{s} \rightarrow \boldsymbol{V}_{h 0}^{d}$ the nodal interpolation operator. Then the commutative diagram (3.16) can be extended to the following three-line commutative diagram

where $I$ is the identity operator.
Lemma 4.1. We have

$$
\begin{equation*}
\inf _{q \in \mathbb{P}_{k+1}(K)}\|\boldsymbol{v}-\nabla q\|_{0, K} \lesssim h_{K}\|\operatorname{curl} \boldsymbol{v}\|_{0, K} \quad \forall \boldsymbol{v} \in \boldsymbol{W}_{k}(K), K \in \mathcal{T}_{h} . \tag{4.11}
\end{equation*}
$$

Proof. Due to (2.5), $\boldsymbol{v}-\mathcal{K}_{K}(\operatorname{curl} \boldsymbol{v}) \in \boldsymbol{W}_{k}(K) \cap \operatorname{ker}(\operatorname{curl})$, which means $\boldsymbol{v}-$ $\mathcal{K}_{K}(\operatorname{curl} \boldsymbol{v}) \in \nabla \mathbb{P}_{k+1}(K)$. Choose $q \in \mathbb{P}_{k+1}(K)$ such that $\boldsymbol{v}-\mathcal{K}_{K}(\operatorname{curl} \boldsymbol{v})=\nabla q$. Apply (2.6) to get

$$
\|\boldsymbol{v}-\nabla q\|_{0, K}=\left\|\mathcal{K}_{K}(\operatorname{curl} \boldsymbol{v})\right\|_{0, K} \lesssim h_{K}\|\operatorname{curl} \boldsymbol{v}\|_{0, K}
$$

which indicates (4.11).
Lemma 4.2. It holds for any $K \in \mathcal{T}_{h}$ that

$$
\begin{equation*}
\left\|\boldsymbol{v}-\boldsymbol{I}_{K}^{c} \boldsymbol{v}\right\|_{0, K} \lesssim h_{K}\|\operatorname{curl} \boldsymbol{v}\|_{0, K} \quad \forall \boldsymbol{v} \in \boldsymbol{W}_{k}(K) . \tag{4.12}
\end{equation*}
$$

Proof. Employing (4.7) and $\nabla \mathbb{P}_{k+1}(K) \subset \boldsymbol{V}_{k}^{c}(K)$, it follows

$$
\boldsymbol{v}-\boldsymbol{I}_{K}^{c} \boldsymbol{v}=(\boldsymbol{v}-\nabla q)-\boldsymbol{I}_{K}^{c}(\boldsymbol{v}-\nabla q) \quad \forall q \in \mathbb{P}_{k+1}(K)
$$

Then we get from (4.9) that

$$
\left\|\boldsymbol{v}-\boldsymbol{I}_{K}^{c} \boldsymbol{v}\right\|_{0, K} \leq\|\boldsymbol{v}-\nabla q\|_{0, K}+\left\|\boldsymbol{I}_{K}^{c}(\boldsymbol{v}-\nabla q)\right\|_{0, K} \lesssim\|\boldsymbol{v}-\nabla q\|_{0, K}
$$

which together with the arbitrariness of $q \in \mathbb{P}_{k+1}(K)$ implies

$$
\left\|\boldsymbol{v}-\boldsymbol{I}_{K}^{c} \boldsymbol{v}\right\|_{0, K} \lesssim \inf _{q \in \mathbb{P}_{k+1}(K)}\|\boldsymbol{v}-\nabla q\|_{0, K}
$$

Thus the inequality (4.12) follows from (4.11).
Lemma 4.3. It holds the discrete Poincaré inequality

$$
\begin{equation*}
\left\|\boldsymbol{v}_{h}\right\|_{0} \lesssim\left\|\operatorname{curl}_{h} \boldsymbol{v}_{h}\right\|_{0} \quad \forall \boldsymbol{v}_{h} \in \mathcal{K}_{h}^{d} \tag{4.13}
\end{equation*}
$$

where $\mathcal{K}_{h}^{d}:=\left\{\boldsymbol{v}_{h} \in \boldsymbol{W}_{h 0}:\left(\boldsymbol{v}_{h}, \nabla q_{h}\right)=0\right.$ for each $\left.q_{h} \in V_{h 0}^{g}\right\}$.

Proof. By the fact that $\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h} \in \boldsymbol{H}_{0}(\operatorname{curl}, \Omega)$, there exists $\boldsymbol{\psi} \in \boldsymbol{H}_{0}^{1}\left(\Omega ; \mathbb{R}^{3}\right)$ such that (cf. $[20,1,15]$ )

$$
\begin{equation*}
\operatorname{curl} \boldsymbol{\psi}=\operatorname{curl}\left(\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right), \quad\|\boldsymbol{\psi}\|_{1} \lesssim\left\|\operatorname{curl}\left(\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right)\right\|_{0} \tag{4.14}
\end{equation*}
$$

Let $\widetilde{\boldsymbol{I}}_{h}^{c}: \boldsymbol{H}_{0}(\operatorname{curl}, \Omega) \rightarrow \boldsymbol{V}_{h 0}^{c}$ and $\widetilde{\boldsymbol{I}}_{h}^{d}: \boldsymbol{H}_{0}(\operatorname{div}, \Omega) \rightarrow \boldsymbol{V}_{h 0}^{d}$ be the $L^{2}$ bounded projection operators devised in [13]. The operators $\widetilde{\boldsymbol{I}}_{h}^{c}$ and $\widetilde{\boldsymbol{I}}_{h}^{d}$ possess the following properties

$$
\begin{equation*}
\operatorname{curl}\left(\widetilde{\boldsymbol{I}}_{h}^{c} \boldsymbol{v}\right)=\widetilde{\boldsymbol{I}}_{h}^{d}(\operatorname{curl} \boldsymbol{v}), \quad\left\|\widetilde{\boldsymbol{I}}_{h}^{c} \boldsymbol{v}\right\|_{0} \lesssim\|\boldsymbol{v}\|_{0} \quad \forall \boldsymbol{v} \in \boldsymbol{H}_{0}(\operatorname{curl}, \Omega) \tag{4.15}
\end{equation*}
$$

By the commuting property of $\widetilde{\boldsymbol{I}}_{h}^{c}$ and $\widetilde{\boldsymbol{I}}_{h}^{d}$, it follows

$$
\operatorname{curl}\left(\widetilde{\boldsymbol{I}}_{h}^{c} \boldsymbol{\psi}\right)=\widetilde{\boldsymbol{I}}_{h}^{d}(\operatorname{curl} \boldsymbol{\psi})=\widetilde{\boldsymbol{I}}_{h}^{d}\left(\operatorname{curl}\left(\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right)\right)=\operatorname{curl}\left(\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right)
$$

By (3.3), there exists $q_{h} \in V_{h 0}^{g}$ such that $\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}-\widetilde{\boldsymbol{I}}_{h}^{c} \boldsymbol{\psi}=\nabla q_{h}$. Because $\left(\boldsymbol{v}_{h}, \nabla q_{h}\right)=0$,

$$
\begin{aligned}
\left\|\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right\|_{0}^{2} & =\left(\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}, \boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}-\widetilde{\boldsymbol{I}}_{h}^{c} \boldsymbol{\psi}\right)+\left(\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}, \widetilde{\boldsymbol{I}}_{h}^{c} \boldsymbol{\psi}\right) \\
& =\left(\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}-\boldsymbol{v}_{h}, \boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}-\widetilde{\boldsymbol{I}}_{h}^{c} \boldsymbol{\psi}\right)+\left(\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}, \widetilde{\boldsymbol{I}}_{h}^{c} \boldsymbol{\psi}\right)
\end{aligned}
$$

Due to (4.15) and (4.14), we get

$$
\begin{aligned}
\left\|\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right\|_{0}^{2} \leq & \left\|\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}-\boldsymbol{v}_{h}\right\|_{0}\left\|\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}-\widetilde{\boldsymbol{I}}_{h}^{c} \boldsymbol{\psi}\right\|_{0}+\left\|\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right\|_{0}\left\|\widetilde{\boldsymbol{I}}_{h}^{c} \boldsymbol{\psi}\right\|_{0} \\
\lesssim & \left\|\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}-\boldsymbol{v}_{h}\right\|_{0}\left(\left\|\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right\|_{0}+\|\boldsymbol{\psi}\|_{1}\right)+\left\|\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right\|_{0}\|\boldsymbol{\psi}\|_{1} \\
\lesssim & \left\|\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}-\boldsymbol{v}_{h}\right\|_{0}\left(\left\|\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right\|_{0}+\left\|\operatorname{curl}\left(\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right)\right\|_{0}\right)+\left\|\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right\|_{0}\| \| \operatorname{curl}\left(\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right) \|_{0} \\
= & \left(\left\|\boldsymbol{v}_{h}-\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right\|_{0}+\left\|\operatorname{curl}\left(\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right)\right\|_{0}\right)\left\|\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right\|_{0} \\
& +\left\|\boldsymbol{v}_{h}-\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right\|_{0}\left\|\operatorname{curl}\left(\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right)\right\|_{0} .
\end{aligned}
$$

Thus we have

$$
\left\|\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right\|_{0} \lesssim\left\|\boldsymbol{v}_{h}-\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right\|_{0}+\left\|\operatorname{curl}\left(\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right)\right\|_{0}
$$

which indicates

$$
\begin{aligned}
\left\|\boldsymbol{v}_{h}\right\|_{0} & \lesssim\left\|\boldsymbol{v}_{h}-\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right\|_{0}+\left\|\operatorname{curl}\left(\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right)\right\|_{0} \\
& \lesssim\left\|\boldsymbol{v}_{h}-\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right\|_{0}+\left\|\operatorname{curl}\left(\boldsymbol{v}_{h}-\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right)\right\|_{0}+\left\|\operatorname{curl}_{h} \boldsymbol{v}_{h}\right\|_{0}
\end{aligned}
$$

Therefore (4.13) follows from (4.12), (4.8) and the inverse inequality.
Lemma 4.4. We have the discrete stability

$$
\begin{align*}
& \left\|\widetilde{\boldsymbol{u}}_{h}\right\|_{H_{h}(\operatorname{grad} \operatorname{curl})}+\left|\widetilde{\lambda}_{h}\right|_{1} \\
\lesssim & \sup _{\left(\boldsymbol{v}_{h}, \mu_{h}\right) \in \boldsymbol{W}_{h 0} \times V_{h 0}^{g}} \frac{\left(\nabla_{h} \operatorname{curl}_{h} \widetilde{\boldsymbol{u}}_{h}, \nabla_{h} \operatorname{curl}_{h} \boldsymbol{v}_{h}\right)+\left(\boldsymbol{v}_{h}, \nabla \widetilde{\lambda}_{h}\right)+\left(\widetilde{\boldsymbol{u}}_{h}, \nabla \mu_{h}\right)}{\left\|\boldsymbol{v}_{h}\right\|_{H_{h}(\text { grad curl })}+\left|\mu_{h}\right|_{1}} \tag{4.16}
\end{align*}
$$

for any $\widetilde{\boldsymbol{u}}_{h} \in \boldsymbol{W}_{h 0}$ and $\widetilde{\lambda}_{h} \in V_{h 0}^{g}$.
Proof. For any $\boldsymbol{v}_{h} \in \mathcal{K}_{h}^{d}$, by using (4.13) and (3.6), we derive the coercivity

$$
\left\|\boldsymbol{v}_{h}\right\|_{H_{h}(\operatorname{grad} \operatorname{curl})} \lesssim\left\|\operatorname{curl}_{h} \boldsymbol{v}_{h}\right\|_{1, h} \lesssim\left|\operatorname{curl}_{h} \boldsymbol{v}_{h}\right|_{1, h}
$$

Since $\nabla V_{h 0}^{g} \subset \boldsymbol{W}_{h 0}$, it holds the discrete inf-sup condition
$\left|\mu_{h}\right|_{1}=\sup _{\boldsymbol{v}_{h} \in \nabla V_{h 0}^{g}} \frac{\left(\boldsymbol{v}_{h}, \nabla \mu_{h}\right)}{\left\|\boldsymbol{v}_{h}\right\|_{0}}=\sup _{\boldsymbol{v}_{h} \in \nabla V_{h 0}^{g}} \frac{\left(\boldsymbol{v}_{h}, \nabla \mu_{h}\right)}{\left\|\boldsymbol{v}_{h}\right\|_{H_{h}(\text { grad curl })}} \leq \sup _{\boldsymbol{v}_{h} \in \boldsymbol{W}_{h 0}} \frac{\left(\boldsymbol{v}_{h}, \nabla \mu_{h}\right)}{\left\|\boldsymbol{v}_{h}\right\|_{H_{h}(\text { grad curl })}}$.
Thus the discrete stability (4.16) follows from the Babuška-Brezzi theory [6].

Thanks to the discrete stability (4.16), the mixed finite element method (4.5)(4.6) are well-posed. As the continuous case, replacing $\boldsymbol{v}_{h}$ in (4.5) with $\nabla \mu_{h}$ for any $\mu_{h} \in V_{h 0}^{g}$, we get $\lambda_{h}=0$ from the fact $\operatorname{div} \boldsymbol{f}=0$ again. As a result, the solution $\boldsymbol{u}_{h} \in \boldsymbol{W}_{h 0}$ satisfies

$$
\begin{equation*}
\left(\nabla_{h} \operatorname{curl}_{h} \boldsymbol{u}_{h}, \nabla_{h} \operatorname{curl}_{h} \boldsymbol{v}_{h}\right)=\left(\boldsymbol{f}, \boldsymbol{v}_{h}\right) \quad \forall \boldsymbol{v}_{h} \in \boldsymbol{W}_{h 0} . \tag{4.17}
\end{equation*}
$$

4.2. Interpolation operator with lower regularity. In this subsection we define an interpolation operator on $\boldsymbol{H}_{0}(\operatorname{grad} \operatorname{curl}, \Omega)$. Since the interpolation operator $\boldsymbol{I}_{h}^{g c}$ is not well-defined on $\boldsymbol{H}_{0}(\operatorname{grad} \operatorname{curl}, \Omega)$, we first present a regular decomposition for the space $\boldsymbol{H}_{0}(\operatorname{grad} \operatorname{curl}, \Omega)$.

Lemma 4.5. It holds the stable regular decomposition

$$
\begin{equation*}
\boldsymbol{H}_{0}(\operatorname{grad} \operatorname{curl}, \Omega)=\boldsymbol{H}_{0}^{2}\left(\Omega ; \mathbb{R}^{3}\right)+\nabla H_{0}^{1}(\Omega) \tag{4.18}
\end{equation*}
$$

Specifically, for any $\boldsymbol{v} \in \boldsymbol{H}_{0}(\operatorname{grad} \operatorname{curl}, \Omega)$, let $\boldsymbol{v}_{2} \in \boldsymbol{H}_{0}^{2}\left(\Omega ; \mathbb{R}^{3}\right)$ and $\boldsymbol{\lambda} \in \operatorname{curl} \boldsymbol{H}_{0}^{2}\left(\Omega ; \mathbb{R}^{3}\right)$ satisfy

$$
\left\{\begin{align*}
\left(\nabla^{2} \boldsymbol{v}_{2}, \nabla^{2} \boldsymbol{\chi}\right)+(\nabla \operatorname{curl} \boldsymbol{\chi}, \nabla \boldsymbol{\lambda}) & =0 & & \forall \boldsymbol{\chi} \in \boldsymbol{H}_{0}^{2}\left(\Omega ; \mathbb{R}^{3}\right)  \tag{4.19}\\
\left(\nabla \operatorname{curl} \boldsymbol{v}_{2}, \nabla \boldsymbol{\mu}\right) & =(\nabla \operatorname{curl} \boldsymbol{v}, \nabla \boldsymbol{\mu}) & & \forall \boldsymbol{\mu} \in \operatorname{curl} \boldsymbol{H}_{0}^{2}\left(\Omega ; \mathbb{R}^{3}\right)
\end{align*}\right.
$$

then there exists $v_{1} \in H_{0}^{1}(\Omega)$ such that $\boldsymbol{v}=\boldsymbol{v}_{2}+\nabla v_{1}$ and

$$
\begin{equation*}
\left\|\boldsymbol{v}_{2}\right\|_{2} \lesssim|\operatorname{curl} \boldsymbol{v}|_{1}, \quad\left\|v_{1}\right\|_{1} \lesssim\|\boldsymbol{v}\|_{0}+\left\|\boldsymbol{v}_{2}\right\|_{0} \tag{4.20}
\end{equation*}
$$

Proof. Recall the de Rham complex with homogeneous boundary conditions [15, the second part of Theorem 1.1]

$$
0 \xrightarrow{\subset} H_{0}^{3}(\Omega) \xrightarrow{\nabla} \boldsymbol{H}_{0}^{2}\left(\Omega ; \mathbb{R}^{3}\right) \xrightarrow{\text { curl }} \boldsymbol{H}_{0}^{1}\left(\Omega ; \mathbb{R}^{3}\right) \xrightarrow{\text { div }} L_{0}^{2}(\Omega) \rightarrow 0,
$$

which is exact for $\Omega$ being contractible. For $\boldsymbol{\mu} \in \operatorname{curl} \boldsymbol{H}_{0}^{2}\left(\Omega ; \mathbb{R}^{3}\right) \subset \boldsymbol{H}_{0}^{1}\left(\Omega ; \mathbb{R}^{3}\right)$, by this complex there exists $\boldsymbol{w} \in \boldsymbol{H}_{0}^{2}\left(\Omega ; \mathbb{R}^{3}\right)$ satisfying

$$
\operatorname{curl} \boldsymbol{w}=\boldsymbol{\mu}, \quad\|\boldsymbol{w}\|_{2} \lesssim|\boldsymbol{\mu}|_{1} .
$$

Then it holds the inf-sup condition

$$
|\boldsymbol{\mu}|_{1}=\frac{(\nabla \boldsymbol{\mu}, \nabla \boldsymbol{\mu})}{|\boldsymbol{\mu}|_{1}} \lesssim \sup _{\boldsymbol{w} \in \boldsymbol{H}_{0}^{2}\left(\Omega ; \mathbb{R}^{3}\right)} \frac{(\nabla \operatorname{curl} \boldsymbol{w}, \nabla \boldsymbol{\mu})}{\|\boldsymbol{w}\|_{2}}
$$

By the Babuška-Brezzi theory [6], problem (4.19) is well-posed, and

$$
\operatorname{curl} \boldsymbol{v}_{2}=\operatorname{curl} \boldsymbol{v}, \quad\left\|\boldsymbol{v}_{2}\right\|_{2}+\|\boldsymbol{\lambda}\|_{1} \lesssim|\operatorname{curl} \boldsymbol{v}|_{1}
$$

Finally we finish the proof by the fact that $\operatorname{curl}\left(\boldsymbol{v}-\boldsymbol{v}_{2}\right)=\mathbf{0}$.
For $\boldsymbol{v} \in \boldsymbol{H}_{0}(\operatorname{grad} \operatorname{curl}, \Omega)$ satisfying $\operatorname{curl} \boldsymbol{v}=\mathbf{0}$, by (4.20), we have $\boldsymbol{v}_{2}=\mathbf{0}$ and $\boldsymbol{v}=\nabla v_{1}$.

Now we apply the regular decomposition (4.18) to $\boldsymbol{v} \in \boldsymbol{H}_{0}(\operatorname{grad} \operatorname{curl}, \Omega)$. Let $I_{h}^{S Z}: H_{0}^{1}(\Omega) \rightarrow V_{h 0}^{g}$ be the Scott-Zhang interpolation operator [37]. Noting that $\boldsymbol{I}_{h}^{g c}$ can be applied to $\boldsymbol{v}_{2}$, define $\boldsymbol{\Pi}_{h}^{g c}: \boldsymbol{H}_{0}(\operatorname{grad} \operatorname{curl}, \Omega) \rightarrow \boldsymbol{W}_{h 0}$ as follows

$$
\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{v}:=\boldsymbol{I}_{h}^{g c} \boldsymbol{v}_{2}+\nabla I_{h}^{S Z} v_{1} .
$$

Clearly we have

$$
\begin{equation*}
\boldsymbol{\Pi}_{h}^{g c}(\nabla v)=\nabla\left(I_{h}^{S Z} v\right) \quad \forall v \in H_{0}^{1}(\Omega) \tag{4.21}
\end{equation*}
$$

We acquire from (3.14) that

$$
\begin{equation*}
\operatorname{curl}_{h}\left(\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{v}\right)=\boldsymbol{I}_{h}^{s}(\operatorname{curl} \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{H}_{0}(\operatorname{grad} \operatorname{curl}, \Omega) . \tag{4.22}
\end{equation*}
$$

Combining (4.21)-(4.22), (3.15) and the complex (3.5) yields the commutative diagram


Lemma 4.6. Assume $\boldsymbol{v} \in \boldsymbol{H}^{1}\left(\Omega ; \mathbb{R}^{3}\right)$ and $\operatorname{curl} \boldsymbol{v} \in \boldsymbol{H}^{2}\left(\Omega ; \mathbb{R}^{3}\right)$. Then

$$
\begin{gather*}
\left\|\boldsymbol{v}-\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{v}\right\|_{0} \lesssim h\left(|\boldsymbol{v}|_{1}+|\operatorname{curl} \boldsymbol{v}|_{1}\right)  \tag{4.23}\\
\left\|\operatorname{curl}_{h}\left(\boldsymbol{v}-\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{v}\right)\right\|_{0}+h\left|\operatorname{curl}_{h}\left(\boldsymbol{v}-\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{v}\right)\right|_{1} \lesssim h^{2}|\operatorname{curl} \boldsymbol{v}|_{2} . \tag{4.24}
\end{gather*}
$$

Proof. Noting that $\nabla v_{1}=\boldsymbol{v}-\boldsymbol{v}_{2}$, it follows

$$
\left|v_{1}\right|_{2} \lesssim\left|\boldsymbol{v}-\boldsymbol{v}_{2}\right|_{1} \lesssim|\boldsymbol{v}|_{1}+|\operatorname{curl} \boldsymbol{v}|_{1} .
$$

Since $\boldsymbol{v}-\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{v}=\boldsymbol{v}_{2}-\boldsymbol{I}_{h}^{g c} \boldsymbol{v}_{2}+\nabla\left(v_{1}-I_{h}^{S Z} v_{1}\right)$, we acquire from (3.10), the error estimate of $I_{h}^{S Z}$ and (4.20) that

$$
\begin{aligned}
\left\|\boldsymbol{v}-\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{v}\right\|_{0} & \leq\left\|\boldsymbol{v}_{2}-\boldsymbol{I}_{h}^{g c} \boldsymbol{v}_{2}\right\|_{0}+\left|v_{1}-I_{h}^{S Z} v_{1}\right|_{1} \\
& \lesssim h^{k+1}\left\|\boldsymbol{v}_{2}\right\|_{2}+h\left|v_{1}\right|_{2} \lesssim h\left(|\boldsymbol{v}|_{1}+|\operatorname{curl} \boldsymbol{v}|_{1}\right) .
\end{aligned}
$$

Employing (3.8), we get from (4.22) that

$$
\begin{gathered}
\left\|\operatorname{curl}_{h}\left(\boldsymbol{v}-\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{v}\right)\right\|_{0}=\left\|\operatorname{curl} \boldsymbol{v}-\boldsymbol{I}_{h}^{s}(\operatorname{curl} \boldsymbol{v})\right\|_{0} \lesssim h^{2}|\operatorname{curl} \boldsymbol{v}|_{2} \\
\left|\operatorname{curl}_{h}\left(\boldsymbol{v}-\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{v}\right)\right|_{1}=\left|\operatorname{curl} \boldsymbol{v}-\boldsymbol{I}_{h}^{s}(\operatorname{curl} \boldsymbol{v})\right|_{1} \lesssim h|\operatorname{curl} \boldsymbol{v}|_{2} .
\end{gathered}
$$

This ends the proof.
4.3. Error analysis. Hereafter we assume $\boldsymbol{u} \in \boldsymbol{H}_{0}(\operatorname{grad}$ curl, $\Omega)$ possesses the regularity curl $\boldsymbol{u} \in \boldsymbol{H}^{2}\left(\Omega ; \mathbb{R}^{3}\right)$, which is true for $\Omega$ being convex (See Lemma A.1.). Applying the integration by parts to the first equation in (4.2), we derive

$$
\begin{equation*}
-(\Delta \operatorname{curl} \boldsymbol{u}, \operatorname{curl} \boldsymbol{v})=(\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{H}_{0}(\operatorname{curl}, \Omega) \tag{4.25}
\end{equation*}
$$

We first present the consistency error of the nonconforming methods (4.5)-(4.6).
Lemma 4.7. We have for any $\boldsymbol{v}_{h} \in \boldsymbol{W}_{h 0}$ that

$$
\begin{equation*}
\left(\nabla \operatorname{curl} \boldsymbol{u}, \nabla_{h} \operatorname{curl}_{h} \boldsymbol{v}_{h}\right)+\left(\Delta \operatorname{curl} \boldsymbol{u}, \operatorname{curl}_{h} \boldsymbol{v}_{h}\right) \lesssim h|\operatorname{curl} \boldsymbol{u}|_{2}\left|\operatorname{curl}_{h} \boldsymbol{v}_{h}\right|_{1, h} \tag{4.26}
\end{equation*}
$$

Proof. Due to (3.1), we get

$$
\begin{aligned}
& \sum_{K \in \mathcal{T}_{h}}\left(\partial_{n}(\operatorname{curl} \boldsymbol{u}), \operatorname{curl}_{h} \boldsymbol{v}_{h}\right)_{\partial K} \\
= & \sum_{K \in \mathcal{T}_{h}} \sum_{F \in \mathcal{F}(K)}\left(\partial_{n}(\operatorname{curl} \boldsymbol{u})-\boldsymbol{Q}_{F}^{0} \partial_{n}(\operatorname{curl} \boldsymbol{u}), \operatorname{curl}_{h} \boldsymbol{v}_{h}\right)_{F} \\
= & \sum_{K \in \mathcal{T}_{h}} \sum_{F \in \mathcal{F}(K)}\left(\partial_{n}(\operatorname{curl} \boldsymbol{u})-\boldsymbol{Q}_{F}^{0} \partial_{n}(\operatorname{curl} \boldsymbol{u}), \operatorname{curl}_{h} \boldsymbol{v}_{h}-\boldsymbol{Q}_{F}^{0} \operatorname{curl}_{h} \boldsymbol{v}_{h}\right)_{F} \\
\lesssim & h|\operatorname{curl} \boldsymbol{u}|_{2}\left|\operatorname{curl}_{h} \boldsymbol{v}_{h}\right|_{1, h} .
\end{aligned}
$$

Thus (4.26) follows from the integration by parts.
Lemma 4.8. We have for any $\boldsymbol{v}_{h} \in \boldsymbol{W}_{h 0}$ that

$$
\begin{equation*}
\left(\nabla \operatorname{curl} \boldsymbol{u}, \nabla_{h} \operatorname{curl}_{h} \boldsymbol{v}_{h}\right)-\left(\boldsymbol{f}, \boldsymbol{v}_{h}\right) \lesssim h\left(|\operatorname{curl} \boldsymbol{u}|_{2}+\|\boldsymbol{f}\|_{0}\right)\left|\operatorname{curl}_{h} \boldsymbol{v}_{h}\right|_{1, h} \tag{4.27}
\end{equation*}
$$

Proof. We get from (4.25) with $\boldsymbol{v}=\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}$ that
$\left(\Delta \operatorname{curl} \boldsymbol{u}, \operatorname{curl}_{h} \boldsymbol{v}_{h}\right)+\left(\boldsymbol{f}, \boldsymbol{v}_{h}\right)=\left(\Delta \operatorname{curl} \boldsymbol{u}, \operatorname{curl}_{h}\left(\boldsymbol{v}_{h}-\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right)\right)+\left(\boldsymbol{f}, \boldsymbol{v}_{h}-\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right)$.
Applying (4.8) and (4.12) gives

$$
\begin{aligned}
& \quad-\left(\Delta \operatorname{curl} \boldsymbol{u}, \operatorname{curl}_{h} \boldsymbol{v}_{h}\right)-\left(\boldsymbol{f}, \boldsymbol{v}_{h}\right) \\
& \lesssim\|\Delta \operatorname{curl} \boldsymbol{u}\|_{0}\left\|\operatorname{curl}_{h}\left(\boldsymbol{v}_{h}-\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right)\right\|_{0}+\|\boldsymbol{f}\|_{0}\left\|\boldsymbol{v}_{h}-\boldsymbol{I}_{h}^{c} \boldsymbol{v}_{h}\right\|_{0} \\
& \lesssim h|\operatorname{curl} \boldsymbol{u}|_{2}\left|\operatorname{curl}_{h} \boldsymbol{v}_{h}\right|_{1, h}+h\|\boldsymbol{f}\|_{0}\left\|\operatorname{curl}_{h} \boldsymbol{v}_{h}\right\|_{0}
\end{aligned}
$$

Together with (4.26), we derive

$$
\begin{aligned}
& \left(\nabla \operatorname{curl} \boldsymbol{u}, \nabla_{h} \operatorname{curl}_{h} \boldsymbol{v}_{h}\right)-\left(\boldsymbol{f}, \boldsymbol{v}_{h}\right) \\
\lesssim & h|\operatorname{curl} \boldsymbol{u}|_{2}\left|\operatorname{curl}_{h} \boldsymbol{v}_{h}\right|_{1, h}+h\|\boldsymbol{f}\|_{0}\left\|\operatorname{curl}_{h} \boldsymbol{v}_{h}\right\|_{0} .
\end{aligned}
$$

Hence (4.27) follows from (3.6).
Now we can show the a priori error estimate.
Theorem 4.9. Let $\boldsymbol{u} \in \boldsymbol{H}_{0}(\operatorname{grad} \operatorname{curl}, \Omega)$ be the solution of the problem (4.1), and $\boldsymbol{u}_{h} \in \boldsymbol{W}_{h 0}$ the solution of the mixed finite element methods (4.5)-(4.6). Assume $\boldsymbol{u} \in \boldsymbol{H}^{1}\left(\Omega ; \mathbb{R}^{3}\right)$ and curl $\boldsymbol{u} \in \boldsymbol{H}^{2}\left(\Omega ; \mathbb{R}^{3}\right)$. We have

$$
\begin{equation*}
\left\|\boldsymbol{u}-\boldsymbol{u}_{h}\right\|_{H_{h}(\operatorname{grad} \operatorname{curl})} \lesssim h\left(\|\operatorname{curl} \boldsymbol{u}\|_{2}+|\boldsymbol{u}|_{1}+\|\boldsymbol{f}\|_{0}\right) \tag{4.28}
\end{equation*}
$$

Proof. It follows from (4.27) that

$$
\begin{aligned}
& \left(\nabla_{h} \operatorname{curl}_{h}\left(\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{u}\right), \nabla_{h} \operatorname{curl}_{h} \boldsymbol{v}_{h}\right)-\left(\boldsymbol{f}, \boldsymbol{v}_{h}\right) \\
= & \left(\nabla_{h} \operatorname{curl}_{h}\left(\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{u}-\boldsymbol{u}\right), \nabla_{h} \operatorname{curl}_{h} \boldsymbol{v}_{h}\right)+\left(\nabla \operatorname{curl} \boldsymbol{u}, \nabla_{h} \operatorname{curl}_{h} \boldsymbol{v}_{h}\right)-\left(\boldsymbol{f}, \boldsymbol{v}_{h}\right) \\
\lesssim & \left.\operatorname{curl}_{h}\left(\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{u}-\boldsymbol{u}\right)\right|_{1, h}\left|\operatorname{curl}_{h} \boldsymbol{v}_{h}\right|_{1, h}+h\left(|\operatorname{curl} \boldsymbol{u}|_{2}+\|\boldsymbol{f}\|_{0}\right)\left|\operatorname{curl}_{h} \boldsymbol{v}_{h}\right|_{1, h} .
\end{aligned}
$$

On the other hand, by the discrete stability (4.16) with $\widetilde{\boldsymbol{u}}_{h}=\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{u}-\boldsymbol{u}_{h}$ and $\widetilde{\lambda}_{h}=0$, we get from (4.17) and the fact $\boldsymbol{u}_{h} \in \mathcal{K}_{h}^{d}$ that

$$
\begin{aligned}
& \left\|\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{u}-\boldsymbol{u}_{h}\right\|_{H_{h}(\operatorname{grad} \operatorname{curl})} \\
\lesssim & \sup _{\left(\boldsymbol{v}_{h}, \mu_{h}\right) \in \boldsymbol{W}_{h 0} \times V_{h 0}^{g}} \frac{\left(\nabla_{h} \operatorname{curl}_{h}\left(\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{u}-\boldsymbol{u}_{h}\right), \nabla_{h} \operatorname{curl}_{h} \boldsymbol{v}_{h}\right)+\left(\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{u}-\boldsymbol{u}_{h}, \nabla \mu_{h}\right)}{\left\|\boldsymbol{v}_{h}\right\|_{H_{h}(\operatorname{grad} \operatorname{curl})}+\left|\mu_{h}\right|_{1}} \\
= & \sup _{\left(\boldsymbol{v}_{h}, \mu_{h}\right) \in \boldsymbol{W}_{h 0} \times V_{h 0}^{g}} \frac{\left(\nabla_{h} \operatorname{curl}_{h}\left(\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{u}\right), \nabla_{h} \operatorname{curl}_{h} \boldsymbol{v}_{h}\right)-\left(\boldsymbol{f}, \boldsymbol{v}_{h}\right)+\left(\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{u}-\boldsymbol{u}, \nabla \mu_{h}\right)}{\left\|\boldsymbol{v}_{h}\right\|_{H_{h}(\operatorname{grad} \operatorname{curl})+\left|\mu_{h}\right|_{1}}} \\
\lesssim & \left\|\boldsymbol{u}-\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{u}\right\|_{0}+\sup _{\boldsymbol{v}_{h} \in \boldsymbol{W}_{h 0}} \frac{\left(\nabla_{h} \operatorname{curl}_{h}\left(\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{u}\right), \nabla_{h} \operatorname{curl}_{h} \boldsymbol{v}_{h}\right)-\left(\boldsymbol{f}, \boldsymbol{v}_{h}\right)}{\left\|\boldsymbol{v}_{h}\right\|_{H_{h}(\operatorname{grad} \operatorname{curl})}} .
\end{aligned}
$$

Hence we get from (4.29) that

$$
\left\|\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{u}-\boldsymbol{u}_{h}\right\|_{H_{h}(\text { grad curl })} \lesssim\left\|\boldsymbol{u}-\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{u}\right\|_{H_{h}(\text { grad curl })}+h\left(|\operatorname{curl} \boldsymbol{u}|_{2}+\|\boldsymbol{f}\|_{0}\right)
$$

Thus

$$
\begin{aligned}
\left\|\boldsymbol{u}-\boldsymbol{u}_{h}\right\|_{H_{h}(\operatorname{grad} \operatorname{curl})} & \lesssim\left\|\boldsymbol{u}-\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{u}\right\|_{H_{h}(\operatorname{grad} \operatorname{curl})}+\left\|\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{u}-\boldsymbol{u}_{h}\right\|_{H_{h}(\text { grad curl })} \\
& \lesssim\left\|\boldsymbol{u}-\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{u}\right\|_{H_{h}(\text { grad curl })}+h\left(|\operatorname{curl} \boldsymbol{u}|_{2}+\|\boldsymbol{f}\|_{0}\right)
\end{aligned}
$$

Finally (4.28) follows from (4.23) and (4.24).

Remark 4.10. As illustrated in [6, Section 7.9], the convergence would deteriorate if using the nonconforming finite element space $\boldsymbol{W}_{h 0}$ to discretize Maxwell equation

$$
\left\{\begin{aligned}
\text { curl curl } \boldsymbol{u}=\boldsymbol{f} & \text { in } \Omega, \\
\operatorname{div} \boldsymbol{u}=0 & \text { in } \Omega, \\
\boldsymbol{u} \times \boldsymbol{n}=\mathbf{0} & \text { on } \partial \Omega
\end{aligned}\right.
$$

Next we estimate $\left\|\operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right)\right\|_{0}$ by the duality argument. To this end, consider the dual problem

$$
\left\{\begin{align*}
-\operatorname{curl} \Delta \operatorname{curl} \widetilde{\boldsymbol{u}} & =\operatorname{curl} \operatorname{curl} \boldsymbol{I}_{h}^{c}\left(\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{u}-\boldsymbol{u}_{h}\right) & & \text { in } \Omega,  \tag{4.30}\\
\operatorname{div} \widetilde{\boldsymbol{u}} & =0 & & \text { in } \Omega, \\
\widetilde{\boldsymbol{u}} \times \boldsymbol{n}=(\operatorname{curl} \widetilde{\boldsymbol{u}}) \times \boldsymbol{n} & =\mathbf{0} & & \text { on } \partial \Omega,
\end{align*}\right.
$$

where $\widetilde{\boldsymbol{u}} \in \boldsymbol{H}_{0}(\operatorname{grad} \operatorname{curl}, \Omega)$. The first equation in the dual problem (4.30) holds in the sense of $\boldsymbol{H}^{-1}(\operatorname{div}, \Omega)$, where $\boldsymbol{H}^{-1}(\operatorname{div}, \Omega):=\left\{\boldsymbol{v} \in \boldsymbol{H}^{-1}\left(\Omega ; \mathbb{R}^{3}\right): \operatorname{div} \boldsymbol{v} \in\right.$ $\left.H^{-1}(\Omega)\right\}$ is the dual space of $\boldsymbol{H}_{0}(\operatorname{curl}, \Omega)$ [11]. Thanks to (4.10) and (4.22), it holds

$$
\begin{aligned}
\operatorname{curl} \boldsymbol{I}_{h}^{c}\left(\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{u}-\boldsymbol{u}_{h}\right) & =\boldsymbol{I}_{h}^{d} \operatorname{curl}_{h}\left(\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{u}-\boldsymbol{u}_{h}\right)=\boldsymbol{I}_{h}^{d}\left(\boldsymbol{I}_{h}^{s} \operatorname{curl} \boldsymbol{u}-\operatorname{curl}_{h} \boldsymbol{u}_{h}\right) \\
& =\boldsymbol{I}_{h}^{d} \operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right)
\end{aligned}
$$

We assume the dual problem (4.30) possesses the following regularity in this section

$$
\begin{equation*}
\|\widetilde{\boldsymbol{u}}\|_{1}+\|\operatorname{curl} \widetilde{\boldsymbol{u}}\|_{2} \lesssim\left\|\operatorname{curl} \operatorname{curl} \boldsymbol{I}_{h}^{c}\left(\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{u}-\boldsymbol{u}_{h}\right)\right\|_{-1} \leq\left\|\boldsymbol{I}_{h}^{d} \operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right)\right\|_{0} \tag{4.32}
\end{equation*}
$$

The regularity (4.32) holds for the domain $\Omega$ being convex (See Lemma A.1.). Similarly as (4.25), it holds from (4.30) that

$$
\begin{equation*}
-(\Delta \operatorname{curl} \widetilde{\boldsymbol{u}}, \operatorname{curl} \boldsymbol{v})=\left(\operatorname{curl} \boldsymbol{I}_{h}^{c}\left(\boldsymbol{\Pi}_{h}^{g c} \boldsymbol{u}-\boldsymbol{u}_{h}\right), \operatorname{curl} \boldsymbol{v}\right) \quad \forall \boldsymbol{v} \in \boldsymbol{H}_{0}(\operatorname{curl}, \Omega) \tag{4.33}
\end{equation*}
$$

Theorem 4.11. Let $\boldsymbol{u} \in \boldsymbol{H}_{0}(\operatorname{grad} \operatorname{curl}, \Omega)$ be the solution of the problem (4.1), and $\boldsymbol{u}_{h} \in \boldsymbol{W}_{h 0}$ the solution of the mixed finite element methods (4.5)-(4.6). Assume the regularity (4.32) holds. We have

$$
\begin{equation*}
\left\|\operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right)\right\|_{0} \lesssim h^{k+1}\|\boldsymbol{f}\|_{0}+h^{2}\left(\|\operatorname{curl} \boldsymbol{u}\|_{2}+|\boldsymbol{u}|_{1}\right) \tag{4.34}
\end{equation*}
$$

Proof. It follows from (3.8) that

$$
\begin{aligned}
& \sum_{K \in \mathcal{T}_{h}}\left(\partial_{n}(\operatorname{curl} \boldsymbol{u}), \boldsymbol{I}_{h}^{s} \operatorname{curl} \widetilde{\boldsymbol{u}}-\operatorname{curl} \widetilde{\boldsymbol{u}}\right)_{\partial K} \\
= & \sum_{K \in \mathcal{T}_{h}} \sum_{F \in \mathcal{F}(K)}\left(\partial_{n}(\operatorname{curl} \boldsymbol{u})-\boldsymbol{Q}_{F}^{0} \partial_{n}(\operatorname{curl} \boldsymbol{u}), \boldsymbol{I}_{h}^{s} \operatorname{curl} \widetilde{\boldsymbol{u}}-\operatorname{curl} \widetilde{\boldsymbol{u}}\right)_{F} \\
\lesssim & h^{2}|\operatorname{curl} \boldsymbol{u}|_{2}|\operatorname{curl} \widetilde{\boldsymbol{u}}|_{2} .
\end{aligned}
$$

Applying (3.8) again, we get

$$
\begin{aligned}
& \left(\nabla \operatorname{curl} \boldsymbol{u}, \nabla_{h}\left(\boldsymbol{I}_{h}^{s} \operatorname{curl} \widetilde{\boldsymbol{u}}-\operatorname{curl} \widetilde{\boldsymbol{u}}\right)\right) \\
= & \sum_{K \in \mathcal{T}_{h}}\left(\partial_{n}(\operatorname{curl} \boldsymbol{u}), \boldsymbol{I}_{h}^{s} \operatorname{curl} \widetilde{\boldsymbol{u}}-\operatorname{curl} \widetilde{\boldsymbol{u}}\right)_{\partial K}-\left(\Delta \operatorname{curl} \boldsymbol{u}, \boldsymbol{I}_{h}^{s} \operatorname{curl} \widetilde{\boldsymbol{u}}-\operatorname{curl} \widetilde{\boldsymbol{u}}\right) \\
\lesssim & h^{2}|\operatorname{curl} \boldsymbol{u}|_{2}|\operatorname{curl} \widetilde{\boldsymbol{u}}|_{2} .
\end{aligned}
$$

Due to (3.10) and the fact $\operatorname{div} \boldsymbol{f}=0$, we have

$$
\left(\boldsymbol{f}, \widetilde{\boldsymbol{u}}-\boldsymbol{\Pi}_{h}^{g c} \widetilde{\boldsymbol{u}}\right)=\left(\boldsymbol{f}, \widetilde{\boldsymbol{u}}_{2}-\boldsymbol{I}_{h}^{g c} \widetilde{\boldsymbol{u}}_{2}\right) \lesssim h^{k+1}\|\boldsymbol{f}\|_{0}\left\|\widetilde{\boldsymbol{u}}_{2}\right\|_{2} \lesssim h^{k+1}\|\boldsymbol{f}\|_{0}|\operatorname{curl} \widetilde{\boldsymbol{u}}|_{1}
$$

Combining the last two inequalities, (4.17) and (4.22) implies

$$
\begin{aligned}
& \left(\nabla_{h} \operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right), \nabla_{h} \operatorname{curl}_{h} \boldsymbol{\Pi}_{h}^{g c} \widetilde{\boldsymbol{u}}\right) \\
= & \left(\nabla \operatorname{curl} \boldsymbol{u}, \nabla_{h} \boldsymbol{I}_{h}^{s} \operatorname{curl} \widetilde{\boldsymbol{u}}\right)-\left(\boldsymbol{f}, \boldsymbol{\Pi}_{h}^{g c} \widetilde{\boldsymbol{u}}\right) \\
= & \left(\nabla \operatorname{curl} \boldsymbol{u}, \nabla_{h}\left(\boldsymbol{I}_{h}^{s} \operatorname{curl} \widetilde{\boldsymbol{u}}-\operatorname{curl} \widetilde{\boldsymbol{u}}\right)\right)+\left(\boldsymbol{f}, \widetilde{\boldsymbol{u}}-\boldsymbol{\Pi}_{h}^{g c} \widetilde{\boldsymbol{u}}\right) \\
\lesssim & h^{2}|\operatorname{curl} \boldsymbol{u}|_{2}|\operatorname{curl} \widetilde{\boldsymbol{u}}|_{2}+h^{k+1}\|\boldsymbol{f}\|_{0}|\operatorname{curl} \widetilde{\boldsymbol{u}}|_{1} .
\end{aligned}
$$

Employing (4.22) and (3.8), we get

$$
\begin{aligned}
& \left(\nabla_{h} \operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right), \nabla_{h} \operatorname{curl}_{h}\left(\widetilde{\boldsymbol{u}}-\mathbf{\Pi}_{h}^{g c} \widetilde{\boldsymbol{u}}\right)\right) \\
= & \left(\nabla_{h} \operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right), \nabla_{h}\left(\operatorname{curl} \widetilde{\boldsymbol{u}}-\boldsymbol{I}_{h}^{s} \operatorname{curl} \widetilde{\boldsymbol{u}}\right)\right) \\
\leq & \left|\operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right)\right|_{1, h}\left|\operatorname{curl} \widetilde{\boldsymbol{u}}-\boldsymbol{I}_{h}^{s} \operatorname{curl} \widetilde{\boldsymbol{u}}\right|_{1, h} \lesssim h\left|\operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right)\right|_{1, h}|\operatorname{curl} \widetilde{\boldsymbol{u}}|_{2} .
\end{aligned}
$$

It holds from the sum of the last two inequalities that

$$
\begin{aligned}
& \left(\nabla_{h} \operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right), \nabla \operatorname{curl} \widetilde{\boldsymbol{u}}\right) \\
\lesssim & \left(h^{2}|\operatorname{curl} \boldsymbol{u}|_{2}+h\left|\operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right)\right|_{1, h}+h^{k+1}\|\boldsymbol{f}\|_{0}\right)\|\operatorname{curl} \widetilde{\boldsymbol{u}}\|_{2} .
\end{aligned}
$$

Thanks to (3.1), we obtain

$$
\begin{aligned}
& -\sum_{K \in \mathcal{T}_{h}}\left(\operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right), \partial_{n} \operatorname{curl} \widetilde{\boldsymbol{u}}\right)_{\partial K} \\
= & -\sum_{K \in \mathcal{T}_{h}} \sum_{F \in \mathcal{F}(K)}\left(\left(\boldsymbol{I}-\boldsymbol{Q}_{F}^{0}\right) \operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right),\left(\boldsymbol{I}-\boldsymbol{Q}_{F}^{0}\right) \partial_{n} \operatorname{curl} \widetilde{\boldsymbol{u}}\right)_{\partial K} \\
\lesssim & h\left|\operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right)\right|_{1, h}|\operatorname{curl} \widetilde{\boldsymbol{u}}|_{2} .
\end{aligned}
$$

Hence we achieve from the last two inequalities that

$$
\begin{aligned}
& -\left(\operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right), \Delta \operatorname{curl} \widetilde{\boldsymbol{u}}\right) \\
\lesssim & \left(h^{2}|\operatorname{curl} \boldsymbol{u}|_{2}+h\left|\operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right)\right|_{1, h}+h^{k+1}\|\boldsymbol{f}\|_{0}\right)\|\operatorname{curl} \widetilde{\boldsymbol{u}}\|_{2} .
\end{aligned}
$$

On the other hand, it follows from (4.31) and(4.33) that

$$
\begin{aligned}
& \left\|\boldsymbol{I}_{h}^{d} \operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right)\right\|_{0}^{2}=-\left(\boldsymbol{I}_{h}^{d} \operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right), \Delta \operatorname{curl} \widetilde{\boldsymbol{u}}\right) \\
= & \left(\left(\boldsymbol{I}-\boldsymbol{I}_{h}^{d}\right) \operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right), \Delta \operatorname{curl} \widetilde{\boldsymbol{u}}\right)-\left(\operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right), \Delta \operatorname{curl} \widetilde{\boldsymbol{u}}\right) \\
\lesssim & \left(h^{2}|\operatorname{curl} \boldsymbol{u}|_{2}+h\left|\operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right)\right|_{1, h}+h^{k+1}\|\boldsymbol{f}\|_{0}\right)\|\operatorname{curl} \widetilde{\boldsymbol{u}}\|_{2},
\end{aligned}
$$

which together with (4.32) yields

$$
\left\|\boldsymbol{I}_{h}^{d} \operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right)\right\|_{0} \lesssim h^{2}|\operatorname{curl} \boldsymbol{u}|_{2}+h\left|\operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right)\right|_{1, h}+h^{k+1}\|\boldsymbol{f}\|_{0} .
$$

Hence

$$
\left\|\operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right)\right\|_{0} \lesssim h^{2}|\operatorname{curl} \boldsymbol{u}|_{2}+h\left|\operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right)\right|_{1, h}+h^{k+1}\|\boldsymbol{f}\|_{0}
$$

Finally (4.34) follows from (4.28).

## 5. Decoupling of the mixed finite element methods

In this section, we will present an equivalent decoupled discretization of the mixed finite element methods (4.5)-(4.6) as the decoupled Morley element method for the biharmonic equation in [29], based on which a fast solver is suggested.
5.1. Decoupling. In the continuous level, the mixed formulation (4.3)-(4.4) of the quad-curl problem (4.2) can be decoupled into the following system [11, 43]: find $\boldsymbol{w} \in \boldsymbol{H}_{0}(\operatorname{curl}, \Omega), \lambda \in H_{0}^{1}(\Omega), \boldsymbol{\phi} \in \boldsymbol{H}_{0}^{1}\left(\Omega ; \mathbb{R}^{3}\right), p \in L_{0}^{2}(\Omega), \boldsymbol{u} \in \boldsymbol{H}_{0}(\operatorname{curl}, \Omega)$ and $\sigma \in H_{0}^{1}(\Omega)$ such that

$$
\begin{aligned}
(\operatorname{curl} \boldsymbol{w}, \operatorname{curl} \boldsymbol{v})+(\boldsymbol{v}, \nabla \lambda) & =(\boldsymbol{f}, \boldsymbol{v}) & & \forall \boldsymbol{v} \in \boldsymbol{H}_{0}(\operatorname{curl}, \Omega) \\
(\boldsymbol{w}, \nabla \tau) & =0 & & \forall \tau \in H_{0}^{1}(\Omega) \\
(\boldsymbol{\nabla} \boldsymbol{\phi}, \nabla \boldsymbol{\psi})+(\operatorname{div} \boldsymbol{\psi}, p) & =(\operatorname{curl} \boldsymbol{w}, \boldsymbol{\psi}) & & \forall \boldsymbol{\psi} \in \boldsymbol{H}_{0}^{1}\left(\Omega ; \mathbb{R}^{3}\right), \\
(\operatorname{div} \boldsymbol{\phi}, q) & =0 & & \forall q \in L_{0}^{2}(\Omega) \\
(\operatorname{curl} \boldsymbol{u}, \operatorname{curl} \boldsymbol{\chi})+(\boldsymbol{\chi}, \nabla \sigma) & =(\boldsymbol{\phi}, \operatorname{curl} \boldsymbol{\chi}) & & \forall \boldsymbol{\chi} \in \boldsymbol{H}_{0}(\operatorname{curl}, \Omega), \\
(\boldsymbol{u}, \nabla \mu) & =0 & & \forall \mu \in H_{0}^{1}(\Omega)
\end{aligned}
$$

Thanks to the discrete Stokes complex (3.5), the mixed finite element methods (4.5)-(4.6) can also be decoupled to find $\boldsymbol{w}_{h} \in \boldsymbol{W}_{h 0}, \lambda_{h} \in V_{h 0}^{g}, \boldsymbol{\phi}_{h} \in \boldsymbol{V}_{h 0}^{s}, p_{h} \in \mathcal{Q}_{h 0}$, $\boldsymbol{u}_{h} \in \boldsymbol{W}_{h 0}$ and $\sigma_{h} \in V_{h 0}^{g}$ such that

$$
\begin{align*}
\left(\operatorname{curl}_{h} \boldsymbol{w}_{h}, \operatorname{curl}_{h} \boldsymbol{v}_{h}\right)+\left(\boldsymbol{v}_{h}, \nabla \lambda_{h}\right) & =\left(\boldsymbol{f}, \boldsymbol{v}_{h}\right) & & \forall \boldsymbol{v}_{h} \in \boldsymbol{W}_{h 0},  \tag{5.1}\\
\left(\boldsymbol{w}_{h}, \nabla \tau_{h}\right) & =0 & & \forall \tau_{h} \in V_{h 0}^{g}  \tag{5.2}\\
\left(\boldsymbol{\nabla}_{h} \boldsymbol{\phi}_{h}, \boldsymbol{\nabla}_{h} \boldsymbol{\psi}_{h}\right)+\left(\operatorname{div}_{h} \boldsymbol{\psi}_{h}, p_{h}\right) & =\left(\operatorname{curl}_{h} \boldsymbol{w}_{h}, \boldsymbol{\psi}_{h}\right) & & \forall \boldsymbol{\psi}_{h} \in \boldsymbol{V}_{h 0}^{s},  \tag{5.3}\\
\left(\operatorname{div}_{h} \boldsymbol{\phi}_{h}, q_{h}\right) & =0 & & \forall q_{h} \in \mathcal{Q}_{h 0}  \tag{5.4}\\
\left(\operatorname{curl}_{h} \boldsymbol{u}_{h}, \operatorname{curl}_{h} \boldsymbol{\chi}_{h}\right)+\left(\boldsymbol{\chi}_{h}, \nabla \sigma_{h}\right) & =\left(\boldsymbol{\phi}_{h}, \operatorname{curl}_{h} \boldsymbol{\chi}_{h}\right) & & \forall \boldsymbol{\chi}_{h} \in \boldsymbol{W}_{h 0},  \tag{5.5}\\
\left(\boldsymbol{u}_{h}, \nabla \mu_{h}\right) & =0 & & \forall \mu_{h} \in V_{h 0}^{g} . \tag{5.6}
\end{align*}
$$

Both (5.1)-(5.2) and (5.5)-(5.6) are mixed finite element methods for the Maxwell equation. From the discrete Poincaré inequality (4.13) and the fact $\nabla V_{h 0}^{g} \subset \boldsymbol{W}_{h 0}^{g}$, we have the discrete stability
$\left\|\widetilde{\boldsymbol{w}}_{h}\right\|_{H_{h}(\text { curl })}+\left|\widetilde{\lambda}_{h}\right|_{1} \lesssim \sup _{\left(\boldsymbol{v}_{h}, \tau_{h}\right) \in \boldsymbol{W}_{h 0} \times V_{h 0}^{g}} \frac{\left(\operatorname{curl}_{h} \widetilde{\boldsymbol{w}}_{h}, \operatorname{curl}_{h} \boldsymbol{v}_{h}\right)+\left(\boldsymbol{v}_{h}, \nabla \widetilde{\lambda}_{h}\right)+\left(\widetilde{\boldsymbol{w}}_{h}, \nabla \tau_{h}\right)}{\left\|\boldsymbol{v}_{h}\right\|_{H_{h}(\operatorname{curl})}+\left|\tau_{h}\right|_{1}}$
for any $\widetilde{\boldsymbol{w}}_{h} \in \boldsymbol{W}_{h 0}$ and $\widetilde{\lambda}_{h} \in V_{h 0}^{g}$, where the squared norm

$$
\left\|\boldsymbol{v}_{h}\right\|_{H_{h}(\text { curl })}^{2}:=\left\|\boldsymbol{v}_{h}\right\|_{0}^{2}+\left\|\operatorname{curl}_{h} \boldsymbol{v}_{h}\right\|_{0}^{2} .
$$

Hence both mixed finite element methods (5.1)-(5.2) and (5.5)-(5.6) are well-posed. The discrete method (5.3)-(5.4) is exactly the nonconforming $P_{1}-P_{0}$ element method for the Stokes equation.

By replacing $\boldsymbol{v}_{h}$ in (5.1) with $\nabla \mu_{h}$ for any $\mu_{h} \in V_{h 0}^{g}$, we obtain $\lambda_{h}=0$ from the fact $\operatorname{div} \boldsymbol{f}=0$. Similarily we achieve $\sigma_{h}=0$ from (5.5). Then (5.1) and (5.5) will be reduced to

$$
\begin{equation*}
\left(\operatorname{curl}_{h} \boldsymbol{w}_{h}, \operatorname{curl}_{h} \boldsymbol{v}_{h}\right)=\left(\boldsymbol{f}, \boldsymbol{v}_{h}\right) \quad \forall \boldsymbol{v}_{h} \in \boldsymbol{W}_{h 0} \tag{5.7}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\operatorname{curl}_{h} \boldsymbol{u}_{h}, \operatorname{curl}_{h} \boldsymbol{\chi}_{h}\right)=\left(\phi_{h}, \operatorname{curl}_{h} \boldsymbol{\chi}_{h}\right) \quad \forall \boldsymbol{\chi}_{h} \in \boldsymbol{W}_{h 0} \tag{5.8}
\end{equation*}
$$

Theorem 5.1. Let $\left(\boldsymbol{w}_{h}, 0, \phi_{h}, p_{h}, \boldsymbol{u}_{h}, 0\right) \in \boldsymbol{W}_{h 0} \times V_{h 0}^{g} \times \boldsymbol{V}_{h 0}^{s} \times \mathcal{Q}_{h 0} \times \boldsymbol{W}_{h 0} \times V_{h 0}^{g}$ be the solution of the discrete methods (5.1)-(5.6). Then $\left(\boldsymbol{u}_{h}, 0\right)$ is the solution of the mixed finite element methods (4.5)-(4.6).

Proof. Since (5.6) and (4.6) are same, we only have to show $\boldsymbol{u}_{h} \in \mathcal{K}_{h}^{d}$ satisfies (4.17). It follows from (5.4) and the complex (3.5) that there exists $\widetilde{\boldsymbol{u}}_{h} \in \mathcal{K}_{h}^{d}$ satisfying $\phi_{h}=\operatorname{curl}_{h} \widetilde{\boldsymbol{u}}_{h}$, which together with (5.8) yields

$$
\left(\operatorname{curl}_{h}\left(\boldsymbol{u}_{h}-\widetilde{\boldsymbol{u}}_{h}\right), \operatorname{curl}_{h} \boldsymbol{\chi}_{h}\right)=0 \quad \forall \boldsymbol{\chi}_{h} \in \boldsymbol{W}_{h 0}
$$

Hence $\widetilde{\boldsymbol{u}}_{h}=\boldsymbol{u}_{h}$ and $\boldsymbol{\phi}_{h}=\operatorname{curl}_{h} \boldsymbol{u}_{h}$. Taking $\boldsymbol{\psi}_{h}=\operatorname{curl}_{h} \boldsymbol{v}_{h}$ in (5.3) with $\boldsymbol{v}_{h} \in \boldsymbol{W}_{h 0}$, we derive from (5.7) that

$$
\left(\boldsymbol{\nabla}_{h} \operatorname{curl}_{h} \boldsymbol{u}_{h}, \boldsymbol{\nabla}_{h} \operatorname{curl}_{h} \boldsymbol{v}_{h}\right)=\left(\operatorname{curl}_{h} \boldsymbol{w}_{h}, \operatorname{curl}_{h} \boldsymbol{v}_{h}\right)=\left(\boldsymbol{f}, \boldsymbol{v}_{h}\right) .
$$

Thus the discrete methods (5.1)-(5.6) and the mixed methods (4.5)-(4.6) are equivalent.
5.2. A fast solver. We discuss a fast solver for the mixed methods (4.5)-(4.6) in this subsection. The equivalence between the mixed methods (4.5)-(4.6) and the mixed methods (5.1)-(5.6) suggests fast solvers for the mixed finite element methods (4.5)-(4.6). We can solve the mixed method (5.1)-(5.2), the mixed method (5.3)(5.4) and the mixed method (5.5)-(5.6) sequentially. The mixed methods (5.1)-(5.2) and (5.5)-(5.6) for the Maxwell equation can be efficiently solved by the solver in [12, Section 4.4]. And for the the mixed method (5.3)-(5.4) of the Stokes equation, we can adopt the block diagonal preconditioner [17] or the approximate blockfactorization preconditioner [10].

Finally we demonstrate the fast solver for the mixed methods (5.1)-(5.2) and (5.5)(5.6). To this end, define the inner product

$$
\left\langle\lambda_{h}, \mu_{h}\right\rangle:=\sum_{i=1}^{n_{g}} \lambda_{i} \mu_{i}\left\|\psi_{i}\right\|_{0}^{2}, \text { where } \lambda_{h}=\sum_{i=1}^{n_{g}} \lambda_{i} \psi_{i}, \mu_{h}=\sum_{i=1}^{n_{g}} \mu_{i} \psi_{i}
$$

with $\left\{\psi_{i}\right\}_{1}^{n_{g}}$ being the basis functions of $V_{h 0}^{g}$. The matrix of $\left\langle\lambda_{h}, \mu_{h}\right\rangle$ is just the diagonal of the mass matrix of $\left(\lambda_{h}, \mu_{h}\right)$. Then we introduce the following two mixed methods

$$
\begin{align*}
\left(\operatorname{curl}_{h} \boldsymbol{w}_{h}, \operatorname{curl}_{h} \boldsymbol{v}_{h}\right)+\left(\boldsymbol{v}_{h}, \nabla \lambda_{h}\right) & =\left(\boldsymbol{f}, \boldsymbol{v}_{h}\right) & & \forall \boldsymbol{v}_{h} \in \boldsymbol{W}_{h 0},  \tag{5.9}\\
\left(\boldsymbol{w}_{h}, \nabla \tau_{h}\right)-\left\langle\lambda_{h}, \tau_{h}\right\rangle & =0 & & \forall \tau_{h} \in V_{h 0}^{g} \tag{5.10}
\end{align*}
$$

and

$$
\begin{align*}
\left(\operatorname{curl}_{h} \boldsymbol{u}_{h}, \operatorname{curl}_{h} \boldsymbol{\chi}_{h}\right)+\left(\boldsymbol{\chi}_{h}, \nabla \sigma_{h}\right) & =\left(\boldsymbol{\phi}_{h}, \operatorname{curl}_{h} \boldsymbol{\chi}_{h}\right) & & \forall \boldsymbol{\chi}_{h} \in \boldsymbol{W}_{h 0},  \tag{5.11}\\
\left(\boldsymbol{u}_{h}, \nabla \mu_{h}\right)-\left\langle\sigma_{h}, \mu_{h}\right\rangle & =0 & & \forall \mu_{h} \in V_{h 0}^{g} . \tag{5.12}
\end{align*}
$$

The well-posedness of the mixed methods (5.9)-(5.10) and (5.11)-(5.12) follows from the stability of the mixed methods (5.1)-(5.2) and (5.5)-(5.6).

Lemma 5.2. The mixed method (5.9)-(5.10) is equivalent to the mixed method (5.1)-(5.2). And the mixed method (5.11)-(5.12) is equivalent to the mixed method (5.5)-(5.6).

Proof. Suppose $\left(\boldsymbol{w}_{h}, 0\right) \in \boldsymbol{W}_{h 0} \times V_{h 0}^{g}$ is the solution of the mixed method (5.1)(5.2). By the fact $\lambda_{h}=0$, apparently $\left(\boldsymbol{w}_{h}, 0\right)$ is also the solution of the mixed method (5.9)-(5.10). The equivalence between the mixed method (5.11)-(5.12) and the mixed method (5.5)-(5.6) follows similarly.

Such equivalence in matrix form has been revealed in [12, (77)-(78)]. The matrix form of the mixed finite element method (5.9)-(5.10) is

$$
\left(\begin{array}{cc}
A & B^{\boldsymbol{\top}} \\
B & -D
\end{array}\right)\binom{\boldsymbol{w}_{h}}{\lambda_{h}}=\binom{\boldsymbol{f}}{0} .
$$

Here we still use $\boldsymbol{w}_{h}, \lambda_{h}$ and $\boldsymbol{f}$ to represent the vector forms of $\boldsymbol{w}_{h}, \lambda_{h}$ and $\left(\boldsymbol{f}, \boldsymbol{v}_{h}\right)$ for ease of presentation. Noting that $D$ is diagonal, we get

$$
\left(A+B^{\boldsymbol{\top}} D^{-1} B\right) \boldsymbol{w}_{h}=\boldsymbol{f}
$$

The Schur complement $A+B^{\boldsymbol{\top}} D^{-1} B$ corresponds to the symmetric matrix of a discontinuous Galerkin method for the vector Laplacian, which is positive definite and can be solved by the conjugate gradient method with the HX preconditioner in [26].

## 6. Numerical Results

In this section, we perform a numerical experiment to demonstrate the theoretical results of the mixed finite element methods (4.5)-(4.6). Let $\Omega=(0,1)^{3}$. And choose the function $\boldsymbol{f}$ in (4.1) such that the exact solution of (4.1) is

$$
\boldsymbol{u}=\operatorname{curl}\left(\begin{array}{c}
0 \\
0 \\
\sin ^{3}(\pi x) \sin ^{3}(\pi y) \sin ^{3}(\pi z)
\end{array}\right)
$$

We take uniform triangulations on $\Omega$. Set $k=0$.
Numerical results of errors $\left\|\boldsymbol{u}-\boldsymbol{u}_{h}\right\|_{0},\left\|\operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right)\right\|_{0}$ and $\left|\operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right)\right|_{1, h}$ with respect to $h$ for $k=0$ are shown in Table 1, from which we can see that they all achieve the optimal convergence rates numerically and agree with the theoretical error estimates in (4.28) and (4.34). It is also observed from Table 1 that $\| \operatorname{curl}_{h}(\boldsymbol{u}-$ $\left.\boldsymbol{u}_{h}\right) \|_{0}=O\left(h^{2}\right)$ numerically, which is one order higher than the theoretical order in (4.34).

Table 1. Errors $\left\|\boldsymbol{u}-\boldsymbol{u}_{h}\right\|_{0},\left\|\operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right)\right\|_{0}$ and $\mid \operatorname{curl}_{h}(\boldsymbol{u}-$ $\left.\boldsymbol{u}_{h}\right)\left.\right|_{1, h}$ for $k=0$ and different $h$.

| $h$ | $\left\\|\boldsymbol{u}-\boldsymbol{u}_{h}\right\\|_{0}$ | order | $\left\\|\operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right)\right\\|_{0}$ | order | $\left\|\operatorname{curl}_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right)\right\|_{1, h}$ | order |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $2^{-1}$ | $1.025 \mathrm{E}+00$ | - | $1.050 \mathrm{E}+01$ | - | $1.076 \mathrm{E}+02$ | - |
| $2^{-2}$ | $9.687 \mathrm{E}-01$ | 0.08 | $5.306 \mathrm{E}+00$ | 0.98 | $9.099 \mathrm{E}+01$ | 0.24 |
| $2^{-3}$ | $3.767 \mathrm{E}-01$ | 1.36 | $1.618 \mathrm{E}+00$ | 1.71 | $5.374 \mathrm{E}+01$ | 0.76 |
| $2^{-4}$ | $1.640 \mathrm{E}-01$ | 1.20 | $4.311 \mathrm{E}-01$ | 1.91 | $2.820 \mathrm{E}+01$ | 0.93 |
| $2^{-5}$ | $7.828 \mathrm{E}-02$ | 1.07 | $1.097 \mathrm{E}-01$ | 1.97 | $1.428 \mathrm{E}+01$ | 0.98 |

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## Appendix A. Regularity of the quad curl problem on convex domains

We will prove the regularity of problem (4.2) under the assumption $\boldsymbol{f} \in \boldsymbol{H}^{-1}$ (div, $\Omega$ ). Similar regularity can be found in $[43$, Theorem 3.5$]$ when $\boldsymbol{f} \in \boldsymbol{L}^{2}\left(\Omega ; \mathbb{R}^{3}\right)$.
Lemma A.1. Assume domain $\Omega$ is convex. Let $\boldsymbol{u} \in \boldsymbol{H}_{0}(\operatorname{grad} \operatorname{curl}, \Omega)$ be the solution of problem (4.2) with the divergence-free right hand side $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\operatorname{div}, \Omega)$, then

$$
\begin{equation*}
\|\boldsymbol{u}\|_{1}+\|\operatorname{curl} \boldsymbol{u}\|_{2} \lesssim\|\boldsymbol{f}\|_{-1} \tag{A.1}
\end{equation*}
$$

Proof. Due to the framework in [11], the problem (4.2) can be equivalently decoupled into the following system: find $\boldsymbol{w} \in \boldsymbol{H}_{0}(\operatorname{curl}, \Omega), \lambda \in H_{0}^{1}(\Omega), \boldsymbol{\phi} \in \boldsymbol{H}_{0}^{1}\left(\Omega ; \mathbb{R}^{3}\right)$, $p \in L_{0}^{2}(\Omega), \boldsymbol{u} \in \boldsymbol{H}_{0}(\operatorname{curl}, \Omega)$ and $\sigma \in H_{0}^{1}(\Omega)$ such that

$$
\begin{align*}
(\operatorname{curl} \boldsymbol{w}, \operatorname{curl} \boldsymbol{v})+(\boldsymbol{v}, \nabla \lambda) & =\langle\boldsymbol{f}, \boldsymbol{v}\rangle & & \forall \boldsymbol{v} \in \boldsymbol{H}_{0}(\operatorname{curl}, \Omega),  \tag{A.2}\\
(\boldsymbol{w}, \nabla \tau) & =0 & & \forall \tau \in H_{0}^{1}(\Omega),  \tag{A.3}\\
(\boldsymbol{\nabla} \boldsymbol{\phi}, \boldsymbol{\nabla} \boldsymbol{\psi})+(\operatorname{div} \boldsymbol{\psi}, p) & =(\operatorname{curl} \boldsymbol{w}, \boldsymbol{\psi}) & & \forall \boldsymbol{\psi} \in \boldsymbol{H}_{0}^{1}\left(\Omega ; \mathbb{R}^{3}\right),  \tag{A.4}\\
(\operatorname{div} \phi, q) & =0 & & \forall q \in L_{0}^{2}(\Omega),  \tag{A.5}\\
(\operatorname{curl} \boldsymbol{u}, \operatorname{curl} \boldsymbol{\chi})+(\boldsymbol{\chi}, \nabla \sigma) & =(\boldsymbol{\phi}, \operatorname{curl} \boldsymbol{\chi}) & & \forall \boldsymbol{\chi} \in \boldsymbol{H}_{0}(\operatorname{curl}, \Omega),  \tag{A.6}\\
(\boldsymbol{u}, \nabla \mu) & =0 & & \forall \mu \in H_{0}^{1}(\Omega) . \tag{A.7}
\end{align*}
$$

Here $\langle\cdot, \cdot\rangle$ is the dual pair between $\boldsymbol{H}^{-1}(\operatorname{div}, \Omega)$ and $\boldsymbol{H}_{0}(\operatorname{curl}, \Omega)$. Since $\boldsymbol{w}, \boldsymbol{u} \in$ $\boldsymbol{H}_{0}(\operatorname{curl}, \Omega) \cap \boldsymbol{H}(\operatorname{div}, \Omega)$, we have $\boldsymbol{w}, \boldsymbol{u} \in \boldsymbol{H}^{1}\left(\Omega ; \mathbb{R}^{3}\right)[20$, Section I.3.4] and

$$
\begin{align*}
&\|\boldsymbol{w}\|_{1} \lesssim \operatorname{curl} \boldsymbol{w} \|_{0} \\
& \lesssim\|\boldsymbol{f}\|_{-1}  \tag{A.8}\\
&\|\boldsymbol{u}\|_{1} \lesssim \operatorname{curl} \boldsymbol{u}\left\|_{0} \lesssim\right\| \boldsymbol{\phi} \|_{0}
\end{align*}
$$

By the regularity of the Stokes problem (A.4)-(A.5) [20, Remark I.5.6], we have

$$
\begin{equation*}
\|\boldsymbol{\phi}\|_{2} \lesssim\|\operatorname{curl} \boldsymbol{w}\|_{0} \lesssim\|\boldsymbol{f}\|_{-1} \tag{A.9}
\end{equation*}
$$

Finally we conclude (A.1) from (A.8)-(A.9) and the fact $\boldsymbol{\phi}=\operatorname{curl} \boldsymbol{u}$.

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