Nonconsensus opinion model on directed networks

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Dynamic social opinion models have been widely studied on undirected networks, and most of them are based on spin interaction models that produce a consensus. In reality, however, many networks such as Twitter and the World Wide Web are directed and are composed of both unidirectional and bidirectional links. Moreover, from choosing a coffee brand to deciding who to vote for in an election, two or more competing opinions often coexist. In response to this ubiquity of directed networks and the coexistence of two or more opinions in decision-making situations, we study a nonconsensus opinion model introduced by Shao *et al.* [Phys. Rev. Lett. **103**, 018701 (2009)] on directed networks. We define directionality ξ as the percentage of unidirectional links in a network, and we use the linear correlation coefficient ρ between the in-degree and out-degree of a node to quantify the relation between the in-degree and out-degree. We introduce two degree-preserving rewiring approaches which allow us to construct directed networks that can have a broad range of possible combinations of directionality ξ and linear correlation coefficient ρ and to study how ξ and ρ impact opinion competitions. We find that, as the directionality ξ or the in-degree and out-degree correlation ρ increases, the majority opinion becomes more dominant and the minority opinion's ability to survive is lowered.

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I. INTRODUCTION

Network theory based on graph theory uses a graph to represent symmetric or asymmetric relations between objects shown by undirected and directed links, respectively. The study of social networks is one of the most important applications of graph theory. Social scientists began refining the empirical study of networks in the 1970s, and many of the mathematical and physical tools currently used in network science were originally developed by them [1]. Social network science has been used to understand the diffusion of innovations, news, and rumors as well as the spread of disease and health-related human behavior [2–9]. The decades-old hot topic of opinion dynamics continues to be a central focus among researchers attempting to understand the opinion formation process. Although it may seem that treating opinion as a variable or a set of variables is too reductive and the complexity of human behavior makes such an approach inappropriate, often human decisions are in response to limited options: to buy or not to buy, to choose Windows or Linux, to buy Procter & Gamble or Unilever, to vote for the Republican or the Democrat.

Treating opinion as a variable allows us to model patterns of opinion formation as a dynamic process on a complex network with nodes as agents and links as interactions between agents. Although the behavior dynamics of human opinion are complex, statistical physics can be used to describe the "opinion states" within a population and also the underlying processes that control any transitions between them [10–16]. Over the past decade, numerous opinion models have combined complex network theory and statistical physics. Examples include the Sznajd model [17], the voter model [18–20], the majority rule model [21,22], the social impact model [23,24], and the bounded confidence model [25,26]. All of these models ultimately produce a consensus state in which all agents share the same opinion. In most real-world scenarios, however, the final result is not consensus but the coexistence of at least two differing opinions.

Shao *et al.* [27] proposed a nonconsensus opinion (NCO) model that achieves a steady state in which two opinions can coexist. Their model reveals that when the initial population of a minority opinion is above a certain critical threshold, a large steady-state spanning cluster with a size proportional to the total population is formed [27]. This NCO complex network model belongs to the same universality class as percolation [27–29], and has received much attention. Among the variants are a NCO model with inflexible contrarians [30] and a NCO model on coupled networks [31,32].

To date, the model has not been applied on directed networks. Directed networks are important because many real-world networks, e.g., Twitter, Facebook, and email networks, are directed [33]. In contrast to undirected networks, directed networks contain unidirectional links. In opinion models, a unidirectional link between two nodes indicates that the influence passing between the two nodes is one way. A real-world example might be a popular singer who influences the opinions the fans hold, but the fans do not influence the singer's opinion. In contrast, bidirectional links occur when the influence between two agents is both ways. Real-world unidirectional links are ubiquitous and strongly influence opinion formation, i.e., widespread one-way influence has a powerful effect on opinion dynamics within a society.

Our goal here is to examine how the NCO model behaves on directed networks. We compare the results of different networks in which we vary the proportion of unidirectional links. We also measure the influence of asymmetry between in-degree and out-degree. We find that when the in-degree and out-degree of each node are the same, an increase in the number of unidirectional links helps the majority opinion spread and when the fraction of unidirectional links is at a certain level, increasing the asymmetry between in-degree and out-degree

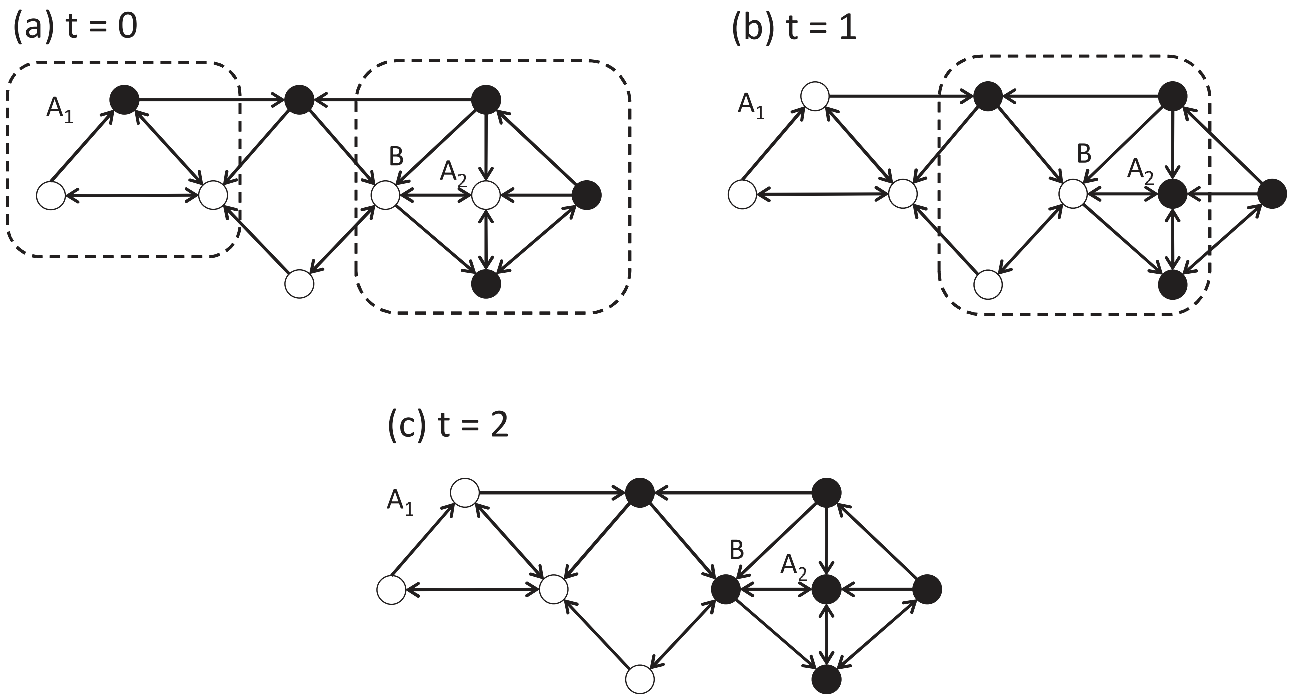


FIG. 1. Schematic plot of the dynamics of the NCO model on a directed graph with 10 nodes.

increases the minority opinion's ability to survive. We also observe that changing the proportion of the unidirectional links or the relationship between the in-degree and out-degree of the nodes causes phase transitions.

II. BASIC DEFINITIONS AND NOTATIONS

A. NCO model

In a NCO model [27] on a single network with N nodes, each with binary opinions, a fraction f of nodes has opinion σ_+ and a fraction $1-f$ has opinion σ_-. The opinions are initially randomly assigned to each node. At each time step, each node adopts the majority opinion, when considering both its own opinion and the opinions of its nearest neighbors (the agent's friends). A node's opinion does not change if there is a tie. Following this opinion formation rule, at each time step the opinion of each node is updated. The updates occur simultaneously and in parallel until a steady state is reached. Note that when the initial fraction f is above a critical threshold, $f \equiv f_c$ (even minority), both opinions continue to exist in the final steady state.

Figure 1 shows an example of the dynamic process of the NCO model on a small directed network with 10 nodes. Here, we consider the in-neighbors of a node as the friends influencing the node, and the out-neighbors as the friends influenced by the node. At time $t=0$, five nodes are randomly assigned the opinion σ_+ (empty circle), and the other five nodes the opinion σ_-. At time $t=0$, nodes A_1 and A_2 have opinions σ_- and σ_+, respectively, but are in a local minority and thus updating it means changing their opinions to σ_+ and σ_-, respectively. At time $t=1$, node B belongs to a local minority and thus needs updating. At time $t=3$, all nodes hold the same opinion as their local majority, and the system has reached a final nonconsensus steady state.

B. Directionality ξ and in-degree–out-degree correlation ρ

To quantitatively measure the one-way influence in a network, we define the directionality ξ as the ratio between unidirectional links and all links. The directionality is $\xi = L_{\text{unidirectional}}/L$, where the normalization $L = L_{\text{unidirectional}} + 2L_{\text{bidirectional}}$, because a bidirectional link can be considered as two unidirectional links. Because we want to determine how much one-way influence affects the NCO model, we consider as a variable the fraction of one-way links ξ, where $\xi = 0$ represents undirected networks. Although the sum of in-degree and the sum of out-degree are equal in a directed network, the in-degree and out-degree of a single node are usually not the same. To quantify the possible difference between the node's in-degree and out-degree, we use the linear correlation coefficient ρ between them,

$$\rho = \frac{\sum_{i=1}^{N}(k_{i,\text{in}} - \langle k\rangle)(k_{i,\text{out}} - \langle k\rangle)}{\sqrt{\sum_{i=1}^{N}(k_{i,\text{in}} - \langle k\rangle)^2}\sqrt{\sum_{i=1}^{N}(k_{i,\text{out}} - \langle k\rangle)^2}}, \quad (1)$$

where $k_{i,\text{in}}$ and $k_{i,\text{out}}$ are the in-degree and out-degree of node i, respectively. The average degree $\langle k\rangle$ is the same for both in-degree and out-degree. Note that when $\rho = 1$, the in-degree is linearly dependent on the out-degree for all nodes, and when $\rho = 0$ the in-degree and out-degree are independent of each other. In this paper, we confine ourselves to the case in which the in-degree and out-degree follow the same distribution. In this case, $\rho = 1$ implies that $k_{i,\text{in}} = k_{i,\text{out}}$ holds for every node i.

III. ALGORITHM DESCRIPTION

Inspired by earlier research on directed networks [31,33–37], we propose two algorithms to construct directed networks. One is a rewiring algorithm that can be applied to any existing undirected network to obtain a directed network with

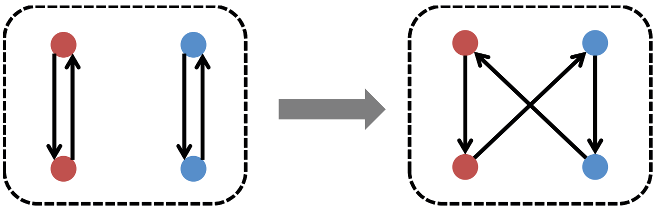


FIG. 2. (Color online) Directionality-increasing rewiring (DIR).

any given directionality, but each node has the same in-degree and out-degree as the original undirected network. The other constructs directed networks with a given directionality and in-degree–out-degree correlation, and with the same given in-degree and out-degree distribution. Note that all networks considered in this paper contain neither self-loops nor multiple links in one direction between two nodes.

A. Directionality-increasing rewiring (DIR)

Here, we introduce a rewiring approach that changes the directionality but does not change the in-degree and out-degree of any node. It was first proposed in Ref. [38], and also employed by Ref. [33]. Here, we improve it to gradually increase the directionality, via a technique we call directionality-increasing rewiring (DIR).

Many undirected network models with various properties have been designed. Examples include the Erdös-Rényi model [39], the Bárabasi-Albert scale-free model [40], and the small-world model [41]. If the links of an undirected graph are considered bidirectional, for an arbitrary undirected graph the in-degree and out-degree correlation will be $\rho = 1$. Figure 2 demonstrates an approach that changes the directionality but does not change the in-degree and out-degree of any node nor ρ. We randomly choose two bidirectional links connecting four nodes and treat them as four unidirectional links. Note that this differs from the technique presented in Ref. [33] in that we choose two bidirectional links instead of two random links that may also contain unidirectional links so that the directionality increases after each step. Then, we choose two unidirectional links, one from each bidirectional link, and rewire them as follows. For both unidirectional links the head of one link is replaced with the head of the other. If this rewiring introduces multiple links in any direction between any two nodes, we discard it and randomly choose two other bidirectional links. We can increase the number of unidirectional links by repeating the rewiring step and increasing the directionality in each step. The directionality ξ can be varied from 0 to 1. In general, DIR can be applied to any directed network to further increase its directionality.

B. Constructing an asymmetric in-degree and out-degree network and rewiring it to decrease its directionality (ANC-DDR)

We have shown how to obtain a desired directionality ξ when the in-degree and out-degree correlation is $\rho = 1$. We further propose an algorithm to construct a network with a given combination of ξ and ρ, where $\rho \neq 1$. Inspired by the work presented in Ref. [36], which focuses on generating directed scale free (SF) networks with correlated in-degree and out-degree sequences, we extend it to a scenario in which the in-degree and out-degree sequences follow a distribution that is arbitrary but the same, and we control not just the correlation between the in-degree and out-degree but also the directionality, which was ignored in Ref. [36]. We generate an in-degree sequence (following a Poisson distribution or power law) and a null out-degree sequence. We then copy a fraction ρ of the in-degree sequence to the out-degree sequence, and shuffle the fraction $1-\rho$ of the in-degree sequence as the rest of the out-degree sequence. We thus create an out-degree sequence, a fraction ρ of which is identical to the corresponding part of the in-degree sequence and a fraction $1-\rho$ of which is independent of the in-degree sequence. After randomly connecting all nodes (given their in-degree and out-degree), as in the configuration model [34], we obtain a network with a directionality[1] $\xi \approx 1$ and an in-degree and out-degree correlation close to ρ. Note that we can further control the in-degree and out-degree correlation in a small range close to ρ by discarding networks with in-degree and out-degree correlations outside the expected range. This enables us to construct a network with the in-degree and out-degree correlation ρ ($0 \lessapprox \rho \leqslant 1$), a technique we call asymmetric in-degree–out-degree network constructing (ANC).

We use the following rewiring steps to further tune the directionality without changing the in-degree and out-degree of each node or the in-degree and out-degree correlation ρ. The goal is to decrease the directionality by repeatedly rewiring two unidirectional links into one bidirectional link. In each step, we choose four nodes linked with at least three directed links as shown on the top half of Fig. 3(a). We rewire these three links to the positions shown at the bottom of Fig. 3(a). If this rewiring introduces multiple links between any two nodes in any direction, we discard the rewiring, select four new nodes, and repeat the step.[2] This rewiring produces at least one more bidirectional link and thus decreases the directionality. We call this procedure directionality-decreasing rewiring (DDR). We combine DDR with ANC and call the entire algorithm ANC-DDR. It seems that ANC-DDR may introduce disconnected components. However, we will see later in Sec. V that the networks generated by all the algorithms are well connected, i.e., almost all the nodes are included in the largest component.

Using ANC, we can construct a network with a specified in-degree and out-degree correlation ρ, where the in-degree and the out-degree follow the same given distribution and, using DDR, we can change the directionality ξ in a range dependent on the given ρ without changing the in-degree and out-degree. The range within which we can tune $\xi \in [\xi_{\min}, 1]$ depends on the given ρ. For example, for binomial networks,[3] ξ can be changed from 0 to 1 when $\rho = 1$, but the minimum value of ξ must be approximately 0.3 and any smaller ξ value is disallowed when $\rho = 0$. We explore the relation between the minimal possible directionality ξ and a given in-degree and

[1] $E(\xi) = 1 - \langle k \rangle^2 N/(N-1)^2$, $\lim_{N\to+\infty} E(\xi) = 1$.

[2] An efficient rewiring program is available upon request.

[3] *Binomial networks* are directed networks with the same Poissonian in-degree and out-degree distributions.

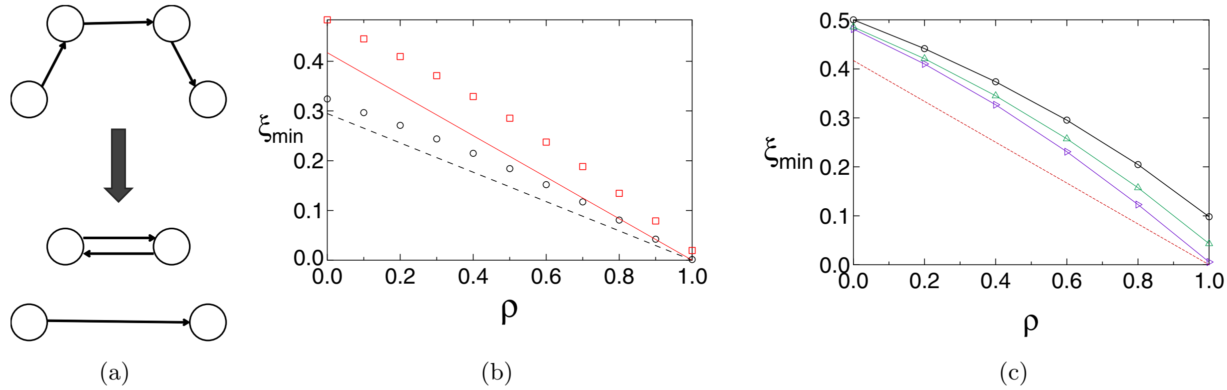


FIG. 3. (Color online) (a) The degree-preserving rewiring for decreasing the directionality. (b) Plot of the minimal directionality $\xi_{\min}$ obtained by simulating ANC-DDR, for binomial networks (○, $\langle k\rangle = 4$, 10^5 nodes) and SF networks (□, $\lambda = 2.63$, 10^5 nodes) with 100 realizations ($M = \langle k\rangle N$), and the theoretical minimum possible directionality $\xi_{\min}$ [Eq. (2)] for binomial networks (the solid line) and for SF networks (the dashed line) as a function of the in-degree and out-degree correlation ρ. (c) Plot of the minimal directionality $\xi_{\min}$ obtained by simulating ANC-DDR with different values of the attempts M: $0.01 * M_0$(○),$0.1 * M_0$(△),$100 * M_0$(◁), where $M_0 = \langle k\rangle N$, and the theoretical minimum possible directionality $\xi_{\min}$ (the dashed line) for SF networks. All the results are the averages of 1000 realizations.

out-degree correlation ρ first via numerical simulations[4] in both binomial and SF networks.[5] Figure 3(b) shows the linear relationship in both types of network. Binomial networks are characterized by a Poisson degree distribution with $P(k) = e^{-\langle k\rangle}\langle k\rangle^k/k!$, where k is the node degree and $\langle k\rangle$ is the average degree. The degree distribution of SF networks is given by $P(k)\sim k^{-\lambda}, k \in [k_{\min},k_{\max}]$, where $k_{\min}$ is the smallest degree, $k_{\max}$ is the degree cutoff, and λ is the exponent characterizing the broadness of the distribution [40]. In this paper, we use the natural cutoff at approximately $N^{1/(\lambda-1)}$ [42] and $k_{\min} = 2$.

For any network constructed using ANC-DDR with an arbitrary given degree distribution $P(k)$ (where the distribution is same in both in-degree and out-degree), we can analytically prove (see Appendix the relationship between the minimal possible directionality $\xi_{\min}$ and the in-degree–out-degree correlation ρ,

$$E(\xi_{\min}) = \frac{1-\rho}{\langle k\rangle}\sum_{k=0}^{N-1} kP(k)\left(\sum_{i=0}^{k} P(i) - \sum_{i=k}^{N-1} P(i)\right). \quad (2)$$

Because of the finite number M of attempts and the random selection process determining the four-node structure, the $\xi_{\min}$ obtained using simulations is slightly larger than the theoretical $\xi_{\min}$. To further understand the gap between the simulation and theoretical results concerning the relationship between $\xi_{\min}$ and ρ, we plot the minimal directionality $\xi_{\min}$ obtained by simulating ANC-DDR with different values of the attempts M in Fig. 3(c). We can see that, as M increases, the simulation results becomes closer to the theoretical results, but the simulation results seem not to converge to the theoretical results. Actually, we find our algorithm may end up at the network in which we cannot further rewire the links to lower the directionality even with exhaustive searching, although the directionality of the network is still larger than the theoretical minimal. According to the Appendix, the theoretical minimal possible directionality can be reached if and only if all the unidirectional links of each node are either in-degree links or out-degree links. However, such structure patterns are mostly unreachable. That is to say, the random process (DDR) almost certainly leads the network to the unrewirable structure patterns which are not the ones inducing the theoretical minimal directionality.

The computational complexity of our algorithm comes mainly from DDR. Here, we will discuss the total rewiring attempts to glimpse the worst case computational complexity of DDR. For a network constructed by ANC, with N nodes, the average degree $\langle k\rangle$, and directionality $\xi = 1$, if we want to rewire it to a desirable directionality ξ_d ($\geqslant\xi_{\min}$), we have to successfully rewire $\langle k\rangle N(1-\xi_d)$ times. This is the maximal possible number of successful rewiring steps since the minimal possible directionality is mostly unreachable. In order to obtain one successful rewiring step, in the worst case, we carry out M rewiring attempts. Hence, in the worst case, the number of the total rewiring attempts including successful and unsuccessful ones is $M^{\langle k\rangle N(1-\xi_d)}$.

Although using ANC-DDR we can construct a network with a given ρ and a given ξ within a corresponding range to ρ, we cannot obtain a network with the directionality ξ close to 0 by DDR. We thus apply DIR to undirected network models in order to generate directed networks with a directionality ranging over [0,1], but with the given in-degree and out-degree correlation $\rho = 1$, to understand the effect of directionality on opinion competitions. We then use ANC-DDR to generate directed networks with a given in-degree and out-degree distribution and correlation, and a given directionality, to explore the effect of both ξ and ρ on the opinion model.

[4]In each realization of the simulations, we apply DDR repeatedly on the network constructed by ANC until the four-node structure in Fig. 3(a) cannot be found after a number M of consecutive attempts, then the directionality ξ is considered the minimal directionality $\xi_{\min}$ corresponding to the given ρ. For each given ρ, we perform 100 realizations and calculate the average of the minimal directionality $\xi_{\min}$.

[5]*SF networks* are directed networks whose in-degree and out-degree distributions follow the same power law.

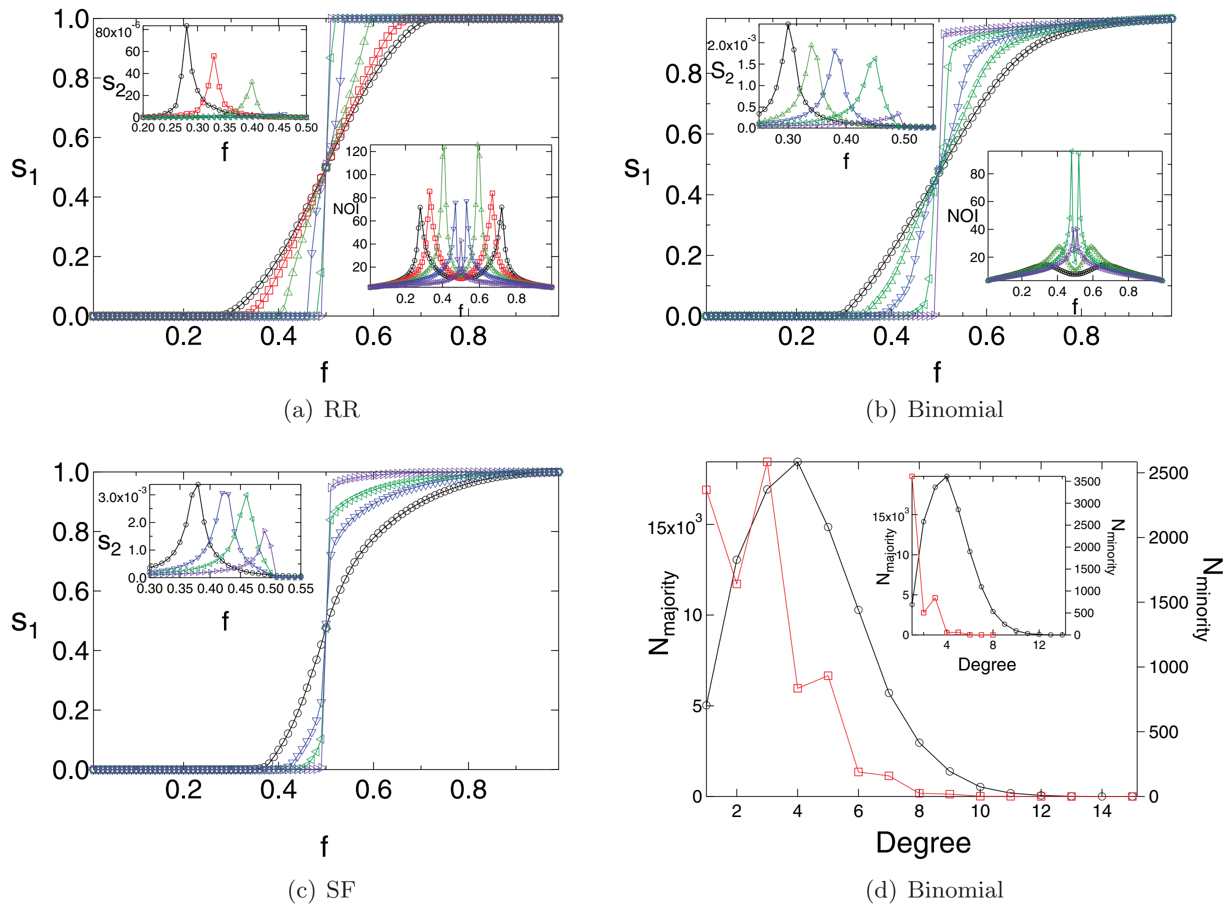


FIG. 4. (Color online) Plot of the normalized largest cluster s_1 of opinion σ_+ as a function of the initial fraction f for different values of the directionality ξ: 0 (○), 0.2 (□),0.4 (△),0.6 (▽),0.8 (◁),1.0 (▷), and for different networks with $N = 10^5$ nodes and $\langle k \rangle = 4$: (a) RR, (b) binomial, (c) SF ($\lambda = 2.63$). In the upper insets of (a), (b), and (c), we plot s_2 as a function of f with the same symbols and for the same networks as in the main figure. In the lower insets of (a) and (b), we plot the number of iterations to the steady-state NOI as a function of f. (d) Plot of the degree distribution of the nodes which keep the majority (○) and the minority opinion (□) in binomial networks (also with $N = 10^5$ nodes and $\langle k \rangle = 4$), when the directionality $\xi = 0.0$ (the main figure) and $\xi = 1.0$ (the inset). All results are based on averaging 1000 realizations.

IV. INFLUENCE OF THE DIRECTIONALITY

In order to examine how the directionality ξ influences the NCO model, we apply DIR to undirected network models to generate directed binomial networks, SF networks, and random regular (RR) networks[6] [43] with directionality ranging over [0,1]. The NCO model is further simulated on each directed network instance. All simulation results are the average of 10^3 networks with $N = 10^5$ nodes and $\langle k \rangle = 4$.

We use S_1 to denote the size of the largest σ_+ cluster in the steady state (where σ_+ is the initial opinion randomly assigned to a fraction f of nodes) and S_2 to denote the size of the second largest cluster. For all three types of networks, we plot $s_1 \equiv S_1/N$ and $s_2 \equiv S_2/N$ as a function of f for different values of the directionality ξ in Figs. 4(a)–4(c). Additionally, we plot the number of iterations to the steady state (NOI) as a function of f for RR and binomial networks in Figs. 4(a) and 4(b), respectively. Note that, depending on the value of ξ, there is a critical threshold $f \equiv f_c$ above which there is a giant steady-state component of opinion σ_+. The peak of s_2 indicates the existence of a second-order phase transition, where s_1 is the order parameter and f is the control parameter. Note that as the value of ξ increases, in all networks f_c shifts to the right, a shift observable from the shift of the peak of s_2. Moreover, the first peak of NOI also shifts to the right as ξ increases, and coincides with the peak of the second largest cluster representing the critical f_c for the minority to emerge as a giant component. The second peak occurs due to symmetry at $1 - f_c$. In RR networks, we lose the peak of s_2 when the directionality ξ is close to 1, which suggests the disappearance of the second-order phase transition. The sharp jump of s_1 around $f = 0.5$ also indicates the appearance of an abrupt phase transition. When these networks contain an increasing one-way influence (increasing directionality), in all cases the minority opinion will need a greater number of initial supporters if they are to survive when the steady state is reached.

To further understand this change, we consider two extreme cases $\xi = 0$ and 1. In the former, an agent influences only those who can influence the agent in return. In the latter, an agent

[6] In this paper, *random regular (RR) networks* are directed networks in which the in-degrees of all nodes and out-degrees of all nodes are the same and the nodes are randomly connected.

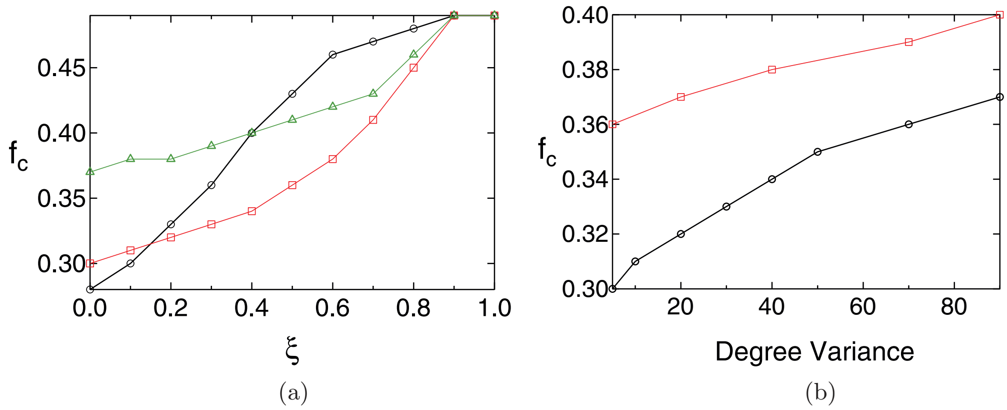


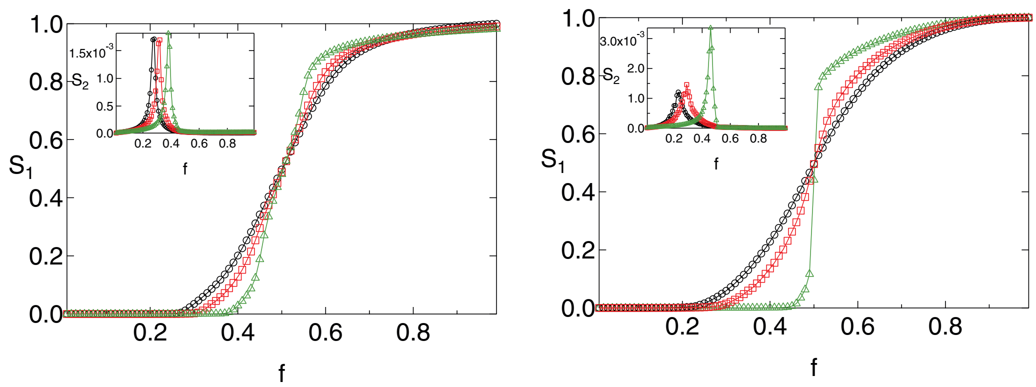
FIG. 5. (Color online) (a) Plot of the critical threshold f_c as a function of the directionality ξ for different networks with $N = 10^5$ nodes and $\langle k \rangle = 4$: RR (○), binomial (□), and SF (△) ($\lambda = 2.63$). All results are based on averaging 1000 realizations. (b) Plot of the critical threshold f_c as a function of the variance of the degree sequence of the networks ($N = 10^5$ nodes and $\langle k \rangle = 4$) with different values of the directionality: $\xi = 0$ (○) and $\xi = 0.5$ (□). All results are based on averaging 100 realizations.

influences only those who cannot influence the agent in return. This latter case allows a much more rapid spread of opinions, each agent interacts with a larger number of agents, each has in-neighbors as well as out-neighbors, and the opinion is diffused over a wider area. Note that both the majority and minority opinions can benefit from this wider diffusion, but there is a higher risk that the minority opinion will be devoured at some point. This is the case because the bidirectional link connecting two minority opinion agents benefits the minority opinion: the two agents can encourage each other to keep the minority opinion. When rewiring this kind of link there is a higher probability that the two agents will interact with the majority opinion and thus a higher probability that their opinion will be changed to the majority opinion. Thus, rewiring makes it more difficult for the minority opinion to form a stable structure.

As directionality ξ increases, it is easier for minority opinion agents to keep their minority opinion if they have fewer neighbors. Figure 4(d) plots the degree distributions (in which the in-degree and out-degree follow the same distribution) of the minority-opinion nodes and majority-opinion nodes, respectively, in the steady state at the critical threshold $f \equiv f_c$ when the directionality is $\xi = 0.0$ and 1.0. Note that the degrees of most of the minority-opinion nodes that keep their minority opinion are equal to 1, 2, or 3. Minority-opinion nodes with a degree larger than 3 can keep their minority opinion when $\xi = 0.0$, but seldom when $\xi = 1.0$, i.e., as the value of ξ increases, the number of nodes following the majority opinion increases, and only low-degree nodes are able to keep the minority opinion. Our results indicate that a lower directionality helps the existence of the minority opinion, which can be desirable, e.g., when the society wants to have different opinions coexisting to inspire or balance each other. We could decrease the directionality of social contact networks by encouraging mutual social interactions, e.g., between friends and family members. Moreover, the minority opinion is likely held by individuals with few social contacts.

It has been shown that network topology may significantly influence such dynamic processes in networks as epidemics or cascading failures [12,44–47]. We thus compare the critical threshold f_c on directed binomial, RR, and SF networks in which the in-degree and out-degree (i) follow the same binomial distribution, (ii) are a constant, and (iii) follow a power-law distribution. Figure 5(a) shows that, as the directionality ξ increases, the critical threshold f_c of the RR networks increases more rapidly than the others. As stated above, as ξ increases, only nodes with degrees less than 4, the average degree, are likely to keep the minority opinion, and in RR networks all nodal degrees are 4. Figure 5(a) also shows that the existence of hubs (extremely high-degree nodes) in SF networks causes them, at $\xi = 0$, to have a much higher critical threshold f_c than the others, and that the critical threshold in binomial networks is slightly larger than the critical threshold in RR networks. The existence of hubs benefits the majority-opinion nodes because the probability that an agent with many friends (i.e., a hub) will follow the majority opinion and influence many others is high. They thus strongly contribute to the diffusion of the majority opinion. This phenomenon suggests that public celebrities, i.e., hubs in social networks, tend to help the propagation of the majority opinion.

References [48,49] describe how a second-order phase transition becomes first order and the critical threshold is higher when the average degree increases. When the average degree is the same, the variance of the degree sequence turns out to be the key factor influencing the critical threshold. In fact, we find that in the networks with the same average degree, the larger the variance of the degree sequence, the larger will be its critical threshold. This is the case because networks with a wider degree variance are more likely to have majority-opinion hubs that can influence many other agents. Figure 5(b) shows simulation results that support this behavior. Note that as the variance of the degree sequence increases, the critical threshold increases. To change the variance in these simulations, we select a SF network with an average degree $\langle k \rangle = 4$, randomly remove an existing link, and randomly add a link between two nodes previously unconnected. As we remove and add links repeatedly, the variance of the degree sequence decreases and we stop at an excepted variance. To obtain the specified directionality, we apply DIR on the networks. This gives us a wide range of degree variance, which allows us to study the relationship between the variance and critical threshold f_c.



(a) Binomial (b) SF

FIG. 6. (Color online) Plot of the normalized largest cluster s_1 of opinion σ_+ as a function of the initial fraction f, when the directionality $\xi = 0.6$, for different values of the in-degree and out-degree correlation ρ: 0 (○),0.5 (□),1 (△), and for different networks with $N = 10^5$ nodes and $\langle k \rangle = 4$: (a) binomial, (b) SF. In the insets, we plot s_2 as a function of f with the same symbols and for the same networks as in the main figure. All results are based on averaging 1000 realizations.

V. INFLUENCE OF IN-DEGREE AND OUT-DEGREE ASYMMETRY

We have discussed how the critical threshold f_c increases as the directionality increases in networks in which the in-degree and out-degree are the same for each node. The number of in-neighbors and out-neighbors of nodes in real-world networks often differ, however. We mentioned above how a popular singer can influence many people and not be influenced in return. The social network of the singer has many more out-neighbors than in-neighbors. Because this real-world phenomenon is so ubiquitous, we now examine how different correlations between the in-degree and out-degree affect opinion competition.

In Sec. III B, we use ANC-DDR to construct a network with an arbitrary but identical in-degree and out-degree distribution, together with a given combination of the directionality ξ and the linear correlation coefficient ρ between the in-degree and out-degree. We perform simulations to study the influence of both the directionality ξ and the correlation coefficient ρ on the critical threshold f_c. Figures 6(a) and 6(b) show that, given the directionality, the critical threshold increases for binomial and SF networks, respectively, as the in-degree and out-degree correlation ρ increases. Figure 7 shows that when the directionality ξ and the correlation coefficient ρ are increased in binomial networks, the critical threshold increases. The same behavior is observed in SF networks. In Fig. 6(a), we can also see that when the initial fraction f is close to 1, the normalized largest cluster s_1 of opinion σ_+ is close to 1, which means that the size of the largest cluster approximately equals to the size of the network. Hence, the algorithm ANC-DDR does not introduce evident disconnected components of the generated networks. Note also that when the in-degree–out-degree correlation $\rho = 1$, the critical thresholds obtained by DIR and ANC-DDR agree with each other for different values of the directionality ξ.

The influence of the in-degree and out-degree correlation ρ on the critical threshold can be understood as follows. A smaller ρ means a clearer inequality or asymmetry between the in-degree and out-degree links. When the in-degree and out-degree links are asymmetrical, a node with more in-neighbors than out-neighbors is more likely to follow the majority opinion and, because it has few out-neighbors, its own opinion will have little influence. Compared with the nodes which have the same number of in-neighbors and out-neighbors and tends to follow as well as spread the majority opinion, such nodes (with fewer out-neighbors) cannot help. Nodes with more out-neighbors than in-neighbors have greater influence and can thus hold the minority opinion and contribute to its spread. Thus, the minority opinion benefits more from an inequality between the in-degree and out-degree or, equivalently, from a smaller ρ, so the lower correlation coefficient ρ leads to a smaller critical value f_c. Such imbalance between the number of friends that influence you and the number of friends that you can influence, actually, helps the existence of the minority opinion.

We now further explore the properties of nodes in the final steady state. We focus on binomial networks in the

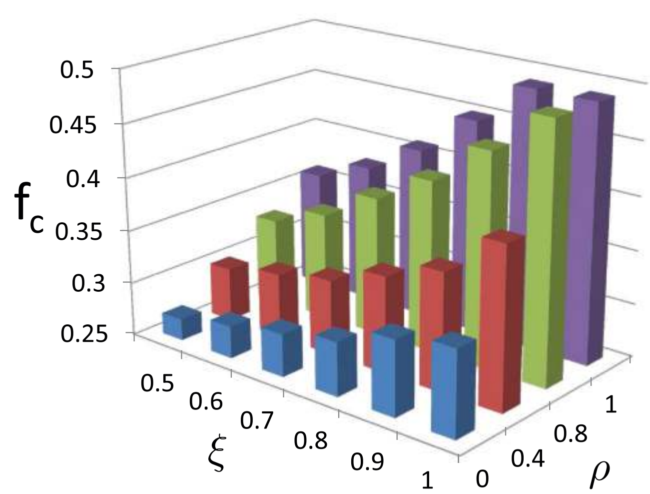


FIG. 7. (Color online) Plot of the critical threshold f_c as a function of the linear correlation coefficient ρ and the directionality ξ for binomial networks. All results are based on averaging 1000 realizations.

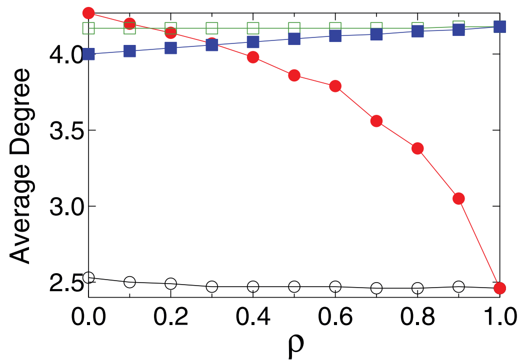


FIG. 8. (Color online) Plot of the average in-degree and out-degree of the nodes in the largest σ_+ and σ_- clusters for binomial networks, when the initial faction f of the opinion σ_+ equals 0.4 as a function of ρ. The representations of the four lines are as follows: the average in-degree (○) and out-degree (●) of the nodes in the largest σ_+ cluster; the average in-degree (□) and out-degree (■) of the nodes in the largest σ_- cluster. All results are based on averaging 100 realizations.

steady state and calculate as a function of ρ the average in-degree and out-degree in the largest σ_+ and σ_- clusters with a directionality $\xi = 1$ (generated by ANC) when the initial fraction f of opinion σ_+ equals 0.4 (minority). As discussed above, and seen in Fig. 8, the out-degree links of a node with the minority opinion in the steady state tend to be larger for all $\rho < 1$ than the in-degree links because nodes with few in-neighbors are less influenced by other nodes and thus can more easily keep their minority opinion. On the contrary, the in-degree of a node with the majority opinion tends to be larger than its out-degree. Note that, when the initial fraction f of the opinion σ_+ is 0.4, the average number of in-degree links is smaller for the nodes in the largest σ_+ cluster compared with the nodes in the largest σ_- cluster. Note also that in the majority clusters (σ^+) both the in-degree and the out-degree are close to 4, which is the average degree of the whole network. This is in marked contrast with the average in-degree of the nodes in the largest minority cluster with degree approximately 2.5. The average out-degree of minority is larger than 4 when the linear correlation coefficient is $\rho = 0$. As ρ increases, there is a higher correlation between the in-degree and out-degree and the average out-degree of minority decreases rapidly.

VI. CONCLUSIONS

Because of the ubiquity of the nonconsensus steady state in real-world opinion competitions and the dominance of unidirectional relationships in real-world social networks, we study a nonconsensus opinion model on directed networks. To quantify the extent to which a network is directed, we use a directionality parameter ξ, defined as the ratio between the number of unidirectional links and the total number of links. We also employ a linear correlation coefficient ρ between the in-degree and out-degree to quantify any asymmetry.

We propose two approaches to construct directed networks. The first is directionality-increasing rewiring (DIR) and is used to rewire the links of an undirected network to obtain a directed network with any directionality value ξ without changing the in-degree and out-degree, i.e., the in-degree–out-degree correlation value is fixed at $\rho = 1$. The second is ANC-DDR, a combination of asymmetric in-degree–out-degree network construction (ANC) and directionality-decreasing rewiring (DDR). Using ANC, we construct a directed network ($\xi \approx 1$) with an arbitrary but identical in-degree and out-degree distribution and a given in-degree–out-degree correlation ρ. We then use DDR to further decrease the directionality ξ of the network.

We use DIR and ANC-DDR to generate directed networks with a given combination of ξ and ρ and investigate how the directionality ξ and the linear correlation coefficient ρ between the in-degree and out-degree links affect the critical threshold f_c of the NCO model. We find that in both binomial and SF networks, increasing ξ or ρ increases the critical threshold f_c. We also find that as ξ and ρ increase, the phase transition becomes abrupt and is no longer second order. We find that as a network becomes more directed, it becomes more difficult for a minority opinion to form a cluster, while increasing the in-degree–out-degree asymmetry makes the minority opinion more stable. Our work indicates that directionality and the asymmetry between in-degree and out-degree play a critical role in real-world opinion competitions.

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APPENDIX: PROOF OF EQ. (2)

Given the in-degree and out-degree of each node, the minimum directionality can be reached if all the unidirectional links of each node are either in-degree links or out-degree links but not both because unidirectional in-degree links and out-degree links of a node may form bidirectional links by rewiring so that the directionality ξ is further reduced. This means that the minimum directionality can be reached if an arbitrary node i has only $|K_{i,\text{in}} - K_{i,\text{out}}|$ unidirectional in-degree links or out-degree links but not both, where $K_{i,\text{in}}$ and $K_{i,\text{out}}$ represent the in-degree and out-degree links of node i, respectively. Hence, the minimum possible directionality, given the number of in-degree and out-degree links of each node, is

$$\xi_{\min} = \frac{\sum_{i=1}^{N} |K_{i,\text{in}} - K_{i,\text{out}}|}{\sum_{i=1}^{N} (K_{i,\text{in}} + K_{i,\text{out}})}. \tag{A1}$$

We denote the in-degree and out-degree sequences by $S_{\text{in}} = \{K_{i,\text{in}} | i = 1,2,\ldots,N\}$ and $S_{\text{out}} = \{K_{i,\text{out}} | i = 1,2,\ldots,N\}$ with the same length N. The in-degree of each node $K_{i,\text{in}}$ is independent and follows the distribution $P(k)$ with the mean $\langle k \rangle$. In order to introduce the in-degree and out-degree correlations, S_{out} is constructed from S_{in} as follows: a fraction ρ of the elements in S_{out} equals that in S_{in} ($K_{i,\text{out}} = K_{i,\text{in}}$, for $i = 1,2,\ldots,\rho N$, without loss of generality, we assume ρN

is an integer), while a fraction $1-\rho$ of S_{out} is obtained by copying and shuffling the rest of S_{in}, such that for $i > \rho N$ and large N, $K_{i,\text{in}}$ and $K_{i,\text{out}}$ are independent but follow the same distribution $Pr[K = k] = P(k)$. Hence,

$$\begin{aligned} E(\xi_{\min}) &= E\left(\frac{\sum_{i=\rho N+1}^{N}|K_{i,\text{in}} - K_{i,\text{out}}|}{\sum_{i=1}^{N}(K_{i,\text{in}} + K_{i,\text{out}})}\right) \\ &= (1-\rho)E\left(\frac{N|K_{\text{in}} - K_{\text{out}}|}{\sum_{i=1}^{N}(K_{i,\text{in}} + K_{i,\text{out}})}\right) \\ &= (1-\rho)E(\xi_{\min,\rho=0}), \end{aligned} \tag{A2}$$

where K_{in} and K_{out} are independent random variables following the same probability distribution $P(k)$, and $\xi_{\min,\rho=0}$ indicates the value of $\xi_{\min}$ when $\rho = 0$.

We then consider the case when $\rho = 0$, i.e.,

$$E(\xi_{\min,\rho=0}) = \frac{E[\text{Max}(K_{\text{in}},K_{\text{out}})] - E[\text{Min}(K_{\text{in}},K_{\text{out}})]}{2\langle k\rangle}, \tag{A3}$$

where Max(...) and Min(...) are the maximum and minimum functions, respectively.

The minimum of random variables $K_{\text{in}}, K_{\text{out}}$ has the distribution

$$\begin{aligned} &Pr[\text{Min}(K_{\text{in}},K_{\text{out}}) = k] \\ &\quad = Pr[K_{\text{in}} = k]Pr[K_{\text{out}} \geqslant k] + Pr[K_{\text{out}} = k]Pr[K_{\text{in}} \geqslant k] \\ &\quad = 2P(k)\sum_{i=k}^{N-1} P(i), \end{aligned} \tag{A4}$$

when the two random variables are independent. In the same way, we have

$$Pr[\text{Max}(K_{\text{in}},K_{\text{out}}) = k] = 2P(k)\sum_{i=1}^{k} P(i). \tag{A5}$$

Hence,

$$E(\xi_{\min,\rho=0}) = \frac{1}{\langle k\rangle}\sum_{k=0}^{N-1} kP(k)\left(\sum_{i=0}^{k} P(i) - \sum_{i=k}^{N-1} P(i)\right). \tag{A6}$$

Combining (A3) and (A2), we have

$$E(\xi_{\min}) = \frac{1-\rho}{\langle k\rangle}\sum_{k=0}^{N-1} kP(k)\left(\sum_{i=0}^{k} P(i) - \sum_{i=k}^{N-1} P(i)\right). \tag{A7}$$

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