NONCOOPERATIVE GENERAL EXCHANGE

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1. BACKGROUND

It is commonly recognized that the classical Walrasian model of pure competitive exchange is not well suited to applications in which the markets for some commodities may be "thin," in the sense that the number of traders interested in buying or selling those commodities is very small. The mathematical representation of the model gives no hint of this weakness; indeed, the formal equilibrium theory goes through beautifully regardless of the number of traders. But when we cross the shadowy watershed between the worlds of "many" and "few," we begin to encounter individual entrepreneurs who are big enough to be able to influence prices market-wide, and so find that the implicit behavioral assumptions on which the competitive model rests are no longer met.

In the 1960s, the theory of games had reached a stage of development where it could begin to throw some light into this shadowy transition zone between "many" and "few." As it happens, the classical model of an exchange economy

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fits very neatly into the format of a multiperson cooperative game. This makes it quite easy to bring to bear a number of different game-theoretic solution concepts and to compare them with the Walrasian competitive equlibrium. Best known of these game solutions, to economists, is the core--the set of coalitionally unimprovable outcomes. The core has been shown to yield substantial agreement with the set of competitive equilibria, in suitably large and homogeneous economies, but to yield qualitatively different results in the presence of "big" traders. A similar situation exists with respect to several other cooperative solution concepts.* Since each of them can be given an economically plausible justification, the possibility is open that the core, or one of the other solutions, might be more faithful to the true state of affairs, with the competitive equilibrium standing merely as a convenient approximation when the situation approaches some limiting ideal of "perfect" competition. At the very least, it seems worthwhile to try to use game theory to build a bridge between that asymptotic ideal, on the one side, and the "imperfections" of oligopolistic behavior and small-group bargaining, on the other.**

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*For example, the "value"; see [2, 3, 15]. The von Neumann-Morgensterm solution behaves rather differently; see [8, 9].

**We feel that the core may have been oversold in this regard. Though "collusive" in a certain sense, the core has little to do with the characteristic tactics of economic collusion: boycotts, strikes, price-manipulation, insincere bidding, restraint of trade. (See [1, 5, 6, 7, 14, 15, 18, 19].) Moreover, like the other cooperative solution concepts, the core is by definition Pareto-optimal, precluding the social inefficiency that is almost the hallmark of monopolistic competition.

These developments may hold great promise of explaining deviations from pure competition that are attributable to coalitional activity in thin markets. But they hardly seem to touch the more inherent and longer-recognized weakness of the classical model, which has little to do with coalitions, namely, the need to assume that entrepreneurs cannot single-handedly influence the price structure. Walrasian prices are given *ex machina* and by definition are not responsive to the traders' buy/sell decisions. The Walrasian equilibrium is tested by conjectural variation of individual bids for goods, but without allowing for any price fluctuation in response to the changed bids. Such a rigid-price assumption cannot be allowed to stand if we hope to build a theoretical bridge between general equilibrium and, say, Cournot oligopoly. The game model that we construct must somehow permit the traders to participate in the formation of prices, and hence must differ radically from the "cooperative game" models that have been offered to date, which virtually ignore the price mechanism while concentrating on coalitional phenomena.

2. A NEW APPROACH WITH APPLICATIONS TO EXTERNALITIES

In this paper we shall describe a new method of "making a game" out of general equilibrium, which we have recently been exploring in collaboration with Martin Shubik. It is based on the "noncooperative" side of game theory, in the spirit of Nash and Cournot. Our traders do not collude, but they do influence prices through their buy and sell orders. The noncooperative equilibria (N.E.) of the game are conceptually not too far removed from the classical competitive equilibria (C.E.), the relationship between the two being especially close when there are many traders of each type and hence relatively stable prices. But when there are "thin" markets, we shall typically find substantial Pareto inefficiency in the N.E., with trading sharply reduced in volume or otherwise displaced from the competitive pattern.

Our previous attempts in this vein had been hampered by the fact that the classical model of an exchange economy does <u>not</u> fit neatly into the <u>non</u>cooperative game format. The noncooperative theory requires a game in "strategic form" rather than "coalitional form." Many <u>ad hoc</u> assumptions were apparently necessary to specify the range of choices for the traders and the rules for price formation in the market; these led to a series of special models of special situations, often interesting in themselves, but with little claim to generality.* Our current approach offers new hope. The decisive step was to meet the problem of money head on--to accept the proposition that, in the world of buying and selling, money is "real." Granting this, the rest falls into place, with remarkably few further assumptions. While some special rules on the handling of money must be made for definiteness (corresponding, indeed, to actual constraints on methods of payment, credit, bankruptcy, etc.), they can be varied widely within our basic framework without upsetting the price-forming mechanism that makes the model work. Thus, what we present here is not a single model of noncooperative general exchange, but a sampling from a class of interrelated models, suitable both for general-purpose microeconomic theory and for studying the phenomena of money and financial institutions.**

There are two ways in which these noncooperative game models may be of service in the study of <u>externalities</u>.*** First, most mathematical theories in this area rely upon some kind of competitive pricing mechanism, both for their negative results, (e.g., nonexistence or nonoptimality of Walrasian prices), and their positive results, (e.g.,

*See Shubik, et al [10, 11, 12, 16, 20, 21, 24].

**See Shubik, et al [22, 23, 25].

***For a "cooperative game" approach to externalities, see [17].

existence or other properties of Lindahlian prices). But most applications of these models must confront "thin" markets.* It would seem prudent, in the light of the above discussion, to consider whether competitive pricing is really suitable in such applications, or whether it it would be better (even leaving aside any questions of collusion) that we recognize the ability of the traders individually to influence price levels.

It might be added that our one major restrictive condition--the requirement of an explicit "money" to serve as a medium of exchange, will not be an added burden in many studies in the field of externalities, since "money" will have to be brought into the model anyway, say as a vehicle for taxation, subsidy or interpersonal compensation--or even as a metric for social value.

To avoid potential misunderstanding, we would stress that the best indications are that the noncooperative equilibrium characteristically "breaks down" in the presence of externalities in much the same way and for much the same reasons as does the classical C.E. An important difference, however, is that when the N.E. fails to exist we are still left with a well-defined game, for which other types of solution

*This occurs automatically when artificially differentiated goods are interjected into the model, each applicable to only one or a few individuals.

may be attempted.* But it is difficult to see how to salvage anything from the wreckage, when the C.E. fails to exist.

The second point at which our approach relates to the question of externalities has to do with the way the market operates. So-called "pecuniary externalities" arising out of the economic system itself can have significant impact, even in the absence of the more familiar and better-understood externalities of production and consumption. Consider, for example, the effect on the fortunes of the small suppliers of a commodity when a large producer "dumps" his surplus on the market. If we choose to regard the rules by which prices are determined as "laws of nature," i.e., as part of the physical or technological environment, then this effect is as much an externality as, say, the lowering of the underground water table by a thirsty industrial plant. The analogy is not perfect; economics is not physics. There is perhaps a semantic question here, but our approach does give one a theoretical grip on these pecuniary "externalities," however one may want to fit them into the general scheme of things.

Another systemic externality that we are able to study is the evaluation of the market itself as a public good, or perhaps, a public "bad." As we have already mentioned, the N.E. typically yields outcomes that are Pareto-

*Including the "Nash equilibrium in mixed strategies," which will exist under very general conditions, though it may be hard to justify in economic terms.

inefficient, in the manner of a Cournot oligopoly. Some of this social loss might be rationalized as a kind of social transactions cost, to be compared with costs that might be associated with other possible methods of price-determination. But when the social loss is great, most of it would better be charged directly to undue monopolistic concentration, which is another kind of public "bad." The use of noncooperative game models may prove to be the key to successful analysis of these questions.

3. THE BASIC MODEL

Let there be n traders and m + 1 commodities. The m + 1st commodity might be called "gold," because it both has intrinsic utility (possibly) and plays a special monetary role in the market. The traders have nonnegative initial bundles:

$$(1) \qquad a^i = (a^i_1, \ldots, a^i_m, a^i_{m+1}), \qquad i = 1, \ldots, n,$$

and concave, nondecreasing utility functions:

$$(2) \qquad u^i(x^i) = u^i(x^i_1, \ldots, x^i_m, x^i_{m+1}), \qquad i = 1, \ldots, n.$$

(We do not exclude the possibility that u^i is independent of x^i_{m+1}.) Horizontal bars will denote summation over the traders; for example, $\bar{a}_{m+1}$ is the total amount of "gold." We assume that all commodities are present somewhere in the economy, i.e., that

$$(3) \qquad \bar{a} > 0.$$

Unlike the usual "general equilibrium game," goods in our model are not freely transferable among consenting traders, but are subject to specific marketing rules. For the present exposition we shall focus first on one version of these rules, pointing out later some possible variations.

Also, in this account, we shall confine ourselves to the one-period case, although several aspects of our approach (for example, the treatment of fiat money, or of loans and interest rates) can only be done justice to in a multi-period setting.

Given just the one period, each trader has a single strategic decision to make. This will be represented by a pair (b^i, s^i) of nonnegative m-vectors, subject to

$$s_j^i \leqq a_j^i, \qquad j = 1, \ldots, m, \tag{4}$$

and

$$\sum_{j=1}^{m} b_j^i \leqq a_{m+1}^i. \tag{5}$$

The letters "b" and "s" are meant to connote "buy" and "sell." To interpret this, imagine that there are m different markets or "trading posts," $M_1, \ldots, M_m$. Trader i sends the amount s_j^i of good j to M_j, to be sold "on consignment," and he sends the amount b_j^i of "gold" to M_j toward the purchase of good j. We could add the constraint $b_j^i s_j^i = 0$, forbidding a trader to enter a market on both sides, but we shall save that for later.*

Once all traders have made their choices, the outcome is readily determined. In fact, if neither $\overline{b}_j$ nor $\overline{s}_j$ are

*See Sec. 5.

zero, then the price

$$p_j = \bar{b}_j/\bar{s}_j > 0, \tag{6}$$

will obviously clear the market M_j. If it happens that $\bar{b}_j = 0$ (no "gold" sent to M_j), then we arbitrarily remove the goods $\bar{s}_j$ from the system. Similarly, if $\bar{s}_j = 0$ (no goods sent to M_j), then the "gold" $\bar{b}_j$ disappears.* Carrying out the indicated transfers, we calculate the final bundles as follows:

$$x_j^i = \begin{cases} a_j^i - s_j^i + (\bar{s}_j/\bar{b}_j)b_j^i, & \text{if } \bar{b}_j > 0, \\ a_j^i - s_j^i, & \text{if } \bar{b}_j = 0 \end{cases} \tag{7}$$

if $j \neq m + 1$, and

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*This proves to be the best way to handle the very real (and realistic) discontinuity in the mechanics of our system, occasioned by the failure to "form a market" for good j. From the individual trader's point of view, it is reasonable to expect (1) that he not get something for nothing, but (2) that he do get nothing for something, when the price of that "something" goes to zero. From the global point of view, it is reasonable to expect that an equilibrium solution will avoid outcomes where goods or money disappear from the system in accordance with this rule.

The reader disturbed by our toying with the law of conservation of matter may be reassured if we postulate an extra trader, or "scavenger," who makes an infinitesimal bid on both sides of every market. Ordinarily, his effect on the system is also infinitesimal. But he can take home a substantial windfall if it happens that one of his infinitesimal bids is not outmatched by a positive bid from one of the regular traders.

$$(8) \qquad x^i_{m+1} = a^i_{m+1} - \sum_{j=1}^{m} b^i_j + \sum_{j:\bar{s}_j>0} (\bar{b}_j/\bar{s}_j)s^i_j.$$

The i-th trader's <u>payoff</u>, as a function of all the trader's strategies, is then given by

$$(9) \qquad P^i = P^i((b^1, s^1), \ldots, (b^n, s^n)) = u^i(x^i).$$

However, note that since the other players' strategies affect i's payoff only in the aggregate, this function of n pairs of m-dimensional variables can be reduced to a function of two pairs of m-dimensional variables, as follows:

$$(10) \qquad P^i = \Pi^i((b^i, s^i), (\bar{b} - b^i, \bar{s} - s^i)).$$

The definition of the noncooperative equilibrium* is now easily stated using (10): it is a pair of matrices $b^{\#}$, $s^{\#}$, such that for each i the expression

$$(11) \qquad \Pi^i((b^i, s^i), (\bar{b}^{\#} - b^{\#i}, \bar{s}^{\#} - s^{\#i}))$$

is maximized, subject to (4) and (5), at $b^i = b^{\#i}$, $s^i = s^{\#i}$. In other words, no trader, given the bids of the others, can improve.

*Robert Wilson has suggested the term "best-response equilibrium." The concept was introduced to game theory by John Nash, who however, admitted the possibility of randomized or "mixed" strategies. Thus, our N.E. is a "Nash equilibrium in pure strategies."

We would like to observe that it is part of the peculiar discipline of game theory that there be players and rules. The players are independent decisionmakers, explicitly constrained, while the rules associate a definite outcome with every set of legal strategy choices. The description of the game (i.e., what can happen) is kept conceptually separate from the solution of the game, (i.e., what will or should happen). This contrasts sharply with the classical model, in which prices appear mysteriously out of the air and in which the overall outcome is undefined in the event any trader "irrationally" steps out of line and fails to conform to the preordained behavior pattern. The contrast fades, however, if one merges description and solution, i.e., considers the equilibrium conditions (N.E. or C.E.) as an integral part of the model, rather than as something applied to the model. Nevertheless, there will be important applications, especially in the study of externalities, in which the N.E. does not exist, or in which it is heuristically inappropriate as a solution concept. We wish to stress that our game model, unlike the competitive model, remains well defined in such situations.

4. THE EDGEWORTH BOX

The case n = 2, m = 1 lends itself to diagrammatic analysis. In the figures that follow, the vertical coordinate is the money commodity, j = 2, and the horizontal coordinate is the other commodity, j = 1. The first trader measures his holdings from O^1, the second from O^2. The point R represents the initial allocation, with coordinates (a_1^1, a_2^1) measured from O^1. A typical strategy for trader 1 is the point S^1 in Fig. 1; it is constrained to lie in the rectangle "southwest" of R. Similarly, S^2 for trader 2. Here, the vectors RS^1 and RS^2 represent what goes to market and O^1S^1 and O^2S^2 what stays home; thus the actual O^1-coordinates of S^1 are $(a_1^1 - s_1^1, a_2^1 - b_1^1)$.

The price p_1 is easily visualized. In fact, it is just the slope $(b_1^1 + b_1^2)/(s_1^1 + s_1^2)$ of the line S^1S^2. The intercepts of this line with the perpendiculars through R determine the coordinates of the final allocation, which is represented by the point X. Note that X must lie either "northwest" or "southeast" of R.

The line RX cooresponds to the "price ray" of the familiar Edgeworth construction; its slope is of course $-p_1$. But in our present model it does not represent a "budget set" from which the trader selects his preferred outcome. Instead, the set of outcomes from which Trader 1 chooses is the curve A^1B^1. This is the trace of the point X, as S^1 varies over its permitted rectangle with S^2

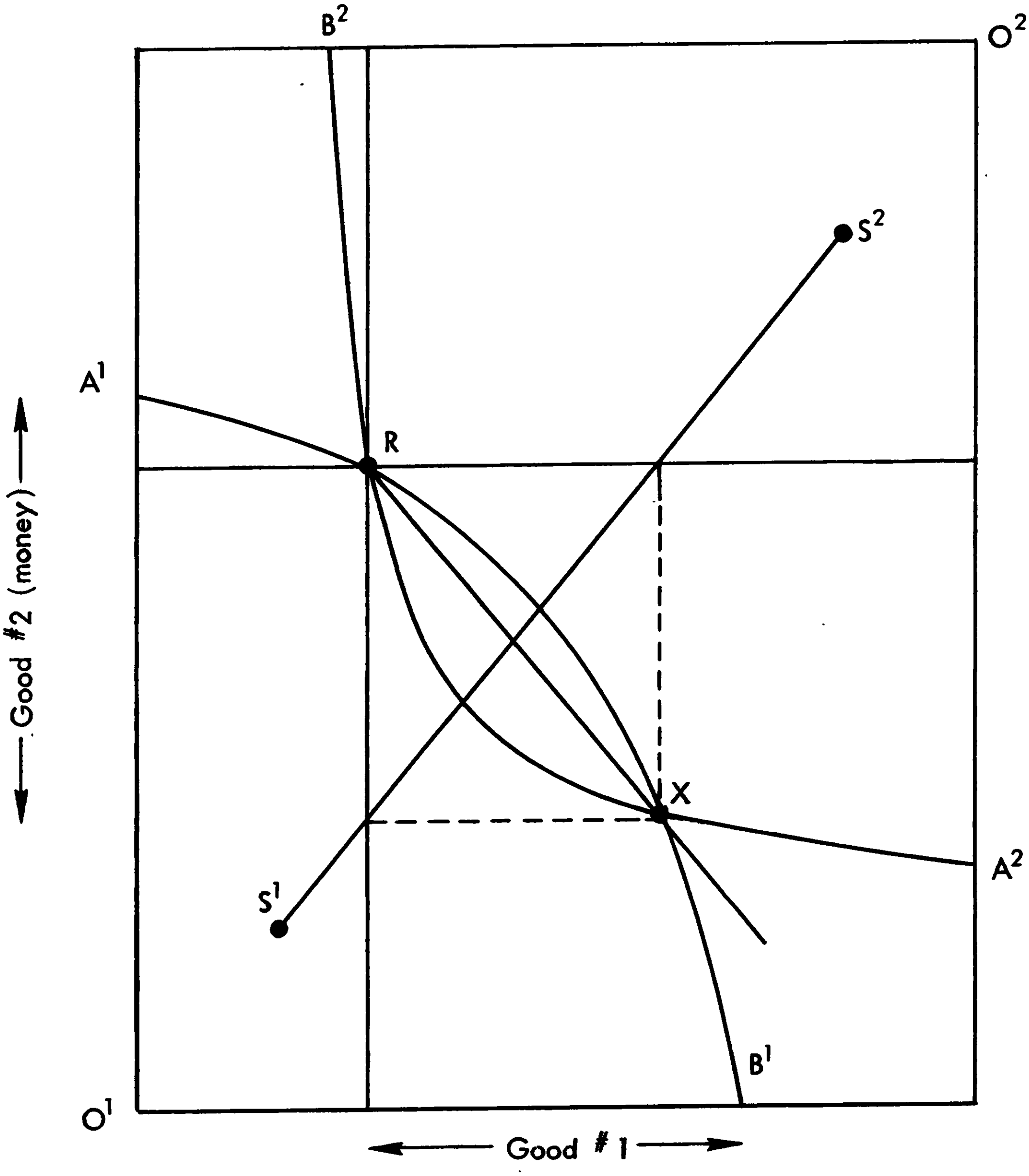


Fig. 1

held fixed. Algebraically it is a segment of a hyperbola, the asymptotes being the perpendiculars through S^2. The analogous set for Trader 2, given S^1 fixed, is the curve A^2B^2.

Observe the contrast between the <u>marginal</u> prices, which are different for the two traders, and the <u>average</u> price p_1. Trader 1's marginal is the (absolute) slope of A^1B^1 at X; it is greater than p_1 in the case illustrated, because he is a net buyer of the commodity when X is in the "southeast" quadrant. Should he try to increase his order, he would drive up the demand and would have to pay a higher market price not merely on the increment, but on his entire purchase. Similarly, Trader 2's marginal, the absolute slope of A^2B^2 at X, is less than p_1, because he happens to be a net seller.

So far we have just considered the machinery of exchange. In the second figure we show Trader 1's utility maximization problem. Note that a unique optimum M exists, because of the convexity of the indifference curves and the opposite curvature of A^1B^1. This optimum is achieved by any legal choice of S^1 on the slanting line through S^2. Note also that M is a continous function of S^2, so long as the vector RS^2 is strictly positive. These properties of uniqueness and continuity continue to hold quite generally in our model, under certain weak nonsaturation assumptions, when there are more traders and/or commodities.

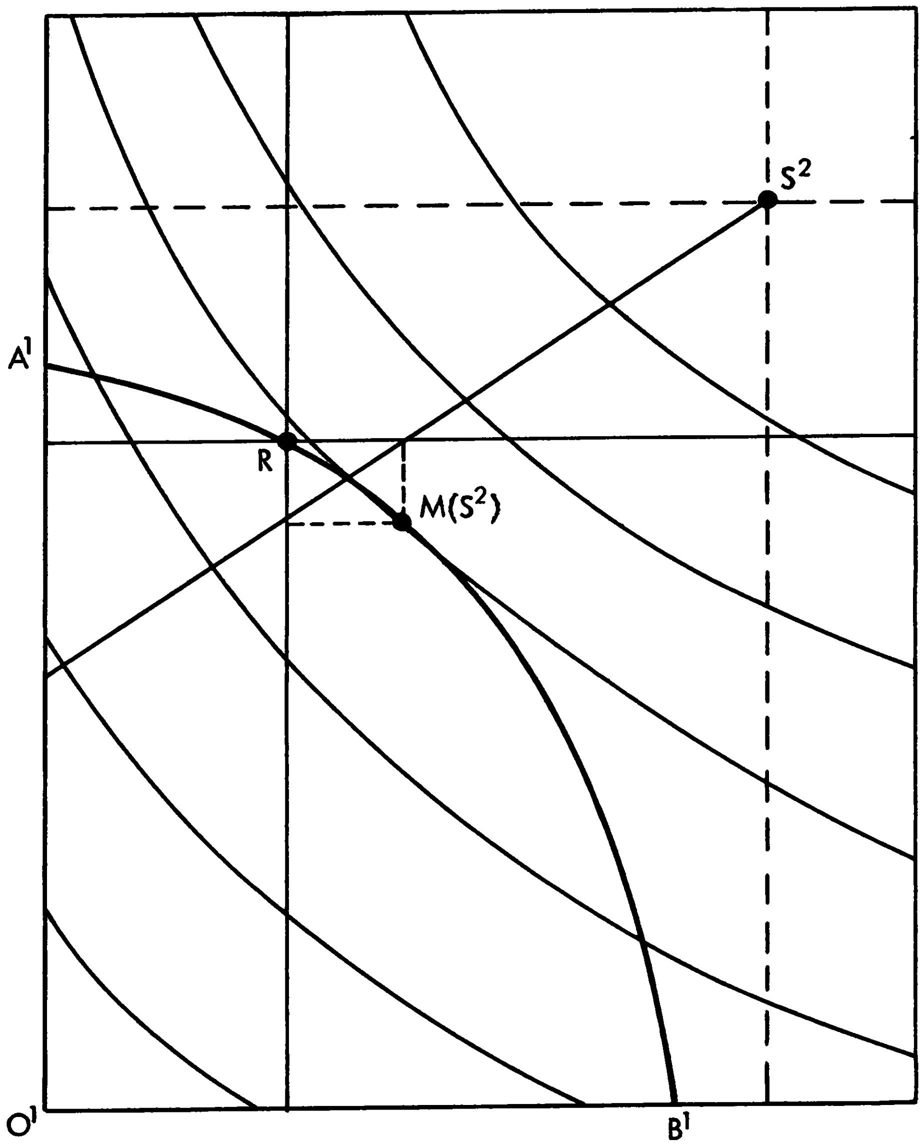


Fig. 2

Figure 3 shows typical indifference curves for both traders, as in the usual Edgeworth diagram, but illustrates a noncooperative equilibrium, with each trader optimizing against the other's strategy. Note that the N.E. allocation, X, is not Pareto efficient. The difference between the two marginal prices leaves room for joint improvement, as indicated by the shaded region.

The Edgeworth "contract curve" of common tangents, CD, passes through this shaded region, as it must. It might appear at first glance that there must always be a C.E. allocation in this region, i.e., a point at which the common tangent, extended back, passes through the point R. A C.E. so situated would not only represent a higher volume of trade than the N.E., but would actually raise the utility of both traders. But matters are not always so straightforward. In Fig. 4 we show how the contract curve, given that there is a N.E. at X, can still twist in such a way that it enters the rectangle between R and X; moreover the unique C.E. can lie in that rectangle, as shown at W. This perverse "twist" is not unrelated to the phenomenon of inferior goods and Giffen's paradox.

On the other hand, given any C.E. allocation, we can show that there is always a N.E. in the rectangle between it and R.* In fact, we can show something stronger. Take any line through R that contains jointly profitable points,

*This is not true for the variants discussed in the next section, however.

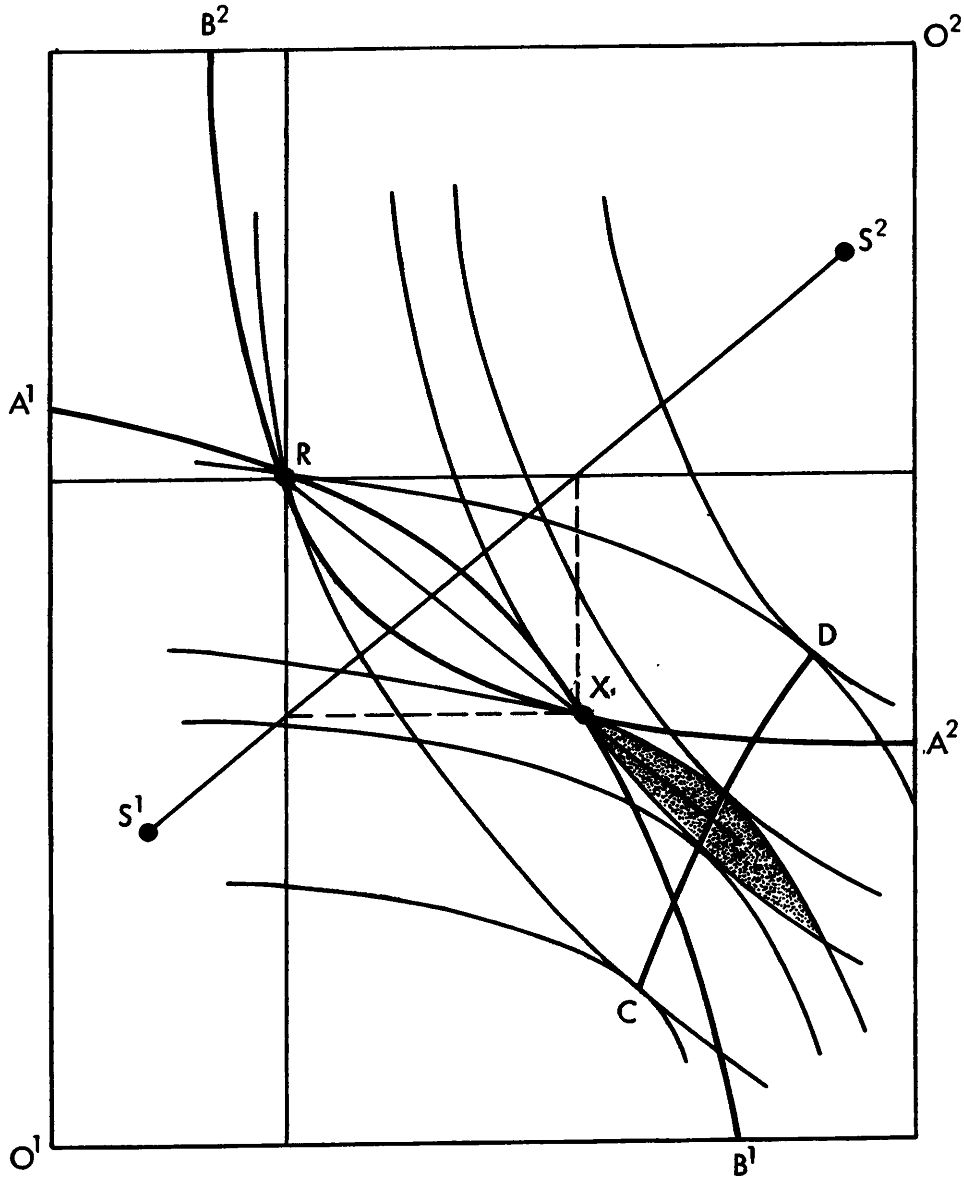
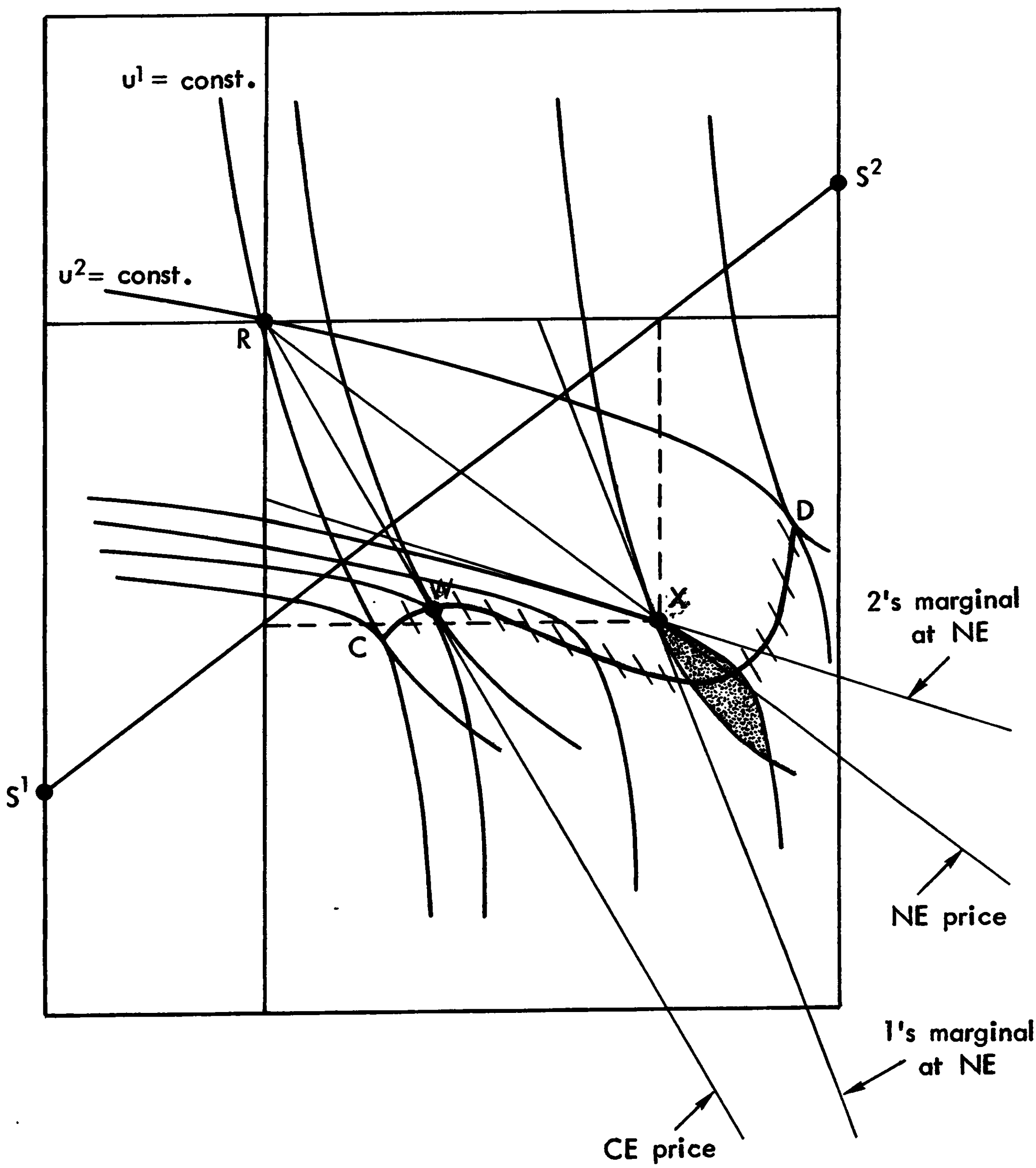


Fig. 3



u¹ = const.
u² = const.
S²
R
D
W
X
C
S¹
2's marginal at NE
NE price
1's marginal at NE
CE price

Fig. 4

for example, the price ray through a C.E. allocation like W in Fig. 4. Then if R is interior to the box, there will be a whole interval of N.E. allocations on this line in the neighborhood of R.

To see why this is so, consider the problem of trying to construct a N.E. that will yield an allocation on such a ray (Fig. 5). Start with the jointly profitable point X, sufficiently close to R so that both utilities increase monotonically from R to X. Then determine G and H as shown, and draw the line through G and H. Then draw XS^2 through X with positive slope matching the negative slope of the u^1 indifference curve through X; this slope will be steeper than XR, and so S^2 will lie in the "northeast" quadrant from R, as shown. Similarly draw XS^1. If S^1 and S^2 happen to lie within the Edgeworth box then they are allowable strategies and constitute a N.E. yielding S, as desired. If not, move X towards R, along the ray RX, and ultimately this will bring the points S^1 and S^2 so close to R that they will be allowable strategies.

The purpose of this construction was not so much to demonstrate an economic theorem, since the result will disappear under the tighter definitions of strategy in the next section. Rather, we wish to highlight the nonuniqueness of the N.E. solution in our basic, expository model and to suggest the desirability of the sharper variants to come.

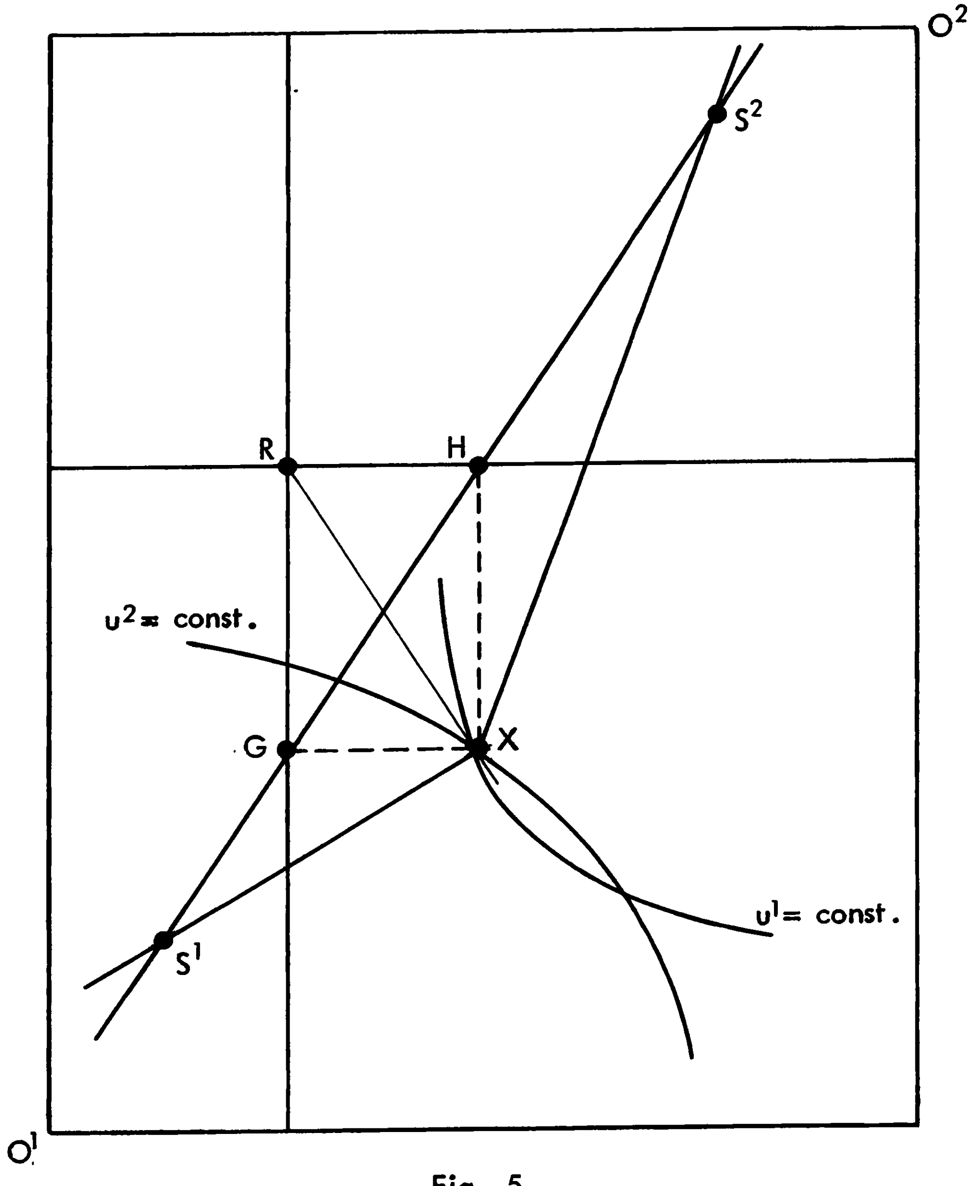


Fig. 5

Much of the foregoing diagrammatic analysis applies when there are more traders and/or goods. As far as traders are concerned, the model has the "aggregation" property. Thus, from trader 1's point of view, the point S^2 can be regarded as the sum of the strategies of traders 2 through n. His personal optimization problem does not depend on the number n, but only on the magnitude and position of the vector RS^2.* As far as goods are concerned, each trading post M_j operates independently of the others. The only links are through the budget constraint (5), whereby the set of allowable choices in any one post depends on how much the trader spends in the others, and through complementarities, etc. in the preferences for different goods, whereby the indifference map for each trading post may depend on what happens at the other posts.

These remarks illustrate vividly the function of money as a decoupler, enabling independent centers of economic initiative (the traders) and independent centers of exchange and price-formation (the trading posts) to operate smoothly within a single system without inconsistency or indeterminateness.

*Note that if RS^2 is very large, then trader 1's choice set A^1B^1 approaches a straight line, approximating what it would be in a fixed-price "competitive" market.

5. VARIANTS

In our basic formulation, a trader does not usually have a unique "best response." This may be seen, e.g., in Fig. 2. Trader 1 is indifferent in his choice of S^1 along any given line through S^2, since the outcome $M(S^2)$ is not affected. All that he accomplishes by moving S^1 along such a line is to change the marginal prices that confront the other trader or traders. The further S^2 is from R, the more stable the price is for them.* While this nonuniqueness of response does not directly account for the nonuniqueness of the N.E. (cf. the construction in the preceding section), it does illustrate the presence of an unnecessary degree of strategic freedom. One would expect that the prospects for a sharp solution concept would be enhanced with that freedom eliminated. Two variants that do this will now be discussed.

In Variant I we insist that the traders not bid on both sides of a trading post simultaneously; in other words, we require

$$(12) \qquad b^i_j s^i_j = 0, \qquad j = 1, \ldots, m,\ i = 1, \ldots, n.$$

In our Edgeworth diagram this means that each S^i is now restricted to the two edges of the previously allowed rectangle that meet at the point R. Under this trading rule, the

*It is accepted practice for a company issuing additional shares of its stock to simultaneously buy or sell in the open market in order to stabilize the price.

two-trader case collapses into a triviality: either the traders just exchange goods for money directly, if they happen to choose opposite sides of the market, or they both lose their goods or money outright, if they happen to choose the same side. But with more traders in the market, a given trader will more likely find both sides active, so that the combined bids of his opponents will not satisfy (12), but rather yield a point like S^2 in Fig. 2, in the interior of the "northeast" quadrant. Thus, Variant I does not have the aggregation property. It

It is of course true that any N.E. for Variant I is also a N.E. for the original model, since in restricting the strategy spaces we have not materially reduced the response capabilities of the traders, even considering the budget constraint (5). Conversely, a solution of the original version that happens to satisfy (12) will also be a solution of the variant, *a fortiori*.

In Variant II, we insist that all goods pass through the market before they are consumed; nothing (except money) may be held back. In other words, we require

$$(13) \qquad s_j^i = a_j^i, \qquad j = 1, \ldots, m, \ i = 1, \ldots, n.$$

This eliminates the s_j^i as strategic variables. Geometrically, the permitted strategies in the Edgeworth box are restricted to the edges O^iG^i (Fig.6). The market price now has an upper bound, as it can never exceed the slope of the diagonal O^1O^2. Certain outcomes are therefore unattainable, and the "response curves" A^iB^i of the previous diagrams are shortened at the "B" end. The shaded area in Fig.6 indicates the full range of outcomes that remain feasible in Variant II, as compared with the two complete quadrants that were possible before. The intuitive reason for this curtailment of the feasible act is that we are requiring the traders to sell all their goods, but are not allowing them to hypothecate the receipts. The terms of trade are "cash in advance."

It is entirely possible that the Edgeworth contract curve will now fail to intersect the feasible set. This raises an interesting point about social efficiency. We have been using the term "Pareto efficient" in the customary way, i.e., with reference to the set of all allocations that could be reached by <u>unrestricted transfers</u> of goods among the traders. If the market rules are regarded as "laws of nature," however, many transfers will be impossible, and a different Pareto set will result inferior to the first. In Variant II it may even be strictly inferior; that is, it may have no points in common with the original. That would be exemplified by the contract curve missing the feasible set in the two-trader Edgeworth market.

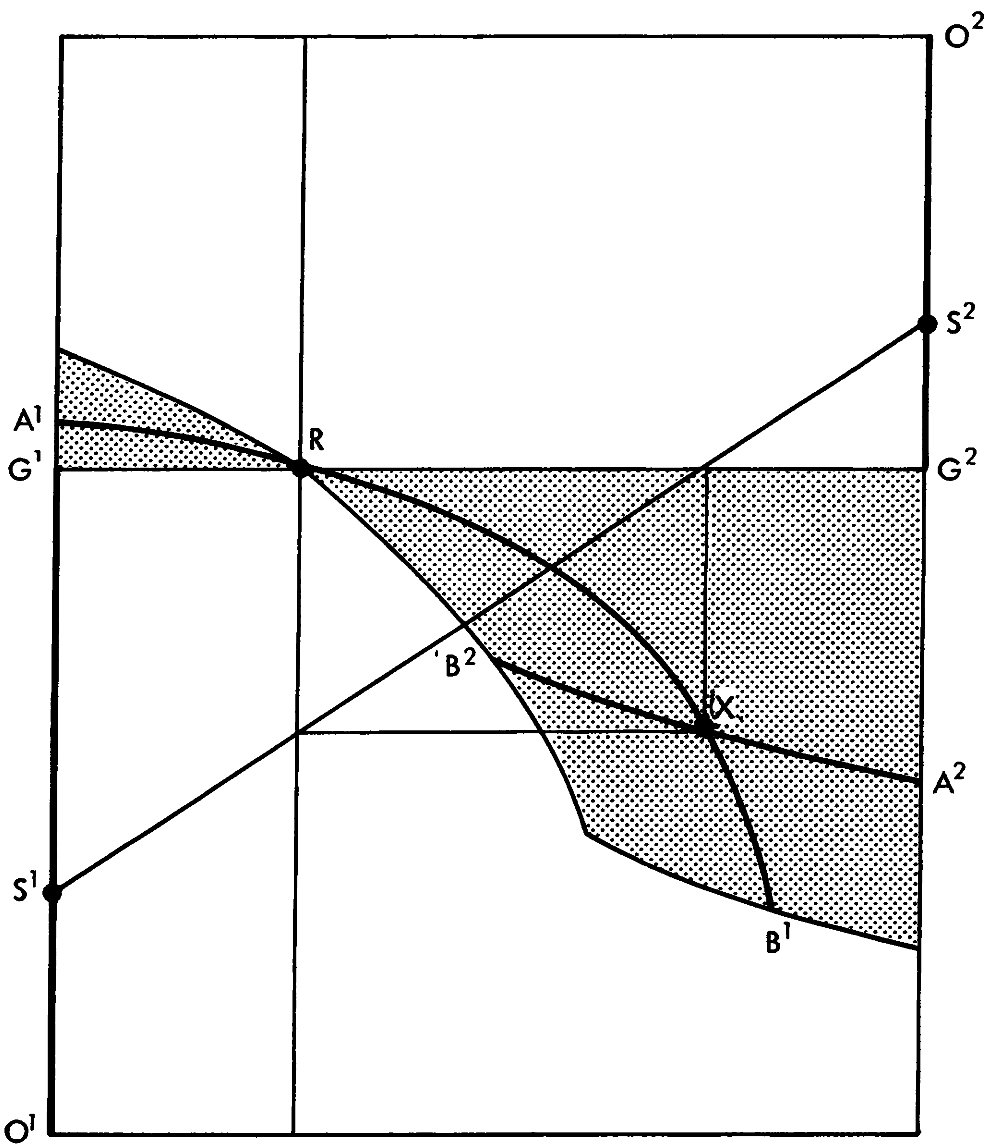


Fig. 6

But even in our basic model (and in Variant I), it will not ordinarily be possible, when there are more than two traders, to reach most of the points on the unrestricted-transfer Pareto surface. This is easy to see if we observe that each feasible point has associated with it a particular market-wide price schedule. Once we have decided, say, what we want Trader 1's final bundle to be, we are burdened with a linear-equation condition on the prices which will limit the possible ways in which the other traders can redistribute the complementary bundle among themselves. Some of the Pareto-efficient outcomes will always be feasible under fixed-price trading, for example, the C.E. allocations, but most of the unrestricted-transfer Pareto surface will in general be inaccessible if $n > 2$. We should emphasize, however, that exactly the same situation prevails with respect to the Walras model.

These matters may be important from the cooperative-game viewpoint: they mean that the several different "strategic forms" that we consider have different characteristic functions, and hence may have different cores, values, bargaining sets, etc.* From the noncooperative viewpoint they are less important. Two points deserve mention, however. First, in all our variants, the N.E. are usually socially inefficient,

*It might surprise some people to learn that the core of the cooperative game that most closely resembles the Walrasian model of exchange (the game in which the players negotiate a market-wide price schedule to govern all transactions) is different from the core usually studied. We do not know of any treatment of this core in the literature.

even with respect to the restricted Pareto set. It usually still pays to collude. Second, the more drastic curtailment possible in Variant II, where the entire original Pareto set becomes infeasible, is most likely to be associated with shortages or maldistribution or near-worthlessness of the "money" commodity. This observation is a sample of the sort of heuristic commentary on the phenomenology of money that we can expect our theory to produce as the analysis proceeds.

The third variant to be discussed points in a different direction; it is an extension rather than a restriction, and may be applied equally well to any of the three versions discussed heretofore. The idea is to introduce <u>credit</u> into the model, i.e., to modify or eliminate the constraint (5). But we wish to avoid (at least at first) any detailed consideration of special rules for repayment, security, interest, bankruptcy, etc. This can be accomplished by the simple expedient of eliminating (5) entirely and extending the consumption set for each trader to include negative as well as positive values of x_{m+1}. "Cash in advance" is out, but a suitable disutility for being caught without sufficient funds at the close of trading is built into the extended utility functions, without any specific modeling of the machinery of insolvency or bankruptcy. Considering the nature of this negative commodity, "debt," it would not be unreasonable to demand concavity of the utility functions over the extended domain. If the disutility of debt is made steep enough,* we can expect the actual N.E. solutions to lie within the positive orthant for each trader, although the spending limit (5) may well be violated in the process of getting there. But, as remarked before, we insist that our game model provide a well-defined "payoff" (final utility vector) for *any* choice of strategies, in or out of equilibrium.

The effect of this extension is easily described in the Edgeworth diagram: simply knock out the top and bottom ends of the box. Among other things, this enlarges the feasible set for Variant II, making it coextensive with that for the other models, since arbitrarily high prices can now be generated, not by withholding goods from market but by simultaneous overbidding by both traders.

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*The extreme case would be to make the penalty for being caught short essentially infinite. The opposite, "easy credit" extreme would be to use the maximal concave extension of the concave utility function originally defined on the positive orthant--such an extension exists and is unique if we adopt the useful viewpoint of Rockafellar [13] wherein convex and concave functions can assume the values $\pm\infty$.

6. TWO THEOREMS

An existence theorem for N.E. is of course a sine qua non of the theory, but it is not easy to prove and we have thus far obtained a satisfactory result only for Variant II. The other cases, however, involve an interesting auxiliary concept, called virtual price, and it is worth taking some space to describe it even though we cannot yet assert a theorem.

We have already alluded to the discontinuity in the system when "no market forms" for some good. Suppose it happens that no bids are received, say, on the money side of M_j, though a positive quantity of goods j is offered for sale.* An arbitrarily small increase in any of the zero bids b_j^i would then suffice to buy up the full supply, $\bar{s}_j$, which otherwise disappears from the system. This type of situation is therefore quite unstable, if we suppose there is at least one trader with money and with a desire for good j. Intuitively, one would hardly expect this sort of unstable situation to have much bearing on the existence of an equilibrium.

It is quite another matter, however, when nothing is offered on either side of a trading post. In this case, the situation is highly stable. A bid by any one person is futile; either he just loses it outright, or (if the rules permit) he may find himself selling his own goods to himself. It

*There is a similar discontinuity if money, but no goods are offered to a trading post.

takes joint action on the part of two or more individuals to start up a profitable market in a good previously untraded. This is surely not just a technical defect in the model--the world is full of "latent" markets awaiting discovery and exploitation. But it has the strange effect of making our existence theorem too easy! There is always the trivial, "null" equilibrium, at which nothing is bought or sold. There will also be numerous other equilibria (usually) that are almost as trivial: one or two trading posts "open for business" and all the others arbitrarily shut down.

While these trivial solutions may not be totally unrealistic, one would hope that there would be at least one full-fledged, "robust" N.E., in which all posts M_j are <u>active</u> in the sense that $\bar{b}_j > 0$ and/or $\bar{s}_j > 0$. This, however, is too much to demand. Some markets may be "legitimately" inactive, the existing distribution being exactly what it "should" be. For example, if the initial allocation of all goods happens to be Pareto optimal, then even the "null" solution will be "legitimate"--and very likely unique.

The following definition points a way through the difficulty. If (b, s) is a N.E. and if M_j is inactive, i.e., if $\bar{b}_j = \bar{s}_j = 0$, then we define the number v_j to be a <u>virtual price</u> for M_j if all the individual strategies (b^i, s^i) would remain "best responses" for their users in the presence of added "external" bids $b_j^0 = \varepsilon v_j$ and $s_j^0 = \varepsilon$, for any $\varepsilon > 0$.

There may be one or many virtual prices, or none, for any given inactive M_j. Intuitively, a virtual price means that a trading post, if it were "open for business" at that price, would attract no business. Our goal is a theorem that asserts (for Variant I, say, and under suitable conditions on the utilities, etc.) that a N.E. exists in which each commodity has either an actual price p_j, if actively traded, or a virtual price v_j, if not.

In Variant II all markets are automatically stocked, so virtual prices do not arise. The possibility of no money being bid for a good still does present a technical problem, since the fixed-point theorems that we rely on require continuity of the mapping from strategies to "best responses." The problem is resolved in our proof by approximating the economy by continuous models having small "external" bids in all markets, and we can report the following theorem:

THEOREM 1. In Variant II, assume that the u^i are continuous, concave and nondecreasing; that $\bar{a} > 0$, and that for each good $j = 1, \ldots, m$, there are at least two traders i such that $a^i_{m+1} > 0$ and u^i is strictly increasing in x_j for x_j in some neighborhood of 0. Then a N.E. exists.

The "two traders" condition in this theorem makes sense: If only one moneyed trader desired good j then he would have no competition in the bidding, and hence no "best" bid $b_j^i > 0$.

Note that we do not require that money have any utility for anyone. But if money is intrinsically worthless (i.e., valuable only strategically), then the Variant II solution is independent of who owns the initial endowments, since no one would care what his goods bring in the marketplace.* The case of "paper" or "fiat" money becomes distinctly more interesting in a multi-period setting.**

The other theorem that we shall discuss expresses a relationship between the two equilibrium concepts--N.E. and C.E.--when there are many traders, each "small" relative to the whole. Our setting will be a "replication sequence" of economies Γ_1, Γ_2, ..., Γ_k ..., in the manner of Scarf and Debreu.*** That is, there are a fixed number, t, of types of traders, characterized by their utilities u^i and endowments a^i, i = 1, ..., t. The economy Γ_k then has n = kt traders, k of each type. Of course, while traders of the same type have identical descriptions we cannot constrain them to behave identically in the game, since each is a free agent. But a N.E. in which traders of the same type do choose the same

*If credit is allowed, we would need to impose a negative utility for negative holdings of this "paper" money, or else the bidding would become unbounded.

**See [22,25]

***See [4]

strategies is called symmetric, and such a N.E. can be represented in a space whose dimension does not depend on the replication number k. Specifically, in Variant II, any symmetric N.E. of Γ_k can be represented by an m-by-t matrix b(k) of bids b_j^i (rather than an m-by-kt matrix). Thus, symmetry permits a straightforward comparison of b(k) for different values of k.

One final concept of interest must be introduced before we can state the theorem. At any N.E. a price is defined for each good, by (6). There is also defined, for each trader, a marginal utility for spending-money, associated with the constraint

$$(14) = (5) \qquad \sum_{j=1}^{m} b_j^i \leqq a_{m+1}^i .$$

In fact, let $\lambda^i \geqq 0$ be the Lagrangian multiplier for (14), i.e., the number (perhaps not unique) such that the choice of b^i would remain optimal for i if the constraint (14) were dropped and the maximand $u^i(x^i)$ replaced by *

*When (14) is dropped it is possible for x_{m+1}^i to go negative; to prevent this from confusing the definition of λ^i (in the no-credit models), it suffices to impose the extra constraint $x_{m+1}^i \geqq 0$, noting that the strategies b^h for $h \neq i$, on which x_{m+1}^i in part depends, are for the time bing considered to be fixed.

$$u^i(x^i) + \lambda^i(a^i_{m+1} - \sum_{j=1}^{m} b^i_j).$$

Thus, λ^i is a kind of "shadow price" for cash; it is in general different for each trader, and different for the same trader at different N.E.s. Of course, when the constraint (14) is slack, as it may well be if money has some value of its own, then the corresponding λ^i will be zero. We note that in a symmetric N.E. the shadow prices for cash can be taken to be the same for all members of the same type, so we can use a t-vector $\lambda(k)$ to denote them all (rather than a tk-vector).

<u>THEOREM 2.</u> <u>Let b(k) represent a symmetric N.E. for the Variant II economy</u> Γ_k, <u>as described above, and let p(k) and λ(k) represent the associated price and shadow price vectors. Suppose that for some increasing subsequence</u> $\{k_\nu\}$ <u>we have</u>

$$p(k_\nu) \to p^* \text{ and } \lambda(k_\nu) \to 0.$$

<u>Then the prices (p*, 1) are competitive for</u> Γ_1, <u>and indeed for each</u> Γ_k. <u>That is, an allocation</u> $x = \{x^i_j\}$ <u>exists such that</u> $x \geq 0$, $\bar{x} = \bar{a}$ <u>and, for each i,</u> $u^i(x^i)$ <u>is maximal subject to the constraint</u>

$$\sum_{j=1}^{m} p^*_j(x^i_j - a^i_j) + x^i_{m+1} - a^i_{m+1} \leq 0.$$

The condition $\lambda \to 0$ in this theorem means that the spending limits (14) are not binding in the limit--everyone eventually has "enough" money. This means in turn that money must have had some intrinsic value for every trader, except that there might be totally saturated individuals who no longer care whether they spend or save.* Thus, we can begin to address the larger question of how to select a suitable commodity to serve as a medium of exchange, with the social aim of achieving or approaching a competitive and hence Pareto-efficient solution. Other properties of the money-commodity, like separability and additivity, appear to have some bearing on the uniqueness of the N.E.; we intend to pursue these questions.

*In the "credit" version of Variant II, there is a similar theorem with the condition $\lambda(k_\nu) \to 0$ replaced by one that states that in the limit no trader is "in the red."

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