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Abstract

We situate the noncrossing partitions associated with a finite Coxeter group within the context of the representation theory of quivers. We describe Reading's bijection between noncrossing partitions and clusters in this context, and show that it extends to the extended Dynkin case. Our setup also yields a new proof that the noncrossing partitions associated with a finite Coxeter group form a lattice. We also prove some new results within the theory of quiver representations. We show that the finitely generated, exact abelian, and extension-closed subcategories of the representations of a quiver Q without oriented cycles are in natural bijection with the cluster tilting objects in the associated cluster category. We also show that these subcategories are exactly the finitely generated categories that can be obtained as the semistable objects with respect to some stability condition.

1. Introduction

A partially ordered set called the *noncrossing partitions* of $\{1, \ldots, n\}$ was introduced by Kreweras [Kre72] in 1972. It was later recognized that these noncrossing partitions should be considered to be connected to the Coxeter group of type A_{n-1} (that is, the symmetric group S_n). In 1997, a version of noncrossing partitions associated with type B_n was introduced by Reiner [Rei97]. The definition of noncrossing partitions for an arbitrary Coxeter group was apparently a part of folklore before it was written down shortly thereafter [Bes03, BW02].

Subsequently, *cluster algebras* were developed by Fomin and Zelevinsky [FZ02]. A cluster algebra has a set of distinguished generators grouped into overlapping sets called *clusters*. It was observed [FZ03] that the number of clusters for the cluster algebra associated with a certain orientation of a Dynkin diagram was the same as the number of noncrossing partitions, the generalized Catalan number. The reason for this was not at all obvious, although somewhat intricate bijections have since been found [ABMW06, Rea07a].

The representation theory of hereditary algebras has proved an extremely fruitful perspective on cluster algebras from [BMRRT06, MRZ03] to the more recent [CK08, CK06]. In this context, clusters appear as the cluster tilting objects in the cluster category. We adopt this perspective on clusters throughout this paper.

Our goal in this paper is to apply the representation theory of hereditary algebras to account for and generalize two properties of the noncrossing partitions in finite type:

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(1) the already-mentioned fact that noncrossing partitions are in natural bijection with clusters;
(2) the noncrossing partitions associated with a Dynkin quiver Q, denoted by NC_Q, form a lattice.

These properties themselves are not our observations. We have already mentioned sources for statement (1). Statement (2) was first established on a type-by-type basis with a computer check for the exceptional types; a proof which does not rely on the classification of Dynkin diagrams was given by Brady and Watt [BW08]. Our hope was that by setting these properties within a new context, we would gain a better understanding of them, and also of what transpires beyond the Dynkin case.

Let k be an algebraically closed field. Let Q be an arbitrary finite quiver without any oriented cycles. Let $\operatorname{rep} Q$ be the category of finite-dimensional representations of Q. We refer to exact abelian and extension-closed subcategories of $\operatorname{rep} Q$ as *wide*. The central object of our researches is $\mathcal{W}_Q$, the set of finitely generated wide subcategories of $\operatorname{rep} Q$. There are a number of sets of algebraic objects which are all in bijection with each other, summarized by the following theorem.

Theorem 1.1. *Let Q be a finite acyclic quiver. Let $\mathcal{C} = \operatorname{rep} Q$. There are bijections between the following sets.*

(1) *Clusters in the acyclic cluster algebra whose initial seed is given by Q.*
(2) *Isomorphism classes of basic cluster tilting objects in the cluster category $\mathcal{D}^b(\mathcal{C})/(\tau^{-1}[1])$.*
(3) *Isomorphism classes of basic exceptional objects in $\mathcal{C}$ which are tilting on their support.*
(4) *Finitely generated torsion classes in $\mathcal{C}$.*
(5) *Finitely generated wide subcategories in $\mathcal{C}$.*
(6) *Finitely generated semistable subcategories in $\mathcal{C}$.*

If Q is Dynkin or extended Dynkin:

(7) *the noncrossing partitions associated with Q.*

If Q is Dynkin:

(8) *the elements of the corresponding Cambrian lattice.*

Some of these results are already known. A surjective map from set (1) to set (2) was constructed in [BMRT07] and a bijection from set (2) to set (1) in [CK06], cf. also the appendix to [BMRT07]. Those from set (2) to set (3) to set (4) are well known but we provide proofs, since we could not find a convenient reference. The bijection from set (4) to set (5) is new. The subcategories in set (6) are included among those contained in set (5) by a result of [Kin94]; the reverse inclusion is new. Bijections from set (8) to set (1) and from set (8) to set (7) were given in the Dynkin case [Rea07a]. Putting these bijections together yields a bijection from set (1) to set (7). A conjectural description of this bijection was given in [RS09]; we prove this conjecture. Another bijection between sets (7) and (8) is also known, although also only in the Dynkin case [ABMW06]. The extension of the bijection between sets (1) and (7) to the extended Dynkin case is new.

The set $\mathcal{W}_Q$ is naturally ordered by inclusion. The inclusion-maximal chains of $\mathcal{W}_Q$ can be identified with the *exceptional sequences* for Q. When Q is of Dynkin type, $\mathcal{W}_Q$ forms a lattice. The map from $\mathcal{W}_Q$ to NC_Q respects the poset structures on $\mathcal{W}_Q$ and NC_Q, which yields a new proof of the lattice property of NC_Q for Q of Dynkin type.

We also gain some new information about the Cambrian lattices: we confirm the conjecture of [Tho06] that they are *trim*, i.e. left modular [BS97] and extremal [Mar92].

2. Wide subcategories of hereditary algebras

2.1 Definitions

In this section we use some standard facts from homological algebra, most of which can be found in [ASS06, §§ A.4 and A.5]. In addition to what can be found there we recall two lemmas. These facts can be proved with straightforward diagram chases. The first lemma is a lesser known variant of the snake lemma.

Lemma 2.1. *If we have maps $A \xrightarrow{\psi} B \xrightarrow{\phi} C$ in an abelian category, then there is a natural exact sequence*

$$0 \to \ker \psi \to \ker \phi\psi \to \ker \phi \to \operatorname{cok} \psi \to \operatorname{cok} \phi\psi \to \operatorname{cok} \phi \to 0.$$

We also use the fact that pushouts preserve cokernels, and pullbacks preserve kernels.

Lemma 2.2. *Given morphisms $g : A \to E$ and $f : A \to B$, consider the pushout*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & E \arrow[d, "f_*"] \\ B \arrow[r, "g_*"'] & E \coprod_A B \end{tikzcd}$$

then $\operatorname{cok} f \simeq \operatorname{cok} f_*$ and $\operatorname{cok} g \simeq \operatorname{cok} g_*$ and the dual statement for pullbacks.

Let k be an algebraically closed field. We will be working with full subcategories of a fixed k-linear abelian category $\mathcal{C}$. In practice $\mathcal{C} = \operatorname{rep} Q$, the category of finite-dimensional modules over kQ where Q is a finite quiver with no oriented cycles. In this section we sometimes prove things in a more general setting. We always assume that $\mathcal{C}$ is small and abelian. We also assume that $\mathcal{C}$ has the following three properties.

- *Artinian.* Every descending chain of subobjects of an object eventually stabilizes.
- *Krull–Schmidt.* Indecomposable objects have local endomorphism rings and every object decomposes into a finite direct sum of indecomposables.
- *Hereditary.* The functor $\operatorname{Ext}^1(X, -)$ is right exact for each object X.

The subcategories we consider will always be full and closed under direct sums and direct summands. So they are determined by their sets of isomorphism classes of indecomposable objects. We abuse notation and occasionally refer to the category as this set. Another way of identifying such a subcategory is by using a single module. We let $\operatorname{add} T$ denote the full subcategory, closed under direct sums, whose indecomposables are all direct summands of T^i for all i. Given a subcategory $\mathcal{A}$ of $\mathcal{C}$, which has only finitely many isomorphism classes of indecomposables, we let $\operatorname{bsc} \mathcal{A}$ be the direct sum over a system of representatives of the isomorphism classes of indecomposables of $\mathcal{A}$. So $\operatorname{add} \operatorname{bsc} \mathcal{A} = \mathcal{A}$. We use the operation bsc on a module as shorthand for $\operatorname{bsc} T = \operatorname{bsc} \operatorname{add} T$. Given a full subcategory $\mathcal{A}$ of $\mathcal{C}$ we let $\operatorname{Gen} \mathcal{A}$ be the full subcategory whose objects are all quotients of objects of $\mathcal{A}$. We also use the same notation $\operatorname{Gen} T$ for an object T in $\mathcal{C}$ as shorthand for $\operatorname{Gen} \operatorname{add} T$.

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Some of the definitions we need for the relevant subcategories include the following.

- *Torsion class*: a full subcategory that is closed under extensions and quotients.
- *Torsion free class*: a full subcategory that is closed under extensions and subobjects.
- *Exact abelian subcategory*: a full abelian subcategory where the inclusion functor is exact, hence closed under kernels and cokernels of the ambient category.
- *Wide subcategory*: an exact abelian subcategory closed under extensions.

2.2 Support tilting modules and torsion classes

In this section we outline the natural bijection between basic support tilting modules and finitely generated torsion classes. We work in the category rep Q of finite-dimensional representations of a finite acyclic quiver Q. Note that this ambient category is Artinian, hereditary and satisfies the Krull–Schmidt property. This material is well known, but we include the results for completeness. Most of the proofs in this section are given by appropriate references.

Definition 2.3. *We say that C is a* partial tilting module *if*:

(1) $\mathrm{Ext}^1(C, C) = 0$;

(2) $\mathrm{pd}\, C \leqslant 1$.

Note that since we are in a hereditary category the second condition will always hold. A *tilting module* C is a partial tilting module such that there is a short exact sequence

$$0 \to kQ \to C' \to C'' \to 0$$

where C', C'' are in $\mathrm{add}\, C$.

We are particularly concerned with partial tilting modules that are tilting on their supports. For a vertex x in the quiver Q, let S_x be the associated simple module of kQ. We say that the support of a module C is the set of simple modules that occur in the Jordan–Holder series for C, up to isomorphism. This also equals the set of simple modules which occur as subquotients of finite sums of copies of C. We need a few lemmas to elucidate the support of a partial tilting module.

Lemma 2.4. *Let C be a partial tilting module and let M be a representation of Q. Then* $\mathrm{supp}\, M \subseteq \mathrm{supp}\, C$ *if and only if M is a subquotient of C^i for some i.*

Proof. Suppose that $\mathrm{supp}\, M \subseteq \mathrm{supp}\, C$. Since the Jordan–Holder series for M is made up of simples which are subquotients of C, the statement will follow once we show that the set of subquotients of C^i for some i is closed under extension. Suppose that x, y are submodules of X, Y which are quotients of C^i for some i. We can map an extension $e \in \mathrm{Ext}^1(x, y) \to \mathrm{Ext}^1(x, Y)$, and then since we are in a hereditary category we can lift via the surjective map $\mathrm{Ext}^1(X, Y) \to \mathrm{Ext}^1(x, Y)$ to obtain an extension E of Y by X. Since C is partial tilting $\mathrm{Gen}\, C$ is a torsion class closed under extensions [ASS06, Lemma VI.2.3], so the extension E is in $\mathrm{Gen}\, C$. The converse is immediate. $\square$

A partial tilting module will be called *support tilting* if it also satisfies one of the following equivalent conditions.

Proposition 2.5. *The following conditions are equivalent for a partial tilting module C:*

(1) *C is tilting as a $kQ/\mathrm{ann}\, C$ module;*

(2) *if M is a subquotient of C^i and $\mathrm{Ext}^1(C, M) = 0$, then M is in $\mathrm{Gen}\, C$;*

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(3) *if* $\operatorname{supp} M \subseteq \operatorname{supp} C$ *and* $\operatorname{Ext}^1(C, M) = 0$, *then* M *is in* $\operatorname{Gen} C$;

(4) *the number of distinct indecomposable direct summands of* C *is the number of distinct simples in its support.*

Proof. The equivalence of (1) and (2) is in [ASS06, Proof of Theorem VI.2.5]. The equivalence of conditions (1) and (4) follows from [ASS06, Theorem VI.4.4.]. The equivalence of conditions (2) and (3) follows from Lemma 2.4. □

The following lemma is not used elsewhere, but clarifies the notion of support tilting.

Lemma 2.6. *Suppose that* C *is a support tilting module. Then the algebra* $kQ/\operatorname{ann} C$ *is the path algebra of the minimal subquiver on which* C *is supported.*

Proof. If a vertex v is not in $\operatorname{supp} C$, then clearly the corresponding idempotent is in $\operatorname{ann} C$ since $e_v C = 0$. Since $\operatorname{ann} C$ is a two-sided ideal, any path x that passes through a vertex not in the support of C is in $\operatorname{ann} C$. So this shows that $kQ/\operatorname{ann} C$ is supported on the minimal subquiver Q' on which C is supported. So we can restrict attention to Q'. Now C is support tilting, and in particular tilting on Q'. Therefore, C is faithful by [ASS06, Theorem VI.2.5] and so its annihilator is zero on Q'. □

We say that an object P in a subcategory $\mathcal{T}$ is $\mathcal{T}$*-split projective* if all surjective morphisms $I \twoheadrightarrow P$ in $\mathcal{T}$ are split. We say that P is $\mathcal{T}$-Ext *projective* if $\operatorname{Ext}^1(P, I) = 0$ for all I in $\mathcal{T}$. We drop the $\mathcal{T}$ in the notation when it is clear from context. The proof of the next lemma follows easily from these definitions.

Lemma 2.7. *If the subcategory* $\mathcal{T}$ *is closed under extensions and* U *is split projective in* $\mathcal{T}$, *then* U *is* Ext *projective.*

We say that a subcategory $\mathcal{T}$ is *generated* by $\mathcal{P} \subseteq \mathcal{T}$ if $\mathcal{T} \subseteq \operatorname{Gen} \mathcal{P}$. We say $\mathcal{T}$ is *finitely generated* if there exists a finite set of indecomposable objects in $\mathcal{T}$ that generate $\mathcal{T}$. We use this notion for torsion classes and wide subcategories.

We say that U is a *minimal generator* if for every direct sum decomposition $U \simeq U' \oplus U''$ we have that U' is not generated by U''. We next show that a finitely generated torsion class has a unique minimal generator.

Lemma 2.8. *A finitely generated torsion class* $\mathcal{T}$ *has a minimal generator, unique up to isomorphism, which is the direct sum of all of its indecomposable split projectives.*

Proof. Since $\mathcal{T}$ is finitely generated, it follows from the Artinian property that $\mathcal{T}$ has a minimal generator. Suppose that $\mathcal{T}$ is finitely generated by the sum of distinct indecomposables $U = \bigoplus U_i$ and suppose that Q in $\mathcal{T}$ is an indecomposable split projective. Since U generates, we can find a surjection $U^i \twoheadrightarrow Q$. This surjection must split so the Krull–Schmidt property allows us to conclude that Q is a summand of U.

For the converse, suppose that $\mathcal{T}$ is a torsion class with a minimal generator U. Let U_0 be an indecomposable summand of U, and consider a surjection $\rho : E \twoheadrightarrow U_0$ in $\mathcal{T}$. We may apply [ASS06, Proof of Lemma IV.6.1] to show that this map must split. Therefore, U is split projective. □

Lemma 2.9. *Let* Q *be a finite acyclic quiver. Let* $\mathcal{T}$ *be a finitely generated torsion class in* $\operatorname{rep} Q$ *and let* C *be the direct sum of its indecomposable* Ext*-projectives. Then* C *is support tilting.*

Proof. Let U be the direct sum of the indecomposable split projectives of $\mathcal{T}$. We know by Lemma 2.8 that U is a minimal generator of $\mathcal{T}$. The proof of Lemma VI.6.4 in [ASS06] shows that there is an exact sequence

$$0 \to kQ/\operatorname{ann} U \to U^i \to U' \to 0$$

where U' is Ext-projective in $\mathcal{T}$, and that $U \oplus U'$ is a tilting module on $kQ/\operatorname{ann} U$. Then [ASS06, Theorem VI.2.5(d)] (as noted in [ASS06, Proof of Lemma VI.6.4]) shows that the Ext-projectives of $\mathcal{T}$ are all summands of $U \oplus U'$. So $\operatorname{bsc} U \oplus U' \simeq \operatorname{bsc} C$ and C is support tilting. □

Given a subcategory $\mathcal{A}$ and an object Q of $\mathcal{C}$, a right $\mathcal{A}$ approximation of Q is a map $f: B \to Q$ where B is in $\mathcal{A}$ and any other morphism from an object in $\mathcal{A}$ to Q factors through f. This is equivalent to the map $f_*: \operatorname{Hom}(X, B) \to \operatorname{Hom}(X, Q)$ being surjective for all X in $\mathcal{A}$. Basic properties of approximations can be found in [AS80].

The next theorem shows that we can recover a basic support tilting object from the torsion class that it generates by taking the sum of the indecomposable Ext-projectives.

Theorem 2.10. *Let C be a support tilting object. Then* $\operatorname{Gen} C$ *is a torsion class and the indecomposable* Ext*-projectives of* $\operatorname{Gen} C$ *are all the indecomposable summands of C. So* $\operatorname{bsc} C$ *is the sum of the indecomposable* Ext*-projectives of* $\operatorname{Gen} C$.

Proof. Let Q be an Ext-projective of $\operatorname{Gen} C$. In particular Q is in $\operatorname{Gen} C$. Let $f: B \to Q$ be an $\operatorname{add} C$ right approximation to Q. Since Q is in $\operatorname{Gen} C$ we know that f is surjective. Apply the functor $\operatorname{Hom}(C, -)$ to the short exact sequence

$$0 \to \ker f \to B \to Q \to 0$$

to obtain the exact sequence

$$\operatorname{Hom}(C, B) \to \operatorname{Hom}(C, Q) \to \operatorname{Ext}^1(C, \ker f) \to \operatorname{Ext}^1(C, B).$$

We know $\operatorname{Ext}^1(C, B) = 0$ since C is partial tilting and B is in $\operatorname{add} C$. We also know that the map $\operatorname{Hom}(C, B) \to \operatorname{Hom}(C, Q)$ is surjective so $\operatorname{Ext}^1(C, \ker f) = 0$. Also $\ker f$ is a subquotient of C so we can conclude that $\ker f \in \operatorname{Gen} C$ since C is support tilting. Now since Q is an Ext-projective in $\operatorname{Gen} C$, the map f must be split and so Q is in $\operatorname{add} C$. So any indecomposable Ext-projective is a direct summand of C. We know that C is Ext-projective in $\operatorname{Gen} C$ since $\operatorname{Ext}^1(C, C) = 0$ and we are in a hereditary category so C can only have Ext-projective summands. This also shows that $\operatorname{Gen} C$ is a torsion class by [ASS06, Corollary VI.6.2]. □

Theorem 2.11. *Let $\mathcal{C} = \operatorname{rep} Q$, where Q is a finite acyclic quiver. Then there is a natural bijection between finitely generated torsion classes and basic support tilting objects given by taking the sum of all indecomposable* Ext*-projectives and its inverse* Gen.

Proof. This follows immediately from the above Theorem 2.10 and Lemma 2.8. □

2.3 Wide subcategories and torsion classes

We now define a bijection between finitely generated torsion classes and finitely generated wide subcategories. Let $\mathcal{T}$ be a torsion class. The wide subcategory corresponding to it is defined by taking those objects of $\mathcal{T}$ such that any morphism in $\mathcal{T}$ whose target is that object, must have its kernel in $\mathcal{T}$. More explicitly, let $\mathfrak{a}(\mathcal{T})$ be the full subcategory whose objects are in the set

$$\{B \in \mathcal{T} \mid \text{for all } (g: Y \to B) \in \mathcal{T}, \ker g \in \mathcal{T}\}.$$

Proposition 2.12. *Let $\mathcal{T}$ be a torsion class. Then $\mathfrak{a}(\mathcal{T})$ is a wide subcategory.*

Proof. We first show that $\mathfrak{a}(\mathcal{T})$ is closed under kernels. Let $f : A \to B$ be a morphism in $\mathfrak{a}(\mathcal{T})$. We know that $\ker f$ is in $\mathcal{T}$ by the definition of $\mathfrak{a}(\mathcal{T})$. Let $i : \ker f \hookrightarrow A$ be the natural injection. Take a test morphism $g : Y \to \ker f$ in $\mathcal{T}$. The composition $ig : Y \to \ker f \hookrightarrow A$ is a morphism in $\mathcal{T}$ with target A in $\mathfrak{a}(\mathcal{T})$. So we know that $\ker(ig)$ is in $\mathcal{T}$, but we also know that $\ker g = \ker(ig)$ since i is injective. So we can conclude that $\ker f$ is in $\mathfrak{a}(\mathcal{T})$.

Next we show that $\mathfrak{a}(\mathcal{T})$ is closed under extensions. Suppose that A, B are in $\mathfrak{a}(\mathcal{T})$ and let $0 \to A \xrightarrow{i} E \xrightarrow{\pi} B \to 0$ be an extension. Take a test map $g : Y \to E$ in $\mathcal{T}$. Using Lemma 2.1 for the composition πg we obtain an induced exact sequence

$$0 \to \ker g \to \ker(\pi g) \xrightarrow{\psi} A.$$

Since B is in $\mathfrak{a}(\mathcal{T})$ and Y is in $\mathcal{T}$ we can conclude that $\ker(\pi g)$ is in $\mathcal{T}$. Since A is in $\mathfrak{a}(\mathcal{T})$ we can use the map ψ of the above sequence to conclude that $\ker g$ is in $\mathcal{T}$.

Lastly we need to show that $\mathfrak{a}(\mathcal{T})$ is closed under cokernels. We take a morphism $f : A \to B$ in $\mathfrak{a}(\mathcal{T})$. Write C for $\operatorname{cok} f$ and let $g : Y \to C$ be a test morphism with Y in $\mathcal{T}$. Let $\pi : B \to C$ be the natural surjection. Note that we know that $\ker \pi = \operatorname{im} f$ is in $\mathcal{T}$ since $\operatorname{im} f$ is a quotient of A. So we form the pullback $Y \prod_C B$, getting an exact sequence

$$0 \to \ker \pi^* \to Y \prod_C B \xrightarrow{\pi^*} Y \to 0.$$

Since $\ker \pi^* \simeq \ker \pi$ and $\mathcal{T}$ is closed under extensions, we see that the pullback $Y \prod_C B$ is in $\mathcal{T}$. Now since B is in $\mathfrak{a}(\mathcal{T})$, the map

$$g^* : Y \prod_C B \to B$$

has kernel in $\mathcal{T}$. So since $\ker g \simeq \ker g^*$, the test map g has kernel in $\mathcal{T}$. □

The map from wide subcategories to torsion classes is described next. We first need to show that wide subcategories generate torsion classes.

Proposition 2.13. *If $\mathcal{A}$ is a wide subcategory of our ambient hereditary category $\mathcal{C}$, then* Gen $\mathcal{A}$ *is a torsion class.*

Proof. We only need to show that Gen $\mathcal{A}$ is closed under extensions. Let a, b be in Gen $\mathcal{A}$ with surjections $\pi : A \to a$ and $\rho : B \to b$ where A, B are in $\mathcal{A}$. Let

$$0 \to a \to e \to b \to 0$$

be an extension. Since we are in a hereditary category the map $\pi_* : \operatorname{Ext}^1(b, A) \to \operatorname{Ext}^1(b, a)$ is surjective. So we can choose a lift of the class of the extension above to obtain an extension

$$0 \to A \to E \to b \to 0$$

such that the pushout $\pi_* E = E \coprod_A a$ is isomorphic to e. Now we can simply pull back the class of E to an extension $\rho^* E = B \prod_b E$ of B by A. Since $\mathcal{A}$ is closed under extensions we see that $\rho^* E$ is in $\mathcal{A}$. The natural map $\pi_* \rho^* : \rho^* E \to e$ is surjective since $\operatorname{cok} \rho^* = \operatorname{cok} \rho = 0 = \operatorname{cok} \pi = \operatorname{cok} \pi_*$. □

The next proposition shows that the operations $\mathfrak{a}$ and Gen are surjective and injective, respectively, and the composition $\mathfrak{a}$ Gen gives the identity. This proposition is more general than we need; we show that once we restrict to finitely generated subcategories we can obtain a bijection.

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Proposition 2.14. *If $\mathcal{A}$ is a wide subcategory, then $\mathcal{A} = \mathfrak{a}(\mathrm{Gen}\,\mathcal{A})$.*

Proof. Suppose that an object B is in $\mathcal{A}$. We wish to show that it is in $\mathfrak{a}(\mathrm{Gen}\,\mathcal{A})$. So we take a test map $g : y \to B$ where y is in $\mathrm{Gen}\,\mathcal{A}$. So there is a surjection $\pi : Y \to y$ with Y in $\mathcal{A}$. Then Lemma 2.1 shows that there is an exact sequence

$$0 \to \ker \pi \to \ker g\pi \to \ker g \to 0.$$

Since $g\pi : Y \to B$ is a map in $\mathcal{A}$ we see that $\ker g\pi$ is in $\mathcal{A}$. So we see that $\ker g$ is in $\mathrm{Gen}\,\mathcal{A}$ and so B is in $\mathfrak{a}(\mathrm{Gen}\,\mathcal{A})$.

Now suppose that b is in $\mathfrak{a}(\mathrm{Gen}\,\mathcal{A})$. Since b is in $\mathrm{Gen}\,\mathcal{A}$, we can find a surjection $\pi : B \to b$ with B in $\mathcal{A}$. Since b is in $\mathfrak{a}(\mathrm{Gen}\,\mathcal{A})$ we know that $\ker \pi$ is in $\mathrm{Gen}\,\mathcal{A}$ and so we can find another surjection $\rho : K \to \ker \pi$ where K is in $\mathcal{A}$. Let $i : \ker \pi \to B$ be the natural inclusion. Now we can conclude that $b \simeq \mathrm{cok}\, i\rho$ and $i\rho : K \to B$ is a map in the wide subcategory $\mathcal{A}$, hence b is in $\mathcal{A}$. □

We need another characterization of the operation $\mathfrak{a}$ in the next proof so we show we can also define $\mathfrak{a}$ using only kernels of surjective maps from split projectives of $\mathcal{T}$.

Proposition 2.15. *Let $\mathcal{T}$ be a finitely generated torsion class in our ambient category $\mathcal{C}$ and define $\mathfrak{a}_s(\mathcal{T}) = \{B \in \mathcal{T} \mid$ for all surjections $g : (Z \to B) \in \mathcal{T}$ with Z split projective, we have $\ker g \in \mathcal{T}\}$. Then $\mathfrak{a}(\mathcal{T}) = \mathfrak{a}_s(\mathcal{T})$.*

Proof. It is clear that $\mathfrak{a}(\mathcal{T}) \subseteq \mathfrak{a}_s(\mathcal{T})$ so take B in $\mathfrak{a}_s(\mathcal{T})$ and a test map $g : Y \to B$ with Y in $\mathcal{T}$. We consider the extension

$$0 \to \ker g \to Y \to \mathrm{im}\, g \to 0$$

and let $i : \mathrm{im}\, g \to B$ be the natural injection. Since we are in a hereditary category, we know that the induced map $i^* : \mathrm{Ext}^1(B, \ker g) \to \mathrm{Ext}^1(\mathrm{im}\, g, \ker g)$ is surjective so we can find Y' such that there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \ker g \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d, "i^*"'] & \mathrm{im}\, g \arrow[r] \arrow[d, "i"'] & 0 \arrow[d, equal] \\
0 \arrow[r] & \ker g \arrow[r] & Y' \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

with $Y \simeq \mathrm{im}\, g \prod_B Y'$. Now B is in $\mathcal{T}$ so $\mathrm{cok}\, g \simeq \mathrm{cok}\, i$ is in $\mathcal{T}$. So we have an exact sequence

$$0 \to \ker i^* \to Y \to Y' \to \mathrm{cok}\, i^* \to 0.$$

Now $\ker i^* = \ker i = 0$ and $\mathrm{cok}\, i^* = \mathrm{cok}\, i$ is in $\mathcal{T}$ and Y is in $\mathcal{T}$ so we may conclude that Y' is in $\mathcal{T}$ since $\mathcal{T}$ is closed under extensions. Now we have a surjection $g' : Y' \to B$ in $\mathcal{T}$ with kernel isomorphic to $\ker g$. Let $h : Z \to Y'$ be a surjection, with Z a split projective. Then $\ker g'h$ is in $\mathcal{T}$, by assumption, and by Lemma 2.1, $\ker g' \simeq \ker g$ is a quotient of $\ker g'h$, so it is also in $\mathcal{T}$. Thus, B is in $\mathfrak{a}(\mathcal{T})$. □

We now are able to prove that we have a bijection from finitely generated torsion classes to finitely generated wide subcategories.

Proposition 2.16. *If $\mathcal{T}$ is a finitely generated torsion class, then $\mathfrak{a}(\mathcal{T})$ is finitely generated and $\mathrm{Gen}\,\mathfrak{a}(\mathcal{T}) = \mathcal{T}$. Furthermore, the projectives of $\mathfrak{a}(\mathcal{T})$ are the split projectives of $\mathcal{T}$.*

Proof. We first show that any $\mathcal{T}$-split projective U is also in $\mathfrak{a}(\mathcal{T})$. Since any surjection $Q \to U$ in $\mathcal{T}$ splits, and $\mathcal{T}$ is closed under direct summands, we know that U is in $\mathfrak{a}(\mathcal{T})$. Also, since U is $\mathcal{T}$-split projective it is also a projective object in $\mathfrak{a}(\mathcal{T})$. Conversely, any object P in $\mathfrak{a}(\mathcal{T})$ admits

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a surjection from some U^i where U is a split projective generator of $\mathcal{T}$, cf. Lemma 2.8. If P is projective in $\mathfrak{a}(\mathcal{T})$, then this surjection must split, so the projectives of $\mathfrak{a}(\mathcal{T})$ and the split projectives of $\mathcal{T}$ coincide.

Now Lemma 2.8 shows that $\mathcal{T}$ is generated by its split projectives, so we see that $\mathfrak{a}(\mathcal{T}) \subseteq \mathcal{T}$ is also finitely generated. □

Combining the above propositions immediately gives one of our main results.

Corollary 2.17. *There is a bijection between finitely generated torsion classes in $\mathcal{C}$ and finitely generated wide subcategories. The bijection is given by $\mathfrak{a}$ and its inverse* Gen.

Lemma 2.18. *Let $\mathcal{C}$ be a subcategory of a hereditary category. If P in $\mathcal{C}$ is* Ext*-projective, then any subobject $Q \hookrightarrow P$ in $\mathcal{C}$ is also* Ext*-projective.*

Proof. If a is in $\mathcal{C}$, then $\mathrm{Ext}^1(P, a) = 0$ and we have a surjection $\mathrm{Ext}^1(P, a) \twoheadrightarrow \mathrm{Ext}^1(Q, a)$. □

Lemma 2.19. *Let $\mathcal{T}$ be a finitely generated torsion class and let Q be a split projective in $\mathcal{T}$. Then any subobject of Q that is in $\mathcal{T}$ is split projective.*

Proof. Let $i : P \to Q$ be an injection in $\mathcal{T}$. Note that $\operatorname{cok} i$ is in $\mathcal{T}$. Since $\mathcal{T}$ is generated by its split projectives we can find a surjection $f : R \to P$ where R is split projective. Since we are in a hereditary category we can lift the extension R in $\mathrm{Ext}^1(P, \ker f)$ to an extension E in $\mathrm{Ext}^1(Q, \ker f)$. So we have an exact sequence

$$0 \to R \to E \to \operatorname{cok} i \to 0$$

which shows that E is in $\mathcal{T}$. Therefore, the surjection $E \to Q$ must split and the class of E in $\mathrm{Ext}^1(Q, \ker f)$ is zero. Therefore the class of R in $\mathrm{Ext}^1(P, \ker f)$ is also zero and so this extension splits. So P is a direct summand of the split projective R. □

Corollary 2.20. *If $\mathcal{A}$ is a finitely generated wide subcategory of* $\operatorname{rep} Q$*, then it is hereditary.*

Proof. We have $\mathcal{A} = \mathfrak{a}(\operatorname{Gen} \mathcal{A})$. The above result combined with Proposition 2.16 shows that this category is hereditary. □

We are also in a position to note that $\mathfrak{a}(\mathcal{T}) \simeq \operatorname{rep} Q'$ for some finite acyclic quiver Q' as in the next corollaries.

Corollary 2.21. *If $\mathcal{A}$ is a finitely generated wide subcategory of* $\operatorname{rep} Q$*, then* $\mathcal{A} \simeq \operatorname{mod} \operatorname{End}(U)$ *where U is the direct sum of the projectives of $\mathcal{A}$.*

Proof. We have $\mathcal{A} = \mathfrak{a}(\operatorname{Gen} \mathcal{A})$. Now Proposition 2.16 shows that the abelian category $\mathcal{A}$ has a projective generator which is the sum of the indecomposable split projectives in $\operatorname{Gen} \mathcal{A}$. So standard Morita theory proves the above equivalence [MR87, § 3.5.5]. □

Corollary 2.22. *If $\mathcal{A}$ is a finitely generated wide subcategory of* $\operatorname{rep} Q$*, then there is a finite acyclic quiver Q' such that* $\mathfrak{a}(\mathcal{T}) \simeq \operatorname{rep}(Q')$.

The proof follows on combining the above statements with the theorem that a finite-dimensional basic hereditary algebra over an algebraically closed ground field is a path algebra of an acyclic quiver [ASS06, Theorem VII.1.7].

We now proceed to give two alternative characterizations of the category $\mathfrak{a}(\mathcal{T})$.

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Proposition 2.23. *The category $\mathfrak{a}(\mathcal{T})$ consists of those objects of $\mathcal{T}$ which can be written as a quotient of a $\mathcal{T}$-split projective by another $\mathcal{T}$-split projective.*

Proof. Suppose that $X \in \mathfrak{a}(\mathcal{T})$. Since $\mathcal{T}$ is generated by split projectives, X can be written as a quotient of a split projective. Now, by the definition of $\mathfrak{a}(\mathcal{T})$, the kernel of this map must be in $\mathcal{T}$. Since it is a subobject of a split projective, it is also a split projective.

Let $X \in \mathcal{T}$, such that $X \simeq P/Q$ for P, Q split projectives. Let $g : S \to X$ be a test morphism, which, by Proposition 2.15, we can assume to be surjective, with S split projective.

From the Hom long exact sequence, we obtain $\operatorname{Hom}(S, P) \to \operatorname{Hom}(S, P/Q) \to \operatorname{Ext}^1(S, Q) = 0$. So g lifts to a map from S to P. We now have a short exact sequence:

$$0 \to \ker g \to S \oplus Q \to P \to 0.$$

Since P is split projective, this splits, and $\ker g$ is a summand of $S \oplus Q$, so is in $\mathcal{T}$. So $X \in \mathfrak{a}(\mathcal{T})$. □

We need the following alternative characterization of the category $\mathfrak{a}(\mathcal{T})$ in the sequel. It describes $\mathfrak{a}(\mathcal{T})$ as the perpendicular of the non-split projectives in $\mathcal{T}$.

Proposition 2.24. *Let $\mathcal{T}$ be a finitely generated torsion class and let P be the direct sum of a system of representatives of the isomorphism classes of indecomposable* Ext*-projectives which are not split projective. Then*

$$\mathfrak{a}(\mathcal{T}) = \{X \in \mathcal{T} \mid \operatorname{Hom}(P, X) = 0\} = \{X \in \mathcal{T} \mid \operatorname{Hom}(P, X) = \operatorname{Ext}^1(P, X) = 0\}.$$

Proof. Let Q be a split projective. We begin by showing that there are no non-zero morphisms from P to Q. Suppose, in contrast, that $f : P \to Q$ is non-zero. Since $\operatorname{im} f$ is a quotient of P, it is in $\mathcal{T}$, so, since it is a subobject of Q, it is split projective. Thus, the short exact sequence

$$0 \to \ker f \to P \to \operatorname{im} f \to 0$$

splits, and P has a split projective direct summand, contradicting the definition of P.

Now suppose we have X in $\mathfrak{a}(\mathcal{T})$. By Proposition 2.23, X can be written as Q/R, for Q, R split projectives. The Hom long exact sequence now gives us

$$0 = \operatorname{Hom}(P, Q) \to \operatorname{Hom}(P, X) \to \operatorname{Ext}^1(P, R) = 0,$$

so $\operatorname{Hom}(P, X) = 0$, as desired.

To prove the converse, we need to recall briefly the notion of minimal approximations. A map $f : R \to X$ is called *right minimal* if any map $g : R \to R$ such that $fg = f$, must be an isomorphism. A map that is right minimal and a right approximation (as defined before Theorem 2.10) is called a minimal right approximation.

Suppose that $X \in \mathcal{T}$ and $\operatorname{Hom}(P, X) = 0$. Let T be the sum of the Ext-projectives of $\mathcal{T}$. Consider the minimal right add T approximation to X; call it $k : R \to X$. Note that R will not include any non-split projective summands, since these admit no morphisms to X. Let K be the kernel of this map. By the properties of minimal approximation, the map $\operatorname{Hom}(T, R) \to \operatorname{Hom}(T, X)$ is surjective, so $\operatorname{Ext}^1(T, K) = 0$. Since the support of K is contained in the support of T, this implies that K is in $\mathcal{T}$ by Proposition 2.5 and Theorem 2.11. Since K is a subobject in $\mathcal{T}$ of a split projective, K is also split projective. Now $X \simeq R/K$ shows that X is in $\mathfrak{a}(\mathcal{T})$, by Proposition 2.23. □

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A *torsion free class* in a category $\mathcal{C}$ is the dual notion to a torsion class: it is a full subcategory closed under direct summands and sums, extensions, and subobjects. In the context of representations of a hereditary algebra A, in which, as we have seen, finitely generated wide subcategories are in bijection with finitely generated torsion classes, it is true dually that finitely cogenerated wide subcategories are in bijection with finitely cogenerated torsion free classes. (Note also that by Corollary 2.22 and its dual, finitely cogenerated wide subcategories coincide with finitely generated wide subcategories.) We do not need to make use of this matter, so we do not pursue it here.

However, we do need certain facts about torsion and torsion-free classes. These facts are well known [ASS06, § VI.1].

Lemma 2.25.

- *If $\mathcal{T}$ is a torsion class in* rep Q*, then the full subcategory $\mathcal{F}$ consisting of all objects admitting no non-zero morphism from an object of $\mathcal{T}$, is a torsion-free class.*
- *Dually, if $\mathcal{F}$ is a torsion-free class, then the full subcategory $\mathcal{T}$ consisting of the objects admitting no non-zero morphism to any object of $\mathcal{F}$ forms a torsion class.*
- *These operations which construct a torsion-free class from a torsion class and vice versa are mutually inverse. Such a pair $(\mathcal{T}, \mathcal{F})$ of reciprocally determining torsion and torsion-free classes is called a torsion pair.*
- *Given a torsion pair $(\mathcal{T}, \mathcal{F})$ and an object $X \in \operatorname{mod} A$, there is a canonical short exact sequence*
$$0 \to t(X) \to X \to X/t(X) \to 0$$
with $t(X) \in \mathcal{T}$ and $X/t(X) \in \mathcal{F}$.

2.4 Support tilting modules and cluster tilting objects

For Q a quiver with no oriented cycles, the most succinct definition of the cluster category is that it is $\mathcal{C}\mathcal{C}_Q = \mathcal{D}^b(Q)/\tau^{-1}[1]$, that is to say, the bounded derived category of representations of Q modulo a certain equivalence.

Fixing a fundamental domain for the action of $\tau^{-1}[1]$, we can identify a set of representatives of the isomorphism classes of the indecomposable objects of $\mathcal{C}\mathcal{C}_Q$ as consisting of a copy of the indecomposable representations of Q together with n objects $P_i[1]$, the shifts of the projective representations.

A cluster tilting object in $\mathcal{C}\mathcal{C}_Q$ is an object T such that $\operatorname{Ext}^1_{\mathcal{C}\mathcal{C}_Q}(T, T) = 0$, and any indecomposable U satisfying $\operatorname{Ext}^1_{\mathcal{C}\mathcal{C}_Q}(T, U) = 0 = \operatorname{Ext}^1_{\mathcal{C}\mathcal{C}_Q}(U, T) = 0$ must be a direct summand of U. Here $\operatorname{Ext}^j_{\mathcal{C}\mathcal{C}_Q}(X, Y)$ is defined as in [Kel05], to be $\bigoplus \operatorname{Ext}^j_{\mathcal{D}^b(Q)}(X, (\tau^{-1}[1])^i(Y))$.

It has been shown [CK06] (cf. also the appendix to [BMRT07]) that there is a bijection from the cluster tilting objects for $\mathcal{C}\mathcal{C}_Q$ to the clusters of the acyclic cluster algebra with initial seed given by Q. The entire structure of the cluster algebra and, in particular, the exchange relations between adjacent clusters, can also be read off from the cluster category [BMR08], although we have no occasion to make use of this here.

To describe the cluster category $\mathcal{C}\mathcal{C}_Q$ in a more elementary way, if X and Y are representations of Q, we have that $\operatorname{Ext}^1_{\mathcal{C}\mathcal{C}_Q}(X, Y) = 0$ iff $\operatorname{Ext}^1_{\mathcal{C}\mathcal{C}_Q}(Y, X) = 0$ iff $\operatorname{Ext}^1_Q(X, Y) = 0 = \operatorname{Ext}^1_Q(Y, X)$. In addition, $\operatorname{Ext}^1_{\mathcal{C}\mathcal{C}_Q}(X, P_i[1]) = 0$ iff $\operatorname{Ext}^1_{\mathcal{C}\mathcal{C}_Q}(P_i[1], X) = 0$ iff $\operatorname{Hom}_Q(P_i, X) = 0$ and, finally, $\operatorname{Ext}^1_{\mathcal{C}\mathcal{C}_Q}(P_i[1], P_j[1]) = 0$ always. Thus, the condition that an object of $\mathcal{C}\mathcal{C}_Q$ is cluster tilting can be expressed in terms of conditions that can be checked within rep Q.

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If T is an object in $\mathcal{CC}_Q$, define $\overline{T}$ to be the maximal direct summand of T which is an object in rep Q. From the above discussion, it is already clear that if T is a cluster tilting object, then $\overline{T}$ is a partial tilting object. In fact, more is true.

PROPOSITION 2.26. *If T is a cluster tilting object in $\mathcal{CC}_Q$, then $\overline{T}$ is support tilting. Conversely, any support tilting object V can be extended to a cluster tilting object in $\mathcal{CC}_Q$ by adding shifted projectives in exactly one way.*

Proof. Let T be a cluster tilting object, which we may suppose to be basic, and thus to have n direct summands. Suppose that p of its indecomposable summands are shifted projectives. So $\overline{T}$ has $n-p$ distinct indecomposable direct summands. Observe that the fact that the p shifted projective summands have no extensions with $\overline{T}$ in $\mathcal{CC}_Q$ implies that $\overline{T}$ is supported away from the corresponding p vertices of Q. Thus, $\overline{T}$ is supported on a quiver with at most $n-p$ vertices. However, $\overline{T}$ is a partial tilting object with $n-p$ different direct summands, so it must actually be support tilting.

Conversely, suppose that V is a support tilting object. Suppose that it has $n-p$ different direct summands. Then its support must consist of $n-p$ vertices. Thus, in $\mathcal{CC}_Q$, the object consisting of the direct sum of V and the shifted projectives corresponding to vertices not in the support of V gives a partial cluster tilting object with n different direct summands, which is therefore a cluster tilting object. Clearly, this is the only way to extend V to a cluster tilting object in $\mathcal{CC}_Q$ by adding shifted projectives (although there will be other ways to extend V to a cluster tilting object in $\mathcal{CC}_Q$, namely, by adding other indecomposable representations of Q). $\square$

2.5 Mutation

An object of $\mathcal{CC}_Q$ is called *almost tilting* if it is partial tilting and has $n-1$ different direct summands. A complement to an almost tilting object S is an indecomposable object M such that $S \oplus M$ is tilting.

LEMMA 2.27 (Buan *et al.* [BMRRT06]).*An almost tilting object S in $\mathcal{CC}_Q$ has exactly two complements (up to isomorphism).*

The procedure which takes a tilting object and removes one of its summands and replaces it by the other complement for the remaining almost tilting object is called *mutation*. It is the analogue in the cluster category of the mutation operation in cluster algebras.

Given an object T in $\mathcal{CC}_Q$, we write $\operatorname{Gen} T$ for the subcategory of rep Q generated by the summands of T which lie in rep Q. When we say that an indecomposable of T is split projective in $\operatorname{Gen} T$, we imply in particular that it is in rep Q.

The main result of this section is the following proposition.

PROPOSITION 2.28. *If S is an almost tilting object in $\mathcal{CC}_Q$ and M and M^* are its two complements in $\mathcal{CC}_Q$, then either M is split projective in* $\operatorname{Gen}(M \oplus S)$ *or M^* is split projective in* $\operatorname{Gen}(M^* \oplus S)$ *and exactly one of these holds.*

Proof. If S contains any shifted projectives, we can remove them and remove the corresponding vertices from Q. So we may assume that S is almost tilting in rep Q. The main tool used in the proof will be the following fact from [HU05].

LEMMA 2.29 (Happel and Unger [HU05]). *Let S be an almost tilting object in* rep Q*. Then either S is not sincere, in which case there is only one complement to S in* rep Q*, or S is sincere,*

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in which case the two complements to S are related by a short exact sequence

$$0 \to M_1 \to B \to M_2 \to 0 \tag{2.30}$$

with B in add S.

Suppose first that S is not sincere, and that M is its complement in rep Q. Since $S \oplus M$ is tilting, and therefore sincere, M admits no surjection from add S. So M is split projective in $\mathrm{Gen}(M \oplus S)$. On the other hand, the other complement M^* to S in $\mathcal{CC}_Q$ is not contained in rep Q, so it is certainly not split projective in $\mathrm{Gen}(M^* \oplus S)$.

Now suppose that S is sincere, and that its complements are M_1 and M_2, which are related as in (2.30). Clearly M_2 is not split projective in $\mathrm{Gen}(M_2 \oplus S)$, since it admits a surjection from B. On the other hand, suppose that there was a surjection $B' \to M_1$ with $B' \in \mathrm{add}\, S$. The non-zero extension of M_2 by M_1 would lift to an extension of M_2 by B', but that is impossible since M_2 is a complement to S. □

An order on basic tilting objects was introduced by Riedtmann and Schofield [RS91]. It was later studied by Happel and Unger in [HU05], in the context of modules over a not necessarily hereditary algebra. Their order is defined in terms of a certain subcategory associated with a basic tilting object:

$$\mathcal{E}(T) = \{M \mid \mathrm{Ext}^i_A(T, M) = 0 \text{ for } i > 0\}.$$

This order on basic tilting objects is defined by $S < T$ iff $\mathcal{E}(S) \subset \mathcal{E}(T)$. We recall the following lemma.

Lemma 2.31 [ASS06, Theorem VI.2.5]. *If T is a tilting object in* rep Q, *then* $\mathcal{E}(T) = \mathrm{Gen}\, T$.

For us, it is natural to consider a partial order on a slightly larger ground set, the set of tilting objects in $\mathcal{CC}_Q$, and to take as our definition that $S \leqslant T$ iff $\mathrm{Gen}\, S \subset \mathrm{Gen}\, T$. This is equivalent to considering the set of all finitely generated torsion classes ordered by inclusion. We show later (in § 4.2) that if Q is a Dynkin quiver, this order is naturally isomorphic to the Cambrian lattice defined by Reading [Rea06].

Lemma 2.32. *Let T be a tilting object in $\mathcal{CC}_Q$, let X be an indecomposable summand of T, and let V be the tilting object obtained by mutation at X. If X is split* Ext*-projective in* Gen T, *then $T > V$; otherwise, $T < V$.*

Proof. Let S be the almost tilting subobject of T which has X as its complement, and let Y be the other complement of S. If X is split Ext-projective in Gen T, then, by Proposition 2.28, Y is not split Ext-projective in Gen V. Thus, Gen V is generated by S, and so $\mathrm{Gen}\, V \subset \mathrm{Gen}\, T$.

On the other hand, if X is not split Ext-projective, then Y is, and the same argument shows that $\mathrm{Gen}\, V \supset \mathrm{Gen}\, S = \mathrm{Gen}\, T$. □

In fact, more is true. It is shown in [HU05] that if T and V are tilting objects in rep Q related by a single mutation, with, say $T > V$, then this is a cover relation in the order, that is to say, there is no other tilting object $R \in \mathrm{rep}\, Q$ with $T > R > V$. The proof in [HU05] extends to the more general setting (tilting objects in $\mathcal{CC}_Q$), but the proof is not simple and as we do not refer to this result again, we do not give a detailed proof here.

2.6 Semistable categories

In this section we show that any finitely generated wide subcategory of $\operatorname{rep} Q$ is a semistable category for some stability condition. (A result in the converse direction also holds, cf. Theorem 2.33.)

Recall that $K_0(kQ)$ is a lattice (i.e. finitely generated free abelian group) with basis naturally indexed by the simple modules. Since the simple modules are in turn indexed by the vertices we use the set of vertices $\{e_i\}$ as a basis of $K_0(kQ)$. We write $\underline{\dim}\, M$ for the class of M in $K_0(kQ)$. We know that $\underline{\dim}\, M = \sum_i \dim_k M_i e_i$. The Euler form on $K_0(kQ)$ is defined to be the linear extension of the pairing:

$$\langle \underline{\dim}\, M, \underline{\dim}\, N\rangle = \dim_k \operatorname{Hom}(M,N) - \dim_k \operatorname{Ext}^1(M,N).$$

For $\alpha = \sum \alpha_i e_i$ and $\beta = \sum \beta_i e_i$ in $K_0(kQ)$ we have

$$\langle \alpha, \beta\rangle = \sum_i \alpha_i\beta_i - \sum_{i\to j} \alpha_i \beta_j.$$

The Euler form is generally not symmetric, but we obtain a pairing on $K_0(kQ)$ by symmetrizing:

$$(\alpha,\beta) = \langle\alpha,\beta\rangle + \langle\beta,\alpha\rangle.$$

A stability condition [Kin94] is a linear function $\theta : K_0(kQ) \to \mathbb{Z}$. A representation V of Q is θ-semistable if $\theta(\underline{\dim}(V)) = 0$ and if $W \subseteq V$ is a subrepresentation, then $\theta(\underline{\dim}(W)) \leqslant 0$. We use the abbreviation $\theta(\underline{\dim}(V)) = \theta(V)$. Let θ_{ss} be the subcategory of representations that are semistable with respect to θ.

The following theorem is in [Kin94].

Theorem 2.33. *Let θ be a stability condition. Then θ_{ss} is wide.*

We need the following easy lemma so we record it here.

Lemma 2.34. *Let θ be a stability condition. Then θ_{ss} can also be described as the representations V such that $\theta(V) = 0$ and for all quotients W of V, we have that $\theta(W) \geqslant 0$.*

Let T be a basic support tilting object with direct summands $T_1, \ldots, T_r$. Since T is support tilting, it is supported on a subquiver Q' of Q with r vertices. Let us number the vertices on which T is supported by $n-r+1$ to n, and number the other vertices from one to $n-r$.

Let d_i be the function on $K_0(kQ)$ defined by

$$d_i(\underline{\dim}(M)) = \langle T_i, M\rangle = \dim_k \operatorname{Hom}(T_i, M) - \dim_k \operatorname{Ext}^1(T_i, M),$$

for $1 \leqslant i \leqslant r$. Let e_j be the function on $K_0(\operatorname{rep} Q)$ defined by $e_j(\underline{\dim}(M)) = \dim_k M_j$, that is, e_j is just the jth component with respect to the usual basis.

Theorem 2.35. *For $T = \bigoplus_{i=1}^r T_i$ a basic support tilting object, the abelian category $\mathfrak{a}(T) = \theta_{ss}$ for θ satisfying:*

$$\theta = \sum_{i=1}^{r} a_i d_i + \sum_{j=1}^{n-r} b_j e_j,$$

where $a_i = 0$ if T_i is split projective in $\operatorname{Gen} T$, *$a_i > 0$ if T_i is non-split projective, and $b_j < 0$.*

Proof. Suppose that θ is of the form given. Let us write $\mathcal{T}$ for $\operatorname{Gen} T$ and $\mathcal{A}$ for $\mathfrak{a}(\mathcal{T})$. First, we prove some statements about the value of θ on various objects in $\mathcal{T}$, then we put the pieces together.

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If $X \in \mathcal{A}$, then X does not admit any homomorphisms from non-split projectives by Proposition 2.24. However, since X is also in $\mathcal{T}$, $\mathrm{Ext}^1(T_i, X) = 0$ for all i. Thus, $\theta(X) = 0$.

If Y is in $\mathcal{T} \setminus \mathcal{A}$, then, by Proposition 2.24 again, X admits some homomorphism from a nonsplit projective. As before, $\mathrm{Ext}^1(T_i, X) = 0$ for all i. It follows that $\theta(Y) > 0$.

If Z is torsion free, on the other hand, we claim that $\theta(Z) < 0$. Since Z is torsion free, $\mathrm{Hom}(T_i, Z) = 0$ for all i. If $\mathrm{supp}(Z)$ is not contained in $\mathrm{supp}(T)$, then some $b_j e_j(Z) < 0$, and we are done. So suppose that $\mathrm{supp}(Z) \subset \mathrm{supp}(T)$. We restrict our attention to the quiver Q' where T is tilting. Now all we need to do is show that $\mathrm{Ext}^1(T_i, Z) \neq 0$ for some non-split projective T_i.

The torsion-free class corresponding to T is cogenerated by $\tau(T)$, so Z admits a homomorphism to τT_i for some i. In fact, we can say somewhat more. There is a dual notion to split projectives for torsion-free classes, namely split injectives, and a torsion-free class is cogenerated by its split injectives. So Z admits a morphism to some split injective τT_i. We must show that T_i is a non-split projective.

Now observe that (in $\mathcal{C}\mathcal{C}_{Q'}$) τT is a tilting object. Let S be the direct sum of all of the T_j other than T_i. So τS is almost tilting. By the dual version of Proposition 2.28, if V is the complement to τS other than τT_i, then either V is a shifted projective or V is non-split injective in $\mathrm{Cogen}\, \tau S$. Applying τ, we find that the complement to S other than T_i is $\tau^{-1} V$. It follows that the short exact sequence of Lemma 2.29 must be

$$0 \to \tau^{-1} V \to B \to T_i \to 0,$$

where B is in $\mathrm{add}\, S$. Since T_i admits a non-split surjection from an element of $\mathrm{add}\, S$, it must be that T_i is non-split projective. The morphism from Z to τT_i shows that $\mathrm{Ext}^1(T_i, Z) \neq 0$, so $\theta(Z) < 0$.

We now put together the pieces. If $X \in \mathcal{A}$, then $\theta(X) = 0$, while any quotient Y of X will be in $\mathcal{T}$, so will have $\theta(Y) \geqslant 0$. This implies that $X \in \theta_{ss}$. Now suppose that we have some $V \notin \mathcal{A}$. If $V \in \mathcal{T}$, then $\theta(V) > 0$, so $V \notin \theta_{ss}$. If $V \notin \mathcal{T}$, then V has some torsion-free quotient Z, and $\theta(Z) < 0$, so $V \notin \theta_{ss}$. Thus, $\theta_{ss} = \mathcal{A}$, as desired. □

3. Noncrossing partitions

3.1 Exceptional sequences and factorizations of the Coxeter element

For this section, we need to introduce the Coxeter group associated with Q, and the notion of exceptional sequences. Let $V = K_0(kQ) \otimes \mathbb{R}$ and recall that (α, β) is the symmetrized Euler form.

If $v \in V$ in non-zero, we can define a reflection

$$s_v(w) = w - \frac{2(v, w)}{(v, v)} v.$$

Let W be the group of transformations generated by the reflections $s_i = s_{e_i}$. The pair $(W, \{s_i\})$ forms a Coxeter system [Hum90, § 5.1].

The elements of V of the form $w(e_i)$ for some $w \in W$ are called (real) roots. The positive roots are those roots which are a non-negative integral combination of the $\{e_i\}$. If v is a positive root, the reflection s_v defined above is contained in W.

For later use, we recall some facts about *reflection functors*. Let Q be a quiver, and let v be a sink in Q. Let $\widetilde{Q}$ be obtained by reversing all of the arrows incident with v. Then there is a functor

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$R_v^+ : \operatorname{rep} Q \to \operatorname{rep} \widetilde{Q}$ such that, if we write P_v for the simple projective module supported at v, then $R_v^+(P_v)=0$, and R_v^+ gives an equivalence of categories from the full subcategory $\mathcal{S}$ of rep Q formed by the objects which do not admit P_v as a direct summand, to the full subcategory $\widetilde{\mathcal{S}}$ of rep $\widetilde{Q}$ formed by the objects which do not admit I_v as a direct summand. The effect of R_v^+ on dimension vectors is closely related to the simple reflection corresponding to v: specifically, if M does not contain any copies of P_v as indecomposable summands, then $\underline{\dim}\, R_v(M) = s_v(\underline{\dim}\, M)$. Dually, there is a reflection functor R_v^- from rep $\widetilde{Q}$ to rep Q. The functors R_v^+ and R_v^- induce mutually inverse equivalences between the full subcategories $\mathcal{S}$ and $\widetilde{\mathcal{S}}$. The functor R_v^+ is left exact and R_v^- is right exact.

The interaction between reflection functors and torsion pairs can be described as follows.

Lemma 3.1. *Let Q be a quiver with a sink at v. Let $(\mathcal{T},\mathcal{F})$ be a torsion pair where the simple projective P_v is in $\mathcal{F}$. We apply the reflection functor R_v^+ and write $\tilde{\mathcal{F}}$ and $\widetilde{\mathcal{T}}$ for the images of $\mathcal{F}$ and $\mathcal{T}$ in rep $\widetilde{Q}$. Then $\widetilde{\mathcal{F}}$ is a torsion-free class and the indecomposables in its corresponding torsion class are the simple injective $\widetilde{I}_v$ and the indecomposables of $\widetilde{\mathcal{T}}$.*

Proof. Suppose that x is in $\widetilde{\mathcal{F}}$ and we have a injection $f : y \to x$. If y has $\widetilde{I}_v$ as a direct summand, then so does x, but $\widetilde{I}_v$ is not in $\widetilde{\mathcal{F}}$, so this is impossible. If we apply the reflection functor R_v^- we obtain a morphism $R_v^-(f) : R_v^-(y) \to R_v^-(x)$. Let z be its kernel, so we have the following sequence exact on the left:

$$0 \to z \to R_v^-(y) \to R_v^-(x).$$

Applying R^+, which is left exact, we obtain

$$0 \to R_v^+(z) \to R_v^+ R_v^-(y) \to R_v^+ R_v^-(x).$$

Noting that since x and y do not have $\widetilde{I}_v$ as a direct summand, $R_v^+ R_v^-(f)$ is an injection, we see that $R_v^+(z)=0$, so z is a sum of copies of P_v, and thus $z \in \mathcal{F}$.

Now consider the short exact sequence

$$0 \to z \to R_v^-(y) \to \operatorname{im}(R_v^-(f)) \to 0.$$

Since $\operatorname{im}(R_v^-(f))$ is a subobject of $R_v^-(x) \in \mathcal{F}$, it is also in $\mathcal{F}$. Since $\mathcal{F}$ is extension closed, it follows that $R^-(y)$ is in $\mathcal{F}$, and thus y is in $\widetilde{\mathcal{F}}$. It is clear that $\widetilde{\mathcal{F}}$ is closed under extensions, so it is a torsion-free class.

Now let x be an indecomposable in its associated torsion class. So $\operatorname{Hom}(x,y)=0$ for all y in $\widetilde{\mathcal{F}}$. Then $\operatorname{Hom}(R_v^- x, R_v^- y)=0$ for all y in $\widetilde{\mathcal{F}}$ and $\operatorname{Hom}(R_v^- x, P_v)=0$. Since P_v and the indecomposables of $R_v^- \widetilde{\mathcal{F}}$ make up all indecomposables of $\mathcal{F}$ we see that $R_v^- x$ is in $\mathcal{T}$. So either x is in $\widetilde{\mathcal{T}}$ or $x \simeq \widetilde{I}_v$. □

A Coxeter element for W is, by definition, the product of the simple reflections in some order. We fix a Coxeter element $\operatorname{cox}(Q)$ to be the product of the s_i written from left to right in an order consistent with the arrows in the quiver Q. (If two vertices are not adjacent, then the corresponding reflections commute, so this yields a well-defined element of W.)

An object $M \in \operatorname{rep} Q$ is called *exceptional* if $\operatorname{Ext}^1(M,M)=0$. If M is an exceptional indecomposable of rep Q, then $\underline{\dim}\, M$ is a positive root. Thus, there is an associated reflection, $s_{\underline{\dim}\, M}$, which we also denote by s_M.

An exceptional sequence in rep Q is a sequence $X_1,\ldots,X_r$ such that each X_i is exceptional, and for $i<j$, $\operatorname{Hom}(X_j,X_i)=0$ and $\operatorname{Ext}^1(X_j,X_i)=0$. The maximum possible length of an exceptional sequence is n since the X_i are necessarily independent in $K_0(kQ) \simeq \mathbb{Z}^n$. An

exceptional sequence of length n is called *complete*. The simple representations of Q taken in any linear order compatible with the arrows of Q yield an exceptional sequence.

We recall some facts from [Cra92].

Lemma 3.2 (Crawley-Boevey [Cra92, Lemma 6]).*If (X, Y) is an exceptional sequence in* rep Q*, there are unique well-defined representations $R_Y X$, $L_X Y$ such that $(Y, R_Y X)$, $(L_X Y, X)$ are exceptional sequences in the smallest subcategory containing X and Y and closed under extensions, kernels of epimorphisms, and cokernels of monomorphisms.*

The objects $R_Y X$ and $L_X Y$ are discussed in several sources; see, for example, [Rud90]. They are called *mutations*; note that mutation has a different meaning in this context than in the context of clusters.

Lemma 3.3 (Crawley-Boevey [Cra92, p. 124]).*We have*

$$\underline{\dim}\, R_Y X = \pm s_Y(\underline{\dim}\, X),$$
$$\underline{\dim}\, L_X Y = \pm s_X(\underline{\dim}\, Y).$$

Lemma 3.4 (Crawley-Boevey [Cra92, Lemma 8]).*Let $(X_1, \ldots, X_n)$ be a complete exceptional sequence. Then*

$$(X_1, \ldots, X_{i-1}, X_{i+1}, Y, X_{i+2}, \ldots, X_n)$$

is an exceptional sequence iff $Y \simeq R_{X_{i+1}} X_i$. Similarly, $(X_1, \ldots, X_{i-1}, Z, X_i, \ldots, X_n)$ is an exceptional sequence iff $Z \simeq L_{X_i} X_{i+1}$.

Let $\mathcal{B}_n$ be the braid group on n strings, with generators $\sigma_1, \ldots, \sigma_{n-1}$ satisfying the braid relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i - j| \geqslant 2$, and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. It is straightforward to verify the following result.

Lemma 3.5 (Crawley-Boevey [Cra92, Lemma 9]).*The braid group on n strings $\mathcal{B}_n$ acts on the set of all complete exceptional sequences by*

$$\sigma_i(X_1, \ldots, X_n) = (X_1, \ldots X_{i-1}, X_{i+1}, R_{X_{i+1}} X_i, X_{i+2}, \ldots, X_n).$$

We can now state the main theorem of [Cra92].

Theorem 3.6 (Crawley-Boevey [Cra92, Theorem]).*The action of $\mathcal{B}_n$ on complete exceptional sequences is transitive.*

The next theorem follows from the above results.

Theorem 3.7. *If $(E_1, \ldots, E_n)$ is a complete exceptional sequence in* rep Q*, then $s_{E_1} \ldots s_{E_n} =$* $\operatorname{cox}(Q)$.

Proof. By the definition of $\operatorname{cox}(Q)$, the statement is true for the exceptional sequence consisting of simple modules. Now we observe that the product $s_{E_1} \ldots s_{E_n}$ is invariant under the action of the braid group. Since the braid group action on exceptional sequences is transitive, the theorem is proved. $\square$

3.2 Defining noncrossing partitions

In this section, we introduce the poset of noncrossing partitions. Let W be a Coxeter group. Let T be the set of all of the reflections of W, that is, the set of all conjugates of the simple reflections of W.

For $w \in W$, define the *absolute length* of w, written $\ell_T(w)$, to be the length of the shortest word for w as a product of arbitrary reflections. Note that this is not the usual notion of length, which would be the length of the shortest word for w as a product of simple reflections. We denote that length function, which will appear later, by $\ell_S(w)$.

Define a partial order on W by taking the transitive closure of the relations $u < v$ if $v = ut$ for some $t \in T$ and $\ell_T(v) = \ell_T(u) + 1$. We use the notation $\leqslant$ for the resulting partial order. This order is called *absolute order*.

One can rephrase this definition as saying that $u \leqslant v$ if there is a minimal-length expression for v as a product of reflections in which an expression for u appears as a prefix.

The noncrossing partitions for W are the interval in this absolute order between the identity element and a Coxeter element. (In finite type, the poset is independent of the choice of Coxeter element, but this is not necessarily true in general.) We write NC_Q for the noncrossing partitions in the Coxeter group corresponding to Q with respect to the Coxeter element $\mathrm{cox}(Q)$.

Inside NC_Q, for Q of finite type, there is yet another way of describing the order: for $u, v \in \mathrm{NC}_Q$, we have that $u \leqslant v$ iff the reverse inclusion of fixed spaces holds: $V^v \subseteq V^u$ (see [Bes03, BW02]).

Lemma 3.8. *We have* $\ell_T(\mathrm{cox}(Q)) = n$.

Proof. By definition, $\mathrm{cox}(Q)$ can be written as a product of n reflections. We just have to check that no smaller number will suffice. To do this, we use an equivalent definition of ℓ_T due to Dyer [Dye01]: fix a word for w as a product of simple reflections. Then $\ell_T(w)$ is the minimum number of simple reflections you need to delete from the word to be left with a factorization of e.

It is clear that, if we remove any less than all of the reflections from $\mathrm{cox}(Q) = s_1 \dots s_n$, we do not obtain the identity. So $\ell_T(\mathrm{cox}(Q)) = n$. $\square$

Lemma 3.9. *For $\mathcal{A}$ a finitely generated wide subcategory of* $\mathrm{rep}\, Q$, $\mathrm{cox}(\mathcal{A}) \in \mathrm{NC}_Q$.

Proof. The simple objects $(S_1, \dots, S_r)$ in $\mathcal{A}$ form an exceptional sequence in $\mathcal{A}$, so also in $\mathrm{rep}\, Q$. Extend it to a complete exceptional sequence in $\mathrm{rep}\, Q$. This exceptional sequence yields a factorization for $\mathrm{cox}(Q)$ as a product of n reflections which has $\mathrm{cox}(\mathcal{A})$ as a prefix, so $\mathrm{cox}(\mathcal{A}) \in \mathrm{NC}_Q$. $\square$

Lemma 3.10. *If $(E_1, \dots, E_r)$ is any exceptional sequence for $\mathcal{A}$, then $s_{E_1} \dots s_{E_r} = \mathrm{cox}(\mathcal{A})$.*

Proof. This follows from Theorem 3.7 applied in $\mathcal{A}$. $\square$

Lemma 3.11. *The map* cox *respects the poset structures on $\mathcal{W}_Q$ and NC_Q, in the sense that if $\mathcal{A} \subset \mathcal{B}$ are finitely generated wide subcategories, then* $\mathrm{cox}(\mathcal{A}) < \mathrm{cox}(\mathcal{B})$.

Proof. The exceptional sequence of simples for $\mathcal{A}$ can be extended to an exceptional sequence for $\mathcal{B}$. Thus, $\mathrm{cox}(\mathcal{A})$ is a prefix of what is, by Lemma 3.10, a minimal-length expression for $\mathrm{cox}(\mathcal{B})$. So $\mathrm{cox}(\mathcal{A}) < \mathrm{cox}(\mathcal{B})$. $\square$

We cannot prove that this map is either injective or surjective in general type. However, in finite or affine type, it is a poset isomorphism, as we now proceed to show.

After this paper was distributed in electronic form, the fact that cox is a poset isomorphism was shown for an arbitrary quiver without oriented cycles, based on a version of Lemma 3.15 below (see [IS09] for details).

3.3 The map from wide subcategories to noncrossing partitions in finite and affine type

For the duration of this section, we assume that Q is of finite or affine type.

LEMMA 3.12. *Let* $\operatorname{cox}(\mathcal{A})$ *be the Coxeter element for a finite type wide subcategory of* rep Q *of* rank r. *If* $\operatorname{cox}(\mathcal{A})$ *is written as a product of* r *reflections from* T*, then the reflections must all correspond to indecomposables of* $\mathcal{A}$.

Proof. Let $\beta_1, \ldots, \beta_r$ be the dimension vectors of the simple objects of $\mathcal{A}$. Being a finite type Coxeter element, $\operatorname{cox}(\mathcal{A})$ has no fixed points in the span $\langle \beta_1, \ldots, \beta_r \rangle$. Thus, its fixed subspace exactly consists of $F_{\mathcal{A}} = \bigcap_i \beta_i^{\perp}$, and is of codimension r. A product of r reflections has fixed space of codimension at most r, and if it has codimension exactly r, then the fixed space must be the intersection of the reflecting hyperplanes. Thus, if $\operatorname{cox}(\mathcal{A}) = s_{M_1} \ldots s_{M_r}$, then $\underline{\dim}\, M_j$ must lie in $F_{\mathcal{A}}^{\perp} = \langle \beta_1, \ldots, \beta_r \rangle$. The only positive roots in the span $\langle \beta_1, \ldots, \beta_r \rangle$ are the positive roots corresponding to indecomposable objects of $\mathcal{A}$, proving the lemma. $\square$

Given a subcategory $\mathcal{A}$ of $\mathcal{C}$ we write the perpendicular category as

$$^{\perp}\mathcal{A} = \{M \in \mathcal{C} \mid \operatorname{Hom}(M, V) = \operatorname{Ext}^1(M, V) = 0 \text{ for all } V \in \mathcal{A}\}.$$

If $\mathcal{A}$ is a wide subcategory, so is $^{\perp}A$. This follows from [Sch91, Theorem 2.3], and is easy to check directly.

THEOREM 3.13. *If* Q *is finite or affine, then* cox *is an injection.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be two finitely generated wide subcategories of rep Q such that $\operatorname{cox}(\mathcal{A}) = \operatorname{cox}(\mathcal{B})$. We may extend an exceptional sequence for $\mathcal{A}$ to one for rep Q, and what we add will be an exceptional sequence for $^{\perp}\mathcal{A}$. So $\operatorname{cox}(\mathcal{A}) \operatorname{cox}(^{\perp}\mathcal{A}) = \operatorname{cox}(Q)$. Hence, it follows that $\operatorname{cox}(^{\perp}\mathcal{A}) = \operatorname{cox}(^{\perp}\mathcal{B})$. Now $\mathcal{A}$ is of finite or affine type, and it is affine iff there is an isotropic dimension vector in the span of its dimension vectors. Since V has at most a one-dimensional isotropic subspace, at most one of $\mathcal{A}$ or $^{\perp}\mathcal{A}$ is of affine type. Thus, without loss of generality, we can assume that $\mathcal{A}$ is of finite type. By assumption, $\operatorname{cox} \mathcal{B} = \operatorname{cox} \mathcal{A}$. Note also that $r = \ell_T(\operatorname{cox}(\mathcal{A})) = \ell_T(\operatorname{cox}(\mathcal{B}))$ is the rank of $\mathcal{B}$, so the expression for $\operatorname{cox}(\mathcal{B})$ as the product of the reflections corresponding to the simples of $\mathcal{B}$ is an expression for $\operatorname{cox} \mathcal{B} = \operatorname{cox} \mathcal{A}$ as a product of r reflections. By the previous lemma, the simple objects of $\mathcal{B}$ must be in $\mathcal{A}$. Since the ranks of $\mathcal{A}$ and $\mathcal{B}$ are equal, $\mathcal{B} = \mathcal{A}$. $\square$

The argument that cox is surjective is based on the following lemma.

LEMMA 3.14. *If* Q *is of finite or affine type and* M_i *are indecomposable objects whose dimension vectors are positive roots such that* $\operatorname{cox}(Q) = s_{M_1} \ldots s_{M_n}$*, then at least one of the* M_i *is post-projective or pre-injective.*

Note that any wild-type quiver Q with at least three vertices has tilting objects which are regular (i.e. have no post-projective or pre-injective summand) [Rin88]. Since a tilting object yields an exceptional sequence, and therefore a factorization of $\operatorname{cox}(Q)$, this lemma cannot hold for any such quivers.

Proof. There is nothing to prove in finite type, since in that case every indecomposable is post-projective (and pre-injective). In affine type, consider the affine reflection group description of W as a semi-direct product, $W = W_{\text{fin}} \ltimes \Lambda$ where Λ is a lattice of translations. The Coxeter element has a non-zero translation component, since otherwise it would be of finite order, and we know this is not so because if M is an indecomposable non-projective object in rep Q, then

$\underline{\dim}(\tau M) = \mathrm{cox}(Q)\, \underline{\dim}\, M$ (see [ASS06, Theorem VII.5.8]). Since all of the regular objects are in finite τ-orbits, their reflecting hyperplanes are in finite $\mathrm{cox}(Q)$-orbits. Thus, they must be parallel to the translation component of $\mathrm{cox}(Q)$. Now $\mathrm{cox}(Q)$ cannot be written as a product of reflections in hyperplanes parallel to the translation component of $\mathrm{cox}(Q)$, because such a product would not have the desired translation component. Thus, any factorization of $\mathrm{cox}(Q)$ must include some factor which is pre-injective or post-projective. □

Lemma 3.15. *If Q is of finite or affine type and $\mathrm{cox}(Q) = s_{M_1} \dots s_{M_n}$, then all of the M_i are exceptional.*

Proof. There is nothing to prove in the finite type case. Fix a specific M_i which we wish to show is exceptional. If M_i is post-projective or pre-injective, we are done. So assume otherwise. Then by the previous lemma there is some M_j with $j \neq i$ which is post-projective or pre-injective. By braid operations, we may assume that it is either M_1 or M_n. Assume the latter. Assume further that M_n is post-projective. Now $\mathrm{cox}(Q) s_M \mathrm{cox}(Q)^{-1} = s_{\tau M}$. Conjugating by $\mathrm{cox}(Q)$ clearly preserves the product, and τ preserves exceptionality. Thus, we may assume that M_n is projective. Applying reflection functors, we may assume that M_n is simple projective. (In this step, the orientation of Q and thus the choice of $\mathrm{cox}(Q)$ will change.) Now let $\mathcal{A} = {}^{\perp}M_n$. Note that $\mathcal{A}$ is isomorphic to the representations of Q with the vertex corresponding to M_n removed, so $\mathcal{A}$ is finite type. Thus, $\mathrm{cox}(\mathcal{A}) = \mathrm{cox}(Q) s_{M_n}$ is a Coxeter element of finite type, so any factorization of it into $n-1$ reflections must make use of reflections with dimension vectors in $\mathcal{A}$. Thus, $M_i \in \mathcal{A}$, so it is exceptional.

If M_n was pre-injective instead of post-projective, we would have conjugated by $\mathrm{cox}^{-1}(Q)$ to make M_n injective. The effect of conjugating by $\mathrm{cox}^{-1}(Q)$ one more time is to turn s_{M_n} into a reflection corresponding to an indecomposable projective. Then we proceed as above. □

Theorem 3.16. *In finite or affine type, the map* cox *is a surjection.*

Proof. The argument is by induction on n. Let $w \in \mathrm{NC}_Q$. If w is rank n, the statement is immediate. By the previous lemma, the statement is also true if w is rank $n-1$: we know that $\mathrm{cox}(Q) w^{-1}$ is a reflection corresponding to an exceptional indecomposable object E, so $w = \mathrm{cox}({}^{\perp}E)$. If rank $w < n-1$, there is some v of rank $n-1$ over w. By the above argument, $v = \mathrm{cox}({}^{\perp}E)$. Apply induction to ${}^{\perp}E$. □

4. Finite type

Throughout this section, we assume that Q is an orientation of a simply laced Dynkin diagram. A fundamental result is Gabriel's theorem, which is proved in [ASS06, § VII.5] as well as other sources.

Theorem 4.1. *The underlying graph of Q is a Dynkin diagram iff there is a finite number of isomorphism classes of indecomposable representations of Q. In this case $\underline{\dim}$ is a bijection from indecomposable representations of Q to the positive roots in the root system corresponding to Q expressed with respect to the basis of simple roots.*

In § 4.4, we show how our results extend to non-simply laced Dynkin diagrams.

4.1 Lattice property of NC_Q

Our first theorem in finite type is an immediate corollary of results we have already proved. This theorem was first established by combinatorial arguments in the classical cases, together with a computer check for the exceptionals. It was given a type-free proof by Brady and Watt [BW08].

Theorem 4.2. *In finite type* NC_Q *forms a lattice.*

Proof. If $\mathcal{A},\mathcal{B}\in\mathcal{W}_Q$, then $\mathcal{A}\cap\mathcal{B}\in\mathcal{W}_Q$, since the intersection of two abelian and extension-closed subcategories is again abelian and extension-closed, while the finite generation condition is trivially satisfied because we are in finite type. This shows that $\mathcal{W}_Q$, ordered by inclusion, has a meet operation. Since it also has a maximum element, and it is a finite poset, this suffices to show that it is a lattice. Now cox is a poset isomorphism from $\mathcal{W}_Q$ ordered by inclusion to NC_Q, so NC_Q is also a lattice. □

Note that if Q is not of finite type, NC_Q need not form a lattice. (There are non-lattices already in $\widetilde{A}_n$ for some choices of (acyclic) orientation [Dig06].) This seems natural from the point of view of $\mathcal{W}_Q$, since the intersection of two finitely generated subcategories of rep Q need not be finitely generated.

4.2 Reading's bijection from noncrossing partitions to clusters

Our second main finite type result concerns bijections between noncrossing partitions and clusters. One such bijection in finite type was constructed by Reading [Rea07a], and another subsequently by Athanasiadis *et al.* [ABMW06]. We show that the bijection we have already constructed between clusters and noncrossing partitions specializes in finite type to that constructed by Reading.

We first need to introduce Reading's notion of a c-sortable element of W, where c is a Coxeter element for W. There are several equivalent definitions; we give the inductive characterization, as that proves to be the most useful for our purposes.

A simple reflection s is called *initial* in c if there is a reduced word for c which begins with s. (Note, therefore, that there may be more than one simple reflection which is initial in c, but there is certainly at least one.) If s is initial in c, then scs is another Coxeter element for W, and sc is a Coxeter element for a reflection subgroup of W, namely, the subgroup generated by the simple reflections other than s.

By [Rea07a, Lemmas 2.4 and 2.5], and the comment after them, the c-sortable elements can be characterized by the following properties:

- the identity e is c-sortable for any c;
- if s is initial in c, then
  * if $\ell_S(sw)>\ell_S(w)$, then w is c-sortable iff w is in the reflection subgroup of W generated by the simple reflections other than s, and w is sc-sortable;
  * if $\ell_S(sw)<\ell_S(w)$, then w is c-sortable iff sw is scs-sortable.

Let Φ be the root system associated with Q, with Φ^+ the positive roots. For $w\in W$, we write $I(w)$ for the set of positive roots α such that $w^{-1}(\alpha)$ is a negative root. $I(w)$ is called the *inversion set* of w.

Gabriel's theorem tells us that $\underline{\dim}$ is a bijection from indecomposable objects of rep Q to Φ^+. If $\mathcal{A}$ is an additive subcategory of rep Q that is closed under direct summands, we write $\mathrm{Ind}(\mathcal{A})$ for the corresponding set of positive roots. If $\alpha\in\Phi^+$, we write M_α for the corresponding indecomposable objects. If M_α is projective (respectively, injective) we sometimes write P_α (respectively, I_α) to emphasize this fact.

Theorem 4.3. *For Q of finite type, there is a bijection between torsion classes and $\operatorname{cox}(Q)$-sortable elements, $\mathcal{T} \to w_{\mathcal{T}}$, where $w_{\mathcal{T}}$ is defined by the property that $\operatorname{Ind}(\mathcal{T}) = I(w_{\mathcal{T}})$.*

Proof. Let $\mathcal{T}$ be a torsion class. We first prove that $\operatorname{Ind}(\mathcal{T})$ is the inversion set of some $\operatorname{cox}(Q)$-sortable element. The proof is by induction on the number of vertices of Q and $|\operatorname{Ind}(\mathcal{T})|$.

Let α be the positive root corresponding to a simple injective for Q. Let v_α designate the corresponding source of Q. Now s_α is initial in $\operatorname{cox}(Q)$. If $I_\alpha \notin \mathcal{T}$, then $\mathcal{T}$ is supported away from v_α. Let Q' be the subquiver of Q with v_α removed, and let W' be the corresponding reflection group. Then $\operatorname{cox}(Q') = s_\alpha \operatorname{cox}(Q)$ and, by induction, $\operatorname{Ind}(\mathcal{T})$ is the inversion set of a $\operatorname{cox}(Q')$-sortable element w. Now $\ell_S(s_\alpha w) > \ell_S(w)$, and w is $s_\alpha \operatorname{cox}(Q)$-sortable, so w is $\operatorname{cox}(Q)$-sortable, as desired.

Now suppose that $I_\alpha \in \mathcal{T}$. In this case, we apply the reflection functor $R^-_{v_\alpha}$. Let $\widetilde{\mathcal{T}}$ be the image of $\mathcal{T}$. It has one fewer indecomposable so, by induction, it corresponds to the inversion set of a $s_\alpha \operatorname{cox}(Q) s_\alpha$-sortable element, say $\widetilde{w}$. Now $s_\alpha \widetilde{w}$ is $\operatorname{cox}(Q)$-sortable and has the desired inversion set.

Next we show that if w is $\operatorname{cox}(Q)$-sortable, then $I(w)$ is $\operatorname{Ind}(\mathcal{T})$ for some torsion class $\mathcal{T}$. Again, we work by induction. If $\ell_S(s_\alpha w) > \ell_S(w)$, then w is $s_\alpha \operatorname{cox}(Q)$-sortable. Thus, by induction, there is a torsion class $\mathcal{T}'$ on Q' with $\operatorname{Ind}(\mathcal{T}') = I(w)$; now $\mathcal{T}'$ is also a torsion class on Q, so we are done.

Suppose, on the other hand, that $\ell_S(s_\alpha w) < \ell_S(w)$. By the induction hypothesis, there is a torsion class $\widetilde{\mathcal{T}}$ on $\widetilde{Q}$, with $\operatorname{Ind}(\widetilde{\mathcal{T}}) = I(s_\alpha w)$. Let $\mathcal{T}$ be the full subcategory additively generated by $R^+_{v_\alpha}(\widetilde{\mathcal{T}})$ and I_α. Now $\operatorname{Ind}(\mathcal{T}) = I(w)$. By Lemma 3.1, $\mathcal{T}$ is a torsion class. □

The c-sortable elements of W, ordered by inclusion of inversion sets, form a lattice, which is isomorphic to the Cambrian lattice $\mathfrak{C}_Q$ (see [Rea07b]). Any readers unfamiliar with Cambrian lattices may take this as the definition. (The original definition of the Cambrian lattice [Rea06] involves some lattice-theoretic notions which we do not require here, so we pass over it.) Thanks to the previous theorem, $\mathfrak{C}_Q$ is also isomorphic to the poset of torsion classes ordered by inclusion.

A *cover reflection* of an element $w \in W$ is a reflection $t \in T$ such that $tw = ws$ where $s \in S$ and $\ell_S(ws) < \ell_S(w)$.

Proposition 4.4. *If s is initial in $\operatorname{cox}(Q)$, and $\mathcal{T}$ is a torsion class such that $\ell_S(sw_{\mathcal{T}}) < \ell_S(w_{\mathcal{T}})$, then s is a cover reflection for $w_{\mathcal{T}}$ iff M_{α_s} is in $\mathfrak{a}(\mathcal{T})$.*

Proof. A reflection $t \in T$ corresponding to a positive root α_t is a cover reflection for $w \in W$ iff $I(w) \backslash \alpha_t$ is also the set of inversions for some element of W. A stronger version of the following lemma (without the simply laced assumption) is [Pil06, Proposition 1]; see also [Bou68, VI§ 1 Exercise 16].

Lemma 4.5. *The sets of roots which arise as inversion sets of elements of W a simply laced finite reflection group, are precisely those whose intersection with any three positive roots of the form $\{\alpha, \alpha + \beta, \beta\}$ is a subset which is neither $\{\alpha, \beta\}$ nor $\{\alpha + \beta\}$.*

We say that a set of positive roots is *good* if it forms the inversion set of an element of W, and *bad* otherwise. Similarly, we speak of good and bad intersections with a given set of positive roots $\{\alpha, \alpha + \beta, \beta\}$.

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Thus, if s is not a cover reflection for $w_{\mathcal{T}}$, then there are some positive roots $R = \{\beta, \beta + \alpha_s, \alpha_s\}$ such that the intersection of $I(w_{\mathcal{T}})$ with R is good, but becomes bad if we remove α_s. Thus, $I(w_{\mathcal{T}}) \cap R = \{\beta + \alpha_s, \alpha_s\}$. So $M_{\beta+\alpha_s} \in \mathcal{T}$. Since s is initial in c, we know that M_{α_s} is a simple injective. Thus, there is a map from $M_{\beta+\alpha_s}$ to M_{α_s}, whose kernel will be some representation of dimension β. In fact, though, a generic representation of dimension $\beta + \alpha_s$ will be isomorphic to $M_{\beta+\alpha_s}$ (see [GR92, Theorem 7.1]), and if we take a generic map from it to M_{α_s}, the kernel will be a generic representation of dimension β, thus isomorphic to M_β. Thus, the kernel of the map from $M_{\beta+\alpha_s}$ to M_{α_s} is M_β. Since $\beta \notin \mathrm{Ind}(\mathcal{T})$, we see that $M_\beta \notin \mathcal{T}$. Thus, by the definition of $\mathfrak{a}(\mathcal{T})$, we have that $M_{\alpha_s} \notin \mathfrak{a}(\mathcal{T})$.

Conversely, suppose $M_{\alpha_s} \notin \mathfrak{a}(\mathcal{T})$. By Proposition 2.15 there is a short exact sequence $0 \to K \to N \to M_{\alpha_s} \to 0$ with $K \notin \mathcal{T}$, $N \in \mathcal{T}$. Choose such a K so that its total dimension is as small as possible.

Let K' be an indecomposable summand of the torsion-free quotient of K (as in Lemma 2.25), with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$ determined by $\mathcal{T}$. Then the pushout N' is a quotient of N, with $0 \to K' \to N' \to M_{\alpha_s} \to 0$.

So by our minimality assumption on K, it must be that K is torsion free and indecomposable. Suppose that N is not indecomposable. Then let N'' be a direct summand of N which maps in a non-zero fashion to M_{α_s}. Let K'' be the kernel of the map from N'' to M_{α_s}. Since K'' is a subobject of K, and $\mathcal{F}$ is closed under subobjects, by minimality, $K'' = K$, so we may assume that both K and N are indecomposables, with dimensions, say, β and $\beta + \alpha_s$. So $\beta \notin \mathrm{Ind}(\mathcal{T})$, while $\beta + \alpha_s \in \mathrm{Ind}(\mathcal{T})$, as desired. □

Reading's map from c-sortable elements to noncrossing partitions can be characterized by the following proposition.

PROPOSITION 4.6 (Reading [Rea07a]). *There is a unique map from the c-sortable elements to* NC_c *characterized by the properties that* $\mathrm{nc}_c(e) = e$, *and, if s is initial in c:*

- *if $\ell_S(sw) > \ell_S(w)$, then $\mathrm{nc}_c(w) = \mathrm{nc}_{sc}(w)$;*
- *if $\ell_S(sw) < \ell_S(w)$ and s is a cover reflection of w, then $\mathrm{nc}_c(w) = \mathrm{nc}_{scs}(sw) \cdot s$;*
- *if $\ell_S(sw) < \ell_S(w)$ and s is not a cover reflection of w, then $\mathrm{nc}_c(w) = s \cdot \mathrm{nc}_{scs}\, w \cdot s$.*

There is also a non-inductive definition of the map, but it is somewhat complicated, and it is not needed here, so we do not give it. The above is essentially [Rea07a, Lemma 6.5].

THEOREM 4.7. *The map* nc *coincides with our map from torsion classes to noncrossing partitions.*

Proof. Our map from torsion classes to noncrossing partitions is $\mathrm{cox} \circ \mathfrak{a}$. The proof amounts to showing that $\mathrm{cox} \circ \mathfrak{a}$ satisfies the characterization of Proposition 4.6. Let s_α be initial in $\mathrm{cox}(Q)$ (and, equivalently, let M_α be a simple injective). Let w be a $\mathrm{cox}(Q)$-sortable element, and let $\mathcal{T}$ be the corresponding torsion class. If $\ell_S(s_\alpha w) > \ell_S(w)$, then, as we have seen, $\mathcal{T}$ is supported on Q'. The desired condition is now trivially true.

Now suppose $\ell_S(s_\alpha w) < \ell_S(w)$. Define $\widetilde{Q}$ to be the reflection of Q at v_α. Let $\widetilde{\mathcal{T}}$ be the image of $\mathcal{T}$ under the reflection functor $R^-_{v_\alpha}$. By Lemma 3.1, $\widetilde{\mathcal{T}}$ is a torsion class for $\mathrm{rep}\,\widetilde{Q}$. $\mathrm{Ind}(\widetilde{\mathcal{T}}) = s_\alpha(\mathrm{Ind}(\mathcal{T}) \backslash \alpha)$.

If s_α is not a cover reflection for w, then $M_\alpha \notin \mathfrak{a}(\mathcal{T})$, so $R^-_{v_\alpha}(\mathfrak{a}(\mathcal{T}))$ is an abelian category which generates $\widetilde{\mathcal{T}}$, and so $\mathfrak{a}(\widetilde{\mathcal{T}}) = R^-_{v_\alpha}(\mathfrak{a}(\mathcal{T}))$, and thus $\mathrm{cox}(\mathfrak{a}(\widetilde{\mathcal{T}})) = s_\alpha \,\mathrm{cox}(\mathfrak{a}(\mathcal{T})) s_\alpha$.

On the other hand, if s_α is a cover reflection for w, then M_α is a simple injective for $\mathfrak{a}(\mathcal{T})$, and so $R^-_{v_\alpha}$ can be restricted to a reflection functor for $\mathfrak{a}(\mathcal{T}) = \operatorname{rep} S$ for some quiver S. Note that $\mathfrak{a}(\widetilde{\mathcal{T}})$ is contained in $R^-_{v_\alpha}(\mathfrak{a}(\mathcal{T})) \subset \operatorname{rep} \widetilde{S}$ so we can restrict our attention to the representations of S and $\widetilde{S}$. The restriction of $\mathcal{T}$ to $\operatorname{rep} S$, though, is all of $\operatorname{rep} S$. Denote the restriction of $\widetilde{\mathcal{T}}$ to $\operatorname{rep} \widetilde{S}$ by $\widetilde{\mathcal{T}}_{\widetilde{S}}$. Now $\operatorname{ind} \widetilde{\mathcal{T}}_{\widetilde{S}}$ consists of all of $\operatorname{ind} \operatorname{rep} \widetilde{S}$ except $\widetilde{M_\alpha}$. This leaves us in a very well-understood situation. In $\operatorname{rep} \widetilde{S}$, $\widetilde{M_\alpha}$ is projective, and if we take P_{v_α} to be the projective corresponding to v_α in $\operatorname{rep} S$, then, in $\operatorname{rep} \widetilde{S}$, we have that $R^-_{v_\alpha}(P_{v_\alpha}) = \tau^{-1}\widetilde{M_\alpha}$, so, in particular, there is a short exact sequence in $\operatorname{rep} \widetilde{S}$, $0 \to \widetilde{M_\alpha} \to \widetilde{P} \to R^-_{v_\alpha}(P_{v_\alpha}) \to 0$, where $\widetilde{P}$ is a sum of indecomposable projectives for $\widetilde{S}$ other than $\widetilde{M_\alpha}$. This shows that $R^-_{v_\alpha}(P_{v_\alpha})$ is not split projective. The other Ext-projectives of $\widetilde{\mathcal{T}}_{\widetilde{S}}$ are projectives of $\operatorname{rep} \widetilde{S}$, so are certainly split projectives. Thus, $\mathfrak{a}(\widetilde{\mathcal{T}}_{\widetilde{S}})$ is the part of $\operatorname{rep} \widetilde{S}$ supported away from $\widetilde{M_\alpha}$, and the same is therefore true of $\mathfrak{a}(\widetilde{\mathcal{T}})$. Thus, $\operatorname{cox}(\mathfrak{a}(\widetilde{\mathcal{T}}))$ can be calculated by taking the product of the reflections corresponding to the injectives of $\operatorname{rep} \widetilde{S}$ other than s_α. The desired result follows. □

Reading also defines a map cl_c from c-sortable elements to 'c-clusters'. We present a version of his map which takes c-sortable elements to support tilting objects, since that fits our machinery better.

Proposition 4.8 (Reading [Rea07a]). *There is a unique map from c-sortable elements to support tilting objects in $\operatorname{rep} Q$ which can be characterized by the following properties:*

- *if s is initial in c and $\ell_S(sw) > \ell_S(w)$, then $\mathrm{cl}_c(w) = \mathrm{cl}_{sc}(w)$;*
- *if s is initial in c and $\ell_S(sw) < \ell_S(w)$, then $\mathrm{cl}_c(w) = \overline{R}^+_{v_s}\,\mathrm{cl}_{scs}(sw)$;*
- $\mathrm{cl}_c(e) = 0$.

In the above proposition $\overline{R}^+_{v_s}$ is a map on objects which is defined by $\overline{R}^+_{v_s}(T) = R^+_{v_s}(T)$ if v_s is in the support of T, but if v_s is not in the support of T, then $\overline{R}^+_{v_s}(T) = R^+_{v_s}(T) \oplus P_{\alpha_s}$.

Theorem 4.9. *The map cl_c corresponds to our map from torsion classes to support tilting objects.*

Proof. Our map from torsion classes to support tilting objects consists of taking the Ext-projectives. Let α be the positive root corresponding to s initial in c, and let v be the corresponding vertex. The image under R^-_v of an Ext-projective for $\mathcal{T}$ will be Ext-projective in $\widetilde{\mathcal{T}}$. Conversely, if M is Ext-projective for $\widetilde{\mathcal{T}}$, then $\operatorname{Ext}^1(M, N) = 0$ for $M, N \in \widetilde{\mathcal{T}}$. It follows that $\operatorname{Ext}^1(R^+_v(M), R^+_v(N)) = 0$, so, in particular, $\operatorname{Ext}^1(R^+_v(M), N') = 0$ for N' any indecomposable of $\mathcal{T}$ except M_α. However, M_α is simple injective, so $\operatorname{Ext}^1(R^+_v(M), M_\alpha) = 0$ as well. The only slight subtlety that can occur is that there might be an Ext-projective of $\mathcal{T}$ that is reflected to zero. (It is not possible for an Ext-projective of $\widetilde{\mathcal{T}}$ to reflect to zero, because $\widetilde{\mathcal{T}}$ is by definition the image of $\mathcal{T}$ under reflection.) This happens precisely if M_α is Ext-projective in $\mathcal{T}$.

Here M_α is Ext-projective in $\mathcal{T}$ iff there are no homomorphisms from $\mathcal{T}$ into $\tau(M_\alpha)$, iff there are no morphisms from $\widetilde{\mathcal{T}}$ into $R^-_v(\tau(M_\alpha))$. Now $R^-_v(\tau(M_\alpha))$ is the injective for $\operatorname{rep} \widetilde{Q}$ which corresponds to the vertex v. There are no morphisms from $\widetilde{\mathcal{T}}$ into $R^-_v(\tau(M_s))$ iff $\widetilde{\mathcal{T}}$ is supported away from the vertex v. □

Conjecture 11.3 of [RS09] describes the composition $\mathrm{NC} \circ \mathrm{cl}^{-1}$. An indecomposable X in a support tilting object T is *upper* if, when we take V to be the cluster obtained by mutating at X,

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we have that $\operatorname{Gen} T \supset \operatorname{Gen} V$. (The definition given in [RS09] is not exactly this, but it is easily seen to be equivalent.) We can now state and prove the conjecture.

THEOREM 4.10 (Reading and Speyer [RS09, Conjecture 11.3]). *For a support tilting object T, the fixed space of $\operatorname{cox}(\mathfrak{a}(\operatorname{Gen}(T)))$ is the intersection of the subspaces perpendicular to the roots α corresponding to upper indecomposables of T.*

(Note that, in the finite type setting, it is known that the fixed subspace of a noncrossing partition determines the noncrossing partition, so this suffices to describe the map fully.)

Proof. By Lemma 2.32, the upper indecomposables of T are exactly the split Ext-projectives of $\operatorname{Gen} T$. The fixed space of $\operatorname{cox}(\mathfrak{a}(\operatorname{Gen}(T)))$ will include the intersection of the subspaces perpendicular to the dimension vectors of the split Ext-projectives, and since the fixed subspace has the same dimension as the intersection of the perpendicular subspaces, we are done. $\square$

4.3 Trimness

All of the lattices which we discuss in this section are assumed to be finite. An element x of a lattice L is said to be *left modular* if, for any $y < z$ in L,

$$(y \vee x) \wedge z = y \vee (x \wedge z).$$

A lattice is called left modular if it has a maximal chain of left modular elements. For more on left modular lattices, see [BS97], where the concept originated, or [MT06].

A *join-irreducible* of a lattice is an element which cannot be written as the join of two strictly smaller elements, and which is not the minimum element of the lattice. A *meet-irreducible* is defined dually. A lattice is called *extremal* if it has the same number of join-irreducibles and meet-irreducibles as the length of the longest chain. (This is the minimum possible number of each.) See [Mar92] for more on extremal lattices.

A lattice is called *trim* if it is both left modular and extremal. Trim lattices have many of the properties of distributive lattices, but need not be graded. This concept was introduced and studied in [Tho06], where it was shown that the Cambrian lattices in types A_n and B_n are trim and conjectured that all Cambrian lattices are trim. We now prove this.

Let Q be a simply laced Dynkin diagram. As we have remarked, the Cambrian lattice $\mathfrak{C}_Q$ can be viewed as the poset of torsion classes of $\operatorname{rep} Q$ ordered by inclusion, which is the perspective which we adopt.

The Auslander–Reiten quiver for $\operatorname{rep} Q$ is a quiver whose vertices are the isomorphism classes of indecomposable representations of Q, and where the number of arrows between the vertices associated with indecomposables L and M equals the dimension of the space of irreducible morphisms from L to M. When Q is Dynkin, this quiver has no oriented cycles. Thus, one can take a total order on the indecomposables of Q which is compatible with this order. We do so, and record our choice by a map $n : \Phi^+ \to \{1, \ldots, |\Phi^+|\}$ so that $n(\alpha)$ records the position of M_α in this total order.

Let $\mathcal{S}_i$ be the full additive subcategory, closed under direct summands, of $\operatorname{rep} Q$ whose indecomposables are the indecomposables $\{M_\alpha \mid n(\alpha) \geqslant i\}$. Each $\mathcal{S}_i$ is a torsion class.

LEMMA 4.11. *For $\mathcal{T}_1, \mathcal{T}_2 \in \mathfrak{C}_Q$, $\mathcal{T}_1 \wedge \mathcal{T}_2 = \mathcal{T}_1 \cap \mathcal{T}_2$.*

Proof. Here $\mathcal{T}_1 \cap \mathcal{T}_2$ is closed under quotients, extensions, and summands, so it is a torsion class, and thus clearly the maximal torsion class contained in both $\mathcal{T}_1$ and $\mathcal{T}_2$. $\square$

For $\alpha \in \Phi^+$, let $\mathcal{T}_\alpha = \mathrm{Gen}(M_\alpha)$. Recall that $\mathrm{Ext}^1(M_\alpha, M_\alpha) = 0$, so M_α is a partial tilting object. Thus, by [ASS06, Lemma VI.2.3], $\mathcal{T}_\alpha$ is a torsion class. We call such torsion classes *principal*.

Lemma 4.12. *For $\alpha \in \Phi^+$, the torsion class $\mathcal{T}_\alpha$ is a join-irreducible in $\mathfrak{C}_Q$.*

Proof. Let $\mathcal{T}'_\alpha = \mathcal{T}_\alpha \cap \mathcal{S}_{n(\alpha)+1}$. This is a torsion class by Lemma 4.11, and its indecomposables are those of $\mathcal{T}_\alpha$ other than M_α itself. Thus, if $\mathcal{T}_1 \vee \mathcal{T}_2 = \mathcal{T}_\alpha$, then at least one of $\mathcal{T}_1, \mathcal{T}_2$ must not be contained in $\mathcal{T}'_\alpha$, so must contain M_α, and thus all of $\mathcal{T}_\alpha$. □

Lemma 4.13. *The only join-irreducible elements of $\mathfrak{C}_Q$ are the principal torsion classes.*

Proof. A non-principal torsion class can be written as the join of the principal torsion classes generated by its split Ext-projectives. □

Proposition 4.14. *The Cambrian lattice $\mathfrak{C}_Q$ is extremal.*

Proof. By the previous lemma, there are $|\Phi^+|$ join-irreducibles of $\mathfrak{C}_Q$. Dualizing, the same is true of the meet-irreducibles. A maximal chain of torsion classes $\mathcal{T}_0 \subset \mathcal{T}_1 \subset \cdots \subset \mathcal{T}_m$ must have $|\mathcal{T}_{i+1}| \geqslant |\mathcal{T}_i| + 1$, so the maximal length of such a chain is $|\Phi^+|$, proving the proposition. □

A torsion class is called splitting if any indecomposable is either torsion or torsion free. The S_i are splitting.

Lemma 4.15. *If $\mathcal{S}$ is a splitting torsion class, and $\mathcal{T}$ is an arbitrary torsion class, then $\mathcal{T} \vee \mathcal{S} = \mathcal{T} \cup \mathcal{S}$.*

Proof. Let $\mathcal{F}$ be the torsion-free class corresponding to $\mathcal{T}$, as in Lemma 2.25, and let $\mathcal{E}$ be the torsion-free class corresponding to $\mathcal{S}$. By the dual of Lemma 4.11, $\mathcal{E} \cap \mathcal{F}$ is a torsion-free class. Clearly, the torsion class corresponding to $\mathcal{E} \cap \mathcal{F}$ contains $\mathcal{S} \cup \mathcal{T}$. We claim that equality holds. Let M be an indecomposable not contained in $\mathcal{S} \cup \mathcal{T}$. Since $M \notin \mathcal{T}$, there is an indecomposable $F \in \mathcal{F}$ which has a non-zero morphism to M. However, since $M \notin \mathcal{S}$, $M \in \mathcal{E}$. Since $(\mathcal{S}, \mathcal{E})$ forms a torsion pair, there are no morphisms from $\mathcal{S}$ to $\mathcal{E}$. Thus, F must not be in $\mathcal{S}$, and so $F \in \mathcal{E}$, since $(\mathcal{S}, \mathcal{E})$ is splitting. We have shown that $F \in \mathcal{E} \cap \mathcal{F}$, and we know there is a non-zero morphism from F to M. So M is not in the torsion class corresponding to $\mathcal{E} \cap \mathcal{F}$. □

Lemma 4.16. *Any splitting torsion class is left modular.*

Proof. Let $\mathcal{S}$ be a splitting torsion class. Let $\mathcal{T} \supset \mathcal{V}$ be two torsion classes. Now

$$\mathcal{T} \wedge (\mathcal{S} \vee \mathcal{V}) = \mathcal{T} \cap (\mathcal{S} \cup \mathcal{V}) = (\mathcal{T} \cap \mathcal{S}) \cup \mathcal{V},$$

by Lemmas 4.11 and 4.15, and the fact that $\mathcal{T} \supset \mathcal{V}$. In particular, this implies that $(\mathcal{T} \cap \mathcal{S}) \cup \mathcal{V}$ is a torsion class. On the other hand, $\mathcal{T} \wedge \mathcal{S} = \mathcal{T} \cap \mathcal{S}$. So $(\mathcal{T} \wedge \mathcal{S}) \vee \mathcal{V} = (\mathcal{T} \cap \mathcal{S}) \vee \mathcal{V}$, the minimal torsion class containing $\mathcal{T} \cap \mathcal{S}$ and $\mathcal{V}$, which is clearly $(\mathcal{T} \cap \mathcal{S}) \cup \mathcal{V}$, as desired. □

Theorem 4.17. *The Cambrian lattice $\mathfrak{C}_Q$ is trim.*

Proof. Lemma 4.16 shows that the S_i are left modular, and clearly they form a maximal chain. We have already shown that $\mathfrak{C}_Q$ is extremal. Thus, it is trim. □

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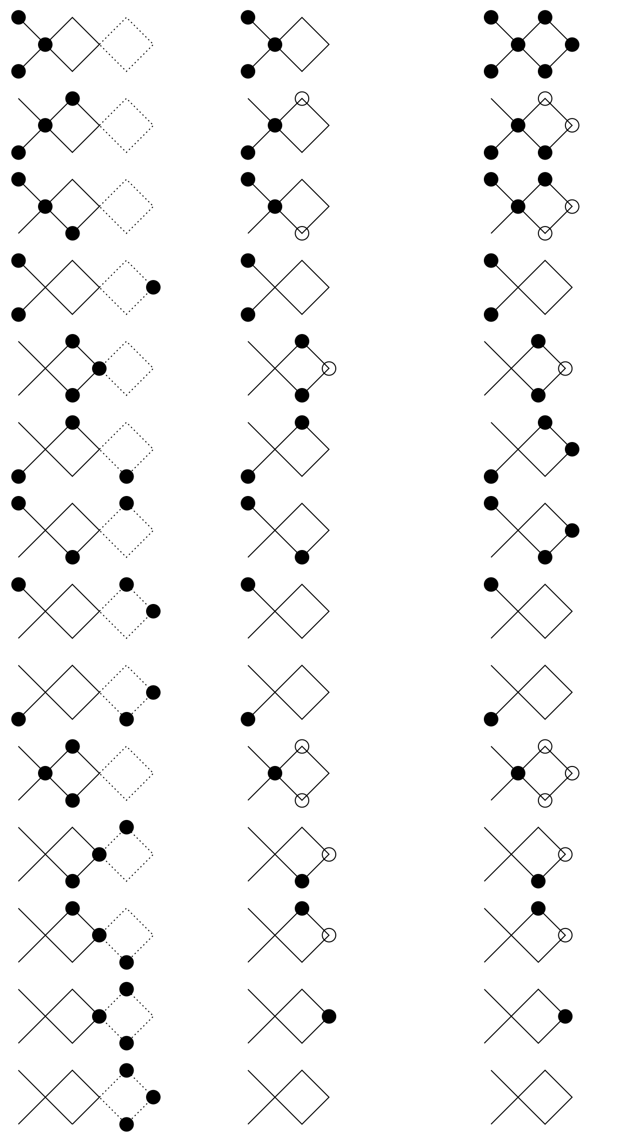
Table 1. Example: A_3.

Cluster tilting objects	Support tilting objects ● Split projective ○ Non–split projective	Torsion classes ● Wide subcategory ○ Rest of torsion class	Noncrossing Partitions



Noncrossing Partitions
(23)(12)(34)
(13)(34)
(24)(12)
(12)(34)
(24)(13)
(23)(34)
(23)(12)
(12)
(34)
(14)
(13)
(24)
(23)
e

4.4 Folding argument

In our consideration of finite type, we have restricted ourselves to simply laced cases. This restriction is not necessary: our conclusions hold without that assumption.

The avenue of proof for non-simply laced cases is to apply a *folding argument* in which we consider a simply laced root system which folds onto the non-simply laced root system.

Let Q be a simply laced quiver with a non-trivial automorphism group. Define the foldable cluster tilting objects for Q to be those cluster tilting objects whose isomorphism class is fixed under the action of the automorphism group of Q on the category of representations, and similarly for foldable support tilting objects. Define foldable torsion classes of Q to be the torsion classes of Q stabilized under the action of the automorphism group, and similarly for foldable wide subcategories. Define foldable c-sortable elements to be those fixed under the action of the automorphism group, and similarly for foldable noncrossing partitions. In each case, the foldable objects for Q correspond naturally to the usual object for the folded root system. All of our bijections preserve foldableness, so all of our results hold. To conclude that all Cambrian lattices are trim, we require the fact that the sublattice of a trim lattice fixed under a group of lattice automorphisms is again trim [Tho06].

5. Example: A_3

In this section we record a few of the correspondences in this paper for the example of A_3 with quiver Q.

$$1 \longleftarrow 2 \longrightarrow 3$$

The Auslander–Reiten quiver of indecomposable representations of Q is as follows, where the dimension vectors are written in the basis given by the simple roots $\alpha_1, \alpha_2, \alpha_3$.

$$\begin{tikzcd}
{[100]} \arrow[dr] & & {[011]} \arrow[dr] & \\
& {[111]} \arrow[ur] \arrow[dr] & & {[010]} \\
{[001]} \arrow[ur] & & {[110]} \arrow[ur] &
\end{tikzcd}$$

In Table 1, each of the 14 noncrossing partitions is listed in the same row as the other objects to which it corresponds: the cluster tilting object, the support tilting object, the torsion class, and the wide subcategory. The subcategories of rep Q are indicated by specifying a subset of the indecomposables of rep Q, arranged as in the Auslander–Reiten quiver. The support tilting objects and cluster tilting objects are indicated by specifying their summands. For the cluster tilting objects, we have drawn a fundamental domain of the indecomposable objects in the cluster category, where the black edges mark the copy of the Auslander–Reiten quiver for rep Q inside the cluster category, and the dashed edges are maps in the cluster category. The cluster tilting objects can also be viewed as clusters when the indecomposable objects in the copy of rep Q are identified with positive roots as in the Auslander–Reiten quiver above, and the three 'extra' indecomposables are identified with the negative simple roots $-\alpha_3, -\alpha_2, -\alpha_1$ reading from top to bottom.

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