Vlastimil Pták

Nondiscrete mathematical induction

In: (ed.): General topology and its relations to modern analysis and algebra IV, Proceedings of the fourth Prague topological symposium, 1976, Part A: Invited papers. Springer, Berlin, 1977. Lecture Notes in Mathematics, 609. pp. [166]--178.

Persistent URL: http://dml.cz/dmlcz/701005

NONDISCRETE MATHEMATICAL INDUCTION

Vlastimil PTÁK
Czechoslovak Academy of Sciences
Institute of Mathematics
Žitná 25, 115 67 Praha 1, Czechoslovakia

The lecture is divided into the following sections:

1. Motivation
2. Statement of the induction theorem
3. Relation to classical theorems
4. Principles of application
5. An illustration: the factorization theorem
6. New results
7. Connections with numerical analysis

1. Motivation

This lecture presents a report about a series of investigations whose aim it is to set up an abstract model for iterative existence proofs and constructions in analysis and numerical analysis.

I intend to show that a model which describes a large class of iteration processes may be based on a certain modification of the closed graph theorem.

Let us start with the following observation. In existence proofs in mathematical analysis and in numerical analysis we often devise iterative procedures in order to construct an element which lies in a certain set or satisfies a given relation. At each stage of the iterative process we are dealing with elements which satisfy the desired relation only approximately, the degree of approximation becoming better at each step.

To describe the abstract model which we shall investigate later, consider the problem of constructing a point x which belongs to a given set W . We start by replacing the given set W by a family $W(r)$ of sets depending on a small positive parameter r ; the inclusion $z \in W(r)$ means - roughly speaking - that the inclusion $z \in W$ is satisfied only approximately, the approximation being measured by the number r . All the $W(r)$ are supposed to be subsets of a complete metric space (E,d) .

In what follows we intend to show that, under suitable hypotheses concerning the relation between the sets $W(.)$ and the metric of the space a simple theorem may be proved which gives the construction of an iterative process converging to a point $x \in W$. The theorem, the so-called induction theorem, is closely related to the closed graph theorem in functional analysis; it could be described as a quantitative strengthening of the closed graph theorem. Indeed, the closed graph theorem can be viewed, in a certain sense, as a limit case of the induction theorem, for an infinitely fast rate of convergence. The proof of the induction theorem is an exercise; the interest of the result lies exclusively in its formulation, which makes it possible to unify a number of theorems in one simple abstract result.

2. Metric spaces and the Induction Theorem

Definition. Let T be an interval of the form $T = \{t;\ 0 < t < t_0\}$ for some positive t_0. A rate of convergence or a small function on T is a function ω defined on T with the following properties

1^o ω maps T into itself

2^o for each $t \in T$ the series $t + \omega(t) + \omega^{(2)}(t) + \ldots$ is convergent.

We use the abbreviation $\omega^{(n)}$ for the n-th iterate of the function ω, so that $\omega^{(2)}(t) = \omega(\omega(t))$ and so on. The sum of the above series will be denoted by σ. The function σ satisfies the following functional equation

$$\sigma(t) - t = \sigma(\omega(t))\ ;$$

one of the consequences of this fact is the possibility of recovering ω if σ is given. Indeed, we have

$$\omega(t) = \sigma^{-1}(\sigma(t) - t)$$

(with the exception of pathological cases).

Given a metric space (E,d) with distance function d, a point $x \in E$ and a positive number r, we denote by $U(x,r)$ the open spherical neighbourhood of x with radius r, $U(x,r) = \{y \in E\ ;\ d(y,x) < r\}$. Similarly, if $M \subset E$, we denote by $U(M,r)$ the set of all $y \in E$ for which $d(y,M) < r$. If we are given, for each sufficiently small positive r, a set $A(r) \subset E$, we define the limit $A(0)$ of the family $A(.)$ as follows

$$A(0) = \bigcap_{s>0} \left(\bigcup_{r \leq s} A(r) \right)^- .$$

Now we may state the Induction Theorem.

Theorem. Let (E,d) be a complete metric space, let T be an interval $\{t \; ; \; 0 < t < t_0\}$ and ω a rate of convergence on T. For each $t \in T$ let $Z(t)$ be a subset of E ; denote by $Z(0)$ the limit of the family $Z(.)$. Suppose that

$$Z(t) \subset U(Z(\omega(t)), t)$$

for each $t \in T$. Then

$$Z(t) \subset U(Z(0, \sigma(t))$$

for each $t \in T$.

Proof. An exercise.

3. Relation to classical theorems

It will be interesting to compare the induction theorem with some classical results. It is almost immediate that the Banach fixed point principle is a simple consequence.

Let E be a complete metric space and f a mapping of E into itself such that

$$d(f(x_1), f(x_2)) \leq \alpha d(x_1,x_2) \quad \text{for all} \quad x_1,x_2 \in E$$

where α is a fixed number, $0 < \alpha < 1$. Then there exists an $x \in E$ such that $x = f(x)$.

Proof. For each $t > 0$ set

$$Z(t) = \{x \; ; \; d(x,f(x)) < t\} \; ;$$

it follows that $Z(0) = \{x \; ; \; x=f(x)\}$. It will be sufficient to show that $Z(t) \subset U(Z(\alpha t),t)$. If $x \in Z(t)$, set $x' = f(x)$ so that $d(x,x') < t$. Let us show that $x' \in Z(\alpha t)$. This, however, is immediate since

$$d(x',f(x')) = d(f(x),f(x')) \leq \alpha d(x,x') = \alpha d(x,f(x)) < \alpha t .$$

The induction theorem applies with $\omega(r) = \alpha r$.

The connection with the closed graph theorem is somewhat less obvious. Roughly speaking, the closed graph theorem consists in the following implication: a mapping which is uniformly almost open is already open. Now the induction theorem can be described as a quantitative refinement of the closed graph theorem. To see that, let us recall the notion of a uniformly almost open mapping [4].

A mapping f from a uniform space E into a uniform space V is said to be uniformly almost open, if, for each entourage U in E, there exists an entourage V in F such that, for each x, we have $f(U(x))^{-} \supset V(f(x))$. This means, that points of $V(f(x))$ may be arbitrarily well approximated by points from $f(U(x))$. The conclusion is that

- under appropriate hypotheses about the spaces and the mapping - that, for a slightly larger $U' \supset U$ we already have the inclusion $f(U'(x)) \supset \supset V(f(x))$ for all x .

It turns out that the same conclusion can be obtained under a weaker assumption. The approximability of V by the elements of $f(U)$ need not be arbitrarily good. It suffices if we are able to approximate to a finite distance, provided the error of the approximation is small as compared with the size of the entourages. Smallness is measured by a small function; the conclusion also gives an information how much larger U' has to be in order to have the inclusion $f(U') \supset V$. For details, see the author's remark [4].

4. Principles of application

Now we should explain why the method has been given the name of nondiscrete mathematical induction. We shall see that the application of the method consists in reducing the given problem to a system of functional inequalities for several indeterminate functions one of which is to be a rate of convergence; this explains the word nondiscrete. The connection with the classical method of mathematical induction is obvious - we investigate the possibility of passing from a point x which approximates the point to be constructed with an error not exceeding r to another point x' close to x for which the approximation is considerably better.

Suppose we are given an approximation of order r , in other words, a point $x \in W(r)$ and are allowed to move from x to a distance not greater than r . Can we find, within $U(x,r)$, an approximation of a much better order r'? A suitable way of giving this a precise meaning is to impose the condition $r' = \omega(r)$ where ω is a small function. The condition that for each $x \in W(r)$ there exists a point $x' \in U(x,r) \cap \cap W(r')$ with $r' = \omega(r)$ may also be expressed in the form $W(r) \subset \subset U(W(\omega(r)), r)$ so that $W(r)$ satisfies the hypotheses of the induction theorem. We have thus

$$W(t) \subset U(W(0), \sigma(t))$$

for sufficiently small t . Hence we shall be able to assert that $W(0)$ is nonvoid provided at least one $W(r)$ is nonvoid.

This corresponds to the first step of an ordinary induction proof; here, as in the discrete case, we have to make sure that the process begins somewhere. There is another point which should be stressed, the heuristic value of the method.

The main advantage consists in the fact that the iterative construction is taken care of by the general theorem so that the application consists in the verification of the hypotheses, the main question being: how much can a given approximation be improved within a given neighbourhood. By separating the hard analysis portion from the construction of the sequence, this method not only yields considerable simplifications of proofs but also evidences more clearly the substance of the problem. Instead of defining an approximation process first and then investigating the degree of approximation at the n-th step the method we propose could be described as exactly the opposite: we begin by looking at the sets $W(r)$ where the degree of approximation is at least r, then choose a suitable rate of convergence; the induction principle gives the construction of an iterative sequence corresponding to that rate of convergence automatically.

In this manner, we are using the relation between the improvement of the approximation and the distance we have to go in order to attain it in the most advantageous manner. There are examples to show that a given system of functional inequalities may be consistent with different rates of convergence. The conclusion obtained from the Induction Theorem may differ according to the choice of these; however, there seems to be (at least in the concrete problems investigated thus far - in particular in the case of the Newton process, which we shall discuss later) a natural rate of convergence which yields the best possible result - in the sense that the estimates are sharp within the class of problems under consideration.

Now let us give all this a more precise formulation.

Let (E,d) be a complete metric space and f a nonnegative continuous function on E. We are looking for a point x for which $f(x) = 0$.

1st observation. Let us assume that, for each x taken from some set $M \subset E$ and each positive $r < r_0$ we can prove an estimate of the form

$$\inf\{f(x'); x' \in U(x,r)\} < h(f(x), r)$$

where h is a suitable function of two variables. Suppose there exists a positive function φ tending to zero with r and a rate of convergence ω such that

$$h(\varphi(r), r) \leq \varphi(\omega(r))$$

Set $W(r) = \{x \in M, f(x) \leq \varphi(r)\}$; then $W(r) \subset U(W(\omega(r)), r)$.

2nd observation. The functional equation connecting ω and σ may be used to obtain information about the distance of the solution from any point u_0 given in advance. Indeed, let u_0 be a fixed point in E. Given a point $x \in E$ and two positive numbers d and r such that

$$d(x,u_o) \leq d - \sigma(r) ,$$

then, for $x' \in U(x,r)$, we have

$$d(x',u_o) \leq d(x,u_o) + d(x',x) \leq d - \sigma(r) + r = d - \sigma(\omega(r)) .$$

It follows that the family

$$Z(r) = \{x \in M ; f(x) \leq \varphi(r) , d(x,u_o) \leq d - \sigma(r)$$

satisfies $Z(r) \subset U(Z(\omega(r)), r)$. It follows that $Z(0)$ will be nonvoid if at least one $Z(r_o)$ is nonvoid since

$$Z(r_o) \subset U(Z(0), \sigma(r_o)) .$$

Summing up: if $h(\varphi(r),r) \leq \varphi(\omega(r))$ and if there exists an $r_o > 0$ and an $x_o \in M$ such that

$$f(x_o) \leq \varphi(r_o) \qquad d(x_o,u_o) \leq d - \sigma(r_o)$$

then there exists an $x_\infty \in M^-$ with the following properties

$$f(x_\infty) = 0$$
$$d(x_\infty,x_o) \leq \sigma(r_o)$$
$$d(x_\infty,u_o) \leq d .$$

We have seen that the first step of the induction method consists in finding a function $h(m,r)$ with the following property: given x with $f(x) \leq m$, there exists, within distance less than r, an x' with $f(x') \leq h(m,r)$. In most cases the estimate for $f(x')$ will not depend on the value of $f(x)$ alone but will require some further characteristics as well; one might think of derivatives or some other additional information.

Suppose, for simplicity, that there is only one such additional characteristic, i.e. that the estimate for $f(x')$ depends also on the value of another positive function f_1 at x so that $\inf \{f(x') , x' \in U(x,r)\} \leq h(f(x), f_1(x), r)$. Consider the case where the estimate h is an increasing function of the second argument. Since we shall need, in the following step of the induction an estimate for $f_1(x')$, we shall need, in fact, a pair of positive functions h and h_1 such that, for each x and r, there exists an $x' \in U(x,r)$ for which

$$f(x') \leq h(f(x), f_1(x),r)$$
$$f_1(x') \geq h_1(f(x),f_1(x),r).$$

In this case h_1 will have to be decreasing in the first argument and increasing in the second argument. It will then be desirable to find a pair of functions φ, φ_1 and a rate of convergence ω such that

$$h(\varphi(r), \varphi_1(r), r) \leq \varphi(\omega(r))$$
$$h_1(\varphi(r), \varphi_1(r), r) \geq \varphi_1(\omega(r)) .$$

Set $W(r) = \{x \in E ; f(x) \leq \varphi(r), f_1(x) \geq \varphi_1(r)\}$; then $W(r) \subset$ $\subset U(W(\omega(r)), r)$.

Let us pass now to examples which illustrate the general principles sketched above.

5. The factorization theorem

The method of nondiscrete mathematical induction has been applied thus far to obtain improvements of selection theorems, transitivity theorems in the theory of C^*-algebras, factorization theorems in Banach algebras and existence theorems in the theory of partial differential equations. The first three are described in the author's paper [3]. The ideas contained there have also been applied successfully by the author's collaborators [10], [11].

Among the many examples which demonstrate the advantages of the method the Rudin-Cohen factorization theorem seems to be the most suitable one; in spite of the fact that the result itself is not new the simplification of the proof is considerable.

If M is a unital Banach algebra, we denote by $G(M)$ the set of its invertible elements. Let A be a Banach algebra without unit and denote by B its unitization. The multiplicative functional on B which has A as its kernel will be denoted by f. Let F be a Banach space which is an A-module. We say that the pair (A,F) possesses an approximate unit of norm β if, for each $a \in A$, $y \in F$ and $\varepsilon > 0$ there exists an $e \in A$ such that

$$|e| \leq \beta \quad , \quad |ea - a| < \varepsilon \quad , \quad |ey - y| < \varepsilon$$

Theorem. Let A be a Banach algebra without a unit and let F be a Banach space which is an A-module. Suppose that (A,F) possesses an approximate unit of norm β. Then, for each $y \in F$ and each $\varepsilon > 0$, there exists an $a \in A$ and a $z \in F$ such that

$$az = y \; , \quad |a| \leq \beta \quad , \quad z \in (Ay)^- \; , \quad |z - y| \leq \varepsilon \quad .$$

Proof. First of all, it is easy to see that the existence of a bounded approximate unit implies $(Ay)^- = (By)^-$ for any $y \in F$, B being the unitization of A.

Consider the space $A \times (By)^-$ equipped with the norm

$$\|p\| = \frac{1}{1 - \omega} \max \left\{ \frac{1}{\beta}|a| \; , \; \frac{1}{\varepsilon}|z| \right\}$$

if $p = [a,z]$; $0 < \omega < 1$ is a constant to be chosen later. For each invertible $b \in B$ let $p(b)$ be the pair $p(b) = [a,z]$, $a = b^{-1} - f(b^{-1})$, $z = by$. For each $r > 0$, set

$$W(r) = \{p(b);\; b \in G(B),\; |f(b^{-1})| \leq r,\; \|p(b) - p(1)\| \leq \frac{1}{1-\omega}(1-r)\} \; .$$

In particular, $[0,y] = p(1) \in W(1)$. Also, observe that $[a,z] \in W(r)$ implies

$$az = (b^{-1} - f(b^{-1}))by = y - f(b^{-1})z$$

so that $[a,z] \in W(0)$ implies $az = y$.

We intend to show that there exists an ω such that

$$W(r) \subset U(W(\omega r),\ r) \quad \text{for each} \quad r > 0 .$$

Having proved that, it will follow from the Induction Principle that $W(1) \subset U(W(0), \frac{1}{1-\omega})$; this means that there exists a pair $p = [a,z] \in W(0)$ with $\|p-p(1)\| \leq \frac{1}{1-\omega}$, in other words $az = y$, $|a| \leq \beta$, $|z-y| \leq \varepsilon$. Given $p(b) \in W(r)$ we intend to show that the pair $p(b')$ corresponding to a slightly perturbed $b' = bc$ will satisfy $p(b') \in W(\omega r)) \cap U(p(b), r)$. For this, clearly it suffices to construct c in such a manner that

(1) $\quad |a' - a| < (1 - \omega)\beta r$

(2) $\quad f(c^{-1}) = \omega$

(3) $\quad |(b'- b)y| \leq (1 - \omega)\varepsilon r$.

We shall show that it is possible to satisfy these three conditions by constructing a c for which

(4) $\quad b'^{-1} - b^{-1}$ is a multiple of $e - 1$ for a suitable e .

Since $f(e-1) = -1$, such a choice of c - if possible - has the following consequences:

$$a'- a = b'^{-1}-b^{-1} - f(b'^{-1}-b^{-1}) = (1 - \omega)f(b^{-1})e$$
$$b'- b = -b'(b'^{-1}-b^{-1})b = -b'(1 - \omega)f(b^{-1})(e-1)b$$
$$b'(1 + (1 - \omega)f(b^{-1})(e-1)b) = b ;$$

for shortness, write $w = (1 - \omega)f(b^{-1})(e-1)b$.

It follows that a suitable choice of c will be $c = (1+w)^{-1}$ provided $|w| < 1$.

We shall need an estimate for w independent of e . Let τ be such that $(1 - \omega)(\beta + 1) = 1 - 2\tau$. Since

$$w = (1 - \omega)f(b)^{-1}(e-1)(b-f(b)) + (1 - \omega)(e-1)$$

we have

$$|w| \leq (1 - \omega)|f(b)^{-1}|\ |(e-1)(b-f(b))| + (1 - \omega)(\beta + 1)$$

and $e \in A$, $|e| \leq \beta$ may be chosen so as to have

(5) $\quad (1 - \omega)|f(b)^{-1}|\ |(e-1)(b-f(b))| + (1 - \omega)(\beta + 1) \leq 1 - \tau$

whence $|w| \leq 1 - \tau$ and $|c| \leq \frac{1}{\tau}$.

Now suppose that e satisfies condition (5) and at the same time

$$|b|\ \tfrac{1}{\tau}(1 - \omega)|f(b)^{-1}|\ |(e-1)by| < (1 - \omega)\varepsilon r$$

then

$$|z' - z| < (1 - \omega)\varepsilon r .$$

The proof is complete.

6. New results

The classical notion of the order of convergence or rate of convergence which reputedly goes back to the last century is defined as follows. Given an iterative process which yields a sequence x_n of elements of a complete metric space (E,d) converging to an element $x \in E$ we say that the convergence is of order p if there exists a constant α such that

$$d(x_{n+1},x) \leq \alpha(d(x_n,x))^p \ .$$

Clearly it is immaterial whether we require this for all n or only asymptotically. Let us point out two difficulties which seem to arise if this point of view is adopted.

1^0 If $p > 1$ then the above inequality contains a certain amount of information about the process; the information, however, is more of a qualitative nature since it relates quantities which we are not able to measure at any finite stage of the process. The obvious meaning of the above inequality seems to consist rather in the fact that, at each stage of the process, the following step of the iteration yields a significant improvement of the estimate.

2^0 Theoretical considerations enable us, in many cases, to establish an inequality of the above type for certain constants α and p ; however, usually this is only possible if we assume n to be larger than a certain bound. We might want, however, to stop the process before this bound is reached - in this case the inequality cannot be used. Of course, it is possible to extend the validity of the estimate to all n by making α sufficiently large - this may invalidate its practical applicability for the initial steps.

It seems therefore reasonable to look for another method of estimating the convergence of iterative processes, one which would satisfy the following requirements.

1^0 It should relate quantities which may be measured or estimated during the actual process.

2^0 It should describe accurately in particular the initial stage of the process, not only its asymptotic behaviour since, after all, we are interested in keeping the number of steps necessary to obtain a good estimate as low as possible.

It is obvious that we cannot expect to have an adequate description of both the beginning and the tail end of the process by any formula as simple as the one we discussed above. In our opinion, a description which fits the whole process, not only an asymptotic one, is only possible by means of suitable functions, not just numbers.

It seems natural to expect that better results may be obtained by looking for small functions ω which relate two consecutive increments of the process by an inequality of the following type

$$d(x_{n+1},x_n) \leq \omega(d(x_n,x_{n-1})) .$$

By allowing a larger class of functions than just those of the type $t \to \alpha t^p$ we have a better chance of getting a closer fit of the estimates even at the beginning of the process.

At the same time this approach measures the rate of convergence at finite stages of the process using only data available at that particular stage of the process, in fact, instead of comparing the two unknown quantities $d(x_n,x)$ and $d(x_{n+1},x)$ it is based on the relation between $d(x_n,x_{n-1})$ and $d(x_{n+1},x_n)$.

Suppose we have a sequence of inequalities

$$d(x_{n+k},x_{n+k-1}) \leq \omega^{(k-1)}(d(x_{n+1},x_n))$$

for $k=1,2,\ldots$ (where $\omega^{(j)}$ stands for the j-th iteration of the function ω) and that the series $\sum_0^\infty \omega^{(j)}(d(x_{n+1},x_n))$ is convergent. Such a sequence of inequalities may be deduced from the above inequality if ω is an increasing function. Then the sequence $x_n,x_{n+1},\ldots$ is a fundamental sequence and, the space (E,d) being complete, converges to a limit x for which

$$d(x_n,x) \leq d(x_{n+1},x_n) + d(x_{n+2},x_{n+1}) + \ldots \leq \sum_0^\infty \omega^{(j)}(d(x_{n+1},x_n)) .$$

As an example, let us mention the rate of convergence of Newton's process recently established by the author. There we have

$$\omega(t) = \frac{t^2}{2(t^2+d)^{1/2}}$$

where d is a positive constant depending on the data of the problem. A closer inspection of this formula shows that, for every small t, the function assumes approximately the form $\frac{t^2}{2d^{1/2}}$ whereas, for large t, the summand t^2 predominates in the denominator so that the function is approximately linear, $\frac{1}{2}t$.

Since ω relates the consecutive steps of Newton's process by the inequality $d(x_{n+1},x_n) \leq \omega(d(x_n,x_{n-1}))$ this shows first that, asymptotically - in other words for small $d(x_n,x_{n-1})$ - the next increment is approximately $\frac{1}{2d^{1/2}}(d(x_n,x_{n-1}))^2$. This phenomenon is usually described by saying that the convergence is quadratic.

However, in the initial stages of the process $d(x_n, x_{n-1})$ is still large so that ω is almost linear. Since it may be shown that the estimates for Newton's process based on ω are sharp at each step, it follows that accurate estimates valid for the whole process - including the initial steps - cannot be based on any simple quadratic monomial.

The precise formulation is as follows.

Let E and F be two Banach spaces, let $x_0 \in E$ and $U = \{x ; |x-x_0| \leq m\}$. Let f be a mapping of U into F twice Fréchet differentiable for each $x \in U$. Suppose the following conditions are satisfied:

1° there exists a constant $k > 0$ such that $|f''(x)| \leq k$ for all $x \in U$,

2° $f'(x_0)$ is invertible and $d(f'(x_0)) = d_0 > 0$

3° $|f'(x_0)^{-1} f(x_0)| \leq r_0$.

If $d_0 \geq 2kr_0$ and if

$$m \geq \frac{d_0}{k} \left(1 - \left(1 - \frac{2kr_0}{d_0}\right)^{1/2}\right)$$

then the Newton process starting at x_0 is meaningful and converges to a point x such that $f(x) = 0$. The function

$$\omega(r) = \frac{r^2}{2(r^2+d)^{1/2}}$$

where $d = \left(\frac{d_0}{k}\right)\left(1 - \frac{2kr_0}{d_0}\right)$, is a rate of convergence and yields the following estimates

$$|x_{n+1} - x_n| \leq \omega^{(n)}(r_0)$$

$$|x - x_0| \leq \frac{d_0}{k}\left(1 - \left(1 - \frac{2kr_0}{d_0}\right)^{1/2}\right).$$

These estimates are sharp in the following sense: for each triple k, d_0, r_0 of positive numbers satisfying the inequality $d_0 \geq 2kr_0$ there exists a mapping f for which these estimates are attained.

The proof is given in [6]. The corresponding σ function is computed in [6] and the finite sums σ_n in [7].

Let us conclude this section by mentioning another example a detailed discussion of which may be found in [5].

If γ and β are positive numbers such that $\gamma > 4\beta$ then

$$\omega(t) = t\,\frac{\gamma + t - ((\gamma + t)^2 - 4\beta)^{1/2}}{\gamma - t + ((\gamma + t)^2 - 4\beta)^{1/2}}$$

is a rate of convergence on the whole positive axis. It has been used in [5] to obtain a result on the spectrum of an almost decomposable operator. The corresponding σ-function is computed in [5] and the finite sums σ_n

in [8].

7. Connections with numerical analysis

Let us turn now to the problem of comparing this new method of measuring convergence with the classical notion described at the beginning. The new method is based on comparing consecutive terms in the sequence

$$d(x_n, x_{n+1})$$

while the classical one compares consecutive terms in the sequence

$$d(x_n, x) .$$

It is thus natural to ask whether estimates using consecutive distances $d(x_n, x_{n-1})$ imply similar estimates for the distances $d(x_n, x)$. More precisely, if $e_{n,n+1}$ stands for an estimate of $d(x_{n+1}, x_n)$ and e_n for an estimate of $d(x_n, x)$ we can ask whether estimates of the form $e_{n+1,n+2} \le \omega(e_{n,n+1})$ imply estimates of the classical type $e_{n+1} \le \le \omega(e_n)$. We intend to show that this is indeed so at least in the case where ω is convex.

To see that, suppose we have a sequence x_n for which the estimate

$$d(x_{n+1}, x_n) \le \omega(d(x_n, x_{n-1}))$$

holds. Hence

$$d(x_n, x) \le d(x_{n+1}, x_n) + d(x_{n+2}, x_{n+1}) + \dots \le$$

$$\le \sum_0^\infty \omega^{(k)}(d(x_{n+1}, x_n)) = \sigma(d(x_{n+1}, x_n)) .$$

Here we have used the fact that ω is nondecreasing; this is a simple consequence of the convexity of ω.

Similarly, $d(x_{n+1}, x) \le \sigma(\omega(d(x_{n+1}, x_n)))$; it follows that the estimates

$$e_{n+p,n+p+1} = \omega^{(p)}(d(x_{n+1}, x_n)) \qquad p=0,1,2,\dots$$

and

$$e_n = \sigma(e_{n,n+1})$$

satisfy the inequalities $e_{n+1} \le \sigma(\omega(e_{n,n+1}))$.

To obtain the desirable estimate $e_{n+1} \le \omega(e_n)$ it would be sufficient to have the inequality $\sigma \circ \omega \le \omega \circ \sigma$ since this yields the following estimates

$$e_{n+1} \le \sigma(\omega(e_{n,n+1})) \le \omega(\sigma(e_{n,n+1})) = \omega(e_n) .$$

This heuristic reasoning should be sufficient to explain the importance of the inequality $\sigma \circ \omega \le \omega \circ \sigma$.

It turns out that such an inequality may be proved in the case of convex rates of convergence ω. The following proposition holds [12].

Suppose ω is a rate of convergence on the interval T . If ω is convex then

$$\omega \circ \sigma \geq \sigma \circ \omega$$

on the interval $T \cap \sigma^{-1}(T)$.

It follows that, in this case, the two ways of estimating convergence discussed above are equivalent.

Detailed proofs and a discussion of the basic principles of the non-discrete induction method may be found in the Gatlinburg Lecture [5].

List of references

[1] V. PTÁK, Some metric aspects of the open mapping theorem, Math. Ann. 163 (1966), 95-104

[2] V. PTÁK, Deux théorèmes de factorisation, Comptes Rendus Acad.Sci. Paris 278 (1974),1091-1094

[3] V.PTÁK, A theorem of the closed graph type, Manuscripta Math. 13 (1974), 109-130

[4] V. PTÁK, A quantitative refinement of the closed graph theorem, Czechoslovak Math.J. 99 (1974), 503-506

[5] V. PTÁK, Nondiscrete mathematical induction and iterative existence proofs, Linear Algebra and Appl. 13 (1976), 223-238

[6] V. PTÁK, The rate of convergence of Newton's process, Numer.Math. 25 (1976), 279-285

[7] V. PTÁK, Concerning the rate of convergence of Newton's process, Comment.Math.Univ.Carolinae 16 (1975), 699-705

[8] V. PTÁK, A rate of convergence, Abh.Math.Sem.Univ. Hamburg (in print)

[9] V. PTÁK, A modification of Newton's method, Časopis pěst.mat. 101 (1976), 188-194

[10] H. PETZELTOVÁ and P. VRBOVÁ, A remark on small divisors problems, Revue Roumaine Math. (in print)

[11] J. ZEMÁNEK, A remark on transitivity of operator algebras, Časopis pěst.mat. 100 (1975), 176-178

[12] V. PTÁK, What should be a rate of convergence, RAIRO, Analyse numérique (in print)