Nondiversification Traps in Catastrophe Insurance Markets

Rustam Ibragimov
Harvard University

Dwight Jaffee
University of California at Berkeley

Johan Walden
University of California at Berkeley

We develop a model for markets for catastrophic risk. The model explains why insurance providers may choose not to offer insurance for catastrophic risks and not to participate in reinsurance markets, even though there is a large enough market capacity to reach full risk sharing through diversification in a reinsurance market. This is a "nondiversification trap." We show that nondiversification traps may arise when risk distributions have heavy left tails and insurance providers have limited liability. When they are present, there may be a coordination role for a centralized agency to ensure that risk sharing takes place.

1. Introduction

Catastrophe insurance provides compensation for losses created by such natural risks as earthquakes, floods, and wind damage, as well as man-created risks including terrorism. Over the past fifteen years, most private-market property and casualty (P&C) insurance firms stopped offering coverage against catastrophe risks, usually in the wake of a major event. Key private market failures include Florida hurricane insurance after Hurricane Andrew in 1992, California earthquake insurance after the Northridge quake of 1994, and, most recently, US terrorism insurance after the 9/11 attack. Catastrophe insurance markets have also failed worldwide in most developed countries. In response, state and federal governments have been forced to try to replace or to revive the private markets, the most recent example being the US Terrorism Risk Insurance

We thank Greg Duffee, William Ellsworth, Christine Parlour, Jacob Sagi, Eric Talley, Gordon Woo, and participants at the NBER Insurance Project workshop for valuable comments. We also thank Steve Evans for help with one of the proofs. Finally, we thank the editor, Matthew Spiegel, and the referee, for valuable suggestions. Send correspondence to Rustam Ibragimov, Department of Economics, Harvard University, 1875 Cambridge St., Cambridge, MA 02138, telephone: 617-496-4795, fax: 617-495-7730. E-mail: ribragim@fas.harvard.edu.

doi:10.1093/rfs/hhn021 Advance Access publication March 29, 2008

Act of 2002 (TRIA) and its 2005 extension. Unfortunately, the government interventions are generally considered to be quite inefficient.[1]

Understanding the source of the private market failures is essential if more effective remedies are to be found. A fundamental question is why catastrophe risks are "uninsurable" for the private insurance firms. Asymmetric information—adverse selection and/or moral hazard—is the common textbook explanation for insurance market failures, but there seems to be little role for asymmetric information with respect to natural disasters or terrorism attacks.[2] Imprecision in estimating the underlying stochastic process is also sometimes suggested as a basis for "uninsurability," but even if parameter imprecision raises the cost of insurance, perhaps due to ambiguity aversion, it is unclear why it would cause the market to fail; see Froot (2001).

A third basis for "uninsurability" is that the possible losses may exceed the capital resources of the P&C insurance industry; see Cummins, Doherty, and Lo (2002). For example, the losses created by war or by terrorist use of weapons of mass destruction (WMD) could readily exceed the capital resources of all P&C firms. The deadweight costs of bankruptcy, including a loss in the value of the managers' human capital, could then motivate the unwillingness of firms to participate in the catastrophe insurance markets. War and WMD risks, however, have long been excluded from most insurance contracts, without jeopardizing the availability of coverage for standard risks.

War and WMD risks aside, size alone does not appear to explain the recent failure of so many different catastrophe insurance markets. Table 1 shows the 10 most costly insurance losses since 1970 as compiled by SwissRe (2006). The losses created by the Katrina hurricane of 2005 were USD 45 billion, followed by the Florida Hurricane Andrew of 1992 (USD 22 billion), and the 9/11 terrorist attack (USD 21 billion). In comparison, the capital resources of the P&C insurance industry at year-end 2005 totaled approximately USD 446 billion, and in each year since 2001, the P&C industry has *increased* its capital resources (net of losses) by at least USD 29 billion.[3]

Although P&C industry resources can cover most catastrophic risks, coverage is provided at the level of individual firms, not the industry. Furthermore, insurers tend to specialize in geographic regions and particular lines of coverage, putting individual firms at potentially high risk to a specific catastrophic event. Regulation is a major cause of the geographic and insurance line specialization, since US insurance firms must be chartered separately in each state

---

[1] See Cummins (2006); Jaffee (2006); and Jaffee and Russell (2006) for recent discussions and references to the literature. OECD (2005, August) and OECD (2005, July) discuss government interventions around the world to reactivate terrorism insurance. Kunreuther and Michel-Kerjan (2006) discuss the specific issue of terrorism insurance in the United States.

[2] There is generally open access to scientific forecasts of natural disasters, much of it provided by governments. Terrorists may be more strategic in their choice of targets, but this does not create a moral hazard on the part of those purchasing terrorism insurance (unless the terrorists particularly target insured properties).

[3] These data are from the Insurance Information Institute; see http://iii.org/media/industry/.

Table 1
The world's 10 costliest insurance events

Date	Event	Insured losses[1] (USD Billion)
August 24, 2005	Hurricane Katrina	45
September 23, 1992	Hurricane Andrew	22
September 11, 2001	World Trade Center	21
January 17, 2004	Northridge Earthquake	18
September 2, 2004	Hurricane Ivan	12
September 20, 2005	Hurricane Rita	10
October 16, 2005	Hurricane Wilma	10
August 11, 2004	Hurricane Charley	8
September 27, 1991	Typhoon Mireille (Japan)	8
January 25, 1990	Storm Daria (Europe)	7

Source: SwissRe (2006).
[1]Property and Business interruption; excludes liability and life insurance.

in which they operate and they face substantial fixed costs for marketing and for developing actuarial expertise for each line and for each state.[4] The result is that relatively few catastrophe insurance firms operate in each state and for each catastrophe line. Risk-averse executives with ties to their own firm may wish to avoid such an undiversified position.

Reinsurance firms exist, of course, precisely to redistribute risks, allowing individual insurers to match their retained risks with their capital resources. Thus, if reinsurance markets function efficiently, then capital adequacy at the industry level is in fact the relevant measure. Unfortunately, the proximate cause of the observed failures of the primary catastrophe insurance markets has been precisely the failure of the associated reinsurance markets. For this reason, the fundamental question is why the reinsurance markets for catastrophe risks have largely failed.

In this paper, we argue that the observed dynamic pattern of widely varying supply conditions for catastrophe insurance and reinsurance could reflect a multiple equilibrium system, with the market sometimes reaching a coordinated reinsurance/diversification equilibrium, but at other times falling into what we call a nondiversification trap. The term is related to poverty traps and development traps in economic growth theory (Barro and Sala-i-Martin, 2004; Azariadis and Stachurski, 2006). It denotes a situation where there are two possible equilibria: a diversification equilibrium in which insurance is offered and there is full risk sharing through the reinsurance market, and a nondiversification equilibrium, in which the reinsurance market is not used, and no insurance is offered at all. A move from the nondiversification equilibrium to the diversification equilibrium has to be coordinated by a large number of

[4] Insurance is unique among US financial services in that it is regulated in the United States *only* at the state level. The structure of a catastrophe insurance market is well illustrated by California's earthquake risk market. As of 2005, 70% of the coverage was provided by the California Earthquake Authority, an entity created by the State of California following the 1994 Northridge quake. With no major quakes since then, private insurers have slowly reentered the market, now representing about 30% of the market. However, still only 35 private insurance groups are offering California earthquake coverage (based on annual written premiums of USD 1 million or more). Furthermore, the top 5, 10, and 20 firms represent 46%, 66%, and 89% of the total private market, respectively.

insurers and reinsurers, which may be difficult to achieve through a market mechanism. Therefore, there may be a role for a centralized agency to ensure that the diversification equilibrium is reached, for example, by mandating that insurance must be offered (as in the case of the US Terrorist Risk Insurance Act of 2002 and in the corresponding government plans in most European countries).

Consequently, our discussion and model focus on reinsurance as the mechanism that could be used to coordinate the diversification equilibrium. A functioning reinsurance industry, however, requires that the primary insurers be willing to write policies in anticipation that other insurers will do the same and that the reinsurers will pool all the risks, to reach the global diversification outcome. Our model will determine the conditions under which such an equilibrium can and cannot occur.

The existence of nondiversification traps depends crucially on there being conditions under which diversification becomes suboptimal for the individual insurers. This is contrary to the traditional framework in which diversification is always preferred (see, e.g., Samuelson 1967, for equity investments and Froot and Posner, 2002, for insurance). The traditional framework uses concave optimization (e.g., via expected utility), with thin-tailed risks (e.g. normal distributions), and without distortions (unlimited liability, no frictions and no fixed costs). If any of these assumptions fails, diversification may not always be preferred.[5] Our discussion focuses, in particular, on the impact of heavy left-tailed distributions (implying a non-negligible probability for large negative outcomes) as the defining property of catastrophic risks.

Figure 1, from Ibragimov and Walden (2007), provides an intuition for how nondiversification traps can arise for insurance. Consider a situation in which there is a maximum number of distinct risks that an individual insurance provider can take on, e.g., $N = 10$. The constraint of the maximum number of risks that an individual firm can accept is in line with our regulatory discussion earlier and can also be motivated by capacity constraints, capital requirements, and segmented markets. The three lines A, B, and C in the figure describe the value (e.g., measured as a certainty equivalent) of holding a diversified portfolio of n risks as a function of n. In this paper, we study a model in which the value is a U-shaped function of the number of risks, corresponding to line B (also studied in a different context in Ibragimov and

[5] Beginning with Mandelbrot (1963) and Fama (1965), numerous papers have studied the presence of heavy-tailedness in economics, finance, and insurance. We mention a sample: heavy-tailedness of return distributions in stock markets has been documented by Jansen and de Vries (1991); Loretan and Phillips (1994); McCulloch (1996, 1997); Rachev and Mittnik (2000); and Gabaix, Gopikrishnan, Plerou, and Stanley (2003). Moreover, Scherer, Harhoff, and Kukies (2000) and Silverberg and Verspagen (2004) report that distributions of financial returns from technological innovations are extremely heavy-tailed and do not have finite means. Nešlehova, Embrechts, and Chavez-Demoulin (2006) discuss empirical results that indicate similar extreme thick-tailedness with infinite first moments for the loss distributions of a number of operational risks. Finally, as was shown in Ibragimov (2004, 2005) in a general context, with heavy-tailed risks diversification may be inferior, regardless of the number of risks available. In Ibragimov and Walden (2007) it was further demonstrated that with heavy tails and limited liability, diversification may be suboptimal up to a certain number of risks, and then become optimal.

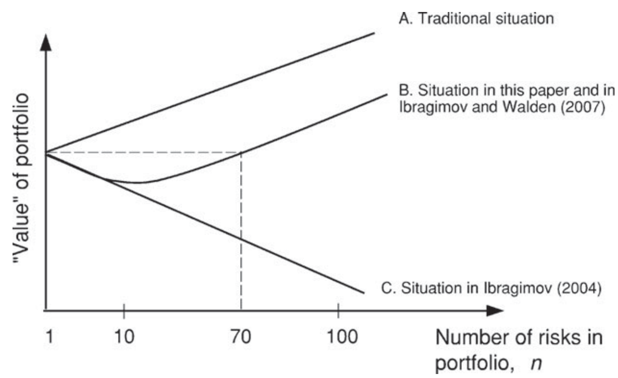

Figure 1
Value of diversification
A: Traditional situation. The value increases monotonically and it is always preferable to add another risk to the portfolio. B: Situation in Ibragimov and Walden (2007). Up to a certain number of assets, value decreases with diversification. C: Situation in Ibragimov (2004, 2005). Value always decreases with diversification.

Walden, 2007). In this case, for any individual insurance provider, diversification will clearly be suboptimal as the value decreases in n for $n \leq N = 10$. However, if there are M insurance providers in the market, they could potentially meet in a reinsurance market, pool the risks, and reach full diversification with NM risks. For this to be preferred to nondiversification, at least $M = 7$ insurance providers must pool the risks. This is a very different situation compared to the traditional situation in line A, in which each individual insurance provider will choose maximal diversification into N risks, and in which two insurance providers can always improve their situation by pooling their risks in a reinsurance market. For line B, there may be a coordination problem.

The objective of this paper is to show how heavy-tailed distributions can lead to nondiversification traps, to provide an understanding of the results and, finally, to interpret markets for catastrophe insurance in light of the results. The paper is organized as follows. In Section 2, we show how traps can arise. Sections 2.1 and 2.2 provide intuition for the results. Sections 2.3 and 2.4 provide a formal game-theoretic setup for a simple reinsurance market, in which nondiversification traps can be analyzed. Moreover, the concept of *genuine* nondiversification traps is introduced. These traps are severe in that they will not disappear, regardless of the capacity of the insurance market. They may, therefore, explain the nonexistence of insurance in markets in which there is a large capacity for risk sharing, but no insurance is offered. In Section 3, we show the existence of nondiversification traps and characterize the conditions under which they can arise. In Section 4, we discuss the implications of our theory for real markets for catastrophe insurance. Finally, we make some concluding remarks in Section 5. All technical details are presented in the Appendix.

2. Nondiversification Traps

2.1 Diversification of heavy-tailed risks

When distributions have heavy tails, diversification may increase risk. This is of course contrary to the traditional case; for example, represented by normal distributions $X \sim$ Normal (μ, σ^2).[6] We show how, using the Lévy distribution concentrated on the left semi-axis. This is mainly for simplicity: the Lévy distribution is one of the few stable distributions for which closed-form expressions exist.[7] The class of stable distributions is a subclass of the class of distributions whose left tails satisfy a Pareto law, i.e., exhibit power-law decay:

$$F(-x) \sim x^{-\alpha}, \qquad x > 0, \ \alpha > 0,$$

where F is a cumulative distribution function (c.d.f.).[8]

The p.d.f. of the Lévy distribution with location parameter μ and spread parameter σ is

$$\phi(x) = \begin{cases} \sqrt{\dfrac{\sigma}{2\pi}} e^{-\sigma/2(\mu-x)}(\mu - x)^{-3/2}, & x < \mu, \\ 0, & x \geq \mu, \end{cases}$$

and the c.d.f. is

$$F(x) = \begin{cases} \mathrm{Erf}\left(\dfrac{\sigma}{\sqrt{2(\mu - x)}}\right), & x < \mu \\ 1, & x \geq \mu \end{cases}. \tag{1}$$

Here, $\mathrm{Erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-t^2} dt$ is the error function; see Abramowitz and Stegun (1970). We denote by $L_{\mu,\sigma}$ the class of random variables (r.v.'s) with the above Lévy distributions.

Similar to normal distributions, a higher value of the spread parameter σ for a Lévy distribution implies higher riskiness (for a fixed μ). In fact, increasing σ for Lévy risks leads to first-order stochastically dominated risk, as seen from Equation (1).

Contrary to normal distributions, diversification of Lévy risks increases σ. For example, if X_1 and X_2 are i.i.d., Lévy risks, both with spread parameters σ, then a portfolio of half of each risk, $(X_1 + X_2)/2$, has a spread parameter of 2σ. This follows from the following general diversification rule for portfolios

---

6 It is also true for general distributions with finite second moments, $E(X^2) < \infty$. This condition is assumed in Samuelson (1967).

7 Stable distributions are those closed under portfolio formation; see, e.g., the reviews in Ibragimov (2004, 2005). Besides Lévy distributions, closed-form expressions for stable densities are available only in the case of normal and Cauchy distributions.

8 Here and throughout the paper, $F(x) \sim G(x)$ denotes that there are constants, c and C, such that $0 < c \leq F(x)/G(x) \leq C < \infty$ for large $x > 0$.

of K independent Lévy distributed risks, with nonnegative portfolio weights, $c_i \in \mathbb{R}_+$:

$$X_i \in L_{\mu_i,\sigma_i},\ i = 1, \ldots, K \qquad \Longrightarrow \qquad \sum_{i=1}^{K} c_i X_i \in L_{\mu,\sigma},$$

$$\mu = \sum_{i=1}^{K} c_i \mu_i,\ \sigma = \left(\sum_{i=1}^{K} (c_i \sigma_i)^{1/2} \right)^2.$$

A special case is uniform diversification,

$$X_i \in L_{\mu,\sigma},\ i = 1, \ldots, K \qquad \Longrightarrow \qquad \frac{\sum_{i=1}^{K} X_i}{K} \in L_{\mu,K\sigma},$$

so uniform diversification of K i.i.d. risks increases the spread parameter from σ to $K\sigma$ in line with the example above.

Another case is the Cauchy distribution with location parameter μ and scale parameter σ, $X \in S_{\mu,\sigma}$, whose p.d.f. is given by

$$\phi(x) = \frac{1}{\pi\sigma} \frac{1}{1 + \left(\frac{x-\mu}{\sigma}\right)^2},$$

and whose c.d.f. has the form

$$F(x) = \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{x - \mu}{\sigma} \right).$$

For independent Cauchy distributions, the portfolio diversification rule is

$$X_i \in S_{\mu_i,\sigma_i},\ i = 1, \ldots, K \qquad \Longrightarrow \qquad \sum_{i=1}^{K} c_i X_i \in S_{\mu,\sigma},$$

$$\text{where} \quad \mu = \sum_{i=1}^{K} c_i \mu_i,\ \sigma = \sum_{i=1}^{K} c_i \sigma_i.$$

Uniform diversification of Cauchy distributed risks therefore has no effect on total risk, i.e.,

$$X_i \in S_{\mu,\sigma},\ i = 1, \ldots, K \qquad \Longrightarrow \qquad \frac{\sum_{i=1}^{K} X_i}{K} \in S_{\mu,\sigma}.$$

The Cauchy case is thus intermediate between the Lévy case and the case with normal distributions.

2.2 Risk pooling

We begin by studying the potential value of risk sharing among multiple risk-takers. We first develop the intuition and then, in the following sections, prove the results rigorously for a model of a reinsurance market.

We study the behavior of risk-takers. Because we focus on the context of risk-taking insurance companies, we will refer to these risk-takers as insurers. We assume that the number of insurers is bounded by M and that all insurers are expected utility optimizers with identical strictly concave utility functions u.

We assume that there is limited liability. Clearly, real-world insurance firms have limited liability and may default in some states of the world. This case is increasingly studied in the insurance literature; see Cummins, Doherty, and Lo (2002); Cummins and Mahul (2003); and Mahul and Wright (2004). For catastrophe insurance, with heavy-tailed distributions, there is an effectively nonzero (although small) probability that such a catastrophic event will create default. Technically, limited liability is needed in the model, because with heavy-tailed distributions, the expected payoffs and values are not otherwise defined. We shall, however, see that the probability for default is small in equilibrium. Moreover, we will show that our results are not driven by the convexity of payoffs introduced by limited liability: For markets with large aggregate risk-bearing capacity, our results will apply only if distributions are heavy-tailed. The assumption of limited liability is modeled by insurers being liable to cover losses only up to a certain amount k. If losses exceed k, an insurer pays k, but defaults on any additional loss.[9] Thus, if an insurer takes on a random risk of X, the effective outcome for the insurer once X is realized is

$$V(X) = \begin{cases} X, & \text{if } X \geq -k \\ -k, & \text{if } X < -k \end{cases}. \tag{2}$$

In the special case when there is no limited liability, i.e., when $k = \infty$, we have $V(X) = X$ for all X. If $k < \infty$, u needs only to be defined on $[-k, \infty)$ and we can, without loss of generality, assume that $u(-k) = 0$.

Assuming i.i.d. risks $X_1, X_2, \ldots$, we wish to study the expected utility of s agents, who share j risks equally. We therefore define the random variable $z_{j,s} = (\sum_{i=1}^{j} X_i)/s$, with c.d.f. $F_{j,s}$. The expected utility of such risk sharing is

$$U_{j,s} \stackrel{\text{def}}{=} Eu(V(z_{j,s})) = \int_{-k}^{\infty} u(x) dF_{j,s}(x). \tag{3}$$

Firms are usually considered to be risk neutral. However, an expected utility setup with concave utility can effectively arise if there are financial imperfections as, for example, assumed in Froot, Scharsfstein, and Stein (1993). If such financial imperfections are present, the value of the firm will be given by a concave transformation of the payoffs, which is effectively identical to our expected utility setup. Another motivation for risk-averse firm behavior is that executives with major financial and human capital investments in their own firm wish to avoid risky positions.

[9] We assume that a third party, perhaps the government, covers the excess losses to policyholders. This avoids the complications of any impact on policyholder demand.

Insurers face two constraints—one that limits the aggregate amount of their risks, and the other that limits the size of individual risks. The aggregate limitation is driven, for example, by capital requirements. This aggregate limit is imposed by assuming that each insurer can bring at most N risks "to the table." Thus, we have $1 \leq s \leq M$, $1 \leq j \leq Ns$. The second constraint is that each risk X_i is indivisible, so it cannot be split between insurers in a primary insurance market. As discussed in the Introduction, real-world catastrophe insurance markets are segmented in this way because relatively few insurers operate in each state and in each line.

When returns are independently normally distributed, it is well known that one can always add an asset to a portfolio and strictly increase the agent's utility via the appropriate selection of weights. In this case, $U_{j,s}$ is strictly increasing in s for each j (an immediate consequence of Samuelson, 1967). In this situation, we can expect a reinsurance market to work well and insurance to be offered for a maximal number of risks, NM. The argument is based on the fact that each insurer will choose to diversify fully, regardless of what the other $M - 1$ insurers do. We call this the *traditional situation*.

The situation is very different when we have limited liability and heavy-tailed distributions. We consider i.i.d. Bernoulli-Cauchy distributed risks $\tilde{X}_i$, i.e.,

$$\tilde{X}_i = \begin{cases} \mu, & \text{with probability } 1 - q \\ X_i,\ X_i \in S_{\nu,\sigma}, & \text{with probability } q \end{cases},$$

where $X_i \in S_{\nu,\sigma}$ are i.i.d. Cauchy r.v.'s with location parameter ν and scale parameter σ. In other words, the r.v.'s X_i are "mixtures" of degenerate and Cauchy r.v.'s. Clearly, the risks $\tilde{X}_i$ can be written as

$$\tilde{X}_i = \mu(1 - \epsilon_i) + X_i\epsilon_i = \mu + (\nu - \mu)\epsilon_i + \sigma Y_i \epsilon_i, \tag{4}$$

where ϵ_i are i.i.d. nonnegative Bernoulli r.v.'s with $P(\epsilon_i = 0) = 1 - q$, $P(\epsilon_i = 1) = q$ and $Y_i \in S_{0,1}$ are i.i.d. symmetric Cauchy r.v.'s with scale parameter $\sigma = 1$ that are independent of ϵ_i's.

For the above distributions, we say that $\tilde{X}_i \in \tilde{S}^q_{\mu,\nu,\sigma}$. Here, μ can be thought of as the premium an insurance provider collects to insure against events that occur with probability q. For $q \ll 1$, this distribution is qualitatively similar to distributions for catastrophic risks: There is a small probability for a catastrophe to occur. However, if it does occur, the loss may be very large due to the heavy left tail of the Cauchy distribution. We use the Cauchy distribution for its analytical tractability (even though it—similar to the normal distribution, used, e.g., in Cummins (2006)—has a nonzero right tail, which does not have a meaningful interpretation for catastrophic events). We assume limited liability ($k < \infty$) and the power utility function $u(x) = (x + k)^\alpha$, $\alpha \in (0, 1)$. Clearly, under the above assumptions, the expected utility for Bernoulli-Cauchy risks always exists.

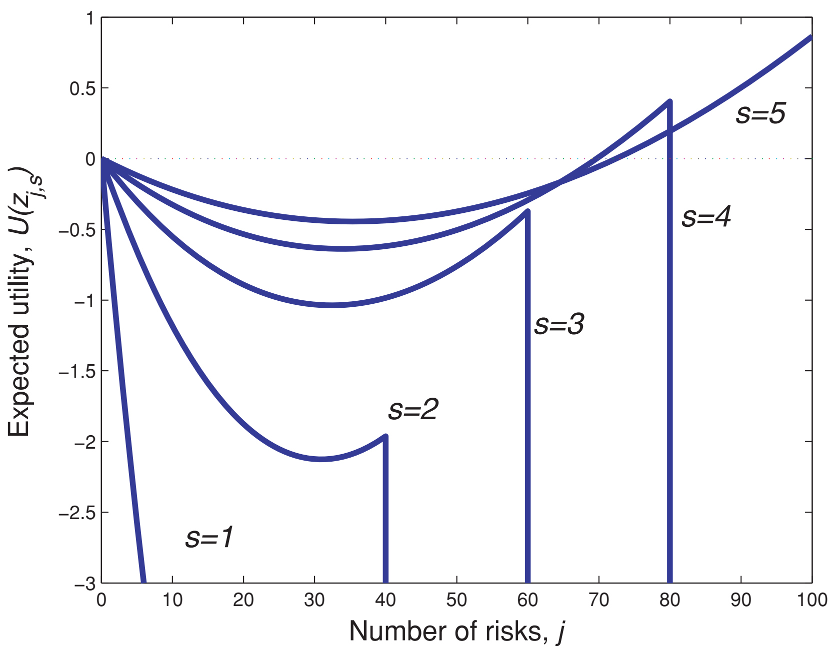

Figure 2
Expected utility for insurers under different risk-sharing alternatives
s denotes the number of insurers sharing risks. j denotes the total number of risks. $U(z_{j,s})$ denotes the expected utility of an insurer as a function of j and s. Parameters: Liability $k = 100$, total number of insurers $M = 5$, maximum number of risks per insurer $N = 20$, risk parameters: $\sigma = 1$, $\nu = -9$, $\mu = 1$, $\alpha = 0.0315$, and $q = 5\%$ [see Equation (4)]. See Appendix for Section 2.2 for closed-form solution.

In Figure 2, we show expected utility for different total numbers of projects, j, and numbers of agents involved in risk sharing, s, with parameters $k = 100$, $\sigma = 1$, $\mu = 1$, $\nu = -9$, $N = 20$, $M = 5$, $\alpha = 0.0315$, and $q = 0.05$. There is a crucial difference compared to the traditional situation. For a moderate number of risks, there is no way to increase expected utility compared to staying away from risks altogether. An insurer has the option of not entering the market and must therefore earn a utility premium to be willing to take on risks (i.e., to offer insurance). No insurer will therefore choose to invest in risks that cannot be pooled. Moreover, if an insurer believes that no other insurer will pool risks, he will not take on risks, whether he can pool them or not. Thus, even though the situation with full diversification and risk sharing ($U_{NM,M}$) is preferred over the no-risk situation ($U_{0,1}$), at least four insurers must agree to pool risk for risk sharing to be worthwhile.

In this situation, there may be a coordination problem: Even though all agents would like to reach $U_{NM,M}$, they may be stuck in $U_{0,1}$. Clearly, the limited liability assumption is important: If liability were unlimited, no agent would ever take on risks. The situation would be as in Ibragimov (2004, 2005), where diversification is always inferior. However, we note that the probability for default in the situation with full pooling and diversification is small: it is approximately 0.3%.

The expected utility assumption is not crucial. Similar results would arise in a value-at-risk (VaR) framework; for example, with agents who trade off VaR versus expected returns for some risk level α. The crucial property of the $U_{j,s}$ curves is that they are "U-shaped" in s. In Ibragimov and Walden (2007), it is shown that similar U-shaped curves occur as a function of diversification when the VaR measure is used. The specification in a VaR framework would be $U_{j,s} = F(\mu, W)$, $\mu = E(V(z_{j,s}))$, $W = VaR_\alpha(V(z_{j,s}))$, with $\partial F/\partial\mu > 0$, and $\partial F/\partial W < 0$, and the analysis would be similar to the analysis we carry out in this paper.

Our argument so far has been informal. We next make these diversification results rigorous by introducing a model of a reinsurance market, where coordination plays a role—the *diversification game*. We will show that in the traditional situation, the only equilibrium is a diversification equilibrium, where NM risks are insured, whereas in the situation with heavy tails there is both a diversification equilibrium and a nondiversification equilibrium in which no insurance is offered.

2.3 A reinsurance market

We analyze a market in which insurance providers sell insurance against risks. For simplicity, we model the market in a symmetric setting: participants in reinsurance markets share risks equally. The setup is a two-stage game that captures the intuitive idea that insurance has to be offered before reinsurance can be pooled. The decision whether to offer primary insurance will be based on beliefs about how well-functioning (the future) reinsurance markets will be. If a critical number of participants is needed for reinsurance markets to take off, then nondiversification traps can occur. As we have already discussed the intuition behind nondiversification traps, Sections 2.3, 2.4, and 3 focus on providing the theoretical foundation for the existence of nondiversification traps.

The two-stage *diversification game* describes the market. In the first stage, agents (insurance providers) simultaneously choose whether to offer insurance against a set of i.i.d. risks. In the second stage, the reinsurance market is formed and each agent chooses whether to participate or not. Agents who choose not to offer primary insurance are allowed to participate in the reinsurance market. Finally, all risks of agents participating in the reinsurance market are pooled and outcomes are realized and shared equally among participating agents.

2.3.1 Insurance market There are $M \geq 2$ agents (also referred to as insurance providers, insurance companies, or insurers). We use m, $1 \leq m \leq M$ to index these agents. There is a set of i.i.d. risks, $\mathcal{X}$, where each risk has c.d.f. $F(x)$. Each agent chooses to take on a specific number of risks, $n_m \in \{0, 1, 2, \ldots, N\}$, where N denotes the maximum insurance capacity, forming a portfolio of risks

$p_m \in \mathcal{P}_m$, where $p_m = \sum_{i=1}^{n_m} X_i$ and $X_i \in \mathcal{X}$. This is the first stage of the market. The risks are atomic (indivisible) and each risk can be chosen by at most one agent. We assume that there are enough risks available to exhaust capacity, i.e., $|\mathcal{X}| = NM$. Here, $|\mathcal{X}|$ denotes the cardinality of $\mathcal{X}$. As risks are i.i.d., only the distributional assumptions of the risks matter and we will not care about which insurance provider chooses which risk. The portfolio p_m is therefore completely characterized by the number of risks, n_m. The total number of risks insured is $\overline{N} = \sum_m n_m$.

Agents have liability to cover losses up to k, where $k \in (0, \infty]$. If losses exceed k for an agent, he defaults and pays k, and a third party, possibly the government, steps in and covers excess losses. The effective outcome under limited liability for agent m, taking on risk z_m, is therefore $V(z_m)$, where V is defined in (2). All agents have identical expected utility over risks, $U_m(z_m) = Eu(V(z_m))$, where u is defined and continuous on $[-k, \infty)$, is strictly concave, is twice continuously differentiable on $(-k, \infty)$ and, if $k < \infty$, satisfies $u(-k) = 0$. The outcome of the first stage is summarized by $p = (p_1, \dots, p_M) \in \mathcal{P} \stackrel{\text{def}}{=} \prod_{m=1}^{M} \mathcal{P}_m$.

2.3.2 Reinsurance market. In the second stage of the game, named the *participation subgame*, the reinsurance market is formed. In this stage, agents have perfect knowledge about p. Each agent, $1 \le m \le M$, sequentially decides whether to participate in the market or not, as follows: First, agent 1 decides whether to participate. This is represented by the binary variable $q_1 \in \{0, 1\}$, where $q_1 = 1$ denotes that agent 1 participates in the reinsurance market and $q_1 = 0$ otherwise. Then, agent 2 decides whether to participate, observing agent 1's decision, etc. This is repeated until all M agents have decided. Previous agents' decisions are observable. If an agent is indifferent between participating and not participating, he will not participate. Agents who offer insurance, and participate, pool all their insurance in the reinsurance market, i.e., $q_m p_m$ is supplied to the reinsurance market by agent m. The total pooled risk is therefore $P = \sum_m q_m p_m$ and the number of risks is $R = \sum_m q_m n_m \in \{0, \dots, NM\}$.

As noted, the two stages separate the choice of offering insurance from the creation of a reinsurance market, which can occur only when the risks are already insured. The total number of participating agents in the reinsurance market is $t = \sum_m q_m$. Finally, the pooled risks are split equally among agents participating in the reinsurance market, i.e., each participating agent receives a fraction $1/t$ of the pooled portfolio, P, with R risks.

The outcome of the participation subgame is summarized by $q = (q_1, q_2, \dots, q_M) \in \{0, 1\}^M$ and the outcome of the total diversification game is thus completely characterized by (p, q). Moreover, the quintuple $\mathcal{G} = (u, F, k, N, M)$ completely characterizes the diversification game.

We study equilibrium outcomes (p, q) of a diversification game $\mathcal{G}$. As the second stage of the market is an M-step sequential game with perfect

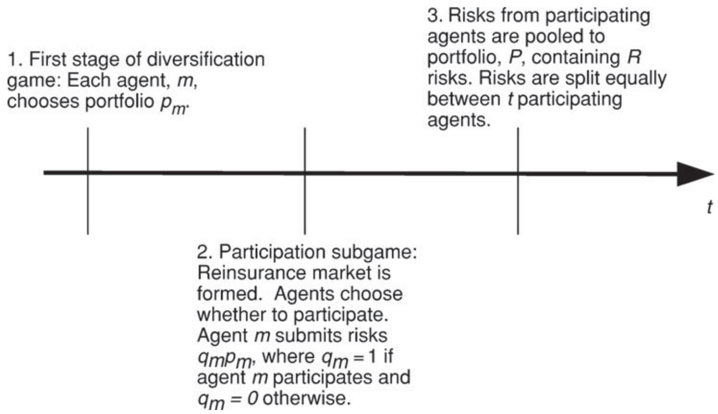

Figure 3
Sequence of events
1. Agents simultaneously choose risk portfolio, p_m. 2. Reinsurance pool $P = \sum_m q_m p_m$ is formed. Agents sequentially choose whether to participate, knowing outcome of step 1 and decision of previous agents. 3. Pooled risk is split between s participating agents, each taking on risk P/t. Agents who do not participate in the reinsurance market take on risk $(1 - q_m)p_m$.

information, it is straightforward to calculate the unique subgame perfect equilibrium by backward induction (existence and uniqueness being guaranteed by Zermelo's theorem and by imposing the assumption that indifferent agents do not participate). A detailed setup for the participation subgame is given in the Appendix. The equilibrium mapping of the participation game, for a specific first-stage realization p, is a vector $q = \mathcal{E}(p) \in \{0, 1\}^M$. We use this mapping to simplify the analysis of the first stage of the diversification game. Specifically, in the first stage, all agents agree on $q = \mathcal{E}(p)$ as the outcome of the participation subgame, and therefore use it directly in their value function. This reduces the size of the strategy space considerably, while not having any effect on the (subgame perfect) equilibrium outcome. The sequence of events is shown in Figure 3.

2.3.3 Strategies. For elements $p \in \mathcal{P}$, we define the first-stage actions of all agents except agent m:

$$p_{-m} = (p_1, \dots, p_{m-1}, p_{m+1}, \dots, p_M) \in \prod_{m' \neq m} \mathcal{P}_{m'} \stackrel{\text{def}}{=} \mathcal{P}_{-m}.$$

A strategy for agent m consists of a pair: $A = (p_m, \eta_m) \in \mathcal{P}_m \times \{0, 1\}^{\mathcal{P}_{-m}}$, where p_m is the chosen portfolio of insurance, and $\eta_m : \mathcal{P}_{-m} \to \{0, 1\}$ is the participation choice, depending on the realization in the first stage.[10]

[10] Here, in line with the previous discussion on reduced strategy space, $\tilde{q}_m$ does not need to be conditioned on the participation choices $q_{m'}$ of agents $m' = 1, \dots, m-1$. This is the case as the equilibrium mapping $q = \mathcal{E}(p) \in \{0, 1\}^M$ is known, so $q_{m'}$ is uniquely implied by p in equilibrium.

2.3.4 Belief sets. Agent m has a belief set about the other agents' first-stage actions, $B_m = p_{-m} \in \mathcal{P}_{-m}$. Agent m's strategy, $A_m = (p_m, q_m)$, conditioned on belief set $B_m = p_{-m}$, is said to be *consistent*, if $\eta_m(p_{-m}) = (\mathcal{E}(\tilde{p}))_m$, where

$$\tilde{p} = ((p_{-m})_1, \ldots, (p_{-m})_{m-1}, p_m, (p_{-m})_{m+1}, \ldots, (p_{-m})_M), \tag{5}$$

and we use the notation $(x)_i$ for the ith element of the ordered set x.

Rational agents will consider only consistent strategies, as inconsistent strategies are suboptimal in the participation phase of the diversification game. The inferred outcome of a consistent strategy, $A_m = (p_m, \eta_m)$, conditioned on a belief set B_m is

$$z_m(p_m|B_m) = \begin{cases} p_m, & \text{if } \eta_m(p_{-m}) = 0 \\ P/t, & \text{if } \eta_m(p_{-m}) = 1 \end{cases},$$

where

$$\tilde{q} = \mathcal{E}(\tilde{p}), \qquad t = \sum_{m'} (\tilde{q})_{m'}, \qquad P = \sum_{m'} (\tilde{p})_{m'} (\tilde{q})_{m'},$$

and $\tilde{p}$ is defined as in (5).

2.3.5 Equilibrium. An M-tuple of strategies, $(A_1, \ldots, A_M)$, and belief sets $(B_1, \ldots, B_M)$, where $A_m = (p_m, \eta_m)$ and $B_m = p_{-m}$, defines an equilibrium of the diversification game $\mathcal{G}$, if

1. Consistent strategies: For each agent m, A_m is consistent, conditioned on belief set B_m.
2. Maximized strategies: For each agent m, $p_m \in \arg\max_{p' \in P} U_m(z_m(p'|B_m))$.
3. Consistent beliefs: For each agent m, for all $m' \neq m$: $(p_{-m})_{m'} = p_{m'}$.

The equilibrium outcome is summarized by $p = (p_1, p_2, \ldots, p_M)$ and $q = (\eta_1(p_{-1}), \eta_2(p_{-2}), \ldots, \eta_M(p_{-M}))$. This concludes the definition of the diversification game.

The diversification game, of course, presents a highly stylized view of how primary markets and reinsurance markets for catastrophic risks work. A natural extension would be to allow the insurance premium (μ) to be defined endogenously by demand and supply. This extension turns out to complicate the analysis severely, so we have avoided it for analytical tractability. However, the nondiversification traps we derive occur for ranges of (fixed) μ's, so an interpretation of our result is that there may be no insurance premium μ for which there is both demand from potential insurance buyers and supply from single insurance providers.[11]

[11] For example, in our calibration to earthquake insurance, in Section 4, we arrive at a nondiversification trap arising for annual insurance premiums, μ, between USD 1840 and USD 2300 per household. Below USD 1840,

Another potential extension of the model would be to allow insurance providers to be able to take on fractions of risks, $x \in \mathcal{X}$, and not just 0 or 1. This type of extension would not qualitatively change our results, except for making the model less tractable.

2.4 Classification of equilibria

We are interested in diversification and nondiversification equilibria to a diversification game $\mathcal{G} = (u, F, k, N, M)$. These formalize the situations that were intuitively described in Section 2.2. We define

Definition 1. A *diversification equilibrium* of a diversification game $\mathcal{G}$ is an equilibrium in which insurance against all risks in $\mathcal{X}$ is offered, i.e., $\overline{N} = NM$.

Definition 2. A *diversification equilibrium* of a diversification game $\mathcal{G}$ is *risk sharing* if all risk insured is pooled in the reinsurance market, i.e., $R = NM$.

Definition 3. A *nondiversification equilibrium* of a diversification game $\mathcal{G}$ is an equilibrium, in which no insurance against risk is offered, i.e., $\overline{N} = 0$.

Definition 4. A *nondiversification trap* exists in a diversification game $\mathcal{G}$, if there is both a nondiversification equilibrium and a risk-sharing diversification equilibrium.

We are especially concerned about cases in which nondiversification traps may arise, even though there is a large risk-bearing capacity of the market as a whole. This might arise if the market is fragmented so that coordination problems may be present, i.e., if M is large. We therefore define

Definition 5. A *genuine nondiversification trap* to the quadruple (u, F, k, N) exists if there exists an M_0, such that for all $M \geq M_0$, the diversification game $\mathcal{G} = (u, F, k, N, M)$ has a nondiversification trap.

In the next section, we analyze when traps can occur in the diversification game. It turns out that we can rigorously classify the conditions under which traps may occur.

3. Existence of Traps

We relate the equilibrium concepts described in Section 2.4 to conditions for the $U_{j,s}$ as defined in Equation (3).

---

the only equilibrium is the nondiversification equilibrium, and above USD 2300, the only equilibrium is the full-diversification equilibrium. The range of μ for which a nondiversification trap arises is thus about 20% of the premium, i.e., $(2300 - 1840)/2340$. With other parameter values, we have derived ranges from a few percent up to an order of magnitude.

Condition 1. $U_{j,1} < U_{0,1}$ for all $j \in \{1, \ldots, N\}$.

Clearly, under Condition 1, an agent would never offer insurance if the reinsurance market were not available:

Condition 2. $U_{j,s} < U_{0,1}$ for all $j \in \{1, \ldots, N\}$ and all $s \in \{1, \ldots, M\}$.

Condition 2 is the stronger requirement that even if there is a reinsurance market, there is no way to increase expected utility by risk sharing if only one agent contributes risk to the reinsurance market. We shall see that a sufficient condition for there to be an equilibrium in which full diversification and risk sharing is achieved is

Condition 3.

- $U_{NM,M} > U_{j,1}$ for all $j \in \{0, \ldots, N\}$ and
- $U_{NM,M} > U_{j,M}$ for all $j \in \{N(M-1), \ldots, NM-1\}$.

Our first set of results relates the existence of nondiversification traps to the expected utilities $\{U_{j,s}\}_{0 \le j \le NM, 1 \le s \le M}$, defined in (3). The results are fully in line with the arguments in Section 2.2. We have:

Proposition 1. *If Condition 2 is satisfied, then there is a nondiversification equilibrium.*

The implication can be almost reversed, as shown in

Proposition 2. *If Condition 2 fails strictly, i.e., if $U_{j,s} > U_{0,1}$ for some $j \in \{1, \ldots, N\}$ and $s \in \{1, \ldots, M\}$, then there is no nondiversification equilibrium.*

Proposition 3. *If Condition 3 is satisfied, then there is a risk-sharing diversification equilibrium.*

Clearly, if $U_{0,1} > U_{j,s}$ for all (j, s) such that $j \in \{1, \ldots, Ns\}$ and $s \in \{1, \ldots, M\}$, then the nondiversification equilibrium is unique. Under these conditions, the risks are by all means uninsurable, which may correspond to the "globally uninsurable" risks mentioned in Cummins (2006). Under such conditions, we can have no hopes for an insurance market to work: The risks are simply too large. Our analysis applies to situations for which risks may be "globally insurable," in that Condition 3 is satisfied but—in the terminology of Cummins (2006)—may be "locally uninsurable." In our model, local uninsurability is similar to Condition 1 being satisfied. For heavy-tailed distributions, Condition 2, which is stronger than Condition 1, may also be satisfied, which makes the "local uninsurability" especially cumbersome, and which may lead to coordination problems and nondiversification traps.

We are now in a position to classify the situations when nondiversification traps can arise. We have

Proposition 4. *In the model in Section 2.2, with Bernoulli-Cauchy distributions with parameters* $N = 20$*,* $M = 5$*,* $\tilde{X} \in \tilde{S}^q_{\mu,\nu,\sigma}$*, with* $\mu = 1$*,* $\nu = -9$*,* $\sigma = 1$*, and* $q = 0.05$*,* $k = 100$*,* $u(x) = (x + k)^\alpha$*, with* $\alpha = 0.0315$*, there is a nondiversification trap. Moreover, the nondiversification trap is genuine.*

As we shall see, the crucial point here is that the trap is *genuine*. We next move on to classifying general distributional properties of the primitive risks that permit traps. It turns out that traps will arise only under quite specific conditions: First, nondiversification traps will not arise in a mean-variance framework with unlimited liability. Thus, in the traditional situation, we will never see nondiversification traps. Second, genuine nondiversification traps can arise only with distributions that have heavy tails (i.e., infinite second moments).

Proposition 5. *If utility is of the form* $Eu(X) = E(X) - \gamma Var(X)$*, and* $k = \infty$*, then a nondiversification trap cannot occur. Moreover, depending on parameter values, only two situations can arise: Either there is a unique nondiversification equilibrium (*$\overline{N} = 0$*,* $j = 0$*,* $t = 0$*) or there is a unique diversification equilibrium with full risk sharing (*$\overline{N} = NM$*,* $R = NM$*,* $t = M$*).*

Nongenuine nondiversification traps can arise under standard conditions, i.e., distributions do not need to be heavy-tailed for nondiversification traps to be possible. For example, the diversification game $\mathcal{G} = (u, F, k, N, M)$, with

$$\begin{aligned} u(x) &= xI(x \leq 0) + \log(1 + x)I(x > 0), \\ F(x) &= I(x \geq -50)/2 + I(x \geq 70)/2, \\ k &= \infty, \\ N &= 20, \\ M &= 5, \end{aligned}$$

where $I(\cdot)$ denotes the indicator function, and $F(x)$ thus is the c.d.f. of a discrete r.v. X, with $\mathbb{P}(X = -50) = \mathbb{P}(X = 70) = 1/2$ having a nondiversification trap. However, *genuine* nondiversification traps arise only if distributions have heavy tails, as shown by the following proposition:

Proposition 6. (i) If $k = \infty$ and the risks $X \in \mathcal{X}$ have finite second moments, i.e., $E(X^2) < \infty$, then a genuine nondiversification trap cannot occur.

(ii) If $k < \infty$, the risks $X \in \mathcal{X}$ have $E(X) \neq 0$, and $E(X^2) < \infty$, then a genuine nondiversification trap cannot occur.

(iii) If $k < \infty$, the risks $X \in \mathcal{X}$ have $E(X) = 0$, and $E(X^{2+\epsilon}) < \infty$, for some arbitrary small $\epsilon > 0$, then a genuine nondiversification trap cannot occur.

Proposition 6 can also be viewed from an approximation perspective. If M is large, but finite, then nondiversification traps can arise only with distributions that have left tails that are "approximately" heavy, i.e., decay slowly until a certain point (even though their real support may be bounded). For details on this type of argument, see Ibragimov and Walden (2007).

4. Traps in Markets for Catastrophic Insurance

In this section, we apply our results to real markets for catastrophic insurance. Obviously, risks vary across product lines and geography, and a full investigation is outside the scope of this paper. Instead, we focus on one type of risk—earthquake insurance in California. Applying the principles of seismology, we show that the distribution of loss sizes indeed follows a Pareto law and that an exponent of unity (the Cauchy case) is by no means unreasonable. Moreover, with a simple calibration, we estimate the value of being able to avoid a trap in residential earthquake insurance in California to be up to USD 3.0 billion per year. This is the direct value effect of a trap. The estimate does not include indirect effects, as, e.g., analyzed in Hubbard, Deal, and Hess (2005). We also discuss how this type of analysis is valid for other types of natural disasters. Finally, we relate our results to several recent events in markets for catastrophic insurance.

4.1 Loss distribution of earthquakes in California

The fact that earthquakes are referred to as catastrophes is suggestive that they have heavy-tailed distributions. In this section, we show more precisely that standard seismic theory leads to loss distributions that follow Pareto laws,

$$h_L(l) \sim l^{-\alpha}.$$

Here, $h_L(l) = \mathbb{P}(L > l)$ is the probability that the economic loss L is larger than l, conditioned on an earthquake occurring. More generally, for an r.v. X, let $h_X(x)$ denote the probability that X exceeds x, conditioned on an earthquake occurring, $\mathbb{P}(X \geq x)$.

Pareto laws arise for the distributions of energy release from earthquakes (see, e.g., Sornette, Knopoff, Kagan, and Vanneste, 1996). We show that *economic loss* also satisfies a Pareto law. For economic loss estimates, it is more natural to work with the Modified Mercalli Intensity (MMI) scale. Let M denote the *moment magnitude* Hanks and Kanamori (1979) of an earthquake.[12] A standard

[12] The moment magnitude is almost the same as the Richter magnitude, M_R, for $M \leq 6.5$, but provides a more accurate measure for earthquakes of larger magnitudes.

model for the distribution of moment magnitudes of earthquakes is

$$h_M(m) = C_1 \mathrm{e}^{-\beta m}, \tag{6}$$

where $\beta = 1.84$ is often used (see McGuire, 2004, pp. 34–40).[13] The exponential distribution is adequate for $M \leq 7$, but for higher M, it *underestimates* the probabilities (McGuire, 2004, pp. 53–54; and Schwartz and Coppersmith, 1984), so the distribution for high levels may in fact have heavier tails than assumed in (6).[14]

An empirical relationship between the MMI and the expected magnitude is given by

$$M = 1.3 + 0.6 I_e \quad \Rightarrow \quad I_e = \frac{M}{0.6} - \frac{1.3}{0.6}, \tag{7}$$

where I_e is the epicentral intensity, i.e., the MMI at the center of the earthquake (McGuire, 2004, p. 44). For simplicity, we assume that this is a deterministic relationship. The MMI at a specific point is directly related to the damage and losses at that point. For example, for an MMI of VIII, the estimates of losses for wooden structures is 5–10% of total value (McGuire, 2004, p. 19).[15] For I_e, we immediately get

$$h_{I_e}(i) = C_1\, \mathrm{e}^{-\beta(1.3+0.6i)} = C_2\, \mathrm{e}^{-1.10i}.^{16}$$

We relate the area A covered by an earthquake to I_e through the *attenuation function*. We use the estimate

$$\begin{aligned} I_d &= I_e + 2.87 + 0.00052D - 1.25 \log_{10}(D + 10) \\ &\geq I_e + 2.87 - 1.25 \log_{10}(D + 10), \end{aligned}$$

where I_d is the MMI at a point of distance D away from the epicenter; see Ho, Sussman, and Veneziano (2001).[17] Let $A_d(I_e, I_d)$ denote the area that experiences an MMI $\geq I_d$ for an earthquake with epicentral intensity I_e. We

---

[13] This is the moment magnitude version of the celebrated Gutenberg-Richter exponential law for the Richter magnitude.

[14] Although for *very* high levels, physical arguments imply that there has to be an upper bound on the energy released; see Knopoff and Kagan (1977), and Kagan and Knopoff (1984). However, even if there is an upper bound, say at $M = 10 - 11$, this still leads to an approximate Pareto law for over 15 magnitudes of energy release. The upper bound is well beyond the limited liability threshold of most insurance markets and is therefore not crucial for our trap argument.

[15] This estimate may be somewhat outdated, as building structures nowadays may be stronger. However, this does not change our general conclusions, only the constants in the formulae (personal communication with William L. Ellsworth, Chief Scientist, Western Region Earthquake Hazards Team, United States Geological Survey).

[16] $1.10 \approx 1.84 \times 0.6$.

[17] Other estimates for the relation are available, e.g., in Bakun, Johnston, and Hopper (2003). However, as with the strength of building structures, they are qualitatively similar and will not change our main conclusions (personal communication with William L. Ellsworth, Chief Scientist, Western Region Earthquake Hazards Team, United States Geological Survey).

also write $A(I_e)$ when I_d is fixed and known. Then, as $A \sim D^2$, it is easy to see that

$$A(I_e, I_d) \geq C_3 \times 10^{1.6(I_e - I_d)} = C_3 \times e^{1.6 \ln(10)(I_e - I_d)} = C_4 \times e^{3.7 I_e},$$

for fixed I_d. Another estimate is obtained by using the results in Hanks and Johnston (1992), for $I_d =$ VI. Their formula is

$$M = 2.38 + 0.96 \log_{10}(A(I_e(M), \text{VI})),$$

which by Equation (7) leads to

$$A(I_e, \text{VI}) = C_5 \times e^{0.6 \times \ln(10) I_e / 0.96} = C_5 \, e^{1.44 I_e}.$$

Under the assumption of uniform geographical population density, it is natural to assume that the economic loss L from an earthquake is *at least* proportional to $A(I_e)$, as this estimate only takes into account area covered, but does not take into account that the higher the I_e, the more the damages close to the epicentrum. This leads to the loss distribution

$$h_L(l) \sim l^{-\alpha}, \qquad \text{where } \alpha \in [0.3, 0.76].$$[18]

The other extreme assumption is that of one-dimensional population density, i.e., that people only live along a one-dimensional coastline. In this case, it is natural to assume that the economic loss L is *at least* proportional to $\sqrt{A(I_e)}$. This leads to the loss distribution

$$h_L(l) \sim l^{-\alpha}, \qquad \text{where } \alpha \in [0.6, 1.5].$$[19]

Thus, overall, the economic loss distribution follows a Pareto law that decays slower than $\alpha = 1.5$, and it may be as slow as $\alpha = 0.3$. An α equal to unity corresponds to the Cauchy case.

Clearly, these calculations are rough. However, the key point is that under standard assumptions, for many orders of magnitude, the distribution of economic loss from earthquakes follows an approximate Pareto law, with a very heavy tail, and that the tail exponent $\alpha = 1$ by no means is unreasonable.

4.2 Value of avoiding a trap

We present an analysis of the value of avoiding a nondiversification trap for California earthquake insurance. Under the assumptions, the value (measured as a certainty equivalent) of being able to avoid a nondiversification trap for residential real-estate earthquake insurance in California may be up to USD 3.0 billion per year. This measures only the direct effect. In addition, the failure of a private insurance market for earthquake risk in California imposes indirect

[18] $0.3 \approx 1.10/3.7$, $0.76 \approx 1.10/1.44$.

[19] $0.6 \approx 1.10/(3.7/2)$, $1.5 \approx 1.10/(1.44/2)$.

costs on important parts of the state's economy, including new construction, home sales, and the mortgage market. Thus, following the Northridge earthquake of 1994, the state felt it important to create the California Earthquake Authority to augment the private market; see Jaffee and Russell (2003).

We assume that the capital of a (large) representative insurance company is USD 10 billion, i.e., that the individual insurance company, not taking reinsurance into account, can cover claims up to USD 10 billion. The total market capacity is USD 400 billion. This is the potential liability available, if the market chooses to provide full liability exposure.[20] It corresponds to 40 companies of equal size. We also assume that the government steps in to cover any additional losses of an earthquake of magnitude above $M = 8.3$, because the losses created by an event of that size or larger would bankrupt the industry.

The convex capital cost for insurance companies is calibrated to fit the assumption that insurance companies will charge a premium of USD 50 million to take on a Bernoulli-type risk that with 50% chance pays +USD 1 billion and with 50% chance costs USD 1 billion (i.e., pays −USD 1 billion). Equivalently, the company would charge a premium of USD 1.05 billion to insure a 50% risk of a catastrophe that, if it occurs, leads to claims of USD 2 billion.[21]

According to US census data, there are 11.5 million households in California, of which 57% are homeowners.[22] For these households, we assume a reconstruction cost of USD 150,000, that the total wealth of a representative homeowner household is USD 200,000 (i.e., USD 50,000 in addition to the value of the house), and that households have CRRA utility with risk-aversion parameter $\gamma = 3$.

We focus on large earthquakes, with $M \geq 6.7$. In line with our analysis in the previous section, we assume that the distribution of earthquakes with $M \geq 6.7$ is Bernoulli-Cauchy distributed. The annual probability for an earthquake in California of magnitude $M \geq 6.7$ is assumed to be 6.3%.[23] The total damage caused by such an event is assumed to be USD 25 billion, of which 50% (USD 12.5 billion) falls upon residential homeowners.[24] In our simplified

[20] Cummins, Doherty, and Lo (2002) assess the market capacity to be USD 350 billion. As noted earlier, the actual capital of the US property and casualty industry at year-end 2005 is estimated to be USD 446 billion.

[21] Another calibration is for a 1% risk of a USD 1 billion event. The premium to insure such a risk under our assumptions would be about USD 11 million, i.e., an additional USD 1 million above the risk-neutral premium. Studies of risk premiums for catastrophe bonds estimate the premium charged by the market to be USD 3 million for such risks (Froot, 2001). Under our assumptions, one-third of this premium would stem from convex capital costs and the other two thirds from other factors, e.g., transaction costs, asymmetric information, and parameter uncertainty.

[22] US Census Bureau, Census 2000. In addition to the 6.5 million owner-occupied housing units, there are 5 million renter-occupied units, and 700,000 vacant units. Our analysis does not cover insurance demand for these units and may thereby underestimate the total value.

[23] According to USGS report 03-0214, "Earthquake probabilities in the San Francisco Bay region: 2002–2031," there is 62% risk for an earthquake with $M \geq 6.7$ in the Bay Area between 2002 and 2031. This corresponds to an average annual probability of 3.2%. Assuming an independent similar total risk in urban areas of the rest of California, including Los Angeles, leads to the total annual probability of 6.3%.

[24] The documented direct losses of the 1994 Northridge earthquake of magnitude $M = 6.7$ were USD 24 billion (Eguchi et al., 1998).

analysis, if an earthquake occurs, a house either suffers no damage, or collapses and realizes a full loss of USD 150,000.[25] The number of houses damaged by an $M = 6.7$ earthquake is therefore 83,000 (USD 12.5 billion/USD 150,000). The maximum residential damage in the model is for an $M = 8.3$ event, under which the total damage is USD 500 billion. In the model, such an event occurs on average once in 630 years.

Under these assumptions, there is an unconditional risk of 0.2% per household per year that the household will have an insured loss. The households' CRRA risk attitudes imply that each household is willing to pay an annual insurance premium of up to USD 2300 to insure this risk. The total willingness to pay is thus about USD 15 billion per year. However, no single insurance company is willing to offer insurance at this premium. This is the nondiversification equilibrium. On the contrary, in a full risk-sharing diversification equilibrium, insurance companies would be willing to offer insurance for an annual premium as low as USD 1840 per year. This gap of USD 460 per year $(2300 - 1840)$ represents the highest amount a household would be willing to pay to avoid the trap. The total gap size is therefore USD 3.0 billion per year $(460 \times 57\% \times 11.5$ million). This is the potential annual value of avoiding a trap for homeowner residential earthuake insurance in California. If a risk-sharing full diversification equilibrium occurs, the market provides full insurance against a California earthquake of magnitude $M = 8.3$. However, a single insurance company will be able to fully cover an event only up to a magnitude of $M = 7.0$.

We have assumed Cauchy distributed tails $(\alpha = 1)$. As mentioned earlier, closed-form solutions are not available for $\alpha \neq 1$. However, it is possible to estimate numerically the conditions under which nondiversification traps arise for other values of α. In the example just discussed, the nondiversification trap arises for $0.76 \leq \alpha \leq 1.46$. If $\alpha > 1.46$, an individual company is willing to offer insurance at a price households are willing to accept. If $\alpha < 0.76$, no diversification equilibrium exists. The latter case corresponds to a globally uninsurable risk.

4.3 Other natural disasters

We have focused on one specific type of catastrophe—earthquakes. A full calibration to each type of catastrophe is outside the scope of this paper. However, the crucial property for our theory is the heavy-tailed distributions, especially Pareto laws. These are generic for natural disasters. As discussed in Woo (1999), heavy-tailedness is intimately connected with the self-similarity of the physical processes underlying natural disasters. For example, the energy distribution released in earthquakes satisfies a Pareto law with an exponent between $\alpha \in (0.8, 1.2)$ (Sornette, Knopoff, Kagan, and Vanneste, 1996). Similarly, the

[25] We have also done calculations where partial damage can occur, with qualitatively similar results to the ones presented here.

energy distribution of extraterrestrial impacts (meteorites and asteroids) satisfies a Pareto law with exponent $\alpha \approx 0.86$, the size distribution of landslides has been estimated to have an exponent of $\alpha \in (1.2, 1.4)$, whereas the area covered by river floods scales with the exponent $\alpha \approx 0.43$ (Woo, 1999). For hurricanes in Florida, estimates of the tail of the loss distribution of $\alpha \approx 1.56$ (Hsieh, 1999) and $\alpha \approx 2.49$ (Hogg and Klugman, 1983) have been made.

In each of these cases, a relationship between the heavy-tailed variable X and economic loss L needs to be established, just like when going from moment magnitude to economic loss for earthquakes in Section 4.1. However, as long as this relationship is of power-type $L(X) \sim X^{\beta}$, the loss distribution will also satisfy a Pareto law, with exponent $\hat{\alpha} = \alpha/\beta$, so our theory can be applied.

4.4 The role of a central agency

As the private catastrophe insurance markets for earthquakes, wind damage, floods, and terrorism have failed, one after the other over the past 15 years, in both the United States and Europe, governments have been forced to intervene. The plans were often created under time pressure and they differ substantially in their details; see OECD (2005, August) and OECD (2005, July) for descriptions of both the European and US plans. So it is intriguing to find that they actually share a fundamental design feature, namely, that each government plan has, in effect, created a mechanism through which a coordinated diversification equilibrium is established.

For example, in the United States following the terrorist attack of September 11, 2001, the US Congress passed the Terrorism Risk Insurance Act (TRIA), which requires all US insurance firms to offer terrorism coverage as a rider to their standard coverage for commercial buildings. The *quid pro quo* is that the government provides reinsurance for the highest layer of risk, although, as shown in Carroll, LaTourrette, Chow, Jones, and Martin (2005), the actual subsidy is very small. Thus, the primary force of TRIA is that it requires a coordinated equilibrium in which all insurers must offer terrorism coverage. The federal government also directly provides most US flood insurance, which, of course, automatically diversifies the risk across all US taxpayers. At the state level, Florida and California have required private firms to continue to cover hurricane and earthquake risks, respectively, while the states support some of the reinsurance. Finally, most European countries have created national catastrophe programs, covering both natural disasters and terrorism, that generally require that catastrophe coverage be offered to all customers, while the government provides a reinsurance facility; see OECD (2005, August) and OECD (2005, July).

Thus, quite systematically, government interventions to support catastrophe insurance markets in both the United States and Europe have, in effect, created coordinated diversification equilibria. This supports a basic conclusion of our paper, that government support to help reach a coordinated diversification equilibrium may play an important role in maintaining functioning markets for catastrophe insurance.

5. Concluding Remarks

Catastrophic risks seem ideal for insurance markets with large aggregate capacity: They are basically independent over types and geography, and there are few, if any, informational asymmetries to hinder well-functioning markets for pooled risks. The limited existence of markets for catastrophe insurance is therefore quite puzzling.

We offer an explanation to this puzzle based on one unique property of catastrophic risks: the non-negligible probability for extremely negative outcomes, i.e., the heavy left tails. The value of diversification decreases drastically when distributions are heavy-tailed. In some cases, it vanishes completely or can even be negative. The heavier the tails, the less we can, therefore, rely on standard mean-variance analysis or normal distributions in our analysis.

In a model of a reinsurance market, we have shown that if insured risks have heavy left tails, there can be nondiversification traps. Nondiversification traps occur when the value of diversification is U-shaped in the number of risks—starting out negative, but eventually becoming positive. In such situations, the value of diversification may be negative on the scale of the individual insurance company, but positive on a market scale.

The welfare loss of a diversification trap may be high. For example, we estimate the direct welfare loss of a trap in California residential earthquake insurance to be up to USD 3.0 billion per year. The effect may be especially severe in fragmented markets with large aggregate risk-bearing capacity. In such markets, diversification must be coordinated by a large number of insurance providers. This could motivate a role for a central agency in coordinating and ensuring that diversification is reached. We make several observations supporting the conclusion that governments are already taking such actions.

Appendix

Results in Section 2.2

With limited liability ($k < \infty$) and the power utility function $u(x) = (x + k)^\alpha$, $\alpha \in (0, 1)$, the expected utility will be

$$\begin{aligned} U_{j,s} &= E\left(\frac{\sum_{i=1}^{j} \tilde{X}_i}{s} + k\right)_+^\alpha \\ &= E\left(\frac{\mu j + (\nu - \mu)\sum_{i=1}^{j} \epsilon_i + \sigma \sum_{i=1}^{j} \epsilon_i Y_i}{M} + k\right)_+^\alpha \\ &= \sum_{n=1}^{j} \binom{j}{n} q^n (1-q)^{j-n} W_{j,n,s} + \left(\frac{j\mu}{s} + k\right)^\alpha, \end{aligned} \tag{8}$$

where

$$W_{j,n,s} = \frac{1}{\pi}\int_{-r/b}^{\infty} \frac{(bx+r)^{\alpha}}{1+x^2}dx, \qquad b = \frac{n\sigma}{s}, \qquad r = k + \frac{(j-n)\mu + n\nu}{s}.$$

The closed-form solution for the integral is

$$W_{j,n,s} = \frac{b^{\alpha}(1+r/b)^{\alpha/2}}{\sin(\pi\alpha)} \times \sin\left(\frac{\pi\alpha}{2} + \alpha\arctan(br)\right).$$

Figure 2 has been plotted using Equation (8).

Results in Section 2.3

The M-person participation subgame. $M \geq 1$ agents decide sequentially whether to participate or not. Previous decisions are observable. The outcome is represented by $q = (q_1, \ldots, q_M) \in \{0, 1\}^M$, where $q_m = 1$ indicates participation. The payoff to not participating (i.e., choosing $q_m = 0$) is $U_{n_m,1}$. The payoff to participating is $U_{R,t}$, where t denotes the number of participating agents, $t = \sum_m q_m$, and $R = \sum_m q_m n_m$. Here n_m is the number of risks chosen by agent m in the first stage of the diversification game. To ensure uniqueness, we assume that if agents are indifferent between participating and not participating ($U_{n_m,1} = U_{R,t}$), then they do not participate. We call this the "laziness" assumptions. The game is shown in Figure 4 for an example with three agents ($M = 3$).

Zermelo's theorem immediately implies that for each realization of the first stage of the diversification game, $p = (p_1, \ldots, p_m)$, there is a unique subgame perfect equilibrium satisfying the laziness assumption, to the participation game, $q \in \{0, 1\}^M$. We can therefore define the equilibrium mapping $\mathcal{E}\colon \mathcal{P}^M \to \{0, 1\}^M$, with $q = \mathcal{E}(p)$. Without loss of generality, we can assume that in the first stage of the diversification game, agents base their strategies on this equilibrium mapping. This reduces the strategy space significantly.

Results in Section 3

Proof of Proposition 1. As Condition 2 implies Condition 1, it is clearly not optimal for any agent who does not participate in the reinsurance market to offer insurance. Moreover, if agent m believes that no other agent will offer nontrivial risks into the pooled market, Condition 2 implies that it is optimal for agent m to not offer nontrivial risk, as any risk sharing with up to N risks is inferior to not taking on risk. Thus, it is an equilibrium for no one to offer insurance. ■

Proof of Proposition 2. If $U_{n,s} > U_{0,1}$ for $n \geq 1$ and $s = 1$, then clearly any agent will strictly improve by taking on n risks. For $U_{n,s} > U_{0,1}$ for some $s > 1$, the proof is a direct consequence of the equilibrium structure of the participation game. For example, agent 1 strictly improves by pooling n risks into the reinsurance market, as agent $2, \ldots, s^*$ will then choose to participate in the participation game for some $s^* = \arg\max_t\{t : U_{n,t} > U_{0,1}\}$, whereas agents $s^* + 1, \ldots, M$ will choose not to participate. This leads to a strict improvement for all agents $1, \ldots, s^*$. Thus, agent 1 will deviate from the assumed nondiversification strategy, so nondiversification cannot be an equilibrium. ■

Proof of Proposition 3. Under Condition 3, if agent m believes that all other agents will participate in the reinsurance market, by choosing N risks and participating, $U_{NM,M}$ can be achieved. This clearly dominates any alternative strategy of not participating in the market, which will lead to $U_{n,1}$ for $1 \leq n \leq M$, or of participating and offering fewer risks, which will lead to $U_{n,M}$, for $N(M-1) \leq n \leq NM - 1$. All alternative strategies are thus strictly dominated by the strategy leading to $U_{NM,M}$.

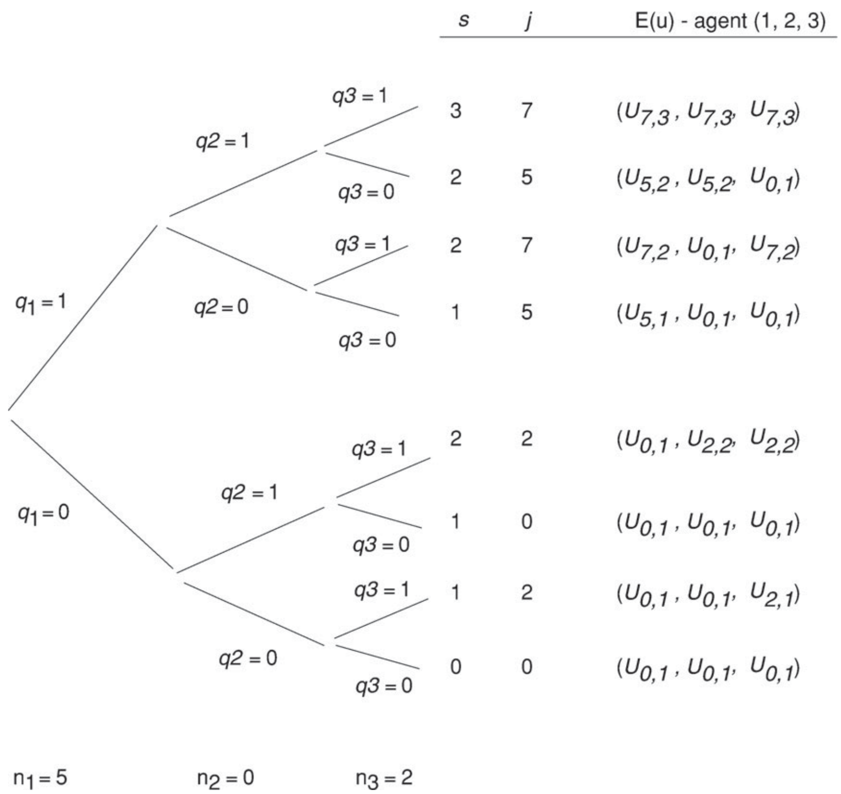

Figure 4
Example of participation subgame with $M = 3$ agents
In the first stage, agents 1, 2, and 3 have chosen to insure $n_1 = 5$, $n_2 = 0$, and $n_3 = 2$ risks, respectively.

■

Proof of Proposition 4.

(i) *Existence of nondiversification trap:* Follows by checking that the example with parameters $N = 20$, $M = 5$, $\tilde{X} \in \tilde{S}^q_{\mu,\nu,\sigma}$, with $\mu = 1$, $\nu = -9$, $\sigma = 1$ and $q = 0.05$, $k = 100$, $u(x) = (x + k)^\alpha$, with $\alpha = 0.0315$, satisfies Conditions 2 and 3 simultaneously.

(ii) *The trap is genuine:*

(ii) (a) We first show that $U_{j,s} < U_{0,1}$ for all $j \in \{1, \ldots, N\}$ and all $s \in \{1, 2, \ldots\}$. This is sufficient for the nondiversification part of the equilibrium to be satisfied. We do this by studying

$$F(j, y) \stackrel{\text{def}}{=} \sum_{n=1}^{j} \binom{j}{n} q^n (1 - q)^{j-n} W_{j,n,1/y} + (j\mu y + k)^\alpha ,$$

for the parameter values $q = 0.05$, $\mu = 1$, $\nu = -9$, $\alpha = 0.0315$, $\sigma = 1$, $k = 100$. By verifying that $F(j, y) < 0$ for $y \in (0, 1]$, for $j = 1, 2, \ldots, 20$, this implies that $U_{j,s} < U_{0,1}$ for all $s \in \{1, 2, \ldots\}$ and each feasible j. In Figure 5, $F(j, y)$ as a function of y is shown for $j = 1, 2, 5, 10, 20$. The lowest curve is for $j = 1$, and the highest for $j = 20$. We have also verified the condition for all other feasible j (not shown in the figure). In fact, F is an increasing function of j for fixed y, so $F(20, y)$ (the highest curve) being strictly negative for $y \in (0, 1]$ is sufficient for the nondiversification part of the equilibrium to be satisfied.

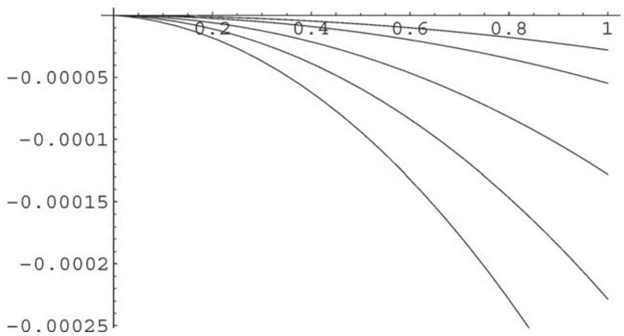

Figure 5
$F(j, y) < 0$ for all $0 < y \leq 1$ and for all $j \in 1, 2, \ldots, 20$.

(b) For the risk-sharing diversification part, we need to show that for some M_0, for all $M \geq M_0$: $U_{NM,M} > U_{0,1}$ and $U_{N(M-1),M} < U_{N(M-1)+1,M} < \cdots < U_{NM-1,M}$. As $U_{0,1} > U_{j,1}$ for $j = 1, \ldots, 20$ (see Figure 2). This is sufficient for the risk-sharing diversification part to hold.

Let us consider the asymptotics of $U_{tM,M}$, where t is a fixed natural number, as $M \to \infty$. In what follows, $\to_P$ denotes convergence in probability and $\to_d$ denotes convergence in distribution. According to (8), we have

$$U_{tM,M} = E\left(\frac{\sum_{j=1}^{tM} \tilde{X}_j}{M} + k\right)_+^{\alpha}$$
$$= E\left(\mu t + (\nu - \mu)\frac{\sum_{j=1}^{tM} \epsilon_j}{M} + \sigma\frac{\sum_{j=1}^{tM} \epsilon_j Y_j}{M} + K\right)_+^{\alpha},$$

where ϵ_j are i.i.d. nonnegative Bernoulli r.v.'s with $P(\epsilon_j = 0) = 1 - q,\ P(\epsilon_j = 1) = q$, and $Y_j \in S_{0,1}$ are i.i.d. symmetric Cauchy r.v.'s with scale parameter $\sigma = 1$ that are independent of ϵ_j's. By the law of large numbers,

$$\frac{\sum_{j=1}^{tM} \epsilon_j}{M} \to_P tE\epsilon_1 = tq \tag{9}$$

as $M \to \infty$.

Because the characteristic function of a symmetric Cauchy r.v. $X \in S_{0,\sigma}$ is given by

$$E\exp(iyX) = \exp(-\sigma|y|), \tag{10}$$

we obtain that the characteristic function $f(y) = E\exp(iyW_M)$ of $W_M = \frac{\sum_{j=1}^{tM} \epsilon_j Y_j}{M}$ satisfies

$$f(y) = \left[E\exp\left(\frac{iy\epsilon_1 Y_1}{M}\right)\right]^{tM} = \left[(1-q) + qE\exp\left(\frac{iyX_1}{M}\right)\right]^{tM}$$
$$= \left[1 + q\left(\exp\left(-\frac{|y|}{M}\right) - 1\right)\right]^{tM} \to \exp(-qt|y|)\ \ \forall y \in \mathbf{R} \tag{11}$$

as $M \to \infty$. Because, according to Equation (10), $\exp(-tq|y|)$ is the characteristic function of the r.v. tqY_1, we conclude from Equation (11) that

$$W_M \to_d tqY_1 \tag{12}$$

as $M \to \infty$. Relations (9) and (12) imply that, as $M \to \infty$,

$$V_M = \left(\mu t + (\nu - \mu)\frac{\sum_{j=1}^{tM} \epsilon_j}{M} + \sigma \frac{\sum_{j=1}^{tM} \epsilon_j Y_j}{M} + K\right)_+^{\alpha}$$
$$\to_d (\mu t + (\nu - \mu)tq + \sigma t q Y_1 + K)_+^{\alpha}. \tag{13}$$

Because, as is not difficult to see, the sequence of r.v.'s V_M is uniformly integrable, from Equation (13) we get that (see, e.g., Ash, 2000, p. 336)

$$U_{tM,M} = E\left(\mu t + (\nu - \mu)\frac{\sum_{j=1}^{tM} \epsilon_j}{M} + \sigma \frac{\sum_{j=1}^{tM} \epsilon_j Y_j}{M} + k\right)_+^{\alpha}$$

converges to

$$\begin{aligned} G(t) &\stackrel{\text{def}}{=} E(\mu t + (\nu - \mu)tq + \sigma t q Y_1 + K)_+^{\alpha} \\ &= \left(t^2 q^2 \sigma^2 + (k + t(\mu - q\mu + q\nu))^2\right)^{\alpha/2} \\ &\quad \times \csc(\pi\alpha) \sin\left(\frac{1}{2}\alpha\left(\pi + 2\arctan\left(\frac{k + t(\mu - q\mu + q\nu)}{qt\sigma}\right)\right)\right) \end{aligned}$$

as $M \to \infty$. In particular,

$$\lim_{M\to\infty} U_{nM,M} = G(n),\ n = 1, \ldots, N.$$

It is straightforward to check that $G(N) > U_{0,1}$.

Let us now show that $U_{N(M-1),M} < U_{N(M-1)+1,M} < \cdots < U_{NM-1,M}$. Let $0 \le m \le M-2$. Note that, for $Y \in S_{0,\sigma}$ and $\alpha \in (0,1)$, $z \in \mathbf{R}$, the expectation $E(Y+z)_+^{\alpha-1}$ is finite and well defined because the integral $\int_{-z}^{\infty} \frac{dx}{\pi\sigma(x+z)^{1-\alpha}(1+x^2)}$ converges. This, by induction and conditioning arguments, implies that the expectation $E\left(\frac{\sum_{j=1}^{N(M-1)+m} X_j}{M} + k\right)_+^{\alpha-1}$ is finite and positive:

$$0 < E\left(\frac{\sum_{j=1}^{N(M-1)+m} X_j}{M} + k\right)_+^{\alpha-1} < \infty. \tag{14}$$

Using a Taylor expansion, it is not difficult to check that the following inequality holds for all $x, z \in \mathbf{R}$ and all $\alpha \in (0,1)$:

$$(x+z)_+^{\alpha} \ge z_+^{\alpha} + \alpha x z_+^{\alpha-1}, \tag{15}$$

with strict inequality for $x, z > 0$. Let $X_1' = \mu + (\nu - \mu)\epsilon_1' + \sigma\epsilon_1' Y_1'$ denote an r.v. with Bernoulli-Cauchy distribution (4), where ϵ_1' is a nonnegative Bernoulli r.v. with $P(\epsilon_i = 0) = 1 - q$, $P(\epsilon_i = 1) = q$ and $Y_1' \in S_{0,1}$ is a symmetric Cauchy r.v. with scale parameter $\sigma = 1$ independent of ϵ_1'. Suppose further that ϵ_1' and Y_1' are independent of the r.v.'s X_j, $j = 1, \cdots, N(M-1)+1$. Using

Equations (14) and (15) we get that, for all $s > 0$,

$$
\begin{aligned}
&E\left(\frac{\sum_{j=1}^{N(M-1)+m} X_j + X_1'}{M} + k\right)_+^{\alpha} I(|Y_1'| < sM) \\
&\quad > E\left(\frac{\sum_{j=1}^{N(M-1)+m} X_j}{M} + k\right)_+^{\alpha} P(|Y_1'| < sM) \\
&\quad + (\alpha/M)E[X_1' I(|Y_1'| < sM)]E\left(\frac{\sum_{j=1}^{N(M-1)+m} X_j}{M} + k\right)_+^{\alpha-1}, \qquad (16)
\end{aligned}
$$

where $I(\cdot)$ is the indicator function.

We further have, by the definition of X_1', ϵ_1', and Y_1' and using the symmetry of the distribution of Y_1',

$$
\begin{aligned}
E[X_1' I(|Y_1'| < sM)] &= (\mu + (\nu - \mu)q)P(|Y_1'| < sM) \\
&\quad + \sigma q E[Y_1' I(|Y_1'| < sM)] \\
&= (\mu + (\nu - \mu)q)P(|Y_1'| < sM) \quad \forall s > 0. \qquad (17)
\end{aligned}
$$

The inequalities (16) and (17) imply that, for all $s > 0$,

$$
\begin{aligned}
&E\left(\frac{\sum_{j=1}^{N(M-1)+m} X_j + X_1'}{M} + k\right)_+^{\alpha} I(|Y_1'| < sM) \\
&\quad > E\left(\frac{\sum_{j=1}^{N(M-1)+m} X_j}{M} + k\right)_+^{\alpha} P(|Y_1'| < sM) \\
&\quad + \frac{\alpha(\mu + (\nu - \mu)q)}{M} P(|Y_1'| < sM). \qquad (18)
\end{aligned}
$$

Letting $s \to \infty$, we obtain

$$
\begin{aligned}
U_{N(M-1)+m+1} &= E\left(\frac{\sum_{j=1}^{N(M-1)+m} X_j + X_1'}{M} + k\right)_+^{\alpha} \\
&\geq E\left(\frac{\sum_{j=1}^{N(M-1)+m} X_j}{M} + k\right)_+^{\alpha} + \frac{\alpha(\mu + (\nu - \mu)q)}{M}. \qquad (19)
\end{aligned}
$$

It is easy to check that, for the values of the parameters chosen in the proposition, $\mu + (\nu - \mu)q = 0.5 > 0$. From (19) we thus conclude that

$$
U_{N(M-1)+m+1} > E\left(\frac{\sum_{j=1}^{N(M-1)+m} X_j}{M} + k\right)_+^{\alpha} = U_{N(M-1)+m}
$$

for all $0 \leq m \leq M - 2$.

Thus, the conditions for a risk-sharing diversification equilibrium are satisfied for all $M \geq M_0$. Together, (i) and (ii) therefore imply that the nondiversification trap is genuine. ■

Proof of Proposition 5.

We first prove the special case, with normal distributions ($X_i \sim \text{Normal}(\mu, \sigma^2)$) and CARA utility ($u(x) = -\exp(-\theta x)$), which is well known to lead to mean-variance optimizing agent behavior.

(i) *No nondiversification trap:* Clearly, for there to be a risk-sharing diversification equilibrium, we must have $U_{NM,M} > U_{0,1}$. However, the strict monotonicity of $U_{j,M}$ in j then implies that $U_{1,M} > U_{0,M} = U_{0,1}$. By Proposition 2, this contradicts there being a nondiversification equilibrium, which would require that $U_{1,M} \le U_{0,1}$. Thus, a nondiversification trap cannot arise.

(ii) *Type of equilibria:* The properties and uniqueness of equilibrium depend on $\lambda_m \stackrel{\text{def}}{=} 2\mu - \theta\sigma^2/m$. Clearly, λ_m is strictly increasing in m. If $\lambda_M < 0$, then $U_{j,s} < U_{0,1}$ for all $j > 0$, for all $1 \le s \le M$. Therefore, nondiversification ($\overline{N} = 0$) is the unique equilibrium. If $\lambda_M = 0$, then a similar argument, together with the laziness assumption, ensures that there is a unique nondiversification equilibrium.

If $\lambda_M > 0$, the conditions of Assumption 3 are satisfied, so there is a risk-sharing diversification equilibrium. Clearly, there cannot be a nondiversification equilibrium in this case, as $U_{1,M} > U_{0,1}$, so Condition 2 fails strictly. We study other candidates for equilibria. Let $s^* = \arg\min_m\{m : \lambda_m > 0\}$:

(i) *Equilibria with no pooling:* If $s^* > 1$, there can clearly be no equilibrium without pooling, as $U_{n,1} < U_{0,1}$ for $1 \le n \le N$ in this case, so this would have to be the nondiversification equilibrium, which is not an equilibrium according to the previous argument. Moreover, if $s^* = 1$, then $U_{n,1}$ is strictly increasing in n, so the only potential equilibrium without pooling is the one in which all agents choose $n_m = N$, $q_m = 0$. However, as λ_m is strictly increasing in m, $U_{NM,M} > U_{j,s}$ for all other feasible $U_{j,s}$, so this is dominating for all agents and it must be that $q = 1^M = \mathcal{E}(p)$. This is a pooling equilibrium, which contradicts the assumption of no pooling. Thus, there can be no equilibrium without pooling of risk.

(ii) *Equilibria with pooling:* Assume that there is an additional equilibrium, with j risks pooled between s agents. For this to be the case, it must be that $s \ge s^*$.

(a) $s^* > 1$. Clearly, agents who do not participate (i.e., have $q_m = 0$) will not take on own risk, so their expected utility of not participating is $U_{0,1}$. However, if $U_{j,s} > U_{0,1}$, then, as $\lambda_{s+1} > \lambda_s > 0$, we have $U_{j,s+1} > U_{0,1}$, so any nonparticipating agent is better off by participating in the reinsurance market. Thus, any equilibrium will have $s = M$. Now, assume that $j < NM$: A similar argument shows that any agent choosing $n_m < N$ would have increased expected utility by choosing $n_m = N$. Thus, the only equilibrium is the risk-sharing diversification equilibrium.

(iii) (b) $s^* = 1$. In this case, agents who do not participate (i.e., have $q_m = 0$) will take on full risk, i.e., their expected utility will be $U_{N,1}$. A similar argument implies that any agent who participates will choose $n_m = N$. Thus, participating agents will have expected utility $U_{Ns,s}$. However, as $\lambda_{s+1} > \lambda_1$, nonparticipating agents will increase their expected utility by deviating and reaching $U_{N(s+1),s+1}$ instead of $U_{N,1}$. Thus, it must be that all agents participate, and once again the only equilibrium can be $j = NM$, $s = M$.

The general case is identical to the proof in case of CARA investors with normal utility. As $U_{j,s} = j/s(\mu - \gamma/s\sigma^2)$, which is monotone in j for each s, and $\lambda_s \stackrel{\text{def}}{=} \mu - \gamma\sigma^2/s$ is monotonously increasing in s, so all the steps of the proof go through. ■

Proof of Proposition 6. (i) Assume that there exists a nondiversification trap to the game (u, F, ∞, N, M) for arbitrarily large M, for a strictly concave, twice-continuously differentiable utility function u and distribution F, satisfying

$$\int_{-\infty}^{\infty} x^2 dF = C < \infty. \tag{20}$$

For notational convenience, we assume that F is differentiable, so that the p.d.f. $\phi = dF/dx$ is well defined. However, the whole proof goes through step by step without this restriction. Without loss of generality, we can assume $u(0) = 0$, $u'(0) = 1$. For a genuine nondiversification trap to exist, it must be the case that for arbitrarily large M,

$$U_{NM,M} > 0, \tag{21}$$

and

$$U_{1,M} < 0. \tag{22}$$

However, a necessary condition for (22) to hold for arbitrarily large M is that $EX \leq 0$, as seen by the following argument: Assume that $\mu = EX > 0$. We define

$$U(\epsilon) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} u(\epsilon x)\phi(x)dx.$$

We decompose

$$u(x) \stackrel{\text{def}}{=} x - t(x) \stackrel{\text{def}}{=} x - x^2 z(x),$$

where $t(0) = t'(0) = 0$, $t'' > 0$, $t(x) < x$, $z(x)$ is continuous, and both t and z are nonnegative. We then have

$$U(\epsilon) = \epsilon\mu - \left(\int_{-\infty}^{-1/\epsilon} t(\epsilon x)\phi(x)dx + \int_{-1/\epsilon}^{1/\epsilon} (\epsilon x)^2 z(\epsilon x)\phi(x)dx \right. \\ \left. + \int_{1/\epsilon}^{\infty} t(\epsilon x)\phi(x)dx\right). \tag{23}$$

The $\int_{1/\epsilon}^{\infty}$-term is clearly of $o(\epsilon)$,[26] as

$$\int_{1/\epsilon}^{\infty} t(\epsilon x)\phi(x)dx \leq \int_{1/\epsilon}^{\infty} \epsilon x\phi(x)dx \leq \epsilon \int_{1/\epsilon}^{\infty} x\phi(x)dx \leq C_2\epsilon^2.$$

Furthermore, as $z(x)$ is continuous, it is bounded on $[-1, 1]$, so Hölder's inequality can be used to bound the $\int_{-1/\epsilon}^{1/\epsilon}$ term by

$$\int_{-1/\epsilon}^{1/\epsilon} (\epsilon x)^2 z(\epsilon x)\phi(x)dx \leq \epsilon^2 \max_{-1\leq y\leq 1} |z(x)| \times C = C_3\epsilon^2,$$

so the second term is also of $o(\epsilon)$. Finally, the $\int_{-\infty}^{-1/\epsilon}$-term is also $o(\epsilon)$, as

$$\int_{-\infty}^{-1/\epsilon} t(\epsilon x)\phi(x)dx = \int_{-\infty}^{-1/\epsilon} \frac{t(\epsilon x)}{t(x)} t(x)\phi(x)dx \leq \epsilon t'(-1) \\ \times \int_{-\infty}^{-1/\epsilon} t(x)\phi(x)dx = o(\epsilon),$$

where we use Hölder's inequality to move the $t(\epsilon x)/t(x)$ outside of the integral, and the inequality

$$\frac{t(\epsilon x)}{t(x)} \leq \epsilon t'(-1),$$

which must hold for $x \geq 1/\epsilon$, as t is convex. Finally,

$$\int_{-\infty}^{-1/\epsilon} t(x)\phi(x)dx = o(1),$$

[26] The term $f(\epsilon) = o(\epsilon)$, denoting that $\lim_{\epsilon \searrow 0} f(\epsilon)/\epsilon = 0$.

as the integral $Eu(X)$ could otherwise not exist. This altogether implies that $U(\epsilon) = \epsilon\mu - o(\epsilon)$, which is strictly positive for small enough ϵ. Therefore, if $EX > 0$, then $U_{1,M}$ will be strictly positive for large enough M, and no genuine nondiversification trap can therefore exist.

However, if $EX \leq 0$, then a nondiversification trap cannot exist, as Jensen's inequality implies that $U_{NM,M}$ is strictly negative for arbitrary $M > 0$ and $N > 0$ and thus $U_{NM,M} < U_{0,1}$. ■

(ii) Assume that there exists a nondiversification trap to the game (u, ϕ, k, N, M) for arbitrarily large M, for a strictly concave, twice-continuously differentiable utility function u and distribution ϕ, satisfying

$$\int_{-\infty}^{\infty} x^2 \phi(x) dx = C < \infty. \tag{24}$$

Without loss of generality, we can assume $u(0) = 0$, $u'(0) = 1$.

If $EX > 0$, then the same argument as in the proof of Proposition 6 rules out a genuine nondiversification trap, as the limited liability *increases* $U_{1,M}$ compared to the unlimited liability case. Thus, for M large enough, $U_{1,M}$ must be strictly positive and a genuine nondiversification trap cannot exist.

If $EX = \mu < 0$, then we use the law of large numbers to show that as M becomes large, $X_{NM} = (NM)^{-1} \sum_{i=1}^{NM} X_i$ converges in distribution to μ. Thus, $\lim_{M\to\infty} E((X_{NM} + kNM)_+ - kNM) = \mu$, so for some large enough M_0, $E((X_{NM} + kNM)_+ - kNM) < 0$ for all $M \geq M_0$. Jensen's inequality therefore again implies that $U_{NM,M}$ is strictly negative for $M \geq M_0$ and thus that $U_{NM,M} < U_{0,1}$, so there can be no genuine nondiversification trap. ■

(iii) Without loss of generality, we can assume $u(0) = 0$, $u'(0) = 1$.

We prove that $U_{NM,M} < U_{0,1} = 0$ for large M. We define

$$\gamma = \min_{x\in[-k/2,k/2]} u''(x).$$

As u is strictly concave and twice continuously differentiable, $\gamma > 0$. We define

$$\tilde{u}(x) = x - \frac{\gamma}{2} x^2 I_{[-k/2,k/2]},$$

where I_A is the indicator function on the set A, implying that $u(x) \leq \tilde{u}(x)$ for all $x \in [-k, \infty)$. Similar to $U_{j,s}$, we define $\tilde{U}_{j,s}$, the "utility" of sharing j risks equally among s agents, for agents with "utility" functions $\tilde{u}$. Clearly, $U_{j,s} \leq \tilde{U}_{j,s}$, so if $\tilde{U}_{NM,M} < 0$ for large M, then $U_{j,s} < 0$ for large M and there cannot be a genuine nondiversification trap.

We next define $Y_1 = \sum_{i=1}^{N} X_i$ and study uniform portfolios of i.i.d. risks $Y_1, \ldots, Y_M$ by defining $\overline{Y}_M \stackrel{\text{def}}{=} (\sum_{m=1}^{M} Y_m)/M$. As $E(\overline{Y}_M) = 0$, the condition $\tilde{U}_{NM,M} < 0$ for large M can be written as

$$\tilde{U}_{NM,M} = E(\overline{Y}_M I_{[-k,\infty)}) - \frac{\gamma}{2} E\left(\overline{Y}_M^2 I_{[-k/2,k/2]}\right) < 0. \tag{25}$$

We begin by bounding $E(\overline{Y}_M^2 I_{[-k/2,k/2]})$ from below. From the central limit theorem, we know that $Z_M \stackrel{\text{def}}{=} \sqrt{M}\,\overline{Y}_M$ converges in distribution to $Z \sim \textit{Normal}(0, \sigma^2)$, so $E(Z_M^2 I_{[-k/2,k/2]}) \to C > 0$, as M grows. As $ME(\overline{Y}_M^2 I_{[-k/2,k/2]}) \geq ME(\overline{Y}_M^2 I_{[-k/2\sqrt{M},k/2\sqrt{M}]}) = E(Z_M^2 I_{[-k/2,k/2]})$, we can therefore conclude that for large M,

$$\frac{\gamma}{2} E\left(\overline{Y}_M^2 I_{[-k/2,k/2]}\right) \geq \frac{C'}{M}, \qquad C' > 0. \tag{26}$$

We next bound $E(\overline{Y}_M I_{[-k,\infty)})$ from above. As $E(\overline{Y}_M) = 0$, we have $E(\overline{Y}_M I_{[-k,\infty)}) = -E(\overline{Y}_M I_{(-\infty,-k)})$. From the Cauchy-Schwarz inequality, we know that

$$-E\left(\overline{Y}_M I_{(-\infty,-k)}\right) \leq E\left(\overline{Y}_M^2\right)^{1/2} E\left(I_{(-\infty,-k)}\right)^{1/2},$$

(as $I^2_{(-\infty,-k)} = I_{(-\infty,-k)}$). Of course, $E(\overline{Y}_M^2) = \sigma^2/M$. Moreover, Rosenthal's inequality (see Rosenthal, 1970; Ibragimov and Sharakhmetov, 1997; de la Peña, Ibragimov, and Sharakhmetov, 2003) implies that $E\left(\overline{Y}_M^{2+\epsilon}\right) \leq C''/M^{1+\epsilon/2}$, and by Markov's inequality, we therefore know that

$$E(I_{(-\infty,-k)}) = P(x < -k) \leq \frac{E\left(\overline{Y}_M^{2+\epsilon}\right)}{k^{2+\epsilon}} \leq \frac{C'' k^{-(2+\epsilon)}}{M^{1+\epsilon/2}}.$$

Overall, this implies that

$$E\left(\overline{Y}_M I_{(-k,\infty)}\right) \leq \sqrt{\frac{\sigma^2}{M}} \sqrt{\frac{C'' k^{-2+\epsilon}}{M^{1+\epsilon/2}}} = \frac{C'''}{M^{1+\epsilon/4}}.$$

The bounds in Equation (25) are therefore

$$\tilde{U}_{NM,M} \leq \frac{C'''}{M^{1+\epsilon/4}} - \frac{C'}{M}, \qquad C' > 0,$$

which is strictly negative for large M. Thus, as $U_{NM,M} \leq \tilde{U}_{NM,M}$, we know that $U_{NM,M} \leq U_{0,1} = 0$ for large M. Therefore, there can be no genuine nondiversification trap in this case either. ■

References

Abramowitz, M., and I. A. Stegun. 1970. *Handbook of Mathematical Functions*. New York: Dover.

Ash, R. B. 2000. *Probability and Mesuare Theory*. Burlington, MA: Academic Press.

Azariadis, C., and J. J. Stachurski. 2006. Poverty Traps, in P. Aghion and S. Durlauf (eds.), *Handbook of Economic Growth*. Amsterdam: Elsevier, pp. 295–384.

Bakun, W. H., A. C. Johnston, and M. G. Hopper. 2003. Estimating Locations and Magnitudes of Earthquakes in Eastern North America from Modified Mercalli Intensities. *Bulletin of the Seismological Society of America* 93:190–202.

Barro, R. J., and X. Sala-i-Martin. 2004. *Economic Growth*. Cambridge, MA: MIT Press.

Carroll, S., T. LaTourrette, B. G. Chow, G. S. Jones, and C. Martin. 2005. Distribution of Losses from Large Terrorist Attacks Under the Terrorism Risk Insurance Act. Santa Monica, CA: Rand Corporation.

Cummins, J. D. 2006. Should the Government Provide Insurance for Catastrophes? *Federal Reserve Bank of St. Louis Review*, forthcoming.

Cummins, J. D., N. Doherty, and A. Lo. 2002. Can Insurers Pay for the Big One? Measuring the Capacity of the Insurance Market to Respond to Catastrophic Losses. *Journal of Banking and Finance* 26:557–83.

Cummins, J. D., and O. Mahul. 2003. Optimal Insurance with Divergent Beliefs About Insurer Total Default Risk. *Journal of Risk and Uncertainty* 27(2):121–38.

de la Peña, V. H., R. Ibragimov, and S. Sharakhmetov. 2003. On Extremal Distributions and Sharp L_p-Bounds for Sums of Multilinear Forms. *Annals of Probability* 31:630–75.

Eguchi, R., J. Goltz, C. Tayloe, S. Chang, P. Flores, L. Johnson, H. Seligson, and N. Blais. 1998. Direct Economic Losses in the Northridge Earthquake: A Three-Year Post-Event Perspective. *Earthquake Spectra* 14: 245–64.

Fama, E. 1965. The Behavior of Stock Market Prices. *Journal of Business* 38:34–105.

Froot, K. A. 2001. The Market for Catastrophe Risk: A Clinical Examination. *Journal of Financial Economics* 60:529–71.

Froot, K. A., and S. Posner. 2002. The Pricing of Event Risks with Parameter Uncertainty. *The Geneva Papers on Risk and Insurance Theory* 27:153–65.

Froot, K. A., D. Scharsfstein, and J. Stein. 1993. Risk Management: Coordinating Corporate Investment and Financing Decisions. *Journal of Finance* 48:1629–58.

Gabaix, X., P. Gopikrishnan, V. Plerou, and H. E. Stanley. 2003. A Theory of Power-Law Distributions in Financial Market Fluctuations. *Nature* 423:267–70.

Hanks, T., and H. Kanamori. 1979. A Moment Magnitude Scale. *Journal of Geophysical Research* 84:2348–50.

Ho, P., J. Sussman, and D. Veneziano. 2001. Estimating the Direct and Indirect Losses from a Midwest Earthquake. Report SE-10, Mid-America Earthquake Center.

Hogg, R., and S. Klugman. 1983. On the Estimation of Long-Tailed Skewed Distributions with Actuarial Applications. *Journal of Econometrics* 23:91–102.

Hsieh, P. 1999. Robustness of Tail Index Estimation. *Journal of Computational Graphical Statistics* 8:318–32.

Hubbard, G., B. Deal, and P. Hess. 2005. The Economic Effect of Federal Participation in Terrorism Risk. *Risk Management and Insurance Review* 8:177–303.

Ibragimov, R. 2004. Portfolio Diversification and Value at Risk Under Thick-Tailedness. Harvard Institute of Economic Research Discussion Paper #2086. Available at http://post.economics.harvard.edu/hier/2005papers/HIER2086.pdf.

Ibragimov, R. 2005. New Majorization Theory in Economics and Martingale Convergence Results in Econometrics. Ph.D. dissertation, Yale University.

Ibragimov, R., and S. Sharakhmetov. 1997. On an Exact Constant for the Rosenthal Inequality. *Theory of Probability and Its Applications* 42:294–302.

Ibragimov, R., and J. Walden. 2007. The Limits of Diversification When Losses May Be Large. *Journal of Banking and Finance* 31:2551–69.

Jaffee, D. 2006. Should the Government Provide Insurance for Catastrophes? Comments on a Paper by J. David Cummins. *Federal Reserve Bank of St. Louis Review*, forthcoming.

Jaffee, D., and T. Russell. 2003. Markets Under Stress: The Case of Extreme Event Insurance, in R. Arnott, B. Greenwald, R. Kanbur, and B. Nalebuff (eds.), *Economics for an Imperfect World: Essays in Honor of Joseph E. Stiglitz*. Cambridge, MA: MIT Press, pp. 35–52.

Jaffee, D., and T. Russell. 2006. Should Governments Provide Catastrophe Insurance? *Economists' Voice* 3(5). Available at http://www.bepress.com/ev/vol3/iss5/art6.

Jansen, D. W., and C. G. de Vries. 1991. On the Frequency of Large Stock Returns: Putting Booms and Busts into Perspective. *Review of Economics and Statistics* 73:18–32.

Kagan, Y., and L. Knopoff. 1984. A Stochastic Model of Earthquake Occurrence. *Proceedings of the Eighth International Conference on Earthquake Engineering* 1:295–302.

Knopoff, L., and Y. Kagan. 1977. Analysis of the Theory of Extremas as Applied to Earthquake Problems. *Journal of Geophysical Research* 82:5647–57.

Kunreuther, H., and E. Michel-Kerjan. 2006. Looking Beyond TRIA: A Clinical Examination of Potential Terrorism Loss Sharing. NBER, Working Paper No. 12069, available at http://www.nber.org/papers/w12069.

Loretan, M., and P. C. B. Phillips. 1994. Testing the Covariance Stationarity of Heavy-Tailed Time Series. *Journal of Empirical Finance* 1:211–48.

Mahul, O., and B. Wright. 2004. Efficient Risk Sharing Within a Catastrophe Insurance Pool. Paper presented at the January 21, 2003, NBER Insurance Project Workshop, available at http://www.arec.umd.edu/Department/Seminars/mahulwright0204.pdf.

Mandelbrot, B. 1963. The Variation of Certain Speculative Prices. *Journal of Business* 36:394–419.

McCulloch, J. H. 1996. Financial Applications of Stable Distributions, in G. S. Maddala and C. R. Rao (eds.), *Handbook of Statistics*. Vol. 14. Amsterdam: Elsevier, pp. 393–425.

McCulloch, J. H. 1997. Measuring Tail Thickness to Estimate the Stable Index Alpha: A Critique. *Journal of Business and Economic Statistics* 15:74–81.

McGuire, R. 2004. *Seismic Hazard and Risk Analysis*. Oakland, CA: Earthquake Engineering Research Institute.

Nešlehova, J., P. Embrechts, and V. Chavez-Demoulin. 2006. Infinite Mean Models and the LDA for Operational Risk. *Journal of Operational Risk* 1:3–25.

OECD. 2005, August. Policy Issues in Insurance No. 08: Catastrophic Risks and Insurance. OECD Publishing, Paris, France.

OECD. 2005, July. Policy Issues in Insurance No. 09: Terrorism Risk Insurance in OECD Countries. OECD Publishing, Paris, France.

Rachev, S. T., and S. Mittnik. 2000. *Stable Paretian Models in Finance*. New York: Wiley.

Rosenthal, H. P. 1970. On the Subspaces of L^p $(p > 2)$ Spanned by Sequences of Independent Random Variables. *Israel Journal of Mathematics* 8:273–303.

Samuelson, P. A. 1967. General Proof That Diversification Pays. *Journal of Financial and Quantitative Analysis* 2:1–13.

Scherer, F. M., D. Harhoff, and J. Kukies. 2000. Uncertainty and the Size Distribution of Rewards from Innovation. *Journal of Evolutionary Economics* 10:175–200.

Schwartz, D., and K. Coppersmith. 1984. Fault Behavior and Characteristic Earthquakes. *Journal of Geophysical Research* 89:5681–98.

Silverberg, G., and B. Verspagen. 2004. The Size Distribution of Innovations Revisited: An Application of Extreme Value Statistics to Citation and Value Measures of Patent Significance. Working Paper, Maastricht University.

Sornette, D., L. Knopoff, Y. Kagan, and C. Vanneste. 1996. Rank-Ordering Statistics of Extreme Events: Application to the Distribution of Large Earthquakes. *Journal of Geophysical Research* 101:13883–94.

SwissRe. 2006. Sigma. *Sigma* 2:267–70.

Woo, G. 1999. *The Mathematics of Natural Catastrophes*. London: Imperial College Press.