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Non-equilibrium spin models with Ising universal behaviour

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Abstract. The properties of a family of non-equilibrium spin models with up–down symmetry on a square lattice are determined by a mean-field pair approximation and by Monte Carlo simulation. The phase diagram in the parameter space displays a critical line that terminates at a first-order critical point. It is found that the critical exponents are the same as those of the equilibrium Ising model.

1. Introduction

It is well known that spin systems can be grouped together according to their universal critical behaviour in classes of universality and that the universal behaviour depends only on such general properties as symmetry, dimension of the system and the dimension of the order parameter. This result is valid for equilibrium and possibly also for non-equilibrium spin systems (systems whose stochastic dynamics does not obey detailed balance) in a steady state [1–3]. Several models have been studied in order to find the classes of universality for non-equilibrium spin models. A well known example is the universality class of the Reggeon field theory [4–6]. Lattice gas models [3, 7, 8] belonging to such a class, as the contact process [9, 10], have the important property of exhibiting an absorbing state. In a steady state these models are always far from equilibrium, no matter what control parameters are varied, since the existence of the absorbing state makes the dynamics irreversible.

Another universality class for non-equilibrium spin systems is the one which includes the equilibrium Ising model. According to Grinstein *et al* [11], any non-equilibrium stochastic spin system with spin-flip dynamics and up–down symmetry falls in this universality class. This result has been found to be valid for specific spin models that do not obey detailed balance [12–20]. With the aim of testing Grinstein *et al* [11] prediction and investigating new non-equilibrium models we analyse here a two-parameter family of spin models with up–down symmetry on a square lattice whose stochastic dynamics is isotropic and short ranged. The model is defined by giving the transition rates between configurations, with the spin independent variables in the transition rates playing the role of control parameters. For particular values of the parameters we recover other familiar models: the equilibrium Ising, voter and majority voter.

As usual, exact calculations are only possible under very restricted conditions, so we have used mean-field-like methods (pair-aproximation) to obtain a qualitative description of the phase diagram. In order to study the critical behaviour and address the question of universality we have complemented the mean-field calculations with Monte Carlo simulations and finite-size scaling analysis.

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2. Model

Consider a square lattice where at each site there is a spin variable $\sigma_i = \pm 1$. The configuration $\sigma = \{\sigma_i\}$ evolves in time according to a one-spin flip stochastic dynamics. The spin-flip probability $w_i(\sigma)$ is of the form [21]

$$w_i(\sigma) = \tfrac{1}{2}[1 - \sigma_i f_i(\sigma)] \tag{1}$$

where $f_i(\sigma)$ is a local function with $|f_i(\sigma)| \leqslant 1$. Here we consider $f_i(\sigma)$ such that $w_i(\sigma)$ has up–down symmetry and is spatially symmetric. That is, $f_i(\sigma) = f(\sum_\delta \sigma_{i+\delta})$, a function of the sum of the nearest-neighbour spin variables with $f(0) = 0$, $f(2) = -f(-2) = x$ and $f(4) = -f(-4) = y$, where x and y are two parameters restricted to $|x| \leqslant 1$ and $|y| \leqslant 1$. This function can also be written in the form

$$f(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4) = a(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4) + b(\sigma_1\sigma_2\sigma_3 + \sigma_1\sigma_2\sigma_4 + \sigma_1\sigma_3\sigma_4 + \sigma_2\sigma_3\sigma_4) \tag{2}$$

where the parameters a and b are related to x and y by $a = (y+2x)/8$ and $b = (y-2x)/8$.

Several known models are special cases of the family of stochastics dynamics defined by equations (1) and (2). The voter model [10] corresponds to $x = 1/2$ and $y = 1$; the isotropic majority-vote model [20] is obtained when $x = y$; and the Glauber model [22], for which $f(S) = \tanh(KS)$ where K is proportional to the inverse temperature, is given parametrically by $x = \tanh(2K)$ and $y = \tanh(4K)$, or even by $y = 2x/(1+x^2)$. With the exception of the x and y values corresponding to the Glauber model, the stochastic dynamics does not obey detailed balance for arbitrary values of x and y. Consider, for instance, a local configuration consisting of a nearest-neighbour pair of up spins, the first being surrounded by up spins and the second having one (or two) down spins as nearest neighbours. From the spin-flip probability, we find that the probability of a closed path of configurations obtained by flipping these two spins in the sequence first-second–first-second is $(1-y)(1+x)^2/16$ whereas the probability of the reversed path is $(1+y)(1-x)^2/16$. The microscopic reversibility (detailed balance) occurs, then, only when these two probabilities are equal, that is when $y = 2x/(1+x^2)$.

Figure 1 shows the phase diagram in the x–y plane. The Monte Carlo and the dynamic pair approximation show that for $x \geqslant 0$ there is a ferromagnetic critical line that crosses the majority-vote model and the Glauber model lines and terminates at the voter model point ($x = 1/2$ and $y = 1$). Along the critical line the order parameter vanishes continuously. Reaching the voter point along the line $y = 1$ (from the ferromagnetic phase) one finds a jump in the order parameter. This point is therefore identified as a first-order critical point [23]. When $b = 0$ ($y = 2x$) the model is linear and can be solved exactly. However, there is no ordering except at the voter point $x = 1/2$, $y = 1$.

Trapping states occur at $|x| = 1$ and at $|y| = 1$. Along the line $y = 1$, there are two absorbing states: all spins up and all spins down. These two states are, however, unstable for $x < 1/2$. Along the line $x = 1$, the absorbing states are vertical or horizontal double stripes of + and − spins, similar to the $\langle 22 \rangle$ phases in equilibrium ANNNI models, and are all unstable; for $x = y = 1$ the model reduces to the 2D Ising model at zero temperature.

3. Pair approximation

In the stationary state the following equations hold

$$\langle \sigma_i \rangle = \langle f_i(\sigma) \rangle \tag{3}$$

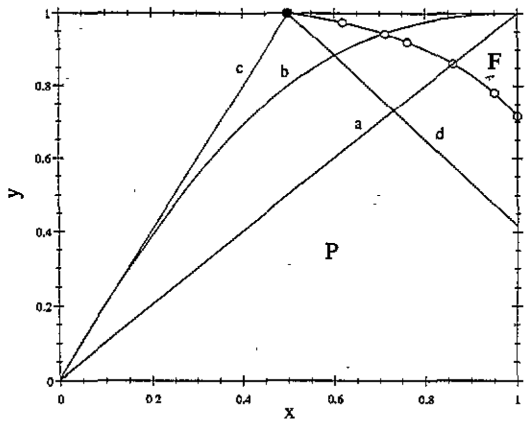

Figure 1. Phase diagram in the x–y parameter space showing the ferromagnetic (F) and the paramagnetic (P) phases. The lines starting at the origin correspond to: a, the majority vote model; b, the Glauber model; c, the linear model. The open circles correspond to the voter model. The line, d, is the critical line in the pair approximation. The full circles are points over the critical line estimated by Monte Carlo simulation. Point 'O' is Onsager's exact solution.

and

$$2\langle\sigma_j\sigma_k\rangle = \langle\sigma_j f_k(\sigma)\rangle + \langle\sigma_k f_j(\sigma)\rangle. \tag{4}$$

These equations do not constitute closed equations for $\langle\sigma_i\rangle$ and $\langle\sigma_j\sigma_k\rangle$, except in the linear case ($b = 0$). One can obtain approximate closed equations for these quantities as follows. Let $P(\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4)$ be the probability of a cluster of spins composed of a central spin σ_0 and its four nearest-neighbour spins σ_1, σ_2, σ_3 and σ_4, which can be written, using conditional probabilities as $P(\sigma_0)P_c(\sigma_1, \sigma_2, \sigma_3, \sigma_4|\sigma_0)$. The pair approximation consists of writing the conditional probability as the product $\prod_{i=1}^4 P_c(\sigma_i|\sigma_0) = \prod_{i=1}^4 P(\sigma_0, \sigma_i)/P(\sigma_0)$. By using this approximation to calculate the right-hand sides of equations (3) and (4), we get the following equations for the magnetization $m = \langle\sigma_i\rangle$ and the nearest-neighbour pair correlation $r = \langle\sigma_j\sigma_k\rangle$

$$m = 4am + 2b\left\{\frac{(r+m)^3}{(1+m)^2} - \frac{(r-m)^3}{(1-m)^2}\right\} \tag{5}$$

and

$$r = a + \tfrac{3}{2}(a+b)\left\{\frac{(r+m)^2}{(1+m)} + \frac{(r-m)^2}{(1-m)}\right\} + b\left\{\frac{(r+m)^4}{(1+m)^3} + \frac{(r-m)^4}{(1-m)^3}\right\}. \tag{6}$$

The paramagnetic solution is given by $m = 0$ and $r = a + 3(a+b)r^2 + br^4$. By a linear analysis this solution becomes unstable for $a + 7b/27 \leqslant 1/4$. This gives rise to a critical line defined by $a + 7b/27 = 1/4$ or $y + 20x/17 = 27/17$. The extremal points of this line are $x = 1/2$, $y = 1$ (the voter point) and $x = 1$, $y = 7/17$.

4. Monte Carlo simulation

We have simulated the model on a square lattice with $N = L \times L$ sites and periodic boundary conditions, for several values of L ranging from $L = 5$ up to $L = 80$. For each simulation we have started with a random configuration of spins (we have checked that the results are not affected by a different choice initial conditions). The quantities of interest were calculated by using a number of Monte Carlo steps (MCS) of the order 10^5, each MCS being equal to N spin-flip trials. The relaxation time for the magnetization in the largest system ($L = 100$) close to the critical point was found to be of order or less than 5×10^3 MCS, thus 10^5 MCS were sufficient to reach the steady state and to obtain enough data for satisfatory statistics. The simulations were performed for points of the x–y plane along several straight lines starting from the point $x = 1$, $y = 1$.

Figure 2 shows the magnetization $M_L = \langle|m|\rangle$, where $m = \sum_{i=1}^{N} \sigma_i/N$, as a function of $q = (1-y)/4$ along the line $x = 1$. Figure 3 shows the magnetization as a function of $1/L$ for the case $x = 1$. From this plot it is possible to locate the critical point. However, a more acurate value is obtained by considering the plot of the reduced fourth-order cumulant

$$U_L = 1 - \frac{\langle m^4\rangle}{3\langle m^2\rangle^2} \tag{7}$$

as a function of q for several values of L. All curves should intersect at the critical point. Along the line $x = 1$, figure 4 gives $q_c = 0.073 \pm 0.001$. By using this technique we have located other points along the critical line. They are all shown in figure 1.

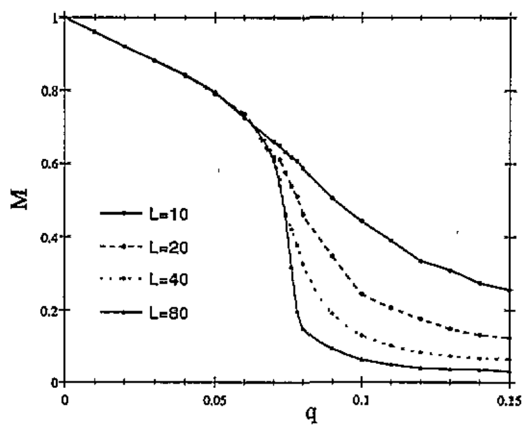

Figure 2. Magnetization $M_L(q)$ as a function of $q = (1-y)/4$ for several values of L along the line $x = 1$.

After having located a critical point the critical exponents are estimated by plotting $\ln M_L$ and $\ln X_L$, where $X_L = N\{\langle m^2\rangle - \langle|m|\rangle^2\}$, as a function of $\ln L$, as shown in figures 5 and 6. The slopes of the straight line fitted to the data points give β/ν and γ/ν. From the figures we have $\beta/\nu = 0.13 \pm 0.01$ and $\gamma/\nu = 1.77 \pm 0.05$. These values compare well with the exact values $\beta/\nu = 1/8$ and $\gamma/\nu = 7/4$ for the equilibrium Ising model. For other values of x and y along the critical line we have also obtained, within the statistical errors, the same critical exponents.

5. The first-order critical point

The Monte Carlo and mean-field results indicate that the critical line in the x–y plane terminates in the voter-model point $x = 1/2$, $y = 1$. In two dimensions, the voter model

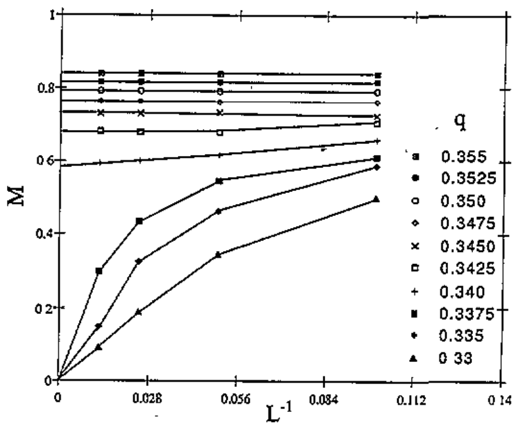

Figure 3. Magnetization $M_L(q)$ as a function of L^{-1} for several values of $q = (1-y)/4$ along the line $x = 1$.

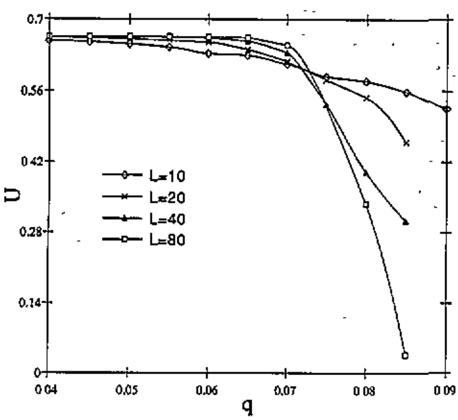

Figure 4. Reduced fourth-order cumulant $U_L(q)$ as a function of $q = (1-y)/4$ for several values of L along the line $x = 1$. The intersection of the curves gives $q_c = 0.073 \pm 0.001$.

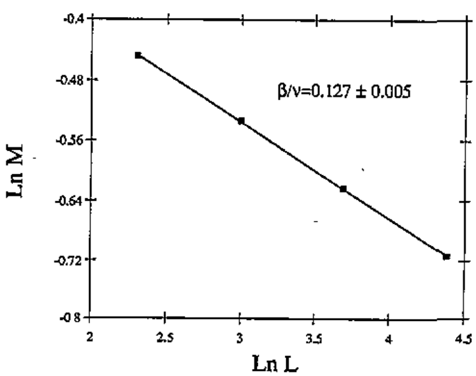

Figure 5. Log–log plot of M_L at the critical point against L for the case $x = 1$.

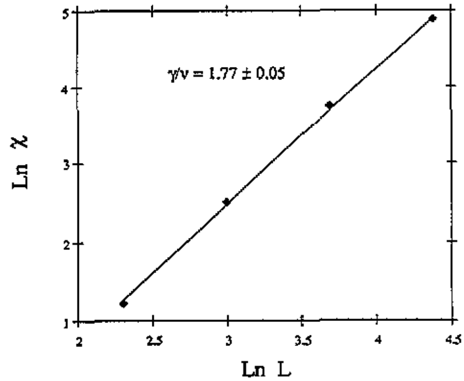

Figure 6. Log–log plot of X_L at the critical point against L for the case $x = 1$.

has only two extremal states, all spins up and all spins down (in contrast to the case $d = 3$, where there is a unique stationary state for every value of the magnetization [14]). Along the line $y = 1$, these two states are absorbing states. However, our results show that they are unstable for $x < 1/2$. In this range the system is disordered, that is the magnetization vanishes. Therefore at $x = 1/2$ there is a jump in the order parameter and the transition is first order.

We have estimated the quantity $X_L = N\{\langle m^2\rangle - \langle |m|\rangle^2\}$, along this line by Monte Carlo. The results reveal that X diverges as one approaches $x = 1/2$, so that the point is, in fact, a first-order critical point [23]. For such a critical point one should have $\gamma = 2\nu$ since $\beta = 0$ and $d = 2$ for the present case. According to finite-size scaling [24], $X_L|x - x_c|L^{-\gamma/\nu}$ is a universal function of $|x - x_c|L^{1/\nu}$, where $x_c = 1/2$. Figure 7 shows that the data points for these two quantities collapse into a single function when $\gamma = 1.25$ which gives $\nu = 0.625$. For comparison we have also considered a fit with $\nu = 0.5$.

Let us consider the linear model defined by $b = 0$, that is the model along the line $y = 2x$. In the stationary state the pair correlation function $\langle\sigma_i\sigma_j\rangle$ satisfies the equation

$$\langle\sigma_i\sigma_j\rangle = a\sum_\delta\langle\sigma_i\sigma_{j+\delta}\rangle \tag{8}$$

for $i \neq j$ and the sumation ($\sum_\delta$) extending to the nearest neighbours of j. For $r \neq 0$, let us define $G(r) = \langle\sigma_0\sigma_r\rangle$. Then

$$\Delta G(r) = \epsilon G(r) \tag{9}$$

where Δ is the discrete Laplacian operator in two dimensions, that is

$$\Delta G(r) = G(r + \hat{x}) + G(r - \hat{x}) + G(r + \hat{y}) + G(r - \hat{y}) - 4G(r) \tag{10}$$

and $\epsilon = 1/a - 4$. For large $|r|$ we get

$$G(r) = Ce^{-|r|/\xi} \tag{11}$$

where the correlation length $\xi = \epsilon^{-1/2}$. Therefore, one obtains $\nu = 1/2$. From the correlation function given by equation (11) we calculate X and obtain the result $X = \epsilon^{-1}$ which gives $\gamma = 1$.

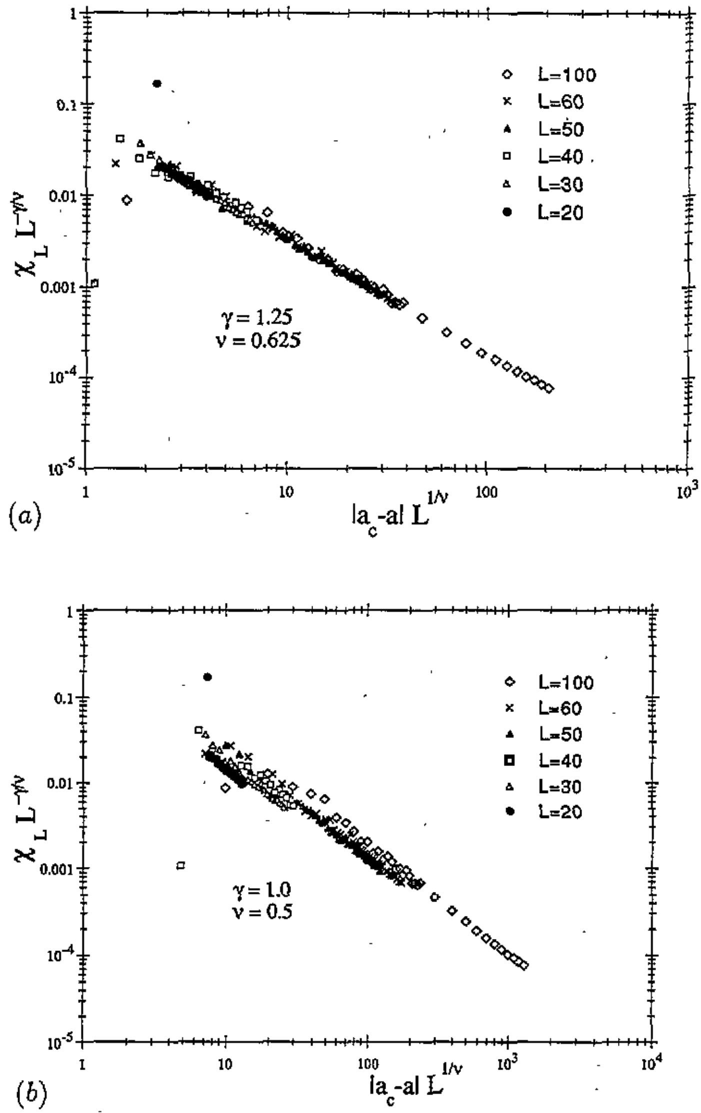

Figure 7. (*a*) Plot of $X_L^{-\gamma/\nu}$ against $(a_c - a)L^{1/\nu}$ along the line $y = 1$. (*b*) Plot of $X_L^{-\gamma/\nu}$ against $(a_c - a)L^{1/\nu}$ along the line $y = 2x$.

6. Conclusion

We have considered a family of non-equilibrium two-dimensional spin models evolving according to single-spin-flip rules and displaying up–down symmetry. The Glauber Ising and the majority voter model are included in this family and recent numerical studies [20] indicate that they belong to the same universality class, the equilibrium Ising class. Our Monte Carlo results show that this type of critical behaviour is also common to models whose rules have only the requirement of being isotropic, short ranged and preserving the up–down symmetry. The voter model displays a distinct characteristic; our simulation results suggest a new universality class ($\gamma = 1.25$) in disagreement with the exact result ($\gamma = 1$). We suspect this may be a consequence of the presence of the absorbing states and are currently investigating this situation.

It will be worth examining the role of the anisotropy, since in the majority-voter case it was found to be irrelevant [25]. In the absence of a general theory for non-equilibrium critical behaviour, is often conjectured that the manifestation of equilibrium Ising behaviour in these systems might be explained in terms of the equilibrium critical properties of some effective (Ising-like) Hamiltonian. The question is then to relate the effective coupling constants in the Hamiltonian to the transition rates of the model, and apart from a few very simple cases [26] this is still an open problem.

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