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Existence of Global Solutions in Time for Reaction-Diffusion Systems with Inhomogeneous Terms in Cones

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Abstract

We consider nonnegative solutions of the initial-boundary value problems in cone domains for the reaction-diffusion systems with inhomogeneous terms dependent on space coordinates and times. In our previous paper the conditions for the nonexistence of global solutions in time were shown. In this paper we show the condition of existence of global solutions in time.

1 Introduction

We consider nonnegative solutions of initial-boundary value problems for the reaction-diffusion systems of the form

$$\begin{cases} u_t = \Delta u + K_1(x,t)v^{p_1}, & x \in D,\ t > 0, \\ v_t = \Delta v + K_2(x,t)u^{p_2}, & x \in D,\ t > 0, \\ u(x,t) = v(x,t) = 0, & x \in \partial D,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in D, \end{cases} \tag{1}$$

where $p_1, p_2 \geq 1$ with $p_1 p_2 > 1$. The domain D is a cone in $\mathbf{R}^N$ such as

$$D = \{x \in \mathbf{R}^N; x \neq 0 \text{ and } x/|x| \in \Omega\}, \tag{2}$$

where Ω is some region on S^{N-1} satisfying $\Omega \neq S^{N-1}$ and $\partial\Omega$ is smooth enough.

The initial data $u_0(x)$ and $v_0(x)$ are nonnegative, bounded and continuous in $\bar{D}$, and $u_0(x) = v_0(x) = 0$ on ∂D. The inhomogeneous terms K_i $(i = 1, 2)$ are nonnegative continuous functions in $D \times (0, \infty)$.

Let Δ_Ω denote the Laplace-Beltrami operator with homogeneous Dirichlet boundary condition in Ω. Let $\psi_n(x/|x|)$ denote the n-th eigenfunction of

$-\Delta_\Omega$ with Dirichlet problem in Ω satisfying $\|\psi_n\|_{L^2(\Omega)} > 0$, where $\|\xi\|_{L^2(\Omega)} = \sqrt{\int_\Omega \xi^2(\phi)d\phi}$. Let $\omega_n > 0$ denote the corresponding eigenvalue to ψ_n. Assume that the sequence $\{\psi_n/\|\psi_n\|_{L^2(\Omega)}\}_{n=1}^\infty$ is a complete orthonormal sequence. Let γ_+ denote the positive root of $\gamma(\gamma + N - 2) = \omega_1$, that is

$$\gamma_+ = \frac{-(N-2) + \sqrt{(N-2)^2 + 4\omega_1}}{2}. \tag{3}$$

We introduce the Green's function $G(x, y, t) = G(r, \theta, \rho, \phi, t)$ for the linear heat equation in the cone D, where

$$r = |x|, \;\; \rho = |y|, \;\; \theta = \frac{x}{|x|} \text{ and } \phi = \frac{y}{|y|} \in \Omega. \tag{4}$$

The Green's function is expressed to

$$G(r, \theta, \rho, \phi, t) = \frac{(r\rho)^{-(N-2)/2}}{2t} \exp\left(-\frac{\rho^2 + r^2}{4t}\right) \sum_{n=1}^\infty c_n I_{\nu_n}\left(\frac{r\rho}{2t}\right) \psi_n(\theta)\psi_n(\phi), \tag{5}$$

where $c_n = 1/\|\psi_n\|^2_{L^2(\Omega)}$, $\nu_n = \left[(N-2)^2/4 + \omega_n\right]^{1/2}$ and I_ν is the modified Bessel function or

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^\infty \frac{(z/2)^{2k}}{k!\Gamma(\nu + k + 1)} \sim \begin{cases} (z/2)^\nu/\Gamma(\nu+1), & \text{as } z \to 0^+ \\ e^z/\sqrt{2\pi z}, & \text{as } z \to +\infty \end{cases} \tag{6}$$

with the Gamma function $\Gamma(z) = \int_0^\infty s^{z-1}e^{-s}ds$ (see Section 4 in detail). The operator $S(t)$ is defined by

$$S(t)\xi(x) = \int_D G(x, y, t)\xi(y)dy = \int_0^\infty \int_\Omega G(r, \theta, \rho, \phi, t)\xi(\rho, \phi)\rho^{N-1}d\phi d\rho \tag{7}$$

with G defined by (5). By using this $S(t)$, the solution (u, v) of (1) is expressed to

$$\begin{cases} u(x,t) &= S(t)u_0(x) + \displaystyle\int_0^t S(t-s)K_1(x,s)v(x,s)^{p_1}ds, \\ v(x,t) &= S(t)v_0(x) + \displaystyle\int_0^t S(t-s)K_2(x,s)u(x,s)^{p_2}ds. \end{cases}$$

Remark. It is easily seen that $\gamma_+ = \nu_1 - (N-2)/2$ by (3).

For given initial values (u_0, v_0), let $T^* = T^*(u_0, v_0)$ be a maximal existence time of the solution of (1). If $T^* = \infty$, the solutions are global in time. On the other hand, if $T^* < \infty$, then the solutions are not global in time. If the solution blows up in finite time such that

$$\limsup_{t\to T^*} \|u(\cdot,t)\|_\infty + \limsup_{t\to T^*} \|v(\cdot,t)\|_\infty = \infty, \tag{8}$$

then the solution is not global, where $\|\cdot\|_\infty$ denotes the L^∞-norm with respect to space variable.

For our theorems we assume that the inhomogeneous terms $K_i (i = 1, 2)$ satisfy

$$K_i(x,t) \le C_U \langle x\rangle^{\sigma_i}(t+1)^{q_i}, \tag{9}$$

or

$$K_i(x,t) \ge C_L |x|^{\sigma_i} t^{q_i} \tag{10}$$

for some $C_U, C_L > 0$, and $\sigma_i, q_i \ge 0$, where

$$\langle x\rangle = \left(|x|^2 + 1\right)^{1/2}.$$

For conditions of the global existence we set

$$\alpha_i = \frac{(2+\sigma_i+2q_i)+(2+\sigma_j+2q_j)p_i}{p_ip_j - 1} \quad ((i,j) = (1,2),(2,1)). \tag{11}$$

Note that (α_1, α_2) satisfies

$$\begin{pmatrix} 1 & -p_1 \\ -p_2 & 1 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = -\begin{pmatrix} 2+\sigma_1+2q_1 \\ 2+\sigma_2+2q_2 \end{pmatrix}.$$

In [10] we considered the case there exists no global nontrivial solution of (1). The result of the global nonexistence for (1) was stated as follows.

Theorem 0 (Theorem 2 of [10]). *Assume that $K_i(x,t)$ $(i = 1,2)$ satisfy* (10). *Suppose that one of the following two conditions holds;*

(i) $\max\{\alpha_1, \alpha_2\} \ge N + \gamma_+$ *with* γ_+ *defined by (3),*

(ii) $u_0 \in H_{a_1}$ *for* $a_1 < \alpha_1$ *or* $v_0 \in H_{a_2}$ *for* $a_2 < \alpha_2$*,*

where

$$H_a = \left\{\xi \in C(\bar{D}) : \xi(x) \ge M\langle x\rangle^{-a}\psi_1\left(\frac{x}{|x|}\right) \text{ for } x \in D \text{ with some } M > 0\right\}.$$

Then there exists no nontrivial nonnegative global solution of (1), *that is* $T^* < \infty$.

On the other hand, the main result of this paper is the following global existence theorem.

Theorem 1. *Assume that* $\max\{\alpha_1, \alpha_2\} < N + \gamma_+$ *with* γ_+ *defined by (3) and* $K_i(x,t)$ $(i = 1, 2)$ *satisfy* (9). *Suppose that*

$$(u_0, v_0) \in H^{a_1} \times H^{a_2} \textit{ for } a_1 > \alpha_1,\ a_2 > \alpha_2, \tag{12}$$

where

$$H^a = \left\{ \xi \in C(\bar{D}) : \xi(x) \le m\langle x\rangle^{-a}\psi_1\left(\frac{x}{|x|}\right) \textit{ for } x \in D \textit{ with small } m > 0 \right\}. \tag{13}$$

Then the solution (u, v) *of* (1) *is global in time, that is* $T^* = \infty$. *Moreover, there exists a positive constant* C *such that*

$$u(x,t) \le CS(t)\langle x\rangle^{-\tilde{a}_1}\psi_1\left(\frac{x}{|x|}\right) \quad \textit{and} \quad v(x,t) \le CS(t)\langle x\rangle^{-\tilde{a}_2}\psi_1\left(\frac{x}{|x|}\right) \tag{14}$$

in $D \times (0, \infty)$, *where* $\tilde{a}_1 \le a_1$ *and* $\tilde{a}_2 \le a_2$ *are chosen to satisfy*

$$p_i \min\{\tilde{a}_j, N + \gamma_+\} - \tilde{a}_i > 2 + \sigma_i + 2q_i \quad ((i,j) = (1,2), (2,1)). \tag{15}$$

¿From Theorems 0 and 1 we may draw up the following table.

	$\max\{\alpha_1, \alpha_2\} \ge N + \gamma_+$	$\max\{\alpha_1, \alpha_2\} < N + \gamma_+$
$a_1 < \alpha_1$ or $a_2 < \alpha_2$	NG	NG
$a_1 > \alpha_1$ and $a_2 > \alpha_2$	NG	G

NG : There exists no global nontrivial solution.

G : There exists a global nontrivial solution for small initial data.

We briefly recall a history of the studies on global existence of solutions to the system (1).

First, the global existence of solutions in the case $D = \mathbf{R}^N$ $(\Omega = S^{N-1})$, $u = v$, $p_i = p$ and $K_i(x,t) = 1$ $(i = 1, 2)$, that is

$$\begin{cases} u_t = \Delta u + u^p, & x \in \mathbf{R}^N,\ t > 0, \\ u(x,0) = u_0(x) \ge 0, & x \in \mathbf{R}^N, \end{cases} \tag{16}$$

was studied by Fujita [3]. Fujita proved that when $p > 1 + 2/N$ the solution of (16) is global in time if $\|u_0\|_\infty$ is small enough and u_0 has an exponential decay. Fujita's results were also extended by some researcher. For the case $p > 1 + 2/N$, Lee-Ni [15] studied that if $\|u_0\|_\infty$ is small enough and $\limsup_{|x|\to\infty} |x|^a u_0(x) < \infty$ with $a > 2/(p-1)$, the solution of (16) is global in time. When D is a cone, that is

$$\begin{cases} u_t = \Delta u + u^p, & x \in D,\ t > 0, \\ u(x,t) = 0, & x \in \partial D,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in D. \end{cases} \tag{17}$$

Levine-Meier [17] proved that if $p > 1+2/(N+\gamma_+)$, nontrivial global solutions of (17) exist.

Fujita's results were extended to the case $D = \mathbf{R}^N$, $u = v$, $p_i = p$ and $K_i(x,t) = K(x,t)$ for $i = 1, 2$, that is

$$\begin{cases} u_t = \Delta u + K(x,t)u^p, & x \in \mathbf{R}^N,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbf{R}^N. \end{cases} \tag{18}$$

In the case $K(x,t) \sim |x|^\sigma$ as $|x| \to \infty$ with $\sigma \in \mathbf{R}$, Suzuki [25] had that if $p > 1 + (2+\sigma)/N$ then a global solution of (18) exists (see also [21]). Thereafter, Qi [23] extended the result to the case $K(x,t) = t^q|x|^\sigma$ with $q \geq 0$, $\sigma \geq 0$. He caught that if $p > 1 + (2+\sigma+2q)/N$, there exists a global solution of (18). When D is a cone, that is

$$\begin{cases} u_t = \Delta u + K(x,t)u^p, & x \in D,\ t > 0, \\ u(x,t) = 0, & x \in \partial D,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in D, \end{cases} \tag{19}$$

in the case $K(x,t) = |x|^\sigma$ with $\sigma \geq 0$, Levine-Meier [17] had that if $p > 1 + (2+\sigma)/(N+\gamma_+)$, there are nontrivial global solutions of (19). For the case $p > 1 + (2+\sigma)/(N+\gamma_+)$, Hamada [7] studied that if $u_0 \in H^a$ with $a > (2+\sigma)/(p-1)$, the solution of (19) is global in time.

In the case $D = \mathbf{R}^N$, our results are reduced to Escobedo-Herrero [2] and Mochizuki [19] with $K_i(x,t) = 1$ $(i = 1, 2)$, to Uda [26] with $K_i(x,t) = t^{q_i}$ $(i = 1, 2)$, and to Mochizuki-Huang [20] with $K_i(x,t) = |x|^{\sigma_i}$ with $\sigma_i \in [0, n(p_i - 1))$ $(i = 1, 2)$. Moreover, when $K_i(x,t)$ $(i = 1, 2)$ satisfy (9) with $D = \mathbf{R}^N$, the system (1) was studied by Igarashi-Umeda [9]. When D is a cone, in the case $K_i(x,t) = 1$, the condition $\max\{\alpha_1, \alpha_2\} < N + \gamma_+$ of Theorem 1 is reduced to Levine [16].

The history for the global nonexistence was stated in [10] (see also [3, 8, 12, 30, 1, 18, 17, 2, 16, 15, 6, 26, 7, 21, 19, 20, 23, 4, 11, 25, 9]).

The rest of the paper is organized as follows. Some preliminary lemmata are given in Section 2. Theorem 1 is proved in Section 3. In Section 4 we confirm the form of the Green function for the heat equation in the cone domain with the Dirichlet condition. In Section 5 we prove Lemma 2.2 in Section 2 of this paper. For the change of variable as (4), we decide $\zeta(x,y,t)=\zeta(r,\theta,\rho,\phi,t)$, $\zeta(x,t)=\zeta(r,\theta,t)$ or $\zeta_0(x)=\zeta_0(r,\theta)$ for any functions.

2 Preliminaries

In this section we prepare a notation and some lemmata for proving Theorem 1.

We define for $a>0$

$$\eta_a(x,t)=S(t)\langle x\rangle^{-a}\psi_1\left(\frac{x}{|x|}\right), \tag{20}$$

with $S(t)$ defined by (7).

Lemma 2.1. *Let η_a be defined in* (20) *with $a>0$. Then we have in $D\times(0,\infty)$,*

$$\eta_a(x,t)^{-1}\le C\max\left\{\langle x\rangle^a,(1+t)^{a/2}\right\}\psi_1\left(\frac{x}{|x|}\right)^{-1},$$

where η_a is defined in (20).

Proof. As well known, $\eta_a(x,t)\to\langle x\rangle^{-a}\psi_1(x/|x|)$ as $t\to0$ locally uniformly in $x\in D$. By (5) we see that

$$\eta_a(x,t)\ge\int_{\frac{2t}{r}}^{\infty}\int_\Omega G(r,\theta,\rho,\phi,t)(1+\rho^2)^{-a/2}\psi_1(\phi)\rho^{N-1}d\phi d\rho.$$

¿From (6), we have

$$I_\nu(z)\ge\begin{cases}Cz^\nu, & 0<z\le1,\\ Cz^{-1/2}e^z, & z>1\end{cases} \tag{21}$$

with some constant $C>0$. Thus we obtain

$$\eta_a(x,t)\ge\frac{C\psi_1(\theta)}{\sqrt{2t}}\int_{\frac{2t}{r}}^{\infty}r^{-(N-1)/2}\rho^{(N-1)/2}(1+\rho^2)^{-a/2}\exp\left(-\frac{(\rho-r)^2}{4t}\right)d\rho.$$

Put $s=\dfrac{\rho-r}{\sqrt{t}}$. Then we see

$$\eta_a(x,t)\ge C\psi_1(\theta)\int_{\frac{2\sqrt{t}}{r}-\frac{r}{\sqrt{t}}}^{\infty}\left(\frac{s\sqrt{t}}{r}+1\right)^{(N-1)/2}\left(1+(s\sqrt{t}+r)^2\right)^{-a/2}e^{-s^2/4}ds.$$

First assume that $0 \le t < 1$. Then it follows that

$$\eta_a(x,t) \ge C\psi_1(\theta) \int_{\frac{2}{r}-r}^{\infty} \left(1+(s+r)^2\right)^{-a/2} e^{-s^2/4} ds.$$

If $|x| > \sqrt{2}$, that is $r > \sqrt{2}$, then we obtain

$$\begin{aligned}
\eta_a(x,t) &\ge C\psi_1(\theta) \int_0^1 \left(1+(s+r)^2\right)^{-a/2} e^{-s^2/4} ds \\
&\ge C\psi_1(\theta)(1+r^2)^{-a/2} \int_0^1 e^{-s^2/4} ds \ge C\psi_1(\theta)\langle x\rangle^{-a}.
\end{aligned}$$

Next, let $t \ge 1$. Then we have

$$\begin{aligned}
&\eta_a(x,t) \\
&\ge C\psi_1(\theta) \int_{\max\left\{\frac{2\sqrt{t}}{r}-\frac{r}{\sqrt{t}},0\right\}}^{\infty} \left(\frac{s\sqrt{t}}{r}+1\right)^{\frac{N-1}{2}} \left(1+(s\sqrt{t}+r)^2\right)^{-\frac{a}{2}} e^{-\frac{s^2}{4}} ds \\
&\ge \frac{C\psi_1(\theta)}{t^{a/2}} \int_{\max\left\{\frac{2\sqrt{t}}{r}-\frac{r}{\sqrt{t}},0\right\}}^{\infty} \left(\frac{s\sqrt{t}}{r}+1\right)^{\frac{N-1}{2}} \left(1+\left(s+\frac{r}{\sqrt{t}}\right)^2\right)^{-\frac{a}{2}} e^{-\frac{s^2}{4}} ds.
\end{aligned}$$

If $r/\sqrt{t} \le 1$, this shows

$$\eta_a(x,t) \ge \frac{C\psi_1(\theta)}{t^{a/2}} \int_{\max\left\{\frac{2\sqrt{t}}{r}-\frac{r}{\sqrt{t}},0\right\}}^{\infty} (1+(s+1)^2)^{-a/2} e^{-s^2/4} ds \ge \frac{C\psi_1(\theta)}{t^{a/2}}.$$

On the other hand, if $\xi = r/\sqrt{t} > 1$, then

$$\begin{aligned}
r^a\eta_a(x,t) &\ge C\psi_1(\theta) \int_{\max\left\{\frac{2}{\xi}-\xi,0\right\}}^{\infty} \left(1+\frac{s}{\xi}\right)^{(N-1)/2} \frac{\xi^a}{(1+(\xi+s)^2)^{a/2}} e^{-s^2/4} ds \\
&\to C\psi_1(\theta) \int_0^{\infty} e^{-s^2/4} ds \quad \text{as } \xi \to \infty.
\end{aligned}$$

Summarizing these results, we obtain the inequality in the lemma. □

Lemma 2.2 (Lemma 3.1 of [7])**.** *Let η_a be defined in (20) with $a > 0$. Assume $0 \le \sigma < \min\{a, N+\gamma_+\}$. Then there exists a positive constant C such that*

$$\langle x\rangle^\sigma \eta_a(x,t) \le \begin{cases} C(1+t)^{(\sigma-a)/2}\psi_1(x/|x|), & \text{if } a < N+\gamma_+, \\ C(1+t)^{\{\sigma-(N+\gamma_+)+\epsilon\}/2}\psi_1(x/|x|), & \text{if } a = N+\gamma_+, \\ C(1+t)^{\{\sigma-(N+\gamma_+)\}/2}\psi_1(x/|x|), & \text{if } a > N+\gamma_+, \end{cases}$$

for any $(x,t) \in D \times (0,\infty)$ and $\epsilon > 0$.

Proof. See Lemma 3.1 of [7] or Section 5 of this paper. □

Lemma 2.3. *Let η_a be defined in* (20) *with $a > 0$. Assume $p \geq 1$, $\sigma \geq 0$, $q \geq 0$ and $b > 0$ and*

$$p \min\{a, N + \gamma_+\} - b > 2 + \sigma + 2q. \tag{22}$$

Then there exists a positive constant C such that

$$\begin{aligned} &(t+1)^q \langle x\rangle^\sigma \eta_a(x,t)^p \\ &\leq \begin{cases} C(1+t)^{(\sigma+2q+b-ap)/2}\eta_b(x,t), & \text{if } a < N+\gamma_+, \\ C(1+t)^{\{\sigma+2q+b-(N+\gamma_+)p+\epsilon\}/2}\eta_b(x,t), & \text{if } a = N+\gamma_+, \\ C(1+t)^{\{\sigma+2q+b-(N+\gamma_+)p\}/2}\eta_b(x,t), & \text{if } a > N+\gamma_+, \end{cases} \end{aligned} \tag{23}$$

for any $(x,t) \in D \times (0,\infty)$ and $\epsilon > 0$.

Proof. By Lemma 2.1, we obtain

$$\begin{aligned} &(t+1)^q \langle x\rangle^\sigma \eta_a(x,t)^p = (t+1)^q \langle x\rangle^\sigma \eta_a(x,t)^p \eta_b(x,t)^{-1}\eta_b(x,t) \\ &\leq C(t+1)^q \langle x\rangle^\sigma \eta_a(x,t)^p \max\{\langle x\rangle^b, (1+t)^{b/2}\}\psi_1(x/|x|)^{-1}\eta_b(x,t). \end{aligned}$$

¿From Lemma 2.2 and (22) we have

$$\begin{aligned} &(t+1)^q \langle x\rangle^\sigma \eta_a(x,t)^p \\ &\leq \begin{cases} C(1+t)^{(\sigma+2q+b-ap)/2}\eta_b(x,t)\psi_1(x/|x|)^{p-1}, & \text{if } a < N+\gamma_+, \\ C(1+t)^{\{\sigma+2q+b-(N+\gamma_+)p+\epsilon\}/2}\eta_b(x,t)\psi_1(x/|x|)^{p-1}, & \text{if } a = N+\gamma_+, \\ C(1+t)^{\{\sigma+2q+b-(N+\gamma_+)p\}/2}\eta_b(x,t)\psi_1(x/|x|)^{p-1}, & \text{if } a > N+\gamma_+, \end{cases} \end{aligned}$$

for any $\epsilon > 0$. If $p \geq 1$, then $\psi_1(x/|x|)^{p-1}$ is bounded. Hence, we obtain (23). □

3 Existence of a global solution

In this section we treat the existence of global solutions in time of (1). Here, we take the same strategy as in [20] and [28].

First note that condition (12) can be replaced by $(u_0, v_0) \in H^{\tilde{a}_1} \times H^{\tilde{a}_2}$ since we have $H^{a_1} \times H^{a_2} \subset H^{\tilde{a}_1} \times H^{\tilde{a}_2}$. Then, to establish Theorem 1, we have only to consider the special case $\tilde{a}_1 = a_1$ and $\tilde{a}_2 = a_2$. As is easily seen, in this case condition (15) is equivalent to

$$p_i \min\{a_j, N + \gamma_+\} - a_i > 2 + \sigma_i + 2q_i \quad ((i,j) = (1,2),(2,1)). \tag{24}$$

If (24) holds, then it is necessarily that $\max\{\alpha_1, \alpha_2\} < N + \gamma_+$, $a_1 > \alpha_1$ and $a_2 > \alpha_2$.

We define the Banach space X as

$$X = \{v : |||v/\eta_{a_2}|||_\infty < \infty\},$$

where η_a is defined in (20) with $a > 0$ and

$$|||w|||_\infty = \sup_{(x,t)\in D\times(0,\infty)} |w(x,t)|.$$

We consider the associated integral system

$$u(x,t) = S(t)u_0(x) + \int_0^t S(t-s)K_1(x,s)v(x,s)^{p_1}ds, \tag{25}$$

$$v(x,t) = S(t)v_0(x) + \int_0^t S(t-s)K_2(x,s)u(x,s)^{p_2}ds, \tag{26}$$

with $S(t)$ defined in (7). Substituting (25) into (26), we have

$$v(x,t) = V(u_0, v_0, v) \tag{27}$$

with

$$\begin{aligned} V(u_0, v_0, v) =& S(t)v_0(x) + \int_0^t S(t-s)K_2(x,s) \\ & \times \left(S(s)u_0(x) + \int_0^s S(s-\tau)K_1(x,\tau)v(x,\tau)^{p_1}d\tau \right)^{p_2} ds. \end{aligned}$$

If V is a strict contraction, then its fixed point yields a solution of (1). Moreover, by the fact $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ for $a > 0$, $b > 0$ and $p \geq 1$, we obtain

$$V(u_0, v_0, v) \leq T(u_0, v_0) + \Gamma(v) \tag{28}$$

with

$$T(u_0, v_0) = S(t)v_0(x) + 2^{p_2-1}\int_0^t S(t-s)K_2(x,s)(S(s)u_0(x))^{p_2}ds,$$

$$\Gamma(v) = 2^{p_2-1}\int_0^t S(t-s)K_2(x,s)\left(\int_0^s S(s-\tau)K_1(x,\tau)v(x,\tau)^{p_1}d\tau\right)^{p_2} ds.$$

Lemma 3.1. *Assume the same hypotheses as in Lemma 2.3. Then there exists a constant $C > 0$ such that*

$$\int_0^t S(t-s)(s+1)^q \langle x\rangle^\sigma \eta_a(x,s)^p ds \leq C\eta_b(x,t) \tag{29}$$

for any $(x,t) \in D \times (0,\infty)$.

Proof. Put
$$\epsilon = \frac{p\min\{a, N+\gamma_+\} - b - 2 - \sigma - 2q}{2}.$$
Then from (22) we see that
$$\sigma + 2q + b - p\min\{a, N+\gamma_+\} + \epsilon \equiv \beta < -2. \tag{30}$$
¿From Lemma 2.3, we have
$$\int_0^t S(t-s)(s+1)^q \langle x\rangle^\sigma \eta_a(x,s)^p ds \le C\eta_b(x,t)\int_0^t (1+s)^{\beta/2} ds.$$
¿From (30) there exists a constant $C' > 0$ such that
$$\int_0^t S(t-s)(s+1)^q \langle x\rangle^\sigma \eta_a(x,s)^p ds \le C'\eta_b(x,t).$$
□

Lemma 3.2. *Let η_a be defined in* (20) *with $a > 0$.*

(i) *Let (u_0, v_0) satisfy* (12). *Then $T(u_0, v_0) \in X$ and*
$$|||T(u_0, v_0)/\eta_{a_2}|||_\infty \le C_a(m + m^{p_2})$$
with some $C_a > 0$, where m is appeared in (13).

(ii) *Let v be the second element of the solution of* (1). *Then Γ maps X into itself and*
$$|||\Gamma(v)/\eta_{a_2}|||_\infty \le C_b|||v/\eta_{a_2}|||_\infty^{p_1p_2}$$
with some $C_b > 0$.

Proof. (i) First, it is easily seen that $S(t)v_0(x) \le m\eta_{a_2}(x,t)$. Next, from Lemma 3.1 and (24), we obtain
$$\begin{aligned}&\int_0^t S(t-s)K_2(x,s)(S(s)u_0(x))^{p_2} ds\\ &\quad\le \int_0^t S(t-s)C_U(s+1)^{q_2}\langle x\rangle^{\sigma_2}(m\eta_{a_1}(x,s))^{p_2} ds \le Cm^{p_2}\eta_{a_2}(x,t).\end{aligned}$$
Thus, we have
$$|T(u_0, v_0)| \le C\eta_{a_2}(x,t)(m + m^{p_2}).$$
This implies assertion (i).

(ii) Similarly as above, it follows from Lemma 3.1 and (24) that

$$\begin{aligned}\Gamma(v) \le & C\||v/\eta_{a_2}|\|_\infty^{p_1p_2} \int_0^t S(t-s)C_U(s+1)^{q_2}\langle x\rangle^{\sigma_2} \\ & \times \left(\int_0^s S(s-\tau)C_U(\tau+1)^{q_1}\langle x\rangle^{\sigma_1}\eta_{a_2}(x,\tau)^{p_1}d\tau\right)^{p_2} ds \\ \le & C\||v/\eta_{a_2}|\|_\infty^{p_1p_2} \int_0^t S(t-s)C_U(s+1)^{q_2}\langle x\rangle^{\sigma_2}\eta_{a_1}(x,s)^{p_2}ds \\ \le & C\||v/\eta_{a_2}|\|_\infty^{p_1p_2}\eta_{a_2}(x,t).\end{aligned}$$

Assertion (ii) thus is concluded. □

Proof of Theorem 1. Let $B_m = \{v \in X; \||v/\eta_{a_2}|\|_\infty \le 3m\}$ and $P = \{v \in X; v \ge 0\}$, where m is appeared in (13). We shall show that $V(u_0, v_0, v)$ is a strict contraction of $B_m \cap P$ into itself provided m is small enough.

¿From (28) and Lemma 3.2 we have

$$\begin{aligned}\||V(u_0,v_0,v)/\eta_{a_2}|\|_\infty &\le \||T(u_0,v_0)/\eta_{a_2}|\|_\infty + \||\Gamma(v)/\eta_{a_2}|\|_\infty \\ &\le C_a(m+m^{p_2}) + C_b(3m)^{p_1p_2} \le 3m.\end{aligned}$$

This proves that V maps $B_m \cap P$ into $B_m \cap P$.

Now, we show that $V(u_0, v_0, v)$ is a strict contraction on $B_m \cap P$. By the definition of V we obtain

$$\begin{aligned}|V(u_0,v_0,v_1) - V(u_0,v_0,v_2)| \le & \int_0^t S(t-s)K_2(x,s) \\ \times & \left|\left(S(s)u_0(x) + \int_0^s S(s-\tau)K_1(x,\tau)v_1(x,\tau)^{p_1}d\tau\right)^{p_2}\right. \\ & \left. - \left(S(s)u_0(x) + \int_0^s S(s-\tau)K_1(x,\tau)v_2(x,\tau)^{p_1}d\tau\right)^{p_2}\right| ds.\end{aligned}$$

Since $|a^p - b^p| \le p(a+b)^{p-1}|a-b|$ for $a \ge 0$, $b \ge 0$ and $p \ge 1$, we can estimate as follows,

$$\begin{aligned}|V(u_0,v_0,v_1) - V(u_0,v_0,v_2)| \le & p_2 \int_0^t S(t-s)K_2(x,s) \\ \times & \left(2S(s)u_0(x) + \int_0^s S(s-\tau)K_1(x,\tau)(v_1(x,\tau)^{p_1} + v_2(x,\tau)^{p_1})d\tau\right)^{p_2-1} \\ & \times \left|\int_0^s S(s-\tau)K_1(x,\tau)(v_1(x,\tau)^{p_1} - v_2(x,\tau)^{p_1})d\tau\right| ds.\end{aligned}$$

Put

$$A(x,s) = \Bigg(2S(s)u_0(x) + \int_0^s S(s-\tau)K_1(x,\tau)(v_1(x,\tau)^{p_1} + v_2(x,\tau)^{p_1})d\tau\Bigg)^{p_2-1},$$

$$B(x,s) = \left| \int_0^s S(s-\tau)K_1(x,\tau)(v_1(x,\tau)^{p_1} - v_2(x,\tau)^{p_1})d\tau \right|.$$

Then we get

$$|V(u_0,v_0,v_1) - V(u_0,v_0,v_2)| \leq p_2 \int_0^t S(t-s)K_2(x,s)A(x,s)B(x,s)ds.$$

Since $(a+b)^p \leq 2^{\max\{p-1,0\}}(a^p+b^p)$ for $a \geq 0$, $b \geq 0$ and $p \geq 0$, we obtain

$$A(x,s) \leq 2^{\max\{p_2-2,0\}} \Bigg\{ (2S(s)u_0(x))^{p_2-1} + \left(\int_0^s S(s-\tau)C_U(\tau+1)^{q_1}\langle x\rangle^{\sigma_1} 2\tilde{v}(x,\tau)^{p_1} d\tau \right)^{p_2-1} \Bigg\}$$

with $\tilde{v} = \max\{v_1, v_2\}$ and

$$\begin{aligned} B(x,s) &\leq \int_0^s S(s-\tau)C_U(\tau+1)^{q_1}\langle x\rangle^{\sigma_1} |v_1(x,\tau)^{p_1} - v_2(x,\tau)^{p_1}| d\tau \\ &\leq \int_0^s S(s-\tau)C_U(\tau+1)^{q_1}\langle x\rangle^{\sigma_1} \\ &\quad \times p_1(v_1(x,\tau)+v_2(x,\tau))^{p_1-1}|v_1(x,\tau)-v_2(x,\tau)|d\tau. \end{aligned}$$

¿From Lemma 3.1 and (24), we have

$$\begin{aligned} A(x,s) &\leq 2^{\max\{p_2-2,0\}} \Bigg\{ (2m\eta_{a_1}(x,s))^{p_2-1} \\ &\quad + \left(2|||\tilde{v}/\eta_{a_2}|||_\infty^{p_1} \int_0^s S(s-\tau)C_U(\tau+1)^{q_1}\langle x\rangle^{\sigma_1}\eta_{a_2}^{p_1}(x,\tau)d\tau \right)^{p_2-1} \Bigg\} \\ &\leq 2^{\max\{p_2-2,0\}} \left\{ (2m)^{p_2-1}\eta_{a_1}^{p_2-1}(x,s) + (2C(3m)^{p_1})^{p_2-1}\, \eta_{a_1}^{p_2-1}(x,s) \right\} \end{aligned}$$

and

$$\begin{aligned} B(x,s) &\leq \int_0^s S(s-\tau)C_U(\tau+1)^{q_1}\langle x\rangle^{\sigma_1} p_1(2v(x,\tau))^{p_1-1}|v_1(x,\tau)-v_2(x,\tau)|d\tau \\ &\leq 2^{p_1-1}C_U \int_0^s S(s-\tau)(\tau+1)^{q_1}\langle x\rangle^{\sigma_1}\eta_{a_2}^{p_1}(x,\tau)(\tilde{v}(x,\tau)/\eta_{a_2}(x,\tau))^{p_1-1} \\ &\qquad \times p_1(|v_1(x,\tau)-v_2(x,\tau)|/\eta_{a_2}(x,\tau))d\tau. \end{aligned}$$

We can take m satisfying $(2m)^{p_2-1}+(2C(3m)^{p_1})^{p_2-1}\le 2^{p_2}m^{(p_2-1)/2}$. Then we have

$$
\begin{aligned}
&|V(u_0,v_0,v_1)-V(u_0,v_0,v_2)|\\
\le& C\int_0^t S(t-s)(s+1)^{q_2}\langle x\rangle^{\sigma_2}\left(2^{p_2}m^{(p_2-1)/2}\eta_{a_1}^{p_2-1}(x,s)\right)\eta_{a_1}(x,s)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\times|||\tilde v/\eta_{a_2}|||_\infty^{p_1-1}|||v_1/\eta_{a_2}-v_2/\eta_{a_2}|||_\infty ds\\
\le& Cm^{p_1+p_2/2-3/2}\int_0^t S(t-s)(s+1)^{q_2}\langle x\rangle^{\sigma_2}\eta_{a_1}^{p_2}(x,s)ds|||v_1/\eta_{a_2}-v_2/\eta_{a_2}|||_\infty\\
\le& Cm^{p_1+p_2/2-3/2}\eta_{a_2}(x,t)|||v_1/\eta_{a_2}-v_2/\eta_{a_2}|||_\infty.
\end{aligned}
$$

Since $p_1,p_2\ge 1$ and $p_1p_2>1$, we obtain for some $\rho<1$

$$
\begin{aligned}
&|||V(u_0,v_0,v_1)/\eta_{a_2}-V(u_0,v_0,v_2)/\eta_{a_2}|\,||_\infty\\
&\le Cm^{p_1+p_2/2-3/2}|||v_1/\eta_{a_2}-v_2/\eta_{a_2}|||_\infty\le\rho|||v_1/\eta_{a_2}-v_2/\eta_{a_2}|||_\infty
\end{aligned}
$$

with m small enough. Then V is a strict contraction of $B_m\cap P$ into itself. Hence, there exists a unique fixed point $v\in X$ which solves (27). Substitute v into (25). Then (u,v) solves (25) and (26). Moreover, since $v\in B_m$, we find

$$
v(x,t)\le CS(t)\langle x\rangle^{-a_2}\psi_1\left(\frac{x}{|x|}\right).
$$

Substituting this into (25), we have

$$
\begin{aligned}
u(x,t)\le& m\eta_{a_1}(x,t)+C\int_0^t S(t-s)C_U(s+1)^{q_1}\langle x\rangle^{\sigma_1}\eta_{a_2}^{p_1}(x,s)ds\\
\le& m\eta_{a_1}(x,t)+C\eta_{a_1}(x,t)\le C\eta_{a_1}(x,t).
\end{aligned}
$$

Then $u\in B_m$; that is,

$$
u(x,t)\le CS(t)\langle x\rangle^{-a_1}\psi_1\left(\frac{x}{|x|}\right).
$$

Then the proof of Theorem 1 is completed. □

4 Appendix A: A Green function in a cone domain

In this section we confirm the form of the Green function for the heat equation in the cone domain with the Dirichlet condition.

We consider the initial-boundary value problem for a heat equation

$$\begin{cases} u_t = \Delta u, & x \in D,\ t > 0, \\ u(x,0) = u_0(x), & x \in D, \\ u = 0, & x \in \partial D,\ t \geq 0, \end{cases} \tag{31}$$

where the domain D is a cone in $\mathbf{R}^N$ such as

$$D = \left\{ x \in \mathbf{R}^N : x \neq 0 \text{ and } \frac{x}{|x|} \in \Omega \right\},$$

where Ω is some region on S^{N-1} smooth enough. We introduce the Green's function $G(x,y,t) = G(r,\theta,\rho,\phi,t)$ for the linear heat equation in the cone D. By the variable transformation (4) the problem (31) is expressed the form

$$\begin{cases} u_t = \Delta u = u_{rr} + \frac{N-1}{r} u_r + \frac{\Delta_\Omega u}{r^2}, & r > 0,\ \theta \in \Omega,\ t > 0, \\ u(r,\theta,0) = u_0(r,\theta), & r > 0,\ \theta \in \Omega, \\ u = 0, & r > 0,\ \theta \in \partial\Omega, \end{cases} \tag{32}$$

where Δ_Ω is Laplace-Beltrami operator on $\Omega \subset S^{N-1}$.

For the Laplace-Beltrami operator with homogeneous Dirichlet boundary condition on $\Omega \in S^{N-1}$, define ω_n as Dirichlet eigenvalues and $\psi_n(\theta)$ as the Dirichlet eigenfunctions corresponding to ω_n which satisfies $\int_\Omega \psi_n^2(\theta) d\theta > 0$. It is following that

$$\int_\Omega \psi_m(\theta)\psi_n(\theta) d\theta = 0$$

for $m \neq n$.

It is known that the Green's function of the first equation of (31) is expressed to

$$G(r,\theta,\rho,\phi,t) = \frac{(r\rho)^{-(N-2)/2}}{2t} \exp\left(-\frac{\rho^2 + r^2}{4t}\right) \sum_{n=1}^{\infty} c_n I_{\nu_n}\left(\frac{r\rho}{2t}\right) \psi_n(\theta)\psi_n(\phi), \tag{33}$$

where $c_n = 1/\|\psi_n\|^2_{L^2(\Omega)}$ and $\nu_n = \left[(N-2)^2/4 + \omega_n\right]^{1/2}$. The function I_ν is the modified Bessel function. The functions satisfy

$$\int_0^\infty e^{-\lambda t} J_\nu(\sqrt{\lambda} r) J_\nu(\sqrt{\lambda}\rho) d\lambda = \frac{1}{t} \exp\left(-\frac{r^2 + \rho^2}{4t}\right) I_\nu\left(\frac{r\rho}{2t}\right) \tag{34}$$

with the Bessel functions J_ν satisfying

$$x^2 J_\nu''(x) + x J_\nu'(x) + (x^2 - \nu^2) J_\nu(x) = 0$$

and

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m}}{m!\Gamma(m+\nu+1)}$$

(see [29, p.p.395]).

In [18] the above fact had been shown. However, the proof is not understood easily for us. Thus in the rest of this section, the fact is confirmed.

¿From (33) and (34) we see that

$$\begin{aligned} &G(r,\theta,\rho,\phi,t) \\ &= \frac{(r\rho)^{-(N-2)/2}}{2} \sum_{n=1}^{\infty} c_n \psi_n(\theta)\psi_n(\phi) \int_0^\infty e^{-\lambda t} J_{\nu_n}(\sqrt{\lambda} r) J_{\nu_n}(\sqrt{\lambda}\rho) d\lambda. \end{aligned} \tag{35}$$

The solution of (31) is expressed to

$$u(x,t) = u(r,\theta,t) = \int_0^\infty \int_\Omega G(r,\theta,\rho,\phi,t) u_0(\rho,\phi)\rho^{N-1} d\phi d\rho. \tag{36}$$

We should confirm the fact.

Let $\tilde{u}$ be the inverse Laplace transformed function of u, i.e.

$$u(r,\theta,t) = \int_0^\infty \tilde{u}(r,\theta,s) e^{-st} ds.$$

Then this $\tilde{u}$ satisfies the following equation of the form

$$-s\tilde{u} = \tilde{u}_{rr} + \frac{N-1}{r}\tilde{u}_r + \frac{\Delta_\Omega \tilde{u}}{r^2}, \quad r>0,\ \theta\in\Omega,\ s>0. \tag{37}$$

Since $\{\psi_n/\|\psi_n\|_{L^2(\Omega)}\}$ is a complete orthonormal system, we have

$$\tilde{u}(r,\theta,s) = \sum_{n=1}^{\infty} \tilde{w}_n(r,s)\psi_n(\theta) \tag{38}$$

with

$$\tilde{w}_n(r,s) = c_n \int_\Omega \tilde{u}(r,\phi,s)\psi_n(\phi) d\phi.$$

¿From (37) and (38) we see that

$$r^2(\tilde{w}_n)_{rr} + (N-1)r(\tilde{w}_n)_r + (r^2 s - \omega_n)\tilde{w}_n = 0. \tag{39}$$

By the Frobenius method we obtain

$$\tilde{w}_n(r,s) = a_n(s) r^{-(N-2)/2} J_{\nu_n}(\sqrt{s}\, r)$$

with some $a_n(s)$. ¿From (38) we see that

$$\tilde{u}(r,\theta,s) = \sum_{n=1}^{\infty} \left\{ a_n(s) r^{-(N-2)/2} J_{\nu_n}(\sqrt{s}\ r)\psi_n(\theta) \right\}. \tag{40}$$

We thus see that

$$u(x,t) = u(r,\theta,t) = \sum_{n=1}^{\infty} \int_0^\infty a_n(s) r^{-(N-2)/2} J_{\nu_n}(\sqrt{s}r) e^{-st} ds \psi_n(\theta).$$

If we let $t = 0$, we have

$$u_0(x) = u_0(r,\theta) = \sum_{n=1}^{\infty} \int_0^\infty a_n(s) r^{-(N-2)/2} J_{\nu_n}(\sqrt{s}r) ds \psi_n(\theta).$$

Then since

$$\begin{aligned}\frac{1}{2}\int_0^\infty \int_0^\infty J_\nu(\sqrt{s}\rho) J_\nu(\sqrt{s}r) f(\rho) ds d\rho &= \int_0^\infty \int_0^\infty \sigma J_\nu(\sigma\rho) J_\nu(\sigma r) f(\rho) d\sigma d\rho \\ &= \frac{1}{r} f(r)\end{aligned}$$

for any $f \in C(0,\infty)$ (see [29, p.p.453], see also [5, §2]) and $\{\psi_n / \|\psi_n\|_{L^2(\Omega)}\}$ is a complete orthonomal system, we see that

$$a_n(s) = \frac{c_n}{2} \int_0^\infty \int_\Omega \rho^{N/2} J_{\nu_n}(\sqrt{s}\rho) u_0(\rho,\phi)\psi_n(\phi) d\phi d\rho.$$

Then we have (36). □

5 Appendix B: Proof of Lemma 2.2

In this section we prove Lemma 2.2 ([7, Lemma 3.1]) in detail. This lemma is equivalent for the following proposition:

Proposition 5.1. *Let η_a be defined in* (20) *with $a > 0$. Assume $0 \le \sigma < \min\{a, N+\gamma_+\}$. Let $\zeta > 0$ be*

$$\begin{array}{lll} (i) & \zeta = a - \sigma, & \text{if } a < N+\gamma_+, \\ (ii) & \zeta < N+\gamma_+ - \sigma, & \text{if } a = N+\gamma_+, \\ (iii) & \zeta = N+\gamma_+ - \sigma, & \text{if } a > N+\gamma_+. \end{array}$$

Then there exists a positive constant C such that

$$\langle x\rangle^\sigma \eta_a(x,t) \le C(1+t)^{-\zeta/2}\psi_1(x/|x|) \quad \text{for } x \in D, t > 0.$$

Proof. ¿From Lemma 2.1 of [10] and the fact $a > \sigma$, there exists a constant $C_1 > 0$ such that

$$\langle x\rangle^\sigma \eta_a(x,t) \le C_1 \quad \text{for } (x,t) \in D \times (0,1).$$

We should only show for $t \ge 1$.

By a direct calculation, we see that

$$r^\sigma \eta_a(x,t) = r^\sigma \int_0^\infty \int_\Omega G(r,\theta,\rho,\phi,t)(1+\rho^2)^{-a/2}\psi_1(\phi)\rho^{N-1} d\phi d\rho.$$

Since $\{\psi_n\}$ is a orthogornal system, we have

$$\begin{aligned}|x|^\sigma \eta_a(x,t) = r^\sigma &\left(\int_0^{2t/r} + \int_{2t/r}^\infty\right) \frac{(r\rho)^{-(N-2)/2}}{2t} \\ &\times \exp\left(-\frac{\rho^2+r^2}{4t}\right) I_{\nu_1}\left(\frac{r\rho}{2t}\right)(1+\rho^2)^{-a/2}\rho^{N-1} d\rho \psi_1(\theta) \\ \equiv\ &(A+B)\psi_1(\theta).\end{aligned}$$

First, we estimate A. ¿From (6) we have for some constant $C_2 > 0$

$$I_\nu(z) \le \begin{cases} C_2 z^\nu, & 0 < z \le 1, \\ C_2 z^{-1/2} e^z, & z > 1. \end{cases} \tag{41}$$

By (41) we obtain

$$\begin{aligned}A \le C_2 r^\sigma \int_0^{2t/r} \frac{(r\rho)^{-(N-2)/2}}{2t} \exp\left(-\frac{\rho^2+r^2}{4t}\right)\left(\frac{r\rho}{2t}\right)^{\nu_1} \\ \times (1+\rho^2)^{-a/2}\rho^{N-1} d\rho.\end{aligned}$$

¿From the definitions of ν_1 and γ_+, we have

$$\begin{aligned}A \le C_2 (2t)^{-N/2-\gamma_+} r^{\sigma+\gamma_+} \exp\left(-\frac{r^2}{4t}\right) \int_0^{2t/r} \exp\left(-\frac{\rho^2}{4t}\right) \\ \times \rho^{N-1+\gamma_+}(1+\rho^2)^{-a/2} d\rho.\end{aligned}$$

Putting $C_3 = 2^{-N/2-\gamma_+} C_2$, we get

$$\begin{aligned}A \le C_3 t^{(\sigma-(N+\gamma_+))/2} \left(\frac{r}{\sqrt{t}}\right)^{\sigma+\gamma_+} \exp\left(-\frac{r^2}{4t}\right) \int_0^{2t/r} \exp\left(-\frac{\rho^2}{4t}\right) \\ \times \rho^{N-1+\gamma_+}(1+\rho^2)^{-a/2} d\rho.\end{aligned}$$

Since $s^{\sigma+\gamma_+}\exp(-s^2)$ is bounded for $s>0$, there exists a constant $C_4>0$ such that

$$
\begin{aligned}
A &\le C_4 t^{(\sigma-(N+\gamma_+))/2}\int_0^{2t/r}\exp\left(-\frac{\rho^2}{4t}\right)\rho^{N-1+\gamma_+}(1+\rho^2)^{-a/2}d\rho \\
&\equiv C_4 t^{(\sigma-(N+\gamma_+))/2}E(r,t).
\end{aligned}
$$

On the hand, the case $a\le N+\gamma_+$ is considered. Since by the assumption (i) and (ii) of Lemma 2.2, $a\ge\zeta+\sigma$, we see that

$$
\begin{aligned}
E(r,t) &\le 2^{a/2}\int_0^{2t/r}\exp\left(-\frac{\rho^2}{4t}\right)\rho^{N+\gamma_+-1}(1+\rho)^{-\zeta-\sigma}d\rho \\
&\le 2^{a/2}\int_0^{2t/r}\exp\left(-\frac{\rho^2}{4t}\right)\rho^{N+\gamma_+-\zeta-\sigma-1}d\rho.
\end{aligned}
$$

Put $\xi=\rho/\sqrt{4t}$. Then we have

$$
\begin{aligned}
E(r,t) &= 2^{a/2}\int_0^{\sqrt{t}/r}\exp(-\xi^2)(\sqrt{4t}\xi)^{N+\gamma_+-\zeta-\sigma-1}\sqrt{4t}d\xi \\
&\le 2^{a/2}(\sqrt{4t})^{N+\gamma_+-\zeta-\sigma-1}\int_0^{\infty}\exp(-\xi^2)\xi^{N+\gamma_+-\zeta-\sigma-1}d\xi.
\end{aligned}
$$

Since $N+\gamma_+-\zeta-\sigma>0$, there exists a constant $C_5>0$ such that

$$
E(r,t)\le C_5 t^{\frac{1}{2}(N+\gamma_+-\zeta-\sigma-1)}.
$$

On the other hand, if $a>N+\gamma_+$,

$$
\begin{aligned}
E(r,t) &\le 2^{a/2}\int_0^{2t/r}\exp\left(-\frac{\rho^2}{4t}\right)(1+\rho)^{N+\gamma_+-a-1}d\rho \\
&\le 2^{a/2}\int_0^{\infty}\rho^{N+\gamma_+-a-1}d\rho\equiv C_6<\infty.
\end{aligned}
$$

Since $\zeta\le N+\gamma_+-\sigma$, we obtain for any $t\ge 1$

$$
A\le\max\{C_5,C_6\}t^{-\zeta/2}.
$$

Next, B is estimated. From (41) we have

$$
\begin{aligned}
B &\le C_2\left\{\int_{[2t/r,\infty)\cap[2r/3,2r]}+\int_{[2t/r,\infty)\setminus[2r/3,2r]}\right\}\left(\frac{1}{2t}\right)^{1/2}\exp\left(-\frac{(\rho-r)^2}{4t}\right) \\
&\qquad\times r^{-(N-1)/2+\sigma}\rho^{(N-1)/2-a}d\rho \\
&\equiv C_2(J+K).
\end{aligned}
$$

On one hand, we compute J. If $t \geq r^2$ then $J = 0$. When $t < r^2$, since $\rho \in [2r/3, 2r]$ we see that

$$\begin{aligned} J &\leq \int_{2r/3}^{2r} \left(\frac{1}{2t}\right)^{1/2} \exp\left(-\frac{(\rho-r)^2}{4t}\right) \left(\frac{\rho}{r}\right)^{(N-1)/2} \left(\frac{r}{\rho}\right)^{a} r^{\sigma-a} d\rho \\ &\leq 2^{(N-1)/2} \left(\frac{3}{2}\right)^{a} t^{(\sigma-a)/2} \int_{-\infty}^{\infty} \left(\frac{1}{2t}\right)^{1/2} \exp\left(-\frac{(\rho-r)^2}{4t}\right) d\rho \\ &\leq C_7 t^{-(a-\sigma)/2} \end{aligned}$$

with some constant $C_7 > 0$. On the other hand, we estimate K. Since $\rho \in [2t/r, \infty)/[2r/3, 2r]$, we have $|\rho - r| > \max\{r/3, \rho/2\}$. We thus obtain

$$-\frac{(\rho-r)^2}{4t} = -\frac{(\rho-r)^2}{8t} - \frac{(\rho-r)^2}{8t} \leq -\frac{\rho^2}{32t} - \frac{r^2}{72t} \tag{42}$$

and

$$\rho \leq \frac{2t}{r} \quad \text{and} \quad r \leq \frac{2t}{\rho}. \tag{43}$$

¿From (42) and (43) we obtain

$$\begin{aligned} K &\leq \int_{[2t/r,\infty)/[2r/3,2r]} \left(\frac{1}{2t}\right)^{1/2} \exp\left(-\frac{\rho^2}{32t} - \frac{r^2}{72t}\right) \left(\frac{2t}{\rho}\right)^{-(N-1)/2} \\ &\qquad\qquad \times r^{\sigma} \rho^{(N-1)/2} \left(\frac{2t}{r}\right)^{-a} d\rho \\ &\leq (2t)^{-(N-1)/2-a} \exp\left(-\frac{r^2}{72t}\right) \left(\frac{r}{\sqrt{t}}\right)^{\sigma+a} (\sqrt{t})^{\sigma+a+N-1} \\ &\qquad \times \int_0^{\infty} \left(\frac{1}{2t}\right)^{1/2} \exp\left(-\frac{\rho^2}{32t}\right) \left(\frac{\rho}{\sqrt{t}}\right)^{N-1} d\rho. \end{aligned}$$

So, there exists a constant $C_8 > 0$ such that

$$K \leq C_8 t^{-(N-1)/2-a+\sigma/2+a/2+(N-1)/2} = C_8 t^{-(a-\sigma)/2}.$$

Then we have

$$B \leq \max\{C_7, C_8\} t^{-(a-\sigma)/2}$$

for any $t \geq 1$. On the other hand from the definition of ζ we have $\zeta \leq a - \sigma$. We thus have

$$\begin{aligned} |x|^{\sigma} \eta_a(x,t) &\leq \max\{C_1, C_5, C_6, C_7, C_8\} \max\{1, t\}^{-\zeta/2} \psi_1(\theta) \\ &\leq C_9 (t+1)^{-\zeta/2} \psi_1(\theta) \end{aligned}$$

with some constant $C_9 > 0$. Since $\langle x \rangle \le |x| + 1$, we see that

$$\langle x \rangle^\sigma \eta_a(x,t) \le C_9(t+1)^{-\zeta/2}\psi_1(x/|x|)$$

for $(x,t) \in D \times (0,\infty)$. $\square$

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The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

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