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Nonexistence of Levi-degenerate hypersurfaces of constant signature in $\mathbf{CP}^n$

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Abstract. Let M be a smooth hypersurface of constant signature in $\mathbf{CP}^n$, $n\geq 3$. We prove the regularity for $\bar\partial_b$ on M in bidegree $(0,1)$. As a consequence, we show that there exists no smooth hypersurface in $\mathbf{CP}^n$, $n\geq 3$, whose Levi form has at least two zero-eigenvalues.

1. Introduction

The question of nonexistence of smooth Levi-degenerate hypersurfaces in $\mathbf{CP}^n$ has attracted a great interest in recent years and takes its origin in the foliation theory, [4] and [9]. Several nonexistence results were obtained for *Levi-flat* CR manifolds. Recall that a CR manifold M is called *Levi flat*, if there exists a local foliation of M by complex manifolds, whose dimension coincides with the CR dimension of M. In the hypersurface case this is equivalent to vanishing of the Levi form of M. The nonexistence of *real-analytic* Levi-flat hypersurfaces in projective spaces for $n\geq 3$ was discussed by Lins Neto in [9]. Siu has shown that there does not exist any C^8*-smooth* Levi-flat hypersurface in $\mathbf{CP}^n$ for $n\geq 2$, [12] and [13]. The regularity assumption on the hypersurface was relaxed to C^4 by Iordan for $n\geq 2$, [8]. The conjecture of Siu on the nonexistence of *higher codimensional* smooth Levi-flat CR manifolds in compact symmetric spaces was proved by Brinkschulte in [2]. Cao and Shaw have proved that there does not exist any *Lipschitz* Levi-flat hypersurface in $\mathbf{CP}^n$ for $n\geq 3$, [3]. The case $n=2$ is still open.

We show that in the more general case of *Levi-degenerate* manifolds (whose Levi form does not necessarily vanish) the nonexistence depends essentially on the signature of the hypersurface. In our paper we use Siu's idea to reduce the problem to the regularity of the tangential Cauchy–Riemann operator $\bar\partial_b$ in bidegree $(0,1)$. One way to prove the regularity is to derive this from L^2-weighted estimates using the Bochner–Kodaira–Nakano inequality. However there are some obstructions to prove the desired estimates in terms of the usual Fubini–Study metric of $\mathbf{CP}^n$. Following Brinkschulte's arguments [2] we construct a new metric in $\mathbf{CP}^n \setminus M$, which provides good estimates for the curvature term in the Bochner–Kodaira–Nakano inequality at points near to the hypersurface. To extend the obtained estimates over the whole projective space we use Ohsawa's method of pseudo-Runge pairs.

Theorem 1.1. *Let M be a smooth real hypersurface in $\mathbf{CP}^n$, $n \geq 3$, having a constant signature (q^-, q^0, q^+) with $q^0 + \min\{q^-, q^+\} \geq 2$ and $q^0 \geq 1$.*

If $f \in C^\infty_{0,1}(M) \cap \operatorname{Ker} \bar\partial_b$, then for every $k \in \mathbf{N}$ there exists $u \in C^k(M)$ such that $\bar\partial_b u = f$ on M.

The above regularity implies our main nonexistence result.

Theorem 1.2. *There exists no smooth real hypersurface in $\mathbf{CP}^n$, $n \geq 3$, whose Levi form has constant signature and satisfies one of the following conditions*:

(i) *the Levi form has at least two zero eigenvalues*;

(ii) *the Levi form has at least one zero eigenvalue and two eigenvalues of opposite signs.*

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2. Construction of the new metrics

We consider the complex projective space $\mathbf{CP}^n$, $n \geq 3$, with its standard Fubini–Study metric ω_{FS}. Let M be a smooth closed real hypersurface in $\mathbf{CP}^n$ represented as

$$M = \{z \in U : \rho(z) = 0\},$$

where $\rho: U \to \mathbf{R}$ is a smooth function in an open neighborhood $U \subset \mathbf{CP}^n$ of M, and $d\rho(z) \neq 0$ on M. The hypersurface M divides $\mathbf{CP}^n$ into two sets Ω^+ and Ω^- with $\Omega^- \cap U = \{z \in U : \rho(z) < 0\}$ and $\Omega^+ \cap U = \{z \in U : \rho(z) > 0\}$. We denote by (q^-, q^0, q^+) the signature of M, that means that the Levi form $\mathcal{L}_z(\rho)$ of ρ has exactly q^- negative, q^0 zero and q^+ positive eigenvalues on $T^{1,0}_z M$ at each point $z \in M$. Clearly, $q^- + q^0 + q^+ = n - 1$. Next, we denote by $\delta_M(z)$ the Fubini distance from a point $z \in \mathbf{CP}^n$ to the hypersurface M and by K a compact set in $\mathbf{CP}^n$ that contains all the points near which $\delta_M(z)$ fails to belong to C^2.

It was proved by Matsumoto [10], that *if Ω is a weakly q-convex set in $\mathbf{CP}^n$, $1 \leq q \leq n$, then the Levi form of $-\log \delta_M$ has at least $q+1$ positive eigenvalues in*

$\Omega\setminus K$. Note that the set Ω^- is weakly (q^0+q^+)-convex, while the set Ω^+ is weakly (q^0+q^-)-convex. Hence there exists a constant $C>0$ such that the Levi form $i\partial\bar\partial(-\log\delta_M(z))$ has at least q^0+q^++1 positive eigenvalues in $\Omega^-\setminus K$, which are not less than C, and the corresponding observation holds for $\Omega^+\setminus K$.

In what follows we consider the problem in Ω^-, unless otherwise indicated. Let us denote by $\gamma_1\le\gamma_2\le...\le\gamma_n$ the eigenvalues of $i\partial\bar\partial(-\log\delta_M(z))$ with respect to ω_{FS}. As

$$i\partial\bar\partial(-\log\delta_M(z))=-\frac{i}{\delta_M}\partial\bar\partial\delta_M+\frac{i}{\delta_M^2}\partial\delta_M\wedge\bar\partial\delta_M,$$

there exists a positive constant N such that

$$\begin{aligned}&\gamma_1\le...\le\gamma_{q^-}\le-\frac{N}{\delta_M},\\ &N\le\gamma_{q^-+1}\le...\le\gamma_{q^-+q^0},\\ &\frac{N}{\delta_M}\le\gamma_{q^-+q^0+1}\le...\le\gamma_n\end{aligned}\tag{1}$$

at points Ω^- near M. By considering a larger compact set K if necessary, we can assume that these inequalities are valid in the whole domain $\Omega^-\setminus K$. Furthermore, it follows from Lemma 2.1 in [1], that for each fixed $z_0\in M$ there is a neighborhood $U_{z_0}\subset\mathbf{CP}^n$ of z_0 and a smooth extension T_z of $T^{1,0}M$ on U_{z_0} such that

$$\mathcal{M}(z):=i\partial\bar\partial\delta_M|_{T_z}=\mathcal{M}^-(z)\oplus\mathcal{M}^0(z)\oplus\mathcal{M}^+(z),\quad z\in U_{z_0},$$

and the eigenvalues of $\mathcal{M}^-(z)$ are the q^- smallest eigenvalues of $\mathcal{M}(z)$ while the eigenvalues of $\mathcal{M}^+(z)$ are the q^+ biggest. Since the Levi form $\mathcal{L}_z(\rho)$ has exactly q^0 zero eigenvalues on $T^{1,0}_zM$ at each point $z\in M$, it follows that $\mathcal{M}^0(z)\equiv0$ for all $z\in M\cap U_{z_0}$. Hence there exists a positive number N' such that the eigenvalues of $\mathcal{M}^0(z)$ satisfy

$$\gamma_{q^-+1}\le...\le\gamma_{q^-+q^0}\le N'$$

near M.

We show below that in order to derive good L^2-estimates one needs to control the sum of some eigenvalues of $i\partial\bar\partial(-\log\delta_M)$. It follows from (1) that the negative eigenvalues of $i\partial\bar\partial(-\log\delta_M)$ with respect to ω_{FS} are not bounded from below and hence one cannot control the above mentioned sum in terms of ω_{FS}. Thus we need to construct a new Hermitian metric.

Let us choose a smooth positive function $\theta\in C^\infty(\mathbf{R},\mathbf{R})$ such that

$$\theta(\gamma)=\begin{cases}-n\gamma & \text{for } \gamma\le-N,\\ N & \text{for } 0\le\gamma\le N',\\ \gamma & \text{for } \gamma\ge N'+1.\end{cases}$$

We consider a new Hermitian metric ω_M defined by the Hermitian endomorphism $\theta(A_{\omega_{\rm FS}}(z))$, where $A_{\omega_{\rm FS}}(z)$ is the Hermitian endomorphism associated with the Hermitian form $i\partial\bar\partial(-\log\delta_M)$ with respect to $\omega_{\rm FS}$. The eigenvalues $\sigma_\nu(z):=\theta(\gamma_\nu(z))$ of $\theta(A_{\omega_{\rm FS}}(z))$ satisfy

$$\begin{aligned} \sigma_1(z)&=n|\gamma_1(z)|, &&..., &\sigma_{q^-}(z)&=n|\gamma_{q^-}(z)|,\\ \sigma_{q^-+1}(z)&=N, &&..., &\sigma_{q^-+q^0}(z)&=N,\\ \sigma_{q^-+q^0+1}(z)&=\gamma_{q^-+q^0+1}(z), &&..., &\sigma_n(z)&=\gamma_n(z)\end{aligned}$$

on $\Omega^-\setminus K$. For the eigenvalues $\lambda_\nu(z)$ of $i\partial\bar\partial(-\log\delta_M)$ with respect to ω_M we have

$$\lambda_\nu(z)=\frac{\gamma_\nu(z)}{\sigma_\nu(z)}\begin{cases}=-1/n & \text{for } \nu\le q^-,\\ \ge 1 & \text{for } \nu>q^-,\end{cases}$$

and hence

$$\lambda_1+...+\lambda_r\ge 1-\frac{q^-}{n}\ge\frac{1}{n}\quad\text{for all } r\ge q^-+1. \tag{2}$$

The corresponding observations yield an analogous estimate in the weakly (q^0+q^-)-convex domain Ω^+ for $r\ge q^++1$.

It follows from the construction that ω_M is comparable with the Fubini–Study metric, i.e. there are positive constants a and b such that

$$a\omega_{\rm FS}\le\omega_M\le b\delta_M^{-2}\omega_{\rm FS}.$$

Moreover, the metric ω_M, constructed in such a way, is complete and $\partial\omega_M$ is bounded with respect to ω_M (see [1]).

All the above considerations lead us to the following result.

Proposition 2.1. *There exists a Hermitian metric ω_M on $\mathbf{CP}^n\setminus M$ with the following properties*:

(i) *there exists $\tau>0$ such that the eigenvalues $\lambda_1\le...\le\lambda_n$ of $i\partial\bar\partial(-\log\delta_M)$ with respect to ω_M satisfy*

$$\lambda_1+...+\lambda_r>\tau$$

on $\mathbf{CP}^n\setminus K$ for all $r\ge\min\{q^-,q^+\}+1$;

(ii) *there are positive constants a and b such that $a\omega_{\rm FS}\le\omega_M\le b\delta_M^{-2}\omega_{\rm FS}$;*

(iii) *there is a positive constant C such that $|\partial\omega_M|_{\omega_M}<C$.*

3. L^2-existence in $\mathbf{CP}^n\backslash M$

We emphasize that (2) holds only outside the exceptional set $K\subset\mathbf{CP}^n\backslash M$. In this section we construct a special family of complete Hermitian metrics, which allow us to extend the estimates over the whole $\mathbf{CP}^n\backslash M$. First, let us recall some definitions from the theory of pseudo-Runge pairs, [11].

Let X be an n-dimensional complex manifold and E be a holomorphic vector bundle over X. Let X_1 and X_2 be two open subsets of X.

The pair (X_1, X_2) is called a *pseudo-Runge pair at bidegree* (p,q) *with respect to* E if $X_1\subset X_2$ and there exists a complete Hermitian metric ω_0 on X_1 and a Hermitian metric φ_0 along the fibers of $E|_{X_1}$, a sequence of complete Hermitian metrics $\{\omega_k\}_{k\in\mathbf{N}}$ on X_2 and a sequence of Hermitian metrics $\{\varphi_k\}_{k\in\mathbf{N}}$ along the fibers of $E|_{X_2}$, such that the following properties are satisfied:

(a) ω_k, φ_k and their derivatives converge respectively to ω_0, φ_0 and to their derivatives uniformly on each compact subset of X_1;

(b) there exist a compact subset $K\subset X$ and a constant $C_1>0$ such that the basic estimate

$$\|f\|^2_{h_k,\omega_k}\le C_1\biggl(\|\bar\partial f\|^2_{h_k,\omega_k}+\|\bar\partial^*_k f\|^2_{h_k,\omega_k}+\int_K |f|^2_{\omega_k}\,dV_{\omega_k}\biggr) \tag{3}$$

holds for any compactly supported $f\in C^\infty_{p,q+1}(X_2,E)$ and any $k\ge1$. Here the subindices mean that we use the metrics h_k and ω_k, and $\bar\partial^*_k$ denotes the adjoint with respect to the metrics ω_k and φ_k (the more precise definition of the adjoint operator will be given below);

(c) there is a constant C_2 such that

$$\|v|_{X_1}\|_{h_0,\omega_0}\le C_2\|v\|_{h_k,\omega_k}$$

for any compactly supported $v\in C^\infty_{p,q+j}(X_2,E)$, $j=1,2$.

Let us fix some $\alpha>0$ that is less than the distance between M and K. We set $X_\alpha:=\{z\in\mathbf{CP}^n:\mathrm{dist}(z,M)>\alpha\}$. Our goal is to show that $(X_\alpha, \mathbf{CP}^n\backslash M)$ is a pseudo-Runge pair at bidegree (n,q) for $q\ge\max\{q^-,q^+\}+1$.

Proposition 3.1. *There exists a complete Hermitian metric* ω_0, *a weight function* φ_0 *on* X_α, *a sequence of complete Hermitian metrics* $\{\omega_k\}_{k\in\mathbf{N}}$ *and a sequence of weight functions* $\{\varphi_k\}_{k\in\mathbf{N}}$ *on* $\mathbf{CP}^n\backslash M$, *satisfying the following conditions*:

(i) ω_k, φ_k *and their derivatives converge respectively to* ω_0, φ_0 *and to their derivatives uniformly on each compact subset of* X_α;

(ii) $\omega_k\le\omega_{k+1}\le\omega_0$ *and* $\varphi_k\le\varphi_{k+1}\le\varphi_0$ *for all* $k\in\mathbf{N}$;

(iii) *there exists a positive constant* τ (*independent of* k) *such that the eigenvalues* $\lambda_1^k \le \ldots \le \lambda_n^k$ *of* $i\partial\bar\partial(-\log\delta_M)$ *with respect to* ω_k *satisfy*

$$\lambda_1^k + \ldots + \lambda_r^k > \tau$$

on $\mathbf{CP}^n \setminus (M \cup K)$ *for all* $r \ge \max\{q^-, q^+\}+1$;

(iv) *there exists a positive constant* A (*independent of* k) *such that the eigenvalues* $c_1^k \le \ldots \le c_n^k$ *of* $i\partial\bar\partial\varphi_k$ *with respect to* ω_k *satisfy*

$$c_1^k + \ldots + c_r^k \ge -A + 2|\partial\omega_k|_{\omega_k}$$

on $\mathbf{CP}^n \setminus M$ *for all* $r \ge \max\{q^-, q^+\}+1$;

(v) *there are positive constants* a *and* b *such that*

$$a\omega_{\mathrm{FS}} \le \omega_k \le \frac{b\omega_{\mathrm{FS}}}{\delta_M^2} \quad \text{and} \quad a \le e^{\varphi_k} \le \frac{b}{\delta_M^2}$$

for all $k \in \mathbf{N}$.

Proof. Let us set $\beta := -\log\alpha$. For every $k \in \mathbf{N}$ we can choose a smooth increasing function $\chi_k(t) \in C^\infty(\mathbf{R})$ satisfying

$$\chi_k(t) = \frac{1}{\beta + 1/k - t}, \quad t \in (-\infty, \beta],$$

and such that $(\chi_k')^2(t) \le \chi_k^3(t)$ on $\mathbf{R}$. Let us set

$$\omega_k := \omega_M + \chi_k(-\log\delta_M)\partial\delta_M \wedge \bar\partial\delta_M$$

and

$$\varphi_k(z) := l \int_{\inf(-\log\delta_M(z))}^{-\log\delta_M(z)} \chi_k(t)\,dt, \tag{4}$$

where $l \in \mathbf{N}$ is large enough. Then the metrics $\{\omega_k\}_{k\in\mathbf{N}}$ are complete on $\mathbf{CP}^n \setminus M$, and on each compact subset of X_α the sequences $\{\omega_k\}_{k\in\mathbf{N}}$, $\{\varphi_k\}_{k\in\mathbf{N}}$ and their derivatives converge uniformly to the complete metric

$$\omega_0 := \omega_M + (\beta + \log\delta_M)^{-2}\partial\delta_M \wedge \bar\partial\delta_M,$$

$$\varphi_0 := \frac{l}{\beta + \log\delta_M} - \frac{l}{\beta - \inf(-\log\delta_M)}$$

and their derivatives, respectively. Thus the properties (i), (ii) and (v) of Proposition 3.1 are verified.

Let $\lambda_1^k \le ... \le \lambda_n^k$ be the eigenvalues of $i\partial\bar\partial(-\log\delta_M)$ with respect to the metric ω_k. By the minimax principle we have $\lambda_n^k \ge \lambda_{n-1}$ and $\lambda_1^k = \lambda_1, ..., \lambda_{n-1}^k = \lambda_{n-1}$ and we get the inequality in (iii).

The eigenvalues of

$$i\partial\bar\partial\varphi_k = l\chi_k(-\log\delta_M)\partial\bar\partial(-\log\delta_M) + l\chi_k'(-\log\delta_M)\partial\log\delta_M \wedge \log\delta_M$$

with respect to ω_M are not less than the eigenvalues

$$l\chi_k(-\log\delta_M)\lambda_1, ..., l\chi_k(-\log\delta_M)\lambda_n$$

of $l\chi_k(-\log\delta_M)\partial\bar\partial(-\log\delta_M)$ with respect to ω_M and hence for the eigenvalues $c_1^k \le ... \le c_n^k$ of $\partial\bar\partial\varphi_k$ with respect to ω_k we have

$$c_1^k + ... + c_r^k \ge l\chi_k(-\log\delta_M)\tau$$

for some $\tau > 0$ and all $r \ge \max\{q^-, q^+\} + 1$.

It follows from the construction of ω_k that

$$\begin{aligned} |\partial\omega_k|_{\omega_k} &= |\partial\omega_M - \chi_k(-\log\delta_M)\partial\bar\partial\delta_M \wedge \partial\delta_M|_{\omega_k} \\ &\le C + |\partial\bar\partial\delta_M|_{\omega_M} |\chi_k\partial\delta_M|_{\chi_k(-\log\delta_M)\bar\partial\delta_M\wedge\partial\delta_M} \\ &\le C_1 + C_2\sqrt{\chi_k(-\log\delta_M)} \end{aligned} \tag{5}$$

for some C_1 and C_2 independent of k. Choosing l large enough we obtain (iv).

Proposition 3.1 is proved. □

Given a domain $\Omega \subset \mathbf{CP}^n$, we denote by $L^2_{p,q}(\Omega, N, k)$, $N \in \mathbf{Z}$, $k = 0, 1, ...,$ the Hilbert space of (p,q)-forms f on Ω with the norm

$$\|f\|^2_{N,k} := \int_\Omega |f|^2_{\omega_k} \delta_M^N e^{-\varphi_k}\, dV_{\omega_k} < +\infty.$$

For each k we denote by $\bar\partial^*_{N,k}$ the Hilbert space adjoint of $\bar\partial$ with respect to the corresponding global inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{N,k}$ of the space $L^2_{p,q}(\Omega, N, k)$. Here we say that $f \in \operatorname{Dom} \bar\partial^*_{N,k}$ if and only if $\bar\partial^*_{N,k} f$, defined in the sense of distribution theory, belongs to $L^2_{p,q}(\Omega, N, k)$.

Taking into account (ii) from Proposition 3.1 and the monotonicity of $|f|^2_{\omega_k}\, dV_{\omega_k}$ (with respect to ω_k) for $p = n$ we obtain

$$|f|^2_{\omega_{k+1}}\, dV_{\omega_{k+1}} \le |f|^2_{\omega_k}\, dV_{\omega_k}$$

for every $k \in \mathbf{N}$. Therefore

$$L^2_{n,q}(\mathbf{CP}^n \setminus M, N, k) \subset L^2_{n,q}(\mathbf{CP}^n \setminus M, N, k+1)$$

and

$$\|f|_{X_\alpha}\|_{N,0} \le \|f\|^2_{N,k}, \quad f \in L^2_{n,q}(\mathbf{CP}^n \setminus M, N, k)$$

for every $k \in \mathbf{N}$.

Thus, in order to show that $(X_\alpha, \mathbf{CP}^n \setminus M)$ is a pseudo-Runge pair at bidegree (n,q) it only remains to prove the basic estimate (3). First note that the curvature of the metric

$$\delta_M^N e^{-\varphi_k} = e^{-(-N \log \delta_M + \varphi_k)}$$

is equal to $-N\partial\bar\partial \log \delta_M + \partial\bar\partial\varphi_k$. In view of the Bochner–Kodaira–Nakano inequality [6], we have

$$\begin{aligned}\tfrac{3}{2}(\|\bar\partial f\|^2_{N,k} + \|\bar\partial^*_{N,k} f\|^2_{N,k}) \ge N\langle\!\langle(\lambda_1^k + \dots + \lambda_q^k)f, f\rangle\!\rangle_{N,k} + \langle\!\langle(c_1^k + \dots + c_q^k)f, f\rangle\!\rangle_{N,k} \\ - \tfrac{1}{2}(\|\tau_k f\|^2_{N,k} + \|\bar\tau_k f\|^2_{N,k} + \|\tau_k^* f\|^2_{N,k} + \|\bar\tau_k^* f\|^2_{N,k})\end{aligned}$$

for all smooth compactly supported (n,q)-forms f in $\mathbf{CP}^n \setminus M$. Here $\tau_k = [\Lambda_k, \partial\omega_k]$ is the torsion operator defined by the torsion form $\partial\omega_k$ and the adjoint operator Λ_k of the left multiplication by ω_k.

The completeness of the metrics ω_k yields the extension of this result to every $f \in \operatorname{Dom} \bar\partial_{N,k} \cap \operatorname{Dom} \bar\partial^*_{N,k}$ by the density of compactly supported forms. Hence it is enough to prove the basic estimate only for compactly supported forms in $\mathbf{CP}^n \setminus M$.

It follows from (iii) and (iv) in Proposition 3.1 that there is a constant $A' > 0$ independent of k such that

$$\tfrac{3}{2}(\|\bar\partial f\|^2_{N,k} + \|\bar\partial^*_{N,k} f\|^2_{N,k}) \ge (\tau N - A')\|f\|^2_{N,k} \tag{6}$$

for all smooth (n,q)-forms f, $q \ge \max\{q^-, q^+\} + 1$, with compact support in $\mathbf{CP}^n \setminus (M \cup \overline{K})$.

In order to get an estimate for forms supported on $\mathbf{CP}^n \setminus M$ we consider compact sets K_1 and K_2 with $K \subset K_1 \subset K_2 \subset X_\alpha$ and a smooth function χ on $\mathbf{CP}^n \setminus M$ satisfying

$$\chi = \begin{cases} 0 & \text{on } K_1, \\ 1 & \text{outside } K_2. \end{cases}$$

Let us apply the inequality (6) to χf. There are some positive constants A'' and N_0, such that for all $N \ge N_0$ and all $k \in \mathbf{N}$ the inequality

$$\|f\|^2_{N,k} \le \|\bar\partial f\|^2_{N,k} + \|\bar\partial^*_{N,k} f\|^2_{N,k} + NA'' \int_{K_2} |f|^2_{\omega_k} \delta_M^N \, dV_{\omega_k} \tag{7}$$

holds for all those smooth (n,q)-forms f, $q \ge \max\{q^-, q^+\} + 1$, that are compactly supported in $\mathbf{CP}^n \setminus M$.

Thus the basic estimate is proved. Together with Proposition 3.1 this implies that $(X_\alpha, \mathbf{CP}^n \setminus M)$ is a pseudo-Runge pair at bidegree (n,q), $q \ge \max\{q^-, q^+\} + 1$.

The next lemma follows from the definition of pseudo-Runge pairs.

Lemma 3.2. *There are integers N_0, k_0 and a positive constant C such that for any $f\in L^2_{n,q}(\mathbf{CP}^n\setminus M,N,k)$, $N>N_0$, $k\geq k_0$ with $q\geq\max\{q^-,q^+\}+1$, satisfying*

$$f|_{X_\alpha}\perp \operatorname{Ker}\bar\partial\cap\operatorname{Ker}\bar\partial^*_{N,k}$$

the following inequality holds:

$$\|f\|^2_{N,k}\leq C(\|\bar\partial f\|^2_{N,k}+\|\bar\partial^*_{N,k}f\|^2_{N,k}). \tag{8}$$

Proof. Assume that the assertion is false. Then there exists a sequence $f_k\in L^2_{n,q}(\mathbf{CP}^n\setminus M,N,k)$ satisfying

$$\|f_k\|_{N,k}=1,\quad \|\bar\partial f_k\|_{N,k}<\frac{1}{k},\quad \|\bar\partial^*_{N,k}f_k\|_{N,k}<\frac{1}{k}\quad\text{and}\quad f_k|_{X_\alpha}\perp\operatorname{Ker}\bar\partial\cap\operatorname{Ker}\bar\partial^*_{N,0}.$$

Since $\|f_k|_{X_\alpha}\|_{N,0}\leq\|f_k\|_{N,k}=1$, there exists a subsequence, which we again denote by f_k, which converges weakly to some f in $L^2_{n,q}(X_\alpha,N,0)$ such that $f_k\to f$ strongly on a common exceptional set K_2 of the basic estimate (7). As

$$f_k|_{X_\alpha}\perp\operatorname{Ker}\bar\partial\cap\operatorname{Ker}\bar\partial^*_{N,0},$$

we have $f\perp\operatorname{Ker}\bar\partial\cap\operatorname{Ker}\bar\partial^*_{N,0}$ by weak L^2-convergence. Moreover, we also have that $\bar\partial f_k\to\bar\partial f$ and $\bar\partial^*_{N,k}f_k\to\bar\partial^*_{N,0}f$ weakly and hence $\bar\partial f=\bar\partial^*_{N,0}f=0$. Thus $f\equiv 0$.

On the other hand, by the strong convergence on K_2 we have

$$\begin{aligned}\int_{K_2}|f|^2_{\omega_0}\delta^N_M\,dV_{\omega_0}&=\lim_{k\to\infty}\int_{K_2}|f_k|^2_{\omega_k}\delta^N_M\,dV_{\omega_k}\\&\geq\lim_{k\to\infty}\frac{1}{NA''}(\|f_k\|^2_{N,k}-\|\bar\partial f_k\|^2_{N,k}-\|\bar\partial^*_{N,k}f_k\|^2_{N,k})\\&\geq\lim_{k\to\infty}\frac{1}{NA''}\left(1-\frac{2}{k^2}\right)\\&=\frac{1}{NA''}.\end{aligned}$$

Therefore $f\neq 0$ and we obtain a contradiction. □

Observe that $f|_{X_\alpha}\perp\operatorname{Ker}\bar\partial\cap\operatorname{Ker}\bar\partial^*_{N,0}$ for every $f\in\bar\partial L^2_{n,q-1}(\mathbf{CP}^n\setminus M,N,k)$ and therefore (8) implies

$$\begin{aligned}&\bar\partial L^2_{n,q-1}(\mathbf{CP}^n\setminus M,N,k)\\&\quad=\operatorname{Ker}\bar\partial\cap\{f\in L^2_{n,q}(\mathbf{CP}^n\setminus M,N,k):\ f|_{X_\alpha}\perp\operatorname{Ker}\bar\partial\cap\operatorname{Ker}\bar\partial^*_{N,k}\}.\end{aligned}\tag{9}$$

This yields the following result.

Proposition 3.3. *For every $N \geq N_0$ there exists an integer k_0 such that for any $k \geq k_0$ and each $f \in \bar{\partial} L^2_{n,q-1}(\mathbf{CP}^n \setminus M, N, k)$, $q \geq \max\{q^-, q^+\}+1$, there exists $u \in L^2_{n,q-1}(\mathbf{CP}^n \setminus M, N, k)$ such that $\bar{\partial} u = f$ and*

$$\|u\|_{N,k} \leq C\|f\|_{N,k}.$$

The next proposition gives the dual version of Proposition 3.3. We denote by $L^2_{0,q}(\mathbf{CP}^n \setminus M, -N, -k)$, $k \in \mathbf{N}$, the Hilbert space of all $(0,q)$-forms u on $\mathbf{CP}^n \setminus M$ with the norm

$$\|u\|^2_{-N,-k} := \int_{\mathbf{CP}^n \setminus M} |u|_{\omega_k} \delta_M^{-N} e^{\varphi_k} \, dV_{\omega_k} < +\infty.$$

Proposition 3.4. *For every $N \geq N_0$ there exists an integer k_0 such that for any $k \geq k_0$ and each $f \in L^2_{0,q}(\mathbf{CP}^n \setminus M, -N, -k) \cap \operatorname{Ker} \bar{\partial}$, with $q \leq \min\{q^-, q^+\}+q^0$ and $f|_{X_\alpha} \equiv 0$, there exists $u \in L^2_{0,q-1}(\mathbf{CP}^n \setminus M, -N, -k)$ such that $\bar{\partial} u = f$.*

Proof. We consider $g = \sharp_{N,k} f \in L^2_{n,n-q}(\mathbf{CP}^n \setminus M, N, k)$, where

$$\sharp_{N,k} \colon \Lambda^{0,q} T^*(\mathbf{CP}^n \setminus M) \longrightarrow \Lambda^{n,n-q} T^*(\mathbf{CP}^n \setminus M)$$

is the conjugate-linear operator defined by

$$\int_{\mathbf{CP}^n \setminus M} u \wedge \sharp_{N,k} v = \langle\langle u, v \rangle\rangle_{N,k}.$$

Then we have $g \in \operatorname{Ker} \bar{\partial}^*_{N,k}$ and $g|_{X_\alpha} \equiv 0$. Hence $g|_{X_\alpha} \perp \operatorname{Ker} \bar{\partial} \cap \operatorname{Ker} \bar{\partial}^*_{N,0}$.

Next we consider the linear form

$$L_g \colon \bar{\partial} L_{n,n-q}(\mathbf{CP}^n \setminus M, N, k) \longrightarrow \mathbf{C},$$
$$\bar{\partial}\varphi \longmapsto \langle\langle g, \varphi \rangle\rangle_{N,k}.$$

To see that L_g is well defined we consider $\varphi \in L^2_{n,n-q}(\mathbf{CP}^n \setminus M, N, k) \cap \operatorname{Ker} \bar{\partial}$. Since $g|_{X_\alpha} \perp \operatorname{Ker} \bar{\partial} \cap \operatorname{Ker} \bar{\partial}^*_{N,0}$ we may also assume that

$$\varphi|_{X_\alpha} \perp \operatorname{Ker} \bar{\partial} \cap \operatorname{Ker} \bar{\partial}^*_{N,0},$$

and hence, taking into account that $q^\pm + q^0 = n - (q^\mp + 1)$, we conclude from Proposition 3.3 that there exists $\psi \in L^2_{n,n-q-1}(\mathbf{CP}^n \setminus M, N, k)$ satisfying $\bar{\partial}\psi = \varphi$. Then for $g \in \operatorname{Ker} \bar{\partial}^*_{N,k}$ we have

$$\langle\langle g, \varphi \rangle\rangle_{N,k} = \langle\langle g, \bar{\partial}\psi \rangle\rangle_{N,k} = \langle\langle \bar{\partial}^*_{N,k} g, \psi \rangle\rangle = 0.$$

Now let $\varphi\in L^2_{n,n-q}(\mathbf{CP}^n\setminus M,N,k)\cap\operatorname{Dom}\bar\partial$. Then by Proposition 3.3 there exists $\widetilde\varphi\in L^2_{n,n-q}(\mathbf{CP}^n\setminus M,N,k)$ such that $\bar\partial\widetilde\varphi=\bar\partial\varphi$ and $\|\widetilde\varphi\|_{N,k}\le C\|\bar\partial\varphi\|_{N,k}$. This implies that L_g is continuous in L^2-norm and its norm is $\le C\|g\|_{N,k}=C\|f\|_{-N,-k}$. Hence there exists $v\in L^2_{n,n-q+1}(\mathbf{CP}^n\setminus M,N,k)$ with $\|v\|_{N,k}\le C\|f\|_{-N,-k}$ such that

$$\langle\!\langle v,\bar\partial\varphi\rangle\!\rangle_{N,k}=\langle\!\langle g,\varphi\rangle\!\rangle_{N,k}$$

for $\varphi\in L^2_{n,n-q}(\mathbf{CP}^n\setminus M,N,k)\cap\operatorname{Dom}\bar\partial$, i.e. $\bar\partial^*_{N,k}v=g$. Setting $u=\sharp^{-1}_{N,k}v$ we obtain $u\in L^2_{0,q-1}(\mathbf{CP}^n\setminus M,-N,-k)$ with $\bar\partial u=f$. $\square$

4. Regularity of $\bar\partial_b$ in bidegree $(0,1)$

Let us first recall some definitions related to foliation theory. Let M be a CR submanifold in $\mathbf{CP}^n$ given locally by a defining function $\rho\colon U\to\mathbf{R}^k$, $U\subset\mathbf{CP}^n$, with $\partial\rho_1\wedge...\wedge\partial\rho_k\neq0$ on $M\cap U$.

A smooth local *foliation* of M is a family of pairwise disjoint complex submanifolds of M, all of the same dimension, which exhaust an open subset $M\cap U$ and varies smoothly with a multivariate real parameter. The existence and uniqueness of such a foliation depends on the geometric properties of the hypersurface. As we know, any *Levi-flat* CR manifold M is locally foliated by complex submanifolds. Furthermore, for each $z\in M$, the space $T^{1,0}_zM$ is the complex tangent space at z of the leaf of the foliation that passes through z. This result can be extended to the more general class of *Levi-degenerate* manifolds as follows.

First, we extend the Levi form to a bilinear map

$$\mathcal{L}_z\rho\colon (T^{1,0}_zM\oplus T^{0,1}_zM)\times(T^{1,0}_zM\oplus T^{0,1}_zM)\longrightarrow\frac{T_zM\otimes\mathbf{C}}{T^{1,0}_zM\oplus T^{0,1}_zM}$$

by setting

$$\mathcal{L}_z\rho(X_z,Y_z)=\frac{1}{2i}\pi_z[X,Y]_z,$$

where π_z is the projection

$$\pi_z\colon T_zM\otimes\mathbf{C}\longrightarrow\frac{T_zM\otimes\mathbf{C}}{T^{1,0}_zM\oplus T^{0,1}_zM}$$

and X, $Y\in T^{1,0}M\oplus T^{0,1}M$ are vector field extensions of the vectors X_z and Y_z. For $z\in M$ let

$$N^{1,0}_zM:=\{v\in T^{1,0}_zM:\ \mathcal{L}_z\rho(u,v)=0 \text{ for each } u\in T^{1,0}_zM\}.$$

$N^{1,0}_zM$ is called the *Levi null set* of M at $z\in M$. In the case when the dimension of $N^{1,0}_zM$ is independent of $z\in M$, $N^{1,0}M=\bigcup_{z\in M}N^{1,0}_zM$ forms a subbundle of $T^{1,0}M$.

Moreover, under this constant rank assumption on $N^{1,0}M$ *there is a unique smooth foliation of* $M\cap U$ *by complex manifolds such that the complex tangent space to the leaf passing through* $z\in M$ *is* $N^{1,0}M$ (see [7]).

Now we return to the main problem. Our manifold M is assumed to be a hypersurface of constant signature, i.e. the number q^0 of zero eigenvalues of its Levi form is constant at each point of M. Let us assume $q^0\geq 1$. Hence $\dim_{\mathbf{C}} N_zM=q^0$ for each $z\in M$ and the hypersurface M can be foliated by q^0-dimensional complex manifolds.

Proof of Theorem 1.1. As before we denote by Ω^- and Ω^+ two open sets that are obtained by the intersection of $\mathbf{CP}^n$ with M. In order to solve the $\bar\partial_b$-problem on M we consider the $\bar\partial$-closed extensions of the $(0,1)$-form f to Ω^- and Ω^+ and then use the L^2-solvability for $\bar\partial$ in each of these domains.

As f is a CR form on the smooth hypersurface M, one can find a smooth extension $\tilde f\in C^\infty_{0,1}(\mathbf{CP}^n)$ of f with support in some tubular neighborhood of M, such that $\bar\partial\tilde f$ vanishes to infinite order along M (cf. [5]). We can assume $\bar\partial\tilde f|_{X_\alpha}\equiv 0$. The assumption $q^0+\min\{q^-,q^+\}\geq 2$ implies that Proposition 3.4 can be applied to the $(0,2)$-form $\bar\partial\tilde f$ and hence there exists a solution g^-, respectively g^+, of the equation $\bar\partial g=\bar\partial\tilde f$ in the domain Ω^-, respectively in Ω^+, that satisfy

$$\int_{\Omega^\pm}|g^\pm|^2_{\omega_{\rm FS}}\delta_M^{-N}\,dV_{\omega_{\rm FS}}<+\infty,\quad N\gg 1. \tag{10}$$

Without loss of generality we can assume that these solutions are minimal, i.e. $\bar\partial^*_{-N}g^\pm=0$. Here we denote by $\bar\partial^*_{-N}$ the Hilbert space adjoint of $\bar\partial$ with respect to the global inner product of the space of all functions u with

$$\|u\|^2_{-N}:=\int_{\mathbf{CP}^n\setminus M}|u|^2_{\omega_{\rm FS}}\delta_M^{-N}\,dV_{\omega_{\rm FS}}<+\infty.$$

Then there exist positive constants K_s, t and T such that

$$\|g^\pm\|^2_{s,\Omega^\pm}\leq K_s(\|\delta_M^{-ts}\bar\partial g^\pm\|^2_{s-2,\Omega^\pm}+\|\delta_M^{-Ts^2}g^\pm\|^2_{0,\Omega^\pm})\quad\text{for all } s\in\mathbf{N}.$$

If N is sufficiently large then by the Sobolev embedding theorem $g^\pm\in C^k_{0,1}(\Omega^\pm)$. As g^- and g^+ vanish to infinite order along M, they can be patched together to form a solution $g\in C^k_{0,1}(\mathbf{CP}^n)$. Let us set $F:=\tilde f-g\in C^k_{0,1}(\mathbf{CP}^n)$. Then $\bar\partial F=0$ on $\mathbf{CP}^n\setminus M$. We show that this is true in the whole $\mathbf{CP}^n$.

Let $\rho(z)$ be the local defining function for M. This means that $d\rho\neq 0$ on M and $M\cap U=\{z\in U:\rho(z)=0\}$ for some open set $U\subset\mathbf{CP}^n$. We have to check that

$(\bar\partial F,\varphi)=(F,\bar\partial^*\varphi)=0$ for any $\varphi\in C^\infty_{0,1}(U)$ with compact support. For $\varepsilon>0$ let us consider a smooth function ψ_ε with

$$\psi_\varepsilon=\begin{cases}1 & \text{on } \{z\in U:\delta_M(z)>\varepsilon\},\\ 0 & \text{on } \{z\in U:\delta_M(z)<\varepsilon/2\},\end{cases}$$

such that $|\bar\partial\psi_\varepsilon|<C/\varepsilon$ for some positive constant C. Then $(F,\bar\partial^*(\varphi\psi_\varepsilon))=0$ for any $\varphi\in C^\infty_{0,1}(U)$ with compact support. Since $\rho^2\bar\partial\psi_\varepsilon\to 0$ in L^2-norm on U as $\varepsilon\to 0$, we also have

$$\rho^2\bar\partial^*(\varphi\psi_\varepsilon)=\rho^2\psi_\varepsilon\bar\partial^*\varphi+(\varphi,\rho^2\bar\partial\psi_\varepsilon)\longrightarrow\rho^2\bar\partial^*\varphi$$

in L^2-norm on U as $\varepsilon\to 0$. Let $\widetilde F=\rho^{-m+4}F$, and thus $\widetilde F$ is L^2 on U. Taking $N\geq 6$ we obtain

$$(F,\bar\partial^*(\varphi\psi_\varepsilon))=(\rho^{N-6}\widetilde F,\rho^2\bar\partial^*(\varphi\psi_\varepsilon))\longrightarrow(\rho^{N-6}\widetilde F,\rho^2\bar\partial^*\varphi)=(F,\bar\partial^*\varphi)$$

as $\varepsilon\to 0$, and hence F is a $\bar\partial$-closed form on U.

Since the Dolbeault cohomology group of the projective space vanishes in bidegree $(0,1)$, there exists $U\in C^k(\mathbf{CP}^n)$ with $\bar\partial U=F$ on $\mathbf{CP}^n$.

Next we show that the restriction $u:=U|_M\in C^k(M)$ solves the equation $\bar\partial_b u=f$ on M. Indeed, if $(\bar\partial_b u-f)(z_0)\neq 0$ for some $z_0\in M$, then $(\bar\partial_b u-f)(z_0)\neq 0$ in a small neighborhood V_{z_0} of z_0 in the leaf of the Levi foliation passing through z_0. Hence there exists a constant $C>0$ such that

$$|\bar\partial U-\tilde f|^2=|g|^2>C \quad \text{in } V_{z_0}.$$

This contradicts (10). □

5. Nonexistence result

In this section we give the proof of the main result of this paper.

Proof of Theorem 1.2. Let M be as in the previous section and let $N^{1,0}_{M,\mathbf{CP}^n}$ be the holomorphic normal line bundle of M. $N^{1,0}_{M,\mathbf{CP}^n}$ can be identified with the quotient bundle $T^{1,0}(\mathbf{CP}^n)/T^{1,0}M$ and can be endowed with the Hermitian metric induced from the standard Fubini–Study metric of $\mathbf{CP}^n$. Since the line bundle $N^{1,0}_{M,\mathbf{CP}^n}$ is trivial on M, the $(1,1)$-curvature form Θ_N of that metric is d-exact on M, i.e. $\Theta_N=d\theta$ as a function on $\Lambda^2(T^{1,0}M\oplus T^{0,1}M)$. Here θ is a smooth 1-form on the holomorphic leaves of the foliation of M, and it can be represented as

$$\theta=\theta^{1,0}+\theta^{0,1},\quad \theta^{1,0}=\overline{\theta^{0,1}},$$

where $\theta^{1,0}$ is a function on $T^{1,0}M$ and $\theta^{0,1}$ is a function on $T^{0,1}M$. After comparing the degrees of the forms we conclude that $\bar{\partial}_b\theta^{0,1}=0$. The regularity theorem of the previous section implies that there exists a smooth function u on M with $\bar{\partial}_b u=\theta^{0,1}$. Hence

$$\Theta_N=\bar{\partial}_b\theta^{1,0}+\partial_b\theta^{0,1}=\partial_b\bar{\partial}_b\psi,$$

where $\psi:=u-\bar{u}$ is a smooth function on M.

Since ψ is smooth, it achieves its maximum at some point $p_0\in M$. Let $z_1,...,z_{q^0}$ be the local holomorphic coordinates at p_0 of the holomorphic leaf through p_0 of the foliation of M, such that $\partial/\partial z_1,...,\partial/\partial z_{q^0}$ are mutually orthogonal unit tangent vectors of $\mathbf{CP}^n$ of type $(1,0)$ at p_0. Then

$$\sum_{j=1}^{q^0}\Theta_N\left(\frac{\partial}{\partial z_j},\frac{\partial}{\partial\bar{z}_j}\right)=\sum_{j=1}^{q^0}\frac{\partial^2\psi}{\partial z_j\partial\bar{z}_j}.$$

On the other hand, if e is the unit tangent vector of $\mathbf{CP}^n$ of type $(1,0)$ at p_0, which is orthogonal to $T^{1,0}M$, then

$$\Theta_N\left(\frac{\partial}{\partial z_j},\frac{\partial}{\partial\bar{z}_j}\right)=\left\langle\Theta_N\left(\frac{\partial}{\partial z_j},\frac{\partial}{\partial\bar{z}_j}\right)e,e\right\rangle\geq\left\langle\Theta_{T^{1,0}_{\mathbf{CP}^n}}\left(\frac{\partial}{\partial z_j},\frac{\partial}{\partial\bar{z}_j}\right)e,e\right\rangle.$$

Here we used the fact that the curvature of the vector bundle is dominated by that of its quotient bundle.

It is well known that the Fubini–Study metric of the complex projective space has a positive holomorphic bisectional curvature. This is equivalent to the Griffiths positivity of the holomorphic tangent bundle $T^{1,0}(\mathbf{CP}^n)$. Hence

$$\sum_{j=1}^{q^0}\left\langle\Theta_{T^{1,0}(\mathbf{CP}^n)}\left(\frac{\partial}{\partial z_j},\frac{\partial}{\partial\bar{z}_j}\right)e,e\right\rangle>0$$

and

$$\sum_{j=1}^{q^0}\frac{\partial^2\psi}{\partial z_j\partial\bar{z}_j}>0$$

at the point p_0. This contradicts the fact that ψ has a maximum at p_0. $\square$

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