# Nonholonomic LR systems as Generalized Chaplygin systems with an Invariant Measure and Geodesic Flows on Homogeneous Spaces *

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**Abstract**

We consider a class of dynamical systems on a compact Lie group $G$ with a left-invariant metric and right-invariant nonholonomic constraints (so called LR systems) and show that, under a generic condition on the constraints, such systems can be regarded as generalized Chaplygin systems on the principle bundle $G \to Q = G/H$, $H$ being a Lie subgroup. In contrast to generic Chaplygin systems, the reductions of our LR systems onto the homogeneous space $Q$ always possess an invariant measure.

We study the case $G = SO(n)$, when LR systems are multidimensional generalizations of the Veselova problem of a nonholonomic rigid body motion, which admit a reduction to systems with an invariant measure on the (co)tangent bundle of Stiefel varieties $V(k, n)$ as the corresponding homogeneous spaces.

For $k = 1$ and a special choice of the left-invariant metric on $SO(n)$, we prove that under a change of time, the reduced system becomes an integrable Hamiltonian system describing a geodesic flow on the unit sphere $S^{n-1}$. This provides a first example of a nonholonomic system with more than two degrees of freedom for which the celebrated Chaplygin reducibility theorem is applicable. In this case we also explicitly reconstruct the motion on the group $SO(n)$.

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# Contents



# 1 Introduction

In classical nonholonomic mechanics a special attention is given to systems whose Lagrangian and the constraints admit symmetries. After an appropriate reduction they take the form of unconstrained Lagrangian systems with some extra (nonholonomic) forces.

Apparently, Appel [1] was the first who proposed changing of time and of momenta in order to eliminate these extra terms and to transform the reduced systems to a canonical (Hamiltonian) form. After that Chaplygin [17] realized this idea in his theory of reducing multiplier for nonholonomic systems with two degrees of freedom. Excellent reviews of the history, various forms and geometric descriptions of the reduced systems, as well as many relevant examples can be found in [33, 13, 5, 39], see also references therein.

The key feature in the Chaplygin approach is the existence of an invariant measure of the reduced system, a rather strong property, which puts the system close to Hamiltonian ones. For reduced generalized Chaplygin systems originated from classical dynamics, this problem was considered in [33]. Later the authors of [14] gave necessary and sufficient conditions for the existence of such a measure in case when the Lagrangian of the system is of a pure kinetic energy type.

On the other hand, numerous attempts to extend the Chaplygin theory of reducing multiplier to systems with more than two degrees of freedom (even having an invariant measure) were ineffective, since in this case several conditions on the metric and constraints are imposed ([22, 28]). To our knowledge, until recently there were no nontrivial examples of multidimensional systems, which are reducible to a Hamiltonian form exactly by the Chaplygin procedure.

As an alternative, much effort has gone into the development of the symplectic and Poisson view of reduced generalized Chaplygin systems with an invariant measure. In particular, for the case Abelian symmetries, Stanchenko [38] showed that such systems can be represented in a Hamilton-like form with respect to an almost symplectic 2-form, which however may be not closed. This observation was extended for generic symmetries in [14].

The importance of the existence of an invariant measure for integrability of nonholonomic systems was also indicated by Kozlov in [34], where various examples were considered. In [40, 41], Veselov and Veselova, inspired by classical problems of nonholonomic dynamics, studied nonholonomic systems on unimodular Lie groups with right-invariant nonintegrable constraints and a left-invariant metric (so called LR systems), and showed that they always possess an invariant measure, whose density can be effectively calculated. In particular, the motion of a rigid body around a fixed point under a nonholonomic constraint (projection of the angular velocity to the fixed vector in space is constant) is described by an integrable LR system [40].

Another method of constructing non-Lagrangian (so called L+R) systems with an invariant measure on Lie groups was proposed in [24]. The kinetic energy of such systems is given by a sum of a left- and right-invariant metrics on the group. It appears that some of L+R systems have natural origins in classical nonholonomic mechanics.

For the related problems concerning the integrability of nonholonomic systems one can see [5, 34, 25, 26, 27, 4, 31] and references therein.

**Contains of the paper.** We study several new geometric aspects of nonholonomic LR systems on a compact Lie group $G$. In Section 2 we show that a class of such systems can be naturally considered as generalized Chaplygin systems on the principle bundle $G \to Q = G/H$, where $H$ is a subgroup of $G$. Such systems are reduced to non-Hamiltonian equations on the cotangent bundle of the homogeneous space $Q$. The latter are described by a Lagrange-d'Alambert equation with extra nonholonomic terms which are explicitly found.

In Section 3 we describe the invariant measure of the original and reduced LR systems. If the homogeneous space is two-dimensional, then, by the Chaplygin reducibility theorem, the existence of such a measure leads to changing of time such that our system becomes Hamiltonian. On the other hand, we prove that if the reduced system is transformable in this way to a Hamiltonian form for any dimension, then it must have invariant measure whose density is prescribed by the corresponding reducing multiplier. We also show that the reduced LR system on $Q$ always possesses an invariant measure, which does not necessarily holds for generic Chaplygin systems.

As a natural example of LR systems, Section 4 describes the classical Veselova problem on nonholonomic rigid body motion and some of its integrable perturbations, as well as its relation to the Neumann system and an integrable geodesic flow on the 2-dimensional sphere.

In Section 5 we consider multidimensional Veselova nonholonomic systems on the Lie group $SO(n)$ characterized by various types of constraints and describe their invariant measure. The constraints allow a reduction of these systems to non-Hamiltonian flows with an invariant measure on the cotangent bundle of Stiefel varieties $V(r, n)$.

In Section 6 we concentrate on the case $r = 1$, which corresponds to reduced flows on the unit sphere $S^{n-1}$. We show that for a special choice of the inertia tensor and after changing of time, the flow reduce to a completely integrable geodesic flow on the sphere. This provides a first example of a nonholonomic system with more than two degrees of freedom for which the celebrated Chaplygin reducibility theorem is applicable.

On the other hand, we prove that, after another change of time, the multidimensional Veselova nonholonomic system on $SO(n)$ reduces to the Neumann system on $S^{n-1}$.

In final Section 7, for the above integrable case, we explicitly solve the reconstruction problem: given a trajectory of the reduced geodesic flow on

$S^{n-1}$, to find the corresponding nonholonomic motion on the group $SO(n)$. To perform this, we made use of the remarkable relations between the Neumann system, the geodesic flow on an $(n-1)$-dimensional ellipsoid, and the evolution of orthogonal frames associated to the geodesics. It appears that the right-invariant distribution $D \subset TSO(n)$ is foliated with invariant tori of generic dimension $n-1$ and the unreduced LR system is integrable.

## 2 Generalized Chaplygin and LR systems on Lie groups

Suppose we are given a nonholonomic Lagrangian system $(M, l, D)$ on the $n$–dimensional configuration space $M$ with (local) coordinates $x$ and Lagrangian $l(x, \dot{x})$ in presence of a $k$–dimensional distribution $D \subset TM$ that describes the kinematics constraints: a curve $x(t)$ is said to satisfy the constraints if $\dot{x}(t) \in D_{x(t)}$ for all $t$. The trajectory of the system $x(t)$ that satisfies the constraints is a solution to the Lagrange-d'Alambert equation

$$\left(\frac{\partial l}{\partial x} - \frac{d}{dt}\frac{\partial l}{\partial \dot{x}}, \eta\right) = 0, \quad \text{for all} \quad \eta \in D_x. \tag{2.1}$$

Here $(\cdot, \cdot)$ denotes pairing between dual spaces.

Now, we assume that there is a bundle structure $\pi : M \to Q$, that is another manifold $Q$ called the base and a map $\pi$ which is a submersion, such that $T_x M = D_x \oplus V_x$ for all $x$. Here $V_x$ is the kernel of $T_x\pi$ and it is called the vertical space at $x$. Then the distribution $D$ can be seen as a collection of horizontal spaces of the *Ehresmann connection* associated to $\pi : M \to Q$.

Given a vector $X_x \in T_x M$ we have decomposition $X_x = X_x^h + X_x^v$, where $X_x^h \in D_x$, $X_x^v \in V_x$. The *curvature* of the connection is the vertical valued two form $B$ on $M$ defined by

$$B(X_x, Y_x) = -[\bar{X}_x^h, \bar{Y}_x^h]_x^v$$

where $\bar{X}$ and $\bar{Y}$ are smooth vector fields on $M$ obtained by extending of $X_x$ and $Y_x$.

With a help of Ehresmann connection the equations of motion can be put into the form (see [5])

$$\left(\frac{\partial l_c}{\partial x} - \frac{d}{dt}\frac{\partial l_c}{\partial \dot{x}}, \eta\right) = \left(\frac{\partial l}{\partial \dot{x}}, B(\dot{x}, \eta)\right), \quad \text{for all} \quad \eta \in D_x, \tag{2.2}$$

where $l_c(x, \dot{x}) = l(x, \dot{x}^h)$ is the constrained Lagrangian.

The form of equations (2.2) is very useful in the presence of some symmetries of the system. Namely, suppose that the configuration space is a principal bundle $\pi : M \to Q = M/\mathfrak{G}$ with respect to the (left) action of a Lie group $\mathfrak{G}$, and $D$ is a principal connection (i.e., $D$ is a $\mathfrak{G}$–invariant distribution). Let the Lagrangian $l$ be also $\mathfrak{G}$–invariant. Then equations (2.2) are $\mathfrak{G}$–invariant and induce well defined *reduced Lagrange-d'Alambert* equation on the tangent bundle $TQ = D/\mathfrak{G}$. The system $(M, l, D)$ is referred to as *a generalized Chaplygin system* (see [33, 5]).

**LR systems.** Now let $M$ be a compact connected Lie group $G$ of dimension $n$ with local coordinates $g$, and $\mathfrak{g} = T_{Id}G$ its Lie algebra with commutator $[\,,\,]$. By $\langle\cdot,\cdot\rangle$ we denote $Ad_G$–invariant scalar product on $\mathfrak{g}$ or bi-invariant scalar product on $G$, and $ds^2_{\mathcal{I}}$ denotes the left-invariant metric on $G$ given by nondegenerate inertia operator $\mathcal{I} : \mathfrak{g} \to \mathfrak{g}$ in the usual way:

$$\forall \eta_1, \eta_2 \in T_gG, \quad (\eta_1, \eta_2)_g = \langle \mathcal{I}(\omega_1), \omega_2 \rangle,$$
$$\text{where} \quad \omega_1 = g^{-1}\eta_1, \quad \omega_2 = g^{-1}\eta_2.$$

Let $y_1, \dots, y_n$ be independent left-invariant vector fields on $G$ generated by some basis vectors $Y_1, \dots, Y_n$ in the algebra. Following [40, 41], one can define an *LR system* on $G$ as a nonholonomic Lagrangian system $(G, l, D)$ where $l = \frac{1}{2}(\dot g, \dot g) - v(g)$ is the Lagrangian with a left-invariant kinetic energy and $D$ is a *right-invariant* (generally nonintegrable) distribution on the tangent bundle $TG$.

The right-invariant distribution is determined by its restriction $\mathfrak{d}$ to the Lie algebra as follows: $D_g = \mathfrak{d} \cdot g = g \cdot (g^{-1} \cdot \mathfrak{d} \cdot g) \subset T_gG$, $\mathfrak{d}$ =const. Let $\mathfrak{h} = \text{span}\,(\mathfrak{h}_1, \dots, \mathfrak{h}_m)$ be the orthogonal complement of $\mathfrak{d}$ with respect to $\langle\cdot,\cdot\rangle$ and $\mathfrak{h}_s$ =const. Then the right-invariant constraints can be written as

$$\omega \in g^{-1} \cdot \mathfrak{d} \cdot g, \quad \text{or} \quad f_s = \langle \omega, g^{-1} \cdot \mathfrak{h}_s \cdot g \rangle = 0, \qquad s = 1, \dots, m, \tag{2.3}$$

where $\omega = g^{-1} \cdot \dot g$.

The LR system $(G, l, D)$ can be described by the Euler–Poincaré equations (also refereed to as the Poincaré–Chetayev or Bolzano–Hamel equations) on the product $\mathfrak{g} \times G$,

$$\begin{aligned} \frac{d}{dt}\mathcal{I}\omega &= [\mathcal{I}\omega, \omega] - y(v(g)) + \sum_{s=1}^{m} \lambda_s \, g^{-1} \cdot \mathfrak{h}_s \cdot g \ , \\ \dot g &= g\omega, \end{aligned} \tag{2.4}$$

where $y(v) = (y_1(v), \dots, y_n(v))^T$ is the vector of Lie derivatives with respect to above left-invariant fields $y_1, \dots, y_n$, and $\lambda_s$ are indefinite multipliers, which can be found by differentiating (2.3).

These equations define a dynamical system on the whole tangent bundle $TG$, and the right-invariant constraint functions $f_s$ in (2.3) are its generic first integrals. Thus, the LR system $(G, l, D)$ itself can be regarded as the restriction of the system (2.4) onto $D \subset TG$. (Also, the LR system with non-homogeneous right-invariant constraints $f_s = c_s \neq 0$ can be considered as a subsystem of (2.4).)

For the case $v(g) = 0$, the system (2.4) can be reduced to the form

$$\begin{aligned} \frac{d}{dt}\mathcal{I}\omega &= [\mathcal{I}\omega, \omega] + \sum_{s=1}^{m} \lambda_s \mathcal{F}_s, \\ \dot{\mathcal{F}}_s &= [\mathcal{F}_s, \omega], \end{aligned} \tag{2.5}$$

where $\mathcal{F}_s(g) = \partial f_s(\omega, g)/\partial \omega = g^{-1} \cdot \mathfrak{h}_s \cdot g$. This forms a closed system on the space $(\omega, \mathcal{F}_1, \dots, \mathcal{F}_s)$.

There is another way of description of LR systems, which is based on the nonholonomic version of the Noether theorem [25, 5]. Namely, as shown in [41], for $v(g)=0$, equations (2.4) have the conservation law $\frac{d}{dt}\operatorname{pr}_{\mathfrak{d}}(g\cdot\mathcal{I}\omega\cdot g^{-1})=0$, which can be rewritten as

$$\frac{d}{dt}(\operatorname{pr}_{g^{-1}\cdot\mathfrak{d}\cdot g}\mathcal{I}\omega)=[\operatorname{pr}_{g^{-1}\cdot\mathfrak{d}\cdot g}\mathcal{I}\omega,\omega]. \tag{2.6}$$

On the other hand, for the case of non-homogeneous constraints $f_s=c_s$, one has $\frac{d}{dt}(\operatorname{pr}_{\mathfrak{h}}(g\omega g^{-1}))=0$, which implies

$$\frac{d}{dt}(\operatorname{pr}_{g^{-1}\mathfrak{h}g}\,\omega)=[\operatorname{pr}_{g^{-1}\mathfrak{h}g}\,\omega,\omega].$$

Combining the above equations, we obtain the momentum equation

$$\dot{\mathcal{M}}=[\mathcal{M},\omega], \tag{2.7}$$

$$\mathcal{M}=\operatorname{pr}_{g^{-1}\cdot\mathfrak{d}\cdot g}\mathcal{I}\omega+\operatorname{pr}_{g^{-1}\cdot\mathfrak{h}\cdot g}\,\omega\,. \tag{2.8}$$

As follows from (2.8), the linear operator sending $\omega$ to $\mathcal{M}$ is nondegenerate, and one can express $\omega$ in terms of $\mathcal{M}$ and the group coordinates $g$ uniquely. Thus (2.7) together with the kinematic equations $\dot g=g\omega$ represent a closed system of differential equations on the space $(\omega,g)$ or $(\mathcal{M},g)$, which is equivalent to the system (2.4). Since on $D\subset TG$, $\operatorname{pr}_{g^{-1}\cdot\mathfrak{d}\cdot g}\mathcal{M}=\operatorname{pr}_{g^{-1}\cdot\mathfrak{d}\cdot g}\mathcal{I}\omega$, on this subvariety the system has the kinetic energy integral $\frac{1}{2}\langle\mathcal{M},\omega\rangle=\frac{1}{2}\langle\mathcal{I}\omega,\omega\rangle$.

Now let

$$\mathfrak{d}=\operatorname{span}\,(\mathfrak{w}_1,\ldots,\mathfrak{w}_{n-m}),\quad \langle\mathfrak{w}_k,\mathfrak{w}_s\rangle=\delta_{ks}$$

and put $\mathcal{W}_k=g^{-1}\cdot\mathfrak{w}_k\cdot g$. Then the above system leads to a closed system of differential equations on the space $(\omega,\mathcal{W}_k)$ or $(\mathcal{M},\mathcal{W}_k)$,

$$\dot{\mathcal{M}}=[\mathcal{M},\omega],\quad \dot{\mathcal{W}}_k=[\mathcal{W}_k,\omega], \tag{2.9}$$

$$\mathcal{M}=\omega+\sum_{k=1}^{n-m}\langle\mathcal{I}\omega-\omega,\mathcal{W}_k\rangle\mathcal{W}_k.$$

The distribution $D$ is represented as invariant subvariety of (2.9) given by the condition

$$\omega-\sum_{k=1}^{n-m}\langle\omega,\mathcal{W}_k\rangle\mathcal{W}_k\equiv\mathcal{M}-\sum_{k=1}^{n-m}\langle\mathcal{M},\mathcal{W}_k\rangle\mathcal{W}_k=0.$$

**Reduction.** Let the linear subspace $\mathfrak{h}$ be the Lie algebra of a subgroup $H\subset G$. Furthermore, we suppose that the potential $v(g)$ is $H$–invariant. Then the Lagrangian $l=\frac{1}{2}(\dot g,\dot g)-v(g)$ and the right-invariant distribution $D$ are also invariant with respect to the left $H$–action. (Notice that for $m>1$ the constraint functions $f_s$ themselves may not be $H$-invariant.) In this case *the LR system $(G,l,D)$ can naturally be regarded as a generalized Chaplygin system.*

Consider homogeneous space $Q=H\backslash G$ of left cosets $\{Hg\}$. The distribution $D$ can be seen as a principal connection of the principal bundle:

$$\begin{array}{ccc} H & \longrightarrow & G \\ & & \downarrow \quad \pi \\ & & Q=H\backslash G \end{array}\ .$$

The Lagrange-d'Alambert equation (2.2) is $H$–invariant and it reduces to a second order equation on $Q$. In order to write the reduced equations in a simple form, we identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by the $Ad_G$–invariant scalar product $\langle\cdot,\cdot\rangle$, and the spaces $TQ$, $T^*Q$ by the *normal* metric, induced by the bi-invariant metric on $G$. Next, consider the moment mappings:

$$\phi : TG \cong T^*G \to \mathfrak{g}, \quad \Phi : TQ \cong T^*Q \to \mathfrak{g},$$

of the natural *right* actions of $G$ on $T^*G$ and $T^*Q$, respectively. We have

$$\phi(X) = g^{-1} \cdot X, \quad X \in T_g G$$

and the moment map $\Phi$ can be considered as a restriction of $\phi$ to $D$.

The *reduced Lagrangian* is by definition the function $l|_D = l_c|_D$

$$l_c(g,\dot g) = \frac{1}{2}\langle \mathrm{pr}_{g^{-1}\mathfrak{d}g}\, \mathcal{I}(\phi(g,\dot g)), \phi(g,\dot g)\rangle - v(q)$$

considered on the orbit space $H\backslash D = T(H\backslash G)$. It follows that the reduced Lagrangian is simply given by:

$$L(q,\dot q) = \frac{1}{2}\langle \mathcal{I}\Phi(q,\dot q), \Phi(q,\dot q)\rangle - V(q),$$

where $q = \pi(g)$ are local coordinates on $Q$ (which may be redundant) and $V(q) = v(g)$. For $v = V = 0$, this is a Lagrangian of the geodesic flow of metric which we shall denote by $ds^2_{\mathcal{I},D}$.

The reduced system on $TQ$ is defined by the following proposition, which appears to be a special case of the general nonholonomic reduction procedure described in [33, 5].

**Proposition 2.1** *The reduced Lagrange–d'Alambert equation describing the motion of the LR system* $(G, l, D)$ *take the following form*

$$\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q}, \xi\right) = \langle \mathcal{I}\Phi(q,\dot q), \mathrm{pr}_{g^{-1}\mathfrak{h}g}[\Phi(q,\dot q), \Phi(q,\xi)]\rangle, \quad \text{for all } \xi \in T_qQ, \tag{2.10}$$

*where* $\mathrm{pr}_{g^{-1}\mathfrak{h}g} : \mathfrak{g} \to g^{-1}\mathfrak{h}g$ *is the orthogonal projection and* $q = \pi(g)$.

As a result, (2.10) leads to a system of Lagrange equations on $TQ$ with some extra terms. Note that this system always has the energy integral

$$E(q,\dot q) = \frac{1}{2}\langle \mathcal{I}\Phi(q,\dot q), \Phi(q,\dot q)\rangle + V(q).$$

*Proof of Proposition* 2.1. First, we need to describe the curvature of the principal connection associated to the distribution $D$. Let $X_1, X_2 \in T_gG$. Then the horizontal and vertical components of $X_i$ have the form

$$X_i^h = g \cdot \mathrm{pr}_{g^{-1}\mathfrak{d}g}\, \phi(X_i), \quad X_i^v = g \cdot \mathrm{pr}_{g^{-1}\mathfrak{h}g}\, \phi(X_i).$$

Also, if $\bar X_1, \bar X_2$ are right invariant extensions of $X_1$ and $X_2$ then $[\bar X_1, \bar X_2]_g = -g \cdot [\phi(X_1), \phi(X_2)]$. (Here the first square bracket denotes the commutator of vector fields, and the second one is the commutator in the algebra $\mathfrak{g}$.) Thus, the curvature is

$$B(X_1, X_2) = -[\bar X_1{}^h, \bar X_2{}^h]_g^v = g \cdot \mathrm{pr}_{g^{-1}\mathfrak{h}g}[\mathrm{pr}_{g^{-1}\mathfrak{d}g}\, \phi(X_1), \mathrm{pr}_{g^{-1}\mathfrak{d}g}\, \phi(X_2)].$$

Therefore the right hand side of (2.2) is equal to

$$\left(\frac{\partial l}{\partial \dot g}, g \cdot \mathrm{pr}_{g^{-1}\mathfrak{h}g}[\omega, \phi(\eta)]\right) = \langle \mathcal{I}\omega, \mathrm{pr}_{g^{-1}\mathfrak{h}g}[\omega, \phi(\eta)]\rangle, \quad \omega = g^{-1}\cdot \dot g = \phi(\dot g).$$

Combining the above expressions, we come to the right hand side of (2.10).

**Reduced momentum equation.** Similarly to the original LR systems, in the absence of potential forces, one can describe reduced LR systems on $T^*Q$ in terms of a momentum equation as well.

Namely, let us now identify$\{p\} = T_q^*Q$ and $\{\dot q\} \in T_qQ$ by the metric $ds^2_{\mathcal{I},D}$, i.e., we put $p = \partial L(q,\dot q)/\partial \dot q$, and also identify the spaces $\mathfrak{g} = \{\omega\}$ and $\mathfrak{g}^* = \{\mathcal{M}\}$ via relation (2.8).

Next, introduce the moment map $\Phi^* : T^*Q \to \mathfrak{g}^*$, $(q,p) \to \mathcal{M}$ by setting

$$\Phi^*(q,p) = \Phi(q, \partial L(q,\dot q)/\partial \dot q),$$

where $\partial L(q,\dot q)/\partial \dot q$ is considered as an element of $T_qQ$ (via identification given by the normal metric).

The map is correctly defined because

$$\Phi^*(q,p)|_{p=\partial L(q,\dot q)/\partial \dot q} = \mathrm{pr}_{g^{-1}\mathfrak{d}g}\, \mathcal{I}\Phi(q,\dot q) = \mathrm{pr}_{g^{-1}\mathfrak{d}g}\, \mathcal{M}. \tag{2.11}$$

Indeed, a preimage of $\partial L(q,\dot q)/\partial \dot q$ in $D_g \subset T_gG$, $g \in \pi^{-1}(q)$ can be chosen in form $\partial l_c(g,\dot g)/\partial \dot g$, $\pi_*(\dot g) = \dot q$. Therefore, we have

$$\Phi(q, \partial L(q,\dot q)/\partial \dot q) = \mathrm{pr}_{g^{-1}\mathfrak{d}g}\, \phi(g, \partial l_c(g,\dot g)/\partial \dot g)$$
$$= \mathrm{pr}_{g^{-1}\mathfrak{d}g}\left(g^{-1}\cdot \frac{\partial l_c(g,\dot g)}{\partial \dot g}\right) = \mathrm{pr}_{g^{-1}\mathfrak{d}g}\, g^{-1}(g\mathcal{I}g^{-1})\dot g = \mathrm{pr}_{g^{-1}\mathfrak{d}g}\, \mathcal{I}\omega,$$

which establishes the first equality in (2.11). The second equality follows from (2.8).

Since the linear subspace $g^{-1}\mathfrak{d}g \subset \mathfrak{g}$ is $H$-invariant, it depends only on $q \in Q$. Thus, the system (2.9) represented in terms of $\omega$ can be regarded as a flow on the quotient manifold $H\backslash TG \cong Q \times \mathfrak{g}$ obtained from $TG \cong G \times \mathfrak{g}$ by factorization by $H$. The same system represented in terms of $\mathcal{M}$ leads to a system on $Q \times \mathfrak{g}^*$.

Relations between the above manifolds are described by the commutative diagram below, where the vertical arrows denote the corresponding inclusions and $\tilde\Phi, \tilde\Phi^*$ are the extensions of the moment maps $\Phi, \Phi^*$ respectively.

$$\begin{array}{ccccc}
TG \cong G\times\mathfrak{g} & \xrightarrow{H\backslash} & Q\times\mathfrak{g} & \xrightarrow{(2.8)} & Q\times\mathfrak{g}^* \\
{\scriptstyle f_s(g,\omega)=0}\uparrow & & {\scriptstyle\tilde\Phi}\uparrow & & {\scriptstyle\tilde\Phi^*}\uparrow \\
D \cong G\times\mathfrak{d} & \xrightarrow{H\backslash} & TQ & \xrightarrow{p=\partial L(q,\dot q)/\partial \dot q} & T^*Q
\end{array}$$

For a fixed $q$, the map $\Phi^*$ establishes a bijection between the subspace $g^{-1}\mathfrak{d}g \subset \mathfrak{g}^*$ and the cotangent space $T_q^*Q$.

Now, applying (2.6) and (2.11), we come to reduced momentum equation

$$\frac{d}{dt}\Phi^*(q,p) = [\Phi^*(q,p), \Phi(q,\dot q)], \tag{2.12}$$

where $\dot q = \dot q(q,p)$ is determined from $p = \partial L(q,\dot q)/\partial \dot q$. This leads to a system of equations on $T^*Q$, which are equivalent to the Lagrange equations on $TQ$ obtained from (2.10).

As a consequence of the momentum equation (2.12), we also obtain the following

**Proposition 2.2** *Apart from the energy integral, in the absence of potential forces the reduced LR system on $T^*Q$ always has the set of first integrals $\mathcal{A} = \{f \circ \Phi^*,\ f \in \mathbb{R}[\mathfrak{g}]^G\}$, where $\mathbb{R}[\mathfrak{g}]^G$ is the algebra of $Ad_G$ invariants on $\mathfrak{g}$.*

The number of independent functions in $\mathcal{A}$ is equal to the number of independent $G$–invariant functions on $T^*Q$, that is to $\dim \mathrm{pr}_{\mathfrak{d}}(\mathrm{ann}(\xi))$, for a generic $\xi \in \mathfrak{d}$ (see [11]). Here $\mathrm{ann}(\xi) = \{\eta \in \mathfrak{g},\ [\xi,\eta] = 0\}$. If $Q = H\backslash G$ is a symmetric space, this number is equal to the rank of $Q$.

# 3 Invariant measure and changing of time

One of the remarkable properties of LR systems is the existence of an invariant measure, which puts them rather close to Hamiltonian systems.

**Theorem 3.1** ([40, 41]). *The LR system (2.5) on the space $(\omega, \mathcal{F}_1, \dots, \mathcal{F}_s)$ possesses an invariant measure with density*

$$\mu = \sqrt{\det(\mathcal{I}^{-1}|_{g^{-1}\mathfrak{h}g})} \equiv \sqrt{\det\langle \mathcal{F}_s, \mathcal{I}^{-1}\mathcal{F}_l\rangle}, \qquad s,l = 1,\dots,m, \tag{3.1}$$

*where $\mathcal{I}^{-1}|_{g^{-1}\mathfrak{h}g}$ is the restriction of the inverse inertia tensor to the linear space $g^{-1}\mathfrak{h}g \subset \mathfrak{g}$.*

The alternative description of LR systems leads to another expression for invariant measure.

**Theorem 3.2** *The LR system defined by the momentum equation (2.9) has the invariant measure*

$$\tilde\mu\, d\omega \wedge d\mathcal{W}_1 \wedge \cdots \wedge d\mathcal{W}_{n-m} = \tilde\mu^{-1}\, d\mathcal{M} \wedge d\mathcal{W}_1 \wedge \cdots \wedge d\mathcal{W}_{n-m} \tag{3.2}$$

$$\tilde\mu = \left|\frac{\partial \mathcal{M}}{\partial \omega}\right|^{1/2} = \sqrt{\det(\mathcal{I}|_{g^{-1}\mathfrak{d}g})} \equiv \sqrt{\det\langle \mathcal{W}_i, \mathcal{I}\mathcal{W}_j\rangle}, \tag{3.3}$$

$$i,j = 1,\dots,n-m,$$

*where $\mathcal{I}|_{g^{-1}\mathfrak{d}g}$ is now the restriction of the inertia tensor to the linear space $g^{-1}\mathfrak{d}g \subset \mathfrak{g}$.*

Expressions (3.1), (3.3) involve complimentary basis vectors in $g^{-1}\mathfrak{g}g$. In this sense the densities $\mu$ and $\tilde\mu$ given by the above theorems are *dual*.

*Proof of Theorem* 3.2. First note that the systems (2.4) and (2.8) can be extended to one and the same system on the space $(\omega, \mathcal{F}_1, \dots, \mathcal{F}_m, \mathcal{W}_1, \dots, \mathcal{W}_{n-m})$. The resulting system has an invariant measure, whose density is the same as those of the original systems. Hence the functions $\mu$ in (3.1) and $\tilde\mu$ in (3.2) can be different only by a constant multiplier.

Next, note that in an appropriate $g$-dependent orthogonal basis in the algebra $\mathfrak{g}$ the Jacobi matrix $\partial\mathcal{M}/\partial\omega$ has the following block structure

$$\frac{\partial\mathcal{M}}{\partial\omega} = \begin{pmatrix} \mathbf{I}_{n-m} & 0 \\ 0 & 0 \end{pmatrix} \mathcal{I} + \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{I}_m \end{pmatrix} \equiv \begin{pmatrix} \mathcal{I}|_{g^{-1}\mathfrak{d}g} & \mathcal{S} \\ 0 & \mathbf{I}_m \end{pmatrix},$$

where $\mathbf{I}_{n-m}, \mathbf{I}_m$ are unit matrices of dimension $(n-m)\times(n-m)$ and $m\times m$ respectively, and $\mathcal{S}$ is a certain $(n-m)\times m$-matrix. In the same basis one has

$$\frac{\partial\mathcal{M}}{\partial\omega}\mathcal{I}^{-1} = \begin{pmatrix} \mathbf{I}_{n-m} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{I}_m \end{pmatrix} \mathcal{I}^{-1} \equiv \begin{pmatrix} \mathbf{I}_{n-m} & 0 \\ \mathcal{U} & \mathcal{I}^{-1}|_{g^{-1}\mathfrak{h}g} \end{pmatrix},$$

with a certain $m\times(n-m)$-matrix $\mathcal{U}$. Comparing the right hand sides of these two expressions with (3.1), we obtain the following chain

$$\mu^2 = \det(\mathcal{I}^{-1}|_{g^{-1}\mathfrak{h}g}) = \left|\frac{\partial\mathcal{M}}{\partial\omega}\mathcal{I}^{-1}\right| = \det(\mathcal{I}^{-1})\left|\frac{\partial\mathcal{M}}{\partial\omega}\right| = \det(\mathcal{I}^{-1})\det(\mathcal{I}|_{g^{-1}\mathfrak{d}g}). \tag{3.4}$$

Hence, we can choose the density $\tilde{\mu}$ in the form (3.3).

Finally, taking into account the relation

$$d\omega\wedge d\mathcal{W}_1\wedge\cdots\wedge d\mathcal{W}_{n-m} = \left|\frac{\partial\mathcal{M}}{\partial\omega}\right|^{-1} d\mathcal{M}\wedge d\mathcal{W}_1\wedge\cdots\wedge d\mathcal{W}_{n-m},$$

and using (3.3), we come to the equality in (3.2). The theorem is proved.

As shown in [41], Theorem 3.1 implies that the original nonholonomic system (2.5) on the left trivialization $\mathfrak{g}\times G$ of $TG$ has the invariant measure $\mu(g)\, d\omega\wedge dg$.

**Reduced invariant measure.** Now we proceed to reduced LR systems. As a natural consequence of the above theorems, we have

**Theorem 3.3** *The reduced LR system (2.10) (or, after the Legendre transformation, the system (3.12) on $T^*(H\backslash G)$) possesses an invariant measure.*

Note that a generic Chaplygin system may not have this property (see [14]).

The proof of Theorem 3.3 consists of two steps. First, it is seen that the restriction of the LR system (2.5) onto the distribution $D\subset TG$ has an invariant measure. Indeed, the volume form on the tangent bundle admits the decomposition

$$d\omega\wedge dg = \theta(g)\, df_1\wedge\cdots\wedge df_m\wedge \mathit{\Pi}\,, \tag{3.5}$$

where $f_s(\dot g, g)$ are the constraint functions in (2.3), $\theta(g)$ is a function, and $\mathit{\Pi}$ is a volume form on $D$. Since the 1-forms $df_s$ are independent on $TG$, $\theta(g)$ does not vanish on $G$.

Let $\mathcal{L}_*$ be the Lie derivative with respect to the nonholonomic flow (2.5). Since the functions $f_s(\dot g, g)$ are its generic first integrals, we have $\mathcal{L}_* d\, f_s = d(\dot{f}_s) = 0$, $s=1,\ldots,m$. As a result, from the condition $\mathcal{L}_*(\mu\, d\omega\wedge dg)=0$ and (3.5) we obtain $df_1\wedge\cdots\wedge df_m\,\mathcal{L}_*(\mu\theta\,\mathit{\Pi})=0$. Hence, the restriction of the flow onto $D$ has the invariant measure $\mu(g)\theta(g)\,\mathit{\Pi}$.

Notice that one can always choose $\mathit{\Pi}$ to be $H$-invariant. In this case, since the form $d\omega\wedge dg$ is $G$-invariant, whereas the wedge product $df_1\wedge\cdots\wedge df_m$ and $\mu(g)$ are $H$-invariant, the density $\mu(g)\theta(g)$ of the restricted measure is also $H$-invariant and goes down to $Q$.

The second step is based on the following general lemma. (Although it is quite natural, we could not find it in the literature.)

**Lemma 3.4** *Suppose a compact group $\mathfrak{G}$ acts freely on a manifold $N$ with local coordinates $z$, and there is a $\mathfrak{G}$–invariant dynamical system $\dot z = Z(z)$ on $N$. If this system has an invariant measure (which is not necessarily $\mathfrak{G}$-invariant), then the reduced system on the quotient manifold $N/\mathfrak{G}$ also has an invariant measure.*

Now, identifying the group $\mathfrak{G}$ and the manifold $N$ with $H$ and $D$ respectively, we arrive at Theorem 3.3.

*Proof of Lemma* 3.4 The manifold $N$ can be locally represented as a direct product $\mathbb{R}^k\{x\} \times \mathfrak{G}$, where $x$ is a local coordinate system on $N/\mathfrak{G}$, so that the $\mathfrak{G}$–action and the dynamical system take the form

$$\begin{aligned} & a \cdot (x, g) = (x, ag), \quad a \in \mathfrak{G}, \\ \text{and} \quad & \dot x = X(x), \quad \dot g = Y = g \cdot \xi(x), \quad \xi(x) \in \mathfrak{g} = T_{Id}\mathfrak{G}, \end{aligned}$$

respectively.

Let $\Theta$ be an invariant measure of the original system on $N$, $\mu$ be a bi-invariant volume form on $\mathfrak{G}$, $\sigma$ be a volume form on $N/\mathfrak{G}$ and $\sigma_x$ be its local representation in $x$–coordinates. Then the invariant measure on $N$ locally has the form $\Theta = f(x,g)\,\mu \wedge \sigma_x$. Thus

$$\mathcal{L}_Z f(x,g)\,\mu \wedge \sigma_x = Z(f)\,\mu \wedge \sigma_x + f \cdot (\mathcal{L}_Z \mu) \wedge \sigma_x + f\mu \wedge (\mathcal{L}_Z \sigma_x) = 0, \tag{3.6}$$

where $\mathcal{L}_Z$ is the Lie derivative with respect to the flow $Z$. Since $d\sigma_x = d_x\sigma_x = 0$ and $d\mu = d_g\mu = 0$, we have

$$\mathcal{L}_Z\sigma_x = (d \circ i_Z)\sigma_x + (i_Z \circ d)\sigma_x = d(i_Z\sigma_x) = d_x(i_X\sigma_x) = \mathcal{L}_X\sigma_x, \tag{3.7}$$

$$\mathcal{L}_Z\mu = (d \circ i_Z)\mu + (i_Z \circ d)\mu = d(i_Z\mu) = d(i_Y\mu) = (d_x + d_g)(i_Y\mu). \tag{3.8}$$

For a fixed $x$, $Y = Y(x)$ is a *left-invariant* vector field on $\mathfrak{G}$, whereas the corresponding flow on $\mathfrak{G}$ is *right-invariant.* Since $\mu$ is bi-invariant, we have $\mathcal{L}_Y\mu = d_g(i_Y\mu) = 0$. Also, it is obvious that $d_x(i_Y\mu)\wedge\sigma = 0$. Therefore, taking into account (3.6–3.8), we get

$$Z(f)\mu \wedge \sigma_x + f\mu \wedge (\mathcal{L}_X\sigma_x) = 0 \tag{3.9}$$

Now we introduce the "averaged" density $\bar f(x) = \int_{\mathfrak{G}} f(x,g)\mu$, which, as we shall see below, has the following property

$$\int_{\mathfrak{G}} Z(f)\mu = X\left(\int_{\mathfrak{G}} f\mu\right) = X(\bar f). \tag{3.10}$$

Then, by integration of (3.9), we obtain $X(\bar f)\sigma_x + \bar f\mathcal{L}_X\sigma_x = 0$. As a result, the reduced system preserves the volume form $\bar f(x)\sigma_x$.

We stress that the above procedure does not depend on the choice of the local coordinates on $N/\mathfrak{G}$. Indeed, let $y = y(x)$ be another coordinate system. Then

$$\Theta = h(y,g)\,\mu \wedge \sigma_y = h(y(x),g)\mu \wedge \det\left(\frac{\partial y}{\partial x}\right)\sigma_x = f(x,g)\,\mu \wedge \sigma_x,$$

and after integration we have $\bar{f}(x)\sigma_x = \bar{h}(y)\sigma_y$.

It remains to prove (3.10). We have $Z(f) = X(f) + Y(f)$ and $\int_{\mathfrak{G}} X(f)\mu = X\left(\int_{\mathfrak{G}} f\mu\right)$. Therefore the relation (3.10) is equivalent to

$$\int_{\mathfrak{G}} Y(f)\mu = 0. \tag{3.11}$$

To check the latter relation, we fix $x$. Then $\mathcal{L}_Y(f\mu) = Y(f)\mu + f\mathcal{L}_Y\mu$ and, on the other hand, $\mathcal{L}_Y(f\mu) = d_g(i_Y(f\mu))$. Since $\mathcal{L}_Y\mu = 0$, we get $Y(f)\mu = d_g(i_Y(f\mu))$, and (3.11) follows from the Stokes theorem. The lemma is proved.

**Chaplygin reducing multiplier.** Here we continue with the reduced LR systems. However, all considerations hold for an arbitrary generalized Chaplygin system with the Lagrangian of the natural mechanical type. Let $q_1, \dots, q_k$ be some local coordinates on $Q$ and $p_1, \dots, p_k$, $p_i = \partial L/\partial \dot{q}_i$ be canonically conjugated momenta, which together form coordinates on the cotangent bundle $T^*Q$. Let also $g_{ij}$ denote metric tensor of $ds^2_{\mathcal{I},D}$ and $g^{ij}$ the dual metric on $T^*Q$.

The reduced Lagrangian is $L(q,\dot{q}) = \frac{1}{2}\sum g_{ij}\dot{q}_i\dot{q}_j - V(q)$. We also introduce the Hamiltonian function $H : T^*Q \to \mathbb{R}$ (the usual Legendre transformation of $L$) $H(q,p) = \frac{1}{2}\sum g^{ij}p_ip_j + V(q)$. Then (2.10) can be rewritten as a first-order dynamical system on $T^*Q$:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} + \Pi_i(q,p), \quad i = 1, \dots, k. \tag{3.12}$$

The functions $\Pi_i$ are quadratic in momenta and can be regarded as non-Hamiltonian perturbations of the equations of motion of a particle on the homogeneous space $Q$.

Now consider changing of time $d\tau = \mathcal{N}(q)dt$, where $\mathcal{N}(q)$ is a differentiable nonvanishing function on $Q$ and denote $q' = dq/d\tau$. Then we have the following commutative diagram:

$$\begin{array}{ccc} TQ\{q,\dot{q}\} & \xrightarrow{q'=\dot{q}/\mathcal{N}(q)} & TQ\{q,q'\} \\ {\scriptstyle p=g\dot{q}}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{p}=\mathcal{N}^2 gq'} \\ T^*Q\{q,p\} & \xrightarrow{\tilde{p}=\ \mathcal{N}p} & T^*Q\{q,\tilde{p}\}. \end{array}$$

The Lagrangian and Hamiltonian functions in the coordinates $\{q,q'\}$ and $\{q,\tilde{p}\}$ take the form

$$L^*(q,q') = \frac{1}{2}\sum \mathcal{N}^2 g_{ij}q'_iq'_j - V(q), \quad H^*(q,\tilde{p}) = \frac{1}{2}\sum \frac{1}{\mathcal{N}^2} g^{ij}\tilde{p}_i\tilde{p}_j + V(q).$$

There is a remarkable relation between the existence of an invariant measure of the reduced system (3.12) and its reducibility to a Hamiltonian form.

**Theorem 3.5** *1). Suppose that after changing of time $d\tau = \mathcal{N}(q)dt$ the equations (3.12) become Hamiltonian,*

$$q'_i = \frac{\partial H^*}{\partial \tilde{p}_i}, \quad \tilde{p}'_i = -\frac{\partial H^*}{\partial q_i}\,. \tag{3.13}$$

*Then the system (3.12) has the invariant measure*

$$\mathcal{N}(q)^{k-1}\, dp_1 \wedge \cdots \wedge dp_k \wedge dq_1 \wedge \cdots \wedge dq_k \equiv \mathcal{N}(q)^{k-1}\, \Omega^k,$$

*where* $\Omega = \sum dp_i \wedge dq_i$ *is the standard symplectic form on* $T^*Q$*.*

*2). For* $k = 2$*, the above statement can also be inverted: the existence of the invariant measure with the density* $\mathcal{N}(q)$ *implies that in the new time* $d\tau = \mathcal{N}(q)dt$*, the system (3.12) gets the Hamiltonian form (3.13).*

In nonholonomic mechanics the factor $\mathcal{N}$ is known as the *reducing multiplier*, item 2) of this theorem is refereed to as the theorem on the *Chaplygin reducing multiplier* or *Chaplygin's reducibility theorem* (see [15, 16, 17] or [36], section III.12). Notice that for $k > 2$, the multiplier $\mathcal{N}(q)$ and the density of the invariant measure of the system (3.12) do not coincide. The procedure of changing of time described above is slightly different from the procedure of changing of time and making the reduced nonholonomic system Hamiltonian with respect to a new symplectic form used in [27, 38, 14]. In this context, item 1) of the theorem was implicitly formulated in [38, 14].

*Proof of item 1) of Theorem* 3.5. For simplicity we shall use the vector notation $p = (p_1, \dots, p_k)$, $q = (q^1, \dots, q^k)$, etc. Let $G$ be the matrix $(g^{ij})$. Then $\dot q = Gp$, $H = \frac{1}{2}(Gp, p)$, $H^* = \frac{\mathcal{N}^2}{2}(G\tilde p, \tilde p)$.

The equations (3.13) in the original time $t$ take the form

$$\dot q = \mathcal{N}\nabla_{\tilde p} H^*(q, \tilde p), \quad \dot{\tilde p} = -\mathcal{N}\nabla_q H^*(q, \tilde p). \tag{3.14}$$

Equations (3.12) have an invariant measure with density $f$ if

$$(\nabla_q, f\nabla_p H) + (\nabla_p, f(-\nabla_q H + \Pi)) = 0. \tag{3.15}$$

For $f$ which depend only on $q$–coordinates, we have $(\nabla_p, \Pi) + (\nabla_q \ln f, Gp) = 0$, or equivalently

$$d(\ln f) + \alpha = d(\ln f) + (A, dq) = 0, \tag{3.16}$$

where $(\nabla_p, \Pi) = (A, \dot q) = \alpha(\dot q)$. In particular, the one-form $\alpha$ is closed.

We shall prove that the function $f(q) = \mathcal{N}^{k-1}(q)$ satisfies equations (3.16). Since $\tilde p = \mathcal{N}p$ we have $\mathcal{N}\dot p + \dot{\mathcal{N}}p = \dot{\tilde p}$. Therefore, using equations (3.14) we obtain

$$\dot p = \mathcal{N}^{-1}\dot{\tilde p} - \dot{\mathcal{N}}(q)p = -\nabla_q H^*(q, \tilde p) - (\nabla_q \mathcal{N}, Gp)p. \tag{3.17}$$

Also, one can easily see that $\nabla_q H^*(q, \tilde p) = \nabla_q H(q, p) - \mathcal{N}^{-1}(Gp, p)\nabla_q \mathcal{N}$. Thus, comparing (3.12) and (3.17) we get

$$\Pi(q, p) = \mathcal{N}^{-1}(Gp, p)\nabla_q \mathcal{N} - \mathcal{N}^{-1}(\nabla N, Gp)p \tag{3.18}$$

Using (3.18) we see

$$(\nabla_p, \Pi) = \frac{1}{\mathcal{N}}\left(2(\nabla_q \mathcal{N}, Gp) - k(\nabla_q \mathcal{N}, Gp) - (\nabla_q \mathcal{N}, Gp)\right) = \frac{1-k}{\mathcal{N}}(\nabla_q \mathcal{N}, Gp).$$

Hence $\alpha = -d\ln(\mathcal{N}^{k-1})$.

As mentioned above, item 2) of the theorem is just a reformulation of the Chaplygin reducibility theorem.

Clearly, the density of an invariant measure of a generic dynamical system depends on the choice of local coordinates on the phase space. However, in case of a system on the cotangent bundle $T^*Q$ the density is invariant with respect to changes of coordinates on $Q$, since the symplectic form $\Omega$ and the measure itself are invariant with respect to contact transformations.

**Remark 3.1** The paper [38] (see also [14]) contains a nontrivial observation about the density of the invariant measure, which in our terms reads as follows. Let a function $f(q,p)$ be a solution of (3.15) in case of absence of the potential ($V(q)=0$). Then one can check that the function $f_0(q)=f(q,0)$ also satisfies the condition (3.15), i.e., it is a solution of (3.16). In other words, if the reduced system (3.12) has an invariant measure for $V=0$, one can take this measure to be of the form $f(q)\Omega^k$. Then, since (3.16) does not depend on the potential, the reduced system (3.12) has the same invariant measure in the presence of the potential field $V(q)$ as well.

# 4 Veselova system on $T\,SO(3)$, the Neumann system and a geodesic flow on $S^2$.

The most descriptive illustration of an LR system is the *Veselova problem* on the motion of a rigid body about a fixed point under the action of nonholonomic constraint

$$(\Omega,\gamma)=0, \tag{4.1}$$

where $\Omega$ is the angular velocity vector, $\gamma$ is a unit vector, which is fixed in a space frame, and $(\;,\;)$ denotes the scalar product in $\mathbb{R}^3$ [40]. Geometrically this means that the projection of the angular velocity of the body to a fixed vector must zero.

This setting should not be confused with the nonholonomic *Suslov problem*, when the analogous constraint is defined by a vector fixed *in the body frame* ([7, 25, 31]).

The equations of motion in the moving frame in the presence of a potential field $V=V(\gamma)$ have the form

$$\begin{aligned} \mathcal{I}\dot{\Omega} &= \mathcal{I}\Omega\times\Omega+\gamma\times\frac{\partial V}{\partial\gamma}+\lambda\gamma, \\ \dot{\gamma} &= \gamma\times\Omega, \end{aligned} \tag{4.2}$$

where $\mathcal{I}$ is the inertia tensor of the rigid body, $\times$ denotes the vector product in $\mathbb{R}^3$, and $\lambda$ is a Lagrange multiplier chosen such that $\Omega(t)$ satisfies the above constraint,

$$\lambda=-\frac{(\mathcal{I}\Omega\times\Omega+\gamma\times\partial V/\partial\gamma,\mathcal{I}^{-1}\gamma)}{(\mathcal{I}^{-1}\gamma,\gamma)}\,. \tag{4.3}$$

The Veselova system (4.1), (4.2) is an LR system on the Lie group $SO(3)$, which is the configuration space of the rigid body motion. After identification of Lie algebras $(\mathbb{R}^3,\times)$ and $(so(3),[\cdot,\cdot])$, the operator $\mathcal{I}$ induces the left-invariant metric $ds^2_I$. The angular velocity correspond to $\Omega=g^{-1}\dot{g}$, the velocity in the left trivialization $TSO(3)\cong SO(3)\times so(3)$, and the Lagrangian function equals $\frac{1}{2}(\Omega,\mathcal{I}\Omega)-V(\gamma)$. The fixed vector in the space corresponds to the right-invariant vector field $\gamma_g=g\cdot(g^{-1}\cdot\mathfrak{h}\cdot g)\in T_gSO(3)$, $\mathfrak{h}\in so(3)$, and the nonholonomic constraint (4.1) has the form $\langle g^{-1}\cdot\mathfrak{h}\cdot g,\Omega\rangle=0$.

On the other hand, equations (4.2), (4.3) define a dynamical system on the space $\{\Omega,\gamma\}=so(3)\times\mathbb{R}^3$, and the constraint function $(\Omega,\gamma)$ appears as its first integral. As noticed in [40], this system has an invariant measure with density $\sqrt{(\mathcal{I}^{-1}\gamma,\gamma)}$. Apart from the above constraint, it always has the geometric integral $(\gamma,\gamma)$. When $V(\gamma)=0$, according to [23], there also exist other two independent integrals

$$\frac{1}{2}(\Omega,\mathcal{I}\Omega)-(\Omega,\gamma)(\mathcal{I}\Omega,\gamma), \quad \frac{1}{2}(\mathcal{I}\Omega-(\mathcal{I}\Omega,\gamma)\gamma+(\Omega,\gamma)\gamma)^2, \tag{4.4}$$

the first expression being an analog of so called *Jacobi–Painlevé* integral. On the constraint subvariety (4.1), these functions reduce to the energy integral $F_1=\frac{1}{2}(\mathcal{I}\Omega,\Omega)$ and an additional integral $F_2=\frac{1}{2}(I\Omega,I\Omega)-\frac{1}{2}(I\Omega,\gamma)^2$ found in [40].

As a result, by the *Euler–Jacobi theorem* (see e.g., [2]), the above system is solvable by quadratures on the whole space $so(3)\times\mathbb{R}^3$. Note that analogous integrable LR systems on the group $SL(2,\,\mathbb{R})$ and the Heisenberg group are studied in [30].

As shown in [41], in case of the absence of the potential the Veselova system (4.2), (4.1) can be explicitly integrated by relating it to the classical Neumann system.

**Theorem 4.1** ([41]). *Let $\gamma(t)$ be a solution of equations (4.2), (4.1) with $V(\gamma)=0$ and with the energy constant $F_1=h$. Then after change of time*

$$d\tau_1=\sqrt{\frac{2h\det\mathcal{I}^{-1}}{(\mathcal{I}^{-1}\gamma,\gamma)}}\,dt$$

*the unit vector $q=\gamma$ is a solution of the Neumann system on the unit sphere $S^2=\{q\in\mathbb{R}^3 \mid q_1^2+q_2^2+q_3^2=1\}$ with the potential $U(q)=\frac{1}{2}(\mathcal{I}q,q)$*

$$\frac{d^2}{d\tau_1^2}q=-\mathcal{I}q+\lambda q, \tag{4.5}$$

*corresponding to the zero value of the integral*

$$\left(\mathcal{I}\left(\frac{d}{d\tau_1}q\times q\right),\frac{d}{d\tau_1}q\times q\right)-\det\mathcal{I}\,(\mathcal{I}^{-1}q,q). \tag{4.6}$$

We notice that for $(\Omega,\gamma)\neq 0$, Theorem 4.1 does not hold, and in this case the procedure of integration of equations (4.2), (4.3) was indicated in [23].

**Reduction.** The above relation between the LR system and the Neumann system, as well as the change of time, appears to be quite natural in view of the fact that the Veselova system is a Chaplygin system on the $SO(2)$–bundle

$$\begin{array}{ccc} SO(2) & \longrightarrow & SO(3) \\ & & \downarrow \pi \\ & & S^2=SO(2)\backslash SO(3) \end{array},$$

where $SO(2)$ is the subgroup generated by rotation about the vector $\gamma$. Indeed, the Lagrangian and the nonholonomic constraint (4.1) are invariant with respect to such rotations. Hence, the Veselova system can be reduced to the (co)tangent bundle of $S^2 = \{q \in \mathbb{R}^3 \mid q_1^2 + q_2^2 + q_3^2 = 1\}$

The moment map $\Phi : TS^2 \to so(3) \cong \mathbb{R}^3$ is simply given by $\Phi(q, \dot q) = \dot q \times q$, hence the reduced Lagrangian is $L(q, \dot q) = \frac{1}{2}(\mathcal{I}(\dot q \times q), \dot q \times q) - V(q)$. Note that the reduced potential is given by the same function $V$, regarded as a function of $q$ instead of $\gamma$.

Next, in view of relation

$$\mathrm{pr}_{g^{-1}\mathfrak{h}g}[\Phi(q, \dot q), \Phi(q, \xi)] = (q, (\dot q \times q) \times (\xi \times q))q = \dot q \times \xi,$$

where $\xi = (\xi_1, \xi_2, \xi_3)^T$ is any tangent vector of $S^2$ at the point $q$, the reduced Lagrange–d'Alambert equation (2.10) takes the form

$$\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q}, \xi\right) = \Psi(q, \dot q, \xi), \qquad \Psi = (\mathcal{I}(\dot q \times q), \dot q \times \xi).$$

Now the reduced LR system on $T^*S^2$ can explicitly be written in terms of local coordinates $q_1, q_2$ on $S^2$ and the corresponding momenta $p_1 = \partial\tilde L/\partial \dot q_1$, $p_2 = \partial\tilde L/\partial \dot q_2$,

$$\frac{\partial \tilde L}{\partial q_k} - \frac{d}{dt}p_k = \frac{\partial \tilde\Psi}{\partial \xi_k}, \qquad k = 1, 2, \tag{4.7}$$

where $\tilde L, \tilde \Psi$ are obtained from $L(q, \dot q)$, $\Psi(q, \dot q, \xi)$ by the substitutions

$$\dot q_3 = -\frac{q_1\dot q_1 + q_2 \dot q_2}{\sqrt{1 - q_1^2 - q_2^2}}, \quad \xi_3 = -\frac{q_1\xi_1 + q_2\xi_2}{\sqrt{1 - q_1^2 - q_2^2}}.$$

A direct (but tedious) calculation shows that the reduced system (4.7) has an invariant measure with density $\mathcal{N}(q) = 1/\sqrt{(q, \mathcal{I}^{-1}q)}$. (As was mentioned above, the latter does not depend on the choice of local coordinates on $S^2$).

Since the reduced system is two-dimensional, Chaplygin's reducibility theorem (item 2 of Theorem 3.5) says that in the new time $d\tau = \mathcal{N} dt$ and new momenta $\tilde p_k = \mathcal{N} p_k$, $k = 1, 2$, equations (4.7) transform to a Hamiltonian system. Equivalently, the latter is described by the following Lagrangian obtained from $L(q, \dot q)$,

$$L^*(q, q') = \frac{1}{2(q, \mathcal{I}^{-1}q)}(\mathcal{I}(q' \times q), q' \times q)) - V(q), \qquad q' = \frac{dq}{d\tau}. \tag{4.8}$$

For $V = 0$, this is a Lagrangian of a geodesic flow on $S^2$.

**Theorem 4.2** *The geodesic flow on $S^2$ with the metric*

$$(q, \mathcal{I}^{-1}q)^{-1} ds^2_{I,D}, \quad ds^2_{I,D} = \det \mathcal{I}\left[(dq, \mathcal{I}^{-1}dq)(\mathcal{I}^{-1}q, q) - (\mathcal{I}^{-1}q, dq)^2\right]$$

*obtained from (4.8), is completely integrable. It has an additional integral, which is quadratic in velocities and corresponds to the integral $F_2$ of the LR system (4.2), (4.1),*

$$F_2^*(q, q') = \frac{1}{2(q, \mathcal{I}^{-1}q)}\left((\mathcal{I}(q' \times q), q' \times q) - (\mathcal{I}(q' \times q), q)^2\right)$$

This theorem, as well as our observations on the reducibility of the Veselova system to Hamiltonian form, is a part of a general integrability theorem for a multi-dimensional Veselova system on the group $SO(n)$, which is described in detail in Section 6. Below we quote some specific properties of the 3-dimensional case.

The classical integrable cases of holonomic rigid body motion were already used to produce integrable geodesic flows on the sphere (see, e.g., [10]). Namely, the Euler–Poisson equations of the motion of the rigid body

$$\mathcal{I}\dot{\Omega} = \mathcal{I}\Omega \times \Omega + \gamma \times \frac{\partial V}{\partial \gamma}, \quad \dot{\gamma} = \gamma \times \Omega \tag{4.9}$$

always have integrals

$$i_1 = (\gamma, \gamma) = 1, \quad i_2 = (\mathcal{I}\Omega, \gamma), \quad f_1 = \frac{1}{2}(\mathcal{I}\Omega, \Omega) + V(\gamma).$$

In the Euler case ($V(\gamma) = 0$) there is an additional integral $f_2 = \frac{1}{2}(\mathcal{I}\Omega, \mathcal{I}\Omega)$, and under the condition $i_2 = 0$ and the substitution $q = \gamma$, equations (4.9) describe the geodesic flow on the sphere $S^2$ with the metric

$$ds^2_{I,P} = \frac{\det \mathcal{I}}{(q, \mathcal{I}q)}(dq, \mathcal{I}^{-1}dq).$$

**Remark 4.1** The metric $ds^2_{I,P}$ can be seen as a submersion metric of the left-invariant metric $ds^2_{\mathcal{I}}$ on $SO(3)$ with respect to the left $SO(2)$–action. In other words, the metric $ds^2_{\mathcal{I},P}$ is induced by the metric $ds^2_{\mathcal{I}}$ as the restriction to the distribution $P \subset TSO(3)$, where $P$ is orthogonal to the leaves of $SO(2)$–action with respect to $ds^2_{\mathcal{I}}$. On the other hand, $ds^2_{\mathcal{I},D}$ is induced by $ds^2_{\mathcal{I}}$ restricted to the distribution, which is orthogonal to the leaves of $SO(2)$–action with respect to the bi-invariant metric on $SO(3)$. On the algebraic level, the metric $ds^2_{\mathcal{I},P}$ has the *Hamiltonian* function of the form of the composition of the function on the Lie algebra with the moment map, while the metric $ds^2_{\mathcal{I},D}$ has the *Lagrangian* of this type. Note that *Hamiltonian functions* of the form of the composition of the function on the Lie algebra with the moment map already appear in the constructions of integrable geodesic flows on spheres, symmetric spaces (see [12, 6]) and other homogeneous spaces (see [11]).

In the presence of a potential the following relation holds.

**Lemma 4.3** *The Veselova system (4.2), (4.1) with the potential $V(\gamma)$ has an additional integral of the form $F = F_2 + F(\gamma)$ (and therefore is integrable by the Euler–Jacobi theorem) if and only if the Euler–Poisson equations (4.9) with the inverse inertia tensor $\mathcal{I}^{-1}$ and the potential $F(\gamma)$ are integrable for $i_2 = 0$ due to the presence of the fourth integral of the form $f_2 + V(\gamma)$.*

The proof is straightforward.

Some integrable polynomial potentials for the Euler–Poisson equations are given in [9]. In a similar way, one can construct integrable polynomial potentials (or Laurent polynomial potentials, such as given in [20]) for the Veselova system. For example, the following proposition holds.

**Proposition 4.4** *Let $\mathcal{I} = diag\,(I_1, I_2, I_3)$. The Veselova system (4.2), (4.1) with potential*

$$V(\gamma) = \alpha_1\left((\mathcal{I}^2\gamma,\gamma) - (\mathcal{I}\gamma,\gamma)^2\right) + \alpha_2(\mathcal{I}\gamma,\gamma) + \frac{\alpha_3}{\gamma_1^2} + \frac{\alpha_4}{\gamma_2^2} + \frac{\alpha_5}{\gamma_3^2}, \tag{4.10}$$

$\alpha_1, \dots, \alpha_5$ *being arbitrary constants, is solvable by quadratures. The additional integral is:*

$$\begin{aligned} F \;&=\; \frac{1}{2}(\mathcal{I}\Omega,\Omega) - \frac{1}{2}(\mathcal{I}\Omega,\gamma)^2 + \alpha_1 \det\mathcal{I}(\mathcal{I}\gamma,\gamma)(\mathcal{I}^{-1}\gamma,\gamma) - \alpha_2 \det\mathcal{I}(\mathcal{I}^{-1}\gamma,\gamma) \\ &\quad +\alpha_3\left(I_2\frac{\gamma_2^2}{\gamma_1^2} + I_3\frac{\gamma_3^2}{\gamma_1^2}\right) + \alpha_4\left(I_3\frac{\gamma_3^2}{\gamma_2^2} + I_1\frac{\gamma_1^2}{\gamma_2^2}\right) + \alpha_5\left(I_1\frac{\gamma_1^2}{\gamma_3^2} + I_2\frac{\gamma_2^2}{\gamma_3^2}\right). \end{aligned}$$

Note that the integrability of the Veselova system with the Clebsch potential $\alpha(\mathcal{I}\gamma,\gamma)$ was already shown in [40, 41].

# 5 Nonholonomic LR systems on $SO(n)$ and their reductions to Stiefel varieties

Now we proceed to a generalization of the Veselova system, which describes the motion of an $n$-dimensional rigid body with a fixed point, that is, the motion on the Lie group $SO(n)$, with certain right-invariant nonholonomic constraints.

For a path $g(t) \in SO(n)$, the angular velocity of the body is given by the left-trivialization $\Omega(t) = g^{-1}\cdot g(t) \in so(n)$. The matrix $g \in SO(n)$ maps a coordinate system fixed in the body to a coordinate system fixed in the space. Therefore, if $e_1 = (e_{11}, \dots, e_{1n})^T, \dots, e_n = (e_{n1}, \dots, e_{nn})^T$ is the orthogonal frame of unit vectors fixed in the space regarded in the moving frame, we have

$$E_1 = g\cdot e_1,\ \dots,\ E_n = g\cdot e_n,$$

where $E_1 = (1, 0, \dots, 0)^T,\ \dots,\ E_n = (0, \dots, 0, 1)^T$. From the conditions $0 = \dot E_i = \dot g\cdot e_i + g\cdot\dot e_i$, we find that the vectors $e_1, \dots, e_n$ satisfy the Poisson equations

$$\dot e_i = -\Omega e_i, \qquad i = 1, \dots, n. \tag{5.1}$$

Below we use the convention $x\wedge y = x\otimes y - y\otimes x = x\cdot y^T - y\cdot x^T$. Also now $\langle\cdot,\cdot\rangle$ denotes the Killing metric on $so(n)$, $\langle X, Y\rangle = -\frac{1}{2}tr(XY)$, $X, Y \in so(n)$. The left-invariant metric on $SO(n)$ is given by non-degenerate inertia operator $\mathcal{I} : so(n) \to so(n)$. Then the Lagrangian of the free motion of the body reads $l = \frac{1}{2}\langle\mathcal{I}\Omega,\Omega\rangle$.

What form may have a multi-dimensional analog of the classical constraint (4.1)? To answer this question, we first note that, instead of rotations *about an axis* in the classical mechanics, in the $n$-dimensional case there are infinitesimal rotations in two-dimensional planes spanned by the basis vectors $e_i, e_j$, $i, j = 1, \dots, n$. Suppose, without loss of generality, that $\gamma = (1, 0, 0)^T$. Then this condition (4.1) can be redefined as follows: only infinitesimal rotations in the planes span$(e_1, e_2)$ and span$(e_1, e_3)$ are allowed. Hence, it is natural to define its $n$-dimensional analog as follows: only infinitesimal rotations in the fixed 2-planes spanned by $(e_1, e_2), \dots, (e_1, e_n)$ (i.e., in the planes containing the vector $e_1$) are allowed. This implies the constraints

$$\langle\Omega, e_i\wedge e_j\rangle = 0, \quad 2 \le i < j \le n. \tag{5.2}$$

Following [26], one can relax these constraints by assuming that the angular velocity matrix *in the space* has the following structure

$$\tilde{\Omega} = g\Omega g^{-1} = \begin{pmatrix} 0 & \cdots & \Omega_{1r} & \cdots & \Omega_{1n} \\ \vdots & \ddots & \vdots & & \vdots \\ -\Omega_{1r} & \cdots & 0 & \cdots & \Omega_{rn} \\ \vdots & & \vdots & \mathbf{O} & \\ -\Omega_{1n} & \cdots & -\Omega_{rn} & & \end{pmatrix},$$

where $\mathbf{O}$ is zero $(n-r)\times(n-r)$ matrix.

Equivalently, consider the right–invariant distribution $D_r$ on $TSO(n)$ whose restriction to the algebra $so(n)$ is given by

$$\mathfrak{d} = \operatorname{span}\{E_j \wedge E_k,\ k = 1, \dots, r,\ j = 1, \dots, n\},$$

where $E_i \wedge E_j$ form the basis in $so(n)$. Since $e_i \wedge e_j = g^{-1} \cdot E_i \wedge E_j \cdot g$, we have that constraints are

$$\begin{gathered} \Omega \in \mathcal{D}_r = g^{-1} \cdot \mathfrak{d} \cdot g = \operatorname{span}\{e_1 \wedge e_i, \dots, e_r \wedge e_i,\ 1 \le i \le n\} \\ \text{that is} \quad \langle \Omega, e_p \wedge e_q \rangle = 0, \quad r < p < q \le n. \end{gathered} \tag{5.3}$$

The LR system on the right-invariant distribution $D_r \subset T\,SO(n)$ can be described by the Euler–Poincaré equations (2.4) on the space $so(n) \times SO(n)$ with indefinite multipliers $\lambda_{pq}$,

$$\begin{gathered} \dot{\mathcal{I}}\Omega + [\Omega, \mathcal{I}\Omega] = \sum_{r<p<q\le n} \lambda_{pq}\, e_p \wedge e_q, \\ \dot{e}_i + \Omega e_i = 0, \qquad i = 1, \dots, n. \end{gathered} \tag{5.4}$$

Here the components of the vectors $e_1, \dots, e_n$ play the role of redundant coordinates on $SO(n)$. For $n = 3, r = 1$, this becomes the usual Veselova system (4.2) with $V = 0$.

Differentiating (5.3), from (5.4) one can obtain a system of linear equations for the determination of the multipliers in terms of the components of $\dot{\Omega}, \Omega$, and $e_i$. Thus, (5.4) contains a closed system of differential equations on the space $(\Omega_{ij}, e_{r+1}, \dots, e_n)$. The latter system has first integrals

$$\langle \Omega, e_p \wedge e_q \rangle = w_{pq}, \quad w_{pq} = \text{const}, \qquad r < p < q \le n. \tag{5.5}$$

and our LR system on $D_r \subset T\,SO(n)$ is the restriction of (5.4) onto the level variety $w_{pq} = 0$.

As follows from Theorem 3.1, the system (5.4) has an invariant measure with density

$$\begin{gathered} \mu = \sqrt{\det(\mathcal{I}^{-1}|_{\perp\mathcal{D}_r})} = \sqrt{|\langle e_p \wedge e_q, \mathcal{I}^{-1}(e_s \wedge e_l)\rangle|}, \\ r < p < q \le n, \quad r < s < l \le n, \end{gathered}$$

where $\perp\mathcal{D}_r \subset so(n)$ is the orthogonal complement of $\mathcal{D}_r$ with respect to the metric $\langle\cdot,\cdot\rangle$ and $\mathcal{I}^{-1}|_{\perp\mathcal{D}_r}$ is the restriction of the inertia tensor to $\perp\mathcal{D}_r$.

In case of the usual Veselova system on $TSO(3)$ one has $e_2 \times e_3 = \gamma \in \mathbb{R}^3$, $\mathcal{I} = I$, $\Omega_{ij} = \varepsilon_{ijk}\omega_k$ and the above expression reduces to the known form $\sqrt{(\gamma, I^{-1}\gamma)}$.

In practice, for a big dimension $n$ and small $r$, the number of constraints (5.3) is large, which leads to rather tedious expressions for the explicit form of the system and the density of its invariant measure. In this case one can make use of the alternative momentum description (the system (2.7)). Namely, in view of the matrix representation

$$\forall\, X \in so^*(n), \quad \mathrm{pr}_{\mathcal{D}_r}(X) = \Gamma X + X\Gamma - \Gamma X \Gamma,$$
$$\Gamma = e_1 \otimes e_1 + \cdots + e_r \otimes e_r \ ,$$

the system (2.9) takes the following form

$$\dot{\mathcal{M}} = [\mathcal{M}, \Omega], \tag{5.6}$$
$$\begin{aligned} \mathcal{M} &= \mathrm{pr}_{\mathcal{D}_r}(\mathcal{I}\Omega) + \mathrm{pr}_{\mathcal{D}_r^\perp}\Omega \\ &\equiv \Omega + (\mathcal{I}\Omega - \Omega)\Gamma + \Gamma(\mathcal{I}\Omega - \Omega) - \Gamma(\mathcal{I}\Omega - \Omega)\Gamma \,. \end{aligned} \tag{5.7}$$

The map $\Omega \to \mathcal{M}$ given by (5.7) is nondegenerate. As a result, equations (5.6), (5.7) together with the Poisson equations (5.1), which are equivalent to

$$\dot{\Gamma} = [\Gamma, \Omega], \tag{5.8}$$

represent a closed system of differential equations on the space $(\Omega, e_1, \ldots, e_r)$ or the space $(\mathcal{M}, e_1, \ldots, e_r)$.

In the classical case $n = 3, r = 1$, using the vector notation of $\Omega$, $\mathcal{M}$ and setting $e_1 = \gamma$, we obtain $\mathcal{M} = \mathcal{I}\Omega - (I\Omega, \gamma)\gamma + (\Omega, \gamma)\gamma$. Then (5.6) and the Poisson equations for $\gamma$ yield explicit equations describing the Veselova LR system (4.2), (4.3) with $V = 0$.

By analogy with (4.2), we will call $(SO(n), l, D_r)$ *multidimensional Veselova system*. As follows from the structure of (5.6), (5.8), this system possesses a family of integrals given by nonzero coefficients of the following polynomial in $\lambda$

$$\mathrm{tr}\ (\mathcal{M} + \lambda\Gamma)^k, \qquad k \in \mathbb{N}.$$

In addition, it has the invariant variety defined by the condition

$$\mathcal{M} \wedge e_1 \wedge \cdots \wedge e_r \equiv \omega \wedge e_1 \wedge \cdots \wedge e_r = 0, \tag{5.9}$$

where $\mathcal{M}, \omega$ are considered as 2-forms and $e_k$ as 1-forms in the same Euclidean space $\mathbb{R}^n$. This gives a set of scalar conditions on the components of $\mathcal{M}$ or $\omega$, which describes the linear subspace $\mathcal{D}_r = g^{-1} \cdot \mathfrak{d} \cdot g \subset so(n)$. Hence, among these conditions only $(n-r)(n-r-1)/2$ are independent.

Next, according to Theorem 3.2, the LR system (5.6), (5.7), (5.8) possesses an invariant measure with the dual density

$$\begin{gathered} \Theta = \tilde{\mu}\, d\Omega \wedge de_1 \wedge \cdots \wedge de_r = \tilde{\mu}^{-1}\, d\mathcal{M} \wedge de_1 \wedge \cdots \wedge de_r \\ \tilde{\mu} = \sqrt{\det(\mathcal{I}|_{\mathcal{D}_r})} = \sqrt{|\langle e_i \wedge e_p, \mathcal{I}(e_j \wedge e_q)\rangle|}, \\ 1 \le p < q \le r, \quad 1 \le i < j \le n, \end{gathered} \tag{5.10}$$

where $\mathcal{I}|_{\mathcal{D}_r}$ is the restriction of the inertia tensor to the subspace $\mathcal{D}_r \subset so(n)$.

**Remark.** Since $\mathcal{D}_r$ is invariant under the action of $SO(n-r)$ on the linear space spanned by $e_{r+1}, \ldots, e_n$, expression (5.10), in fact, does not depend explicitly on the components of these vectors. Moreover, $\mathcal{D}_r$ is also invariant under the

$SO(r)$-action on the space span $(e_1, \dots, e_r)$, hence the above density depends explicitly only on the *Plücker coordinates* of the $r$-form $e_1 \wedge \cdots \wedge e_r$, which are invariants of this action. Clearly, a simplified expression for the density depends on the choice of $\mathcal{I}$.

**The special inertia tensor.** It appears that for some special inertia tensors, the density (5.10) takes an especially simple form, which we shall make use in the sequel. Suppose that the operator $\mathcal{I}$ is defined by a diagonal matrix $A = diag(A_1, \dots, A_n)$ in the following way

$$\mathcal{I}(E_i \wedge E_j) = \frac{A_i A_j}{\det A} E_i \wedge E_j. \tag{5.11}$$

Notice that for $n = 3$ this corresponds to the well known three-dimensional vector formula $I(x \times y) = \frac{1}{\det A} Ax \times Ay$, $A = I^{-1}$ and thus, in this case, defines a *generic* inertia tensor.

**Theorem 5.1** *Under the above choice of* $\mathcal{I}$,

$$\det(\mathcal{I}|_{\mathcal{D}_r}) = \mathcal{P}_{n,r} = (\det A)^\rho \left[ \sum_I A_{i_1} \cdots A_{i_r} (e_1 \wedge \cdots \wedge e_r)_I^2 \right]^{n-r-1}, \tag{5.12}$$

*where $\rho$ is an integer constant and the summation is over all $r$-tuples $I = \{1 \le i_1 < \cdots < i_r \le n\}$.*

*Proof.* It is more convenient to calculate first the dual determinant $|\mathcal{I}^{-1}|_{\perp \mathcal{D}_r}|$, which can be represented in form

$$\left| \det A(e_p, A^{-1}e_s)(e_q, A^{-1}e_l) - \det A(e_p, A^{-1}e_l)(e_q, A^{-1}e_s) \right|, \tag{5.13}$$
$$r < p < q \le n, \quad r < s < l \le n.$$

Since we work with purely algebraic expressions, in this proof one can regard $e_{r+1}, \dots, e_n$ as vectors in the complex space $\mathbb{C}^n$. Next, since the action of $SO(n-r)$ on the linear space $\bar{\Lambda} \subset \mathbb{C}^n$ spanned by $e_{r+1}, \dots, e_n$ does not change $\perp\mathcal{D}_r \subset \wedge^2 \mathbb{C}^n$, the above determinant must depend only on the Plücker coordinates

$$(e_{r+1} \wedge \cdots \wedge e_n)_J, \quad J = \{j_1, \dots, j_{n-r}\}, \quad 1 \le j_1 < \cdots < j_{n-r} \le n.$$

In view of dimension and the structure of the determinant (5.13), it is a homogeneous polynomial in the components of $e_p$ of degree

$$4 \cdot \dim SO(n-r) = 2(n-r)(n-r-1).$$

Hence, it is a homogeneous polynomial of degree $2(n-r-1)$ in the Plücker coordinates.

Suppose that the linear space $\bar{\Lambda}$ is tangent to (possibly imaginary) cone $\mathcal{K} = \{(X, A^{-1}X) = 0\} \subset \mathbb{C}^n$ and, without loss of generality, assume that $e_n$ is directed along the tangent line $\bar{\Lambda} \cap \mathcal{K}$. Then $(e_n, A^{-1}e_p) = 0$ for $p = r+1, \dots, n$, and in this case the last $n-r-1$ rows and columns of the determinant (5.13), and therefore the determinant itself, vanish.

On the other hand, the condition for $\bar\Lambda$ to be tangent to $\mathcal{K}$ has the form

$$\det(A^{-1}|_{\bar\Lambda}) = \begin{vmatrix} (e_{r+1}, A^{-1}e_{r+1}) & \cdots & (e_{r+1}, A^{-1}e_n) \\ \vdots & \ddots & \vdots \\ (e_n, A^{-1}e_{r+1}) & \cdots & (e_n, A^{-1}e_n) \end{vmatrix} = 0 .$$

where $A^{-1}|_{\bar\Lambda}$ is the restriction of the quadratic form $A$ onto $\bar\Lambda$. Expanding the latter determinant we see that it equals $\sum_J A^{-1}_{i_1}\cdots A^{-1}_{i_r}(e_{r+1}\wedge\cdots\wedge e_n)^2_J$, thus it is a quadratic polynomial in the above Plücker coordinates.

Combining our results, we see that when $\bar\Lambda$ is tangent to $\mathcal{K}$, the matrix $A^{-1}|_{\bar\Lambda}$ has corank 1, whereas the matrix $\mathcal{I}^{-1}|_{\perp\mathcal{D}_r}$ has corank $(n-r-1)$. Hence, the determinant (5.13) is divisible by $(n-r-1)$-th power of $\det(A^{-1}|_{\bar\Lambda})$, which is a homogeneous polynomial of degree $2(n-r-1)$ in the coordinates $(e_{r+1}\wedge\cdots\wedge e_n)_J$. Thus the corresponding quotient has zero degree in these coordinates. Since it cannot have poles, it is a constant. An additional study of (5.13) shows that this constant is a positive power of $\det A$. Hence

$$\det(\mathcal{I}^{-1}|_{\perp\mathcal{D}_r}) = (\det A)^{\rho_1}\left[\sum_J A^{-1}_{j_1}\cdots A^{-1}_{j_{n-r}}(e_{r+1}\wedge\cdots\wedge e_n)^2_J\right]^{n-r-1}, \quad \rho_1\in\mathbb{N}.$$

Now, in order to obtain the desired expression (5.12), we can use relations (3.4) and $(e_{r+1}\wedge\cdots\wedge e_n)^2_J = (e_1\wedge\cdots\wedge e_r)^2_I$, where $I$ and $J$ are complimentary multi-indices in the sense that $\{i_1,\dots,i_r,\, j_1,\dots,j_{n-r}\}$ is a permutation of $\{1,\dots,n\}$, as well as the fact that $\det\mathcal{I}$ is a power of $\det A$. This proves the theorem.

From Theorems 3.1, 5.1 we get

**Corollary 5.2** *Under the condition (5.11), the LR system* (5.6)–(5.8) *has an invariant measure*

$$\left[\sum_I A_{i_1}\cdots A_{i_r}(e_1\wedge\cdots\wedge e_r)^2_I\right]^{-(n-r-1)/2} d\mathcal{M}\wedge d\,e_1\wedge\cdots\wedge d\,e_r . \qquad (5.14)$$

*In the particular case $r=1$ the density $\tilde\mu$ in (5.10) is proportional to* $(e_1, Ae_1)^{(n-2)/2}$.

As follows from (5.12) or (5.14), in the opposite extreme case $r=n-1$ (no constraints), $\mathcal{P}_{n,r}$ and $\tilde\mu$ are just constants, as expected.

**Reduction to Stiefel varieties.** Now we notice that in case of the constraints (5.3) the orthogonal complement $\mathfrak{h}$ of $\mathfrak{d}$ is a Lie algebra, namely

$$\mathfrak{h} = \operatorname{span}\{E_p\wedge E_q,\ r<p<q\le n\} \cong so(n-r).$$

Therefore, according to the observations of Section 2, the multidimensional Veselova system can be treated as a generalized Chaplygin system on the principal bundle

$$\begin{array}{ccc} SO(n-r) & \longrightarrow & SO(n) \\ & & \downarrow\ \pi \\ & & V(r,n) = SO(n-r)\backslash SO(n) \end{array} , \qquad (5.15)$$

where $V(r,n)$ is the Stiefel variety, which can be regarded as the variety of ordered sets of $r$ orthogonal unit vectors $e_1,\dots,e_r$ in $\mathbb{R}^n$ ($\mathbb{C}^n$), or, equivalently, the set of $r\times n$ matrices $\mathcal{X}=(e_1\cdots e_r)$ satisfying $\mathcal{X}^T\mathcal{X}=\mathbf{I}_r$, where $\mathbf{I}_r$ is the $r\times r$ unit matrix. Thus $V(r,n)$ is a smooth variety of dimension $rn-r(r+1)/2$ (see e.g., [21]), and the components of the vectors $e_k$ are *redundant* coordinates on it.

The nonholonomic right-invariant distribution $D_r$ is orthogonal to the leaf of the action of $SO(n-r)$ with respect to the bi-invariant metric on $SO(n)$.

The tangent bundle $TV(r,n)$ is the set of pairs $\mathcal{X},\dot{\mathcal{X}}$ subject to constraints

$$\mathcal{X}^T\mathcal{X}=\mathbf{I}_r,\quad \mathcal{X}^T\dot{\mathcal{X}}+\dot{\mathcal{X}}^T\mathcal{X}=0, \tag{5.16}$$

which give $r(r+1)$ independent scalar constraints.

Next, the moment map $\Phi : TV(r,n)\to so(n)$ is given by

$$\begin{aligned}\Omega=\Phi(\mathcal{X},\dot{\mathcal{X}})&=\mathcal{X}\dot{\mathcal{X}}^T-\dot{\mathcal{X}}\mathcal{X}^T+\frac{1}{2}\mathcal{X}[\mathcal{X}^T\dot{\mathcal{X}}-\dot{\mathcal{X}}^T\mathcal{X}]\mathcal{X}^T\\ &=e_1\wedge\dot e_1+\cdots+e_r\wedge\dot e_r+\frac{1}{2}\sum_{1\le\alpha<\beta\le r}[(e_\alpha,\dot e_\beta)-(\dot e_\alpha,e_\beta)]\ e_\alpha\wedge e_\beta.\end{aligned} \tag{5.17}$$

Indeed, we have $\Phi(\mathcal{X},\dot{\mathcal{X}})^T=-\Phi(\mathcal{X},\dot{\mathcal{X}})$ and $\Phi(\mathcal{X},\dot{\mathcal{X}})\in so(n)$. Taking into account constraints (5.16), we obtain

$$-\Phi(\mathcal{X},\dot{\mathcal{X}})\mathcal{X}=\dot{\mathcal{X}}-\mathcal{X}\dot{\mathcal{X}}^T\mathcal{X}-\frac{1}{2}\mathcal{X}(\mathcal{X}^T\dot{\mathcal{X}}-\dot{\mathcal{X}}^T\mathcal{X}),$$

which implies the Poisson equations for $e_i$

$$\dot{\mathcal{X}}=-\Omega\mathcal{X}. \tag{5.18}$$

On the other hand, putting $\dot{\mathcal{X}}=-\Omega\mathcal{X}$ into $\Phi(\mathcal{X},\dot{\mathcal{X}})$, we get

$$\Omega=\Omega\Gamma+\Gamma\Omega-\Gamma\Omega\Gamma=\mathrm{pr}_{\mathcal{D}}(\Omega),\qquad \Gamma=e_1\otimes e_1+\cdots+e_r\otimes e_r.$$

Hence $\Phi(\mathcal{X},\dot{\mathcal{X}})\in\mathcal{D}$ and formula (5.17) describes the momentum mapping.

The reduced Lagrangian $L(\mathcal{X},\dot{\mathcal{X}})$ takes the form

$$L=\frac{1}{2}\langle\mathcal{I}\Phi(\mathcal{X},\dot{\mathcal{X}}),\Phi(\mathcal{X},\dot{\mathcal{X}})\rangle=-\frac{1}{4}\mathbf{tr}\left(\mathcal{I}\Phi(\mathcal{X},\dot{\mathcal{X}})\circ\Phi(\mathcal{X},\dot{\mathcal{X}})\right).$$

Then we introduce $r\times n$ moment matrix

$$\mathcal{P}_{is}=\partial L(\mathcal{X},\dot{\mathcal{X}})/\partial\dot{\mathcal{X}}_{is}. \tag{5.19}$$

Since the Lagrangian is degenerate in the redundant velocities $\dot{\mathcal{X}}_{is}$, from this relation one cannot express $\dot{\mathcal{X}}$ in terms of $\mathcal{X},\mathcal{P}$ uniquely. On the other hand, the cotangent bundle $T^*V(r,n)$ can be realized as the set of pairs $\mathcal{X},\mathcal{P}$ subject to constraints

$$\mathcal{X}^T\mathcal{X}=\mathbf{I}_r,\quad \mathcal{X}^T\mathcal{P}+\mathcal{P}^T\mathcal{X}=0. \tag{5.20}$$

Under conditions (5.20), relation (5.19) can be uniquely inverted, and one can get $\dot{\mathcal{X}}=\dot{\mathcal{X}}(\mathcal{X},\mathcal{P})$ (for $r=1$ see the section below).

The symplectic structure $\mathit{\Omega}$ on $T^*V(r,n)$ is just the restriction of the canonical 2-form on the ambient space $\mathbb{R}^{2nr}=(\mathcal{X},\mathcal{P})$,

$$\sum_{i=1}^n\sum_{s=1}^r d\mathcal{P}_{is}\wedge\, d\mathcal{X}_{is}\,.$$

Next, in view of (2.11), (5.17) and (5.19), the moment map $\Phi^*(\mathcal{X},\mathcal{P})$ has the form

$$\mathcal{I}\Omega|_{\mathcal{D}} = \Phi^*(\mathcal{X},\mathcal{P}) = \mathcal{X}\mathcal{P}^T - \mathcal{P}\mathcal{X}^T + \frac{1}{2}\mathcal{X}[\mathcal{X}^T\mathcal{P} - \mathcal{P}^T\mathcal{X}]\mathcal{X}^T. \tag{5.21}$$

For a fixed $\mathcal{X}$, it establishes a bijection between the linear subspace $\mathcal{D}_r \subset so^*(n) = \{\mathcal{M}\}$ and the cotangent space $T^*_{\mathcal{X}}V(r,n)$.

**Theorem 5.3** *The reduced LR system on $T^*V(r,n)$ is the restriction of the following system on the space $(\mathcal{X},\mathcal{P})$,*

$$\dot{\mathcal{X}} = -\Omega(\mathcal{X},\mathcal{P})\,\mathcal{X}, \quad \dot{\mathcal{P}} = -\Omega(\mathcal{X},\mathcal{P})\,\mathcal{P}, \tag{5.22}$$

*where* $\Omega(\mathcal{X},\mathcal{P}) = \Phi(\mathcal{X},\dot{\mathcal{X}}(\mathcal{X},\mathcal{P}))$.

*Proof.* Substituting the expression (5.21) into the momentum equation (2.12), differentiating its left hand side, then taking into account the Poisson equations (5.18) and the conditions (5.20), we obtain

$$\mathcal{X}\dot{\mathcal{P}}^T - \dot{\mathcal{P}}\mathcal{X}^T - \mathcal{X}(\dot{\mathcal{P}}^T\mathcal{X} - \mathcal{P}^T\Omega\mathcal{X})\mathcal{X}^T = \mathcal{X}\mathcal{P}^T\Omega + \Omega\mathcal{P}\mathcal{X}^T.$$

Multiplying both sides of this relations from the left by $\mathcal{X}^T$ and from the right by $\mathcal{X}$, then using again the conditions (5.20), we arrive at equation

$$-\mathcal{X}^T\dot{\mathcal{P}} = \mathcal{X}^T\Omega\mathcal{P},$$

which implies $\dot{\mathcal{P}} = -\Omega P$. The first equation in this system is just a repetition of (5.18). The theorem is proved.

According to (5.22), apart from the energy integral, the reduced flow on $T^*V(r,n)$ possesses matrix momentum integral $\mathcal{P}^T\mathcal{P}$.

Notice that the form of equations (5.22) is quite similar to those describing geodesic flows on Stiefel and Grassmann varieties (see [6, 11]). However, our system is not Hamiltonian with respect to the symplectic structure $\varOmega$.

**Reduced invariant measure.** Now note that the phase space $(\mathcal{M}, e_1,\dots,e_r)$ of the LR system (5.6), (5.7), (5.8) can naturally be regarded as the dual to the semi-direct Lie algebra product

$$so(n) \ltimes (\underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{r \text{ times}})$$

and it carries the corresponding Lie–Poisson structure $\{\cdot,\cdot\}$. (For $r=1$ this is just the Lie–Poisson bracket on the coalgebra $e^*(n)$.) This Poisson structure is degenerate, and the subvariety $\mathcal{O}_r \subset (\mathcal{M}, e_1,\dots,e_r)$ defined by the constraints (5.9) and the conditions $(e_k,e_k) = \delta_{kl},\ k,l = 1,\dots,r$ is its $2N$-dimensional symplectic leaf ($N = \dim V(r,n) = rn - r(r+1)/2$ : the restriction of $\{\cdot,\cdot\}$ onto $\mathcal{O}_r$ is nondegenerate. One can show that $\mathcal{O}_r$ is a (generally singular) orbit of coadjoint action of the semi-direct group product $SO(n) \ltimes (\mathbb{R}^n \times \cdots \times \mathbb{R}^n)$.

Let $\varSigma$ be the corresponding symplectic structure on $\mathcal{O}_r$. By construction, the extended moment map $\tilde{\Phi}^* : T^*V(r,n) \to \mathcal{O}_r$ preserves the Poisson structure, hence it is a symplectomorphism: the 2-form $\varSigma$ passes to the symplectic structure $\varOmega$ on $T^*V(r,n)$. Thus, $\varSigma^N$, as a volume form on $\mathcal{O}_r$, transforms to the canonical volume form on $T^*V(r,n)$.

As we know from Theorem 3.5, a reduced LR system always has an invariant measure. Using the above property of $\tilde{\Phi}^*$, in our example the measure can be written explicitly.

**Theorem 5.4** *The reduced LR flow on $T^*V(r,n)$ has invariant measure*

$$1/\sqrt{\det(\mathcal{I}|_{\mathcal{D}_r})}\,\Omega^N, \qquad N = \dim V(r,n) = rn - r(r+1)/2.$$

Notice that the density of this measure coincides with that of the invariant measure $\Theta$ of the LR system (5.6)–(5.8) in the coordinates $(\mathcal{M}, e_1, \ldots, e_r)$. In particular, for the special inertia tensor (5.11) the density is the same as in (5.14).

*Sketch of proof of Theorem* 5.4. Let

$$\psi_1(\mathcal{M}, e), \ldots, \psi_m(\mathcal{M}, e), \qquad m = (n-r)(n-r-1)/2$$

be any independent linear combinations of the constraint functions defined by (5.9). Then, at points of $\mathcal{O}_r \subset (\mathcal{M}, e_1, \ldots, e_r)$,

$$\begin{aligned} d\mathcal{M} \wedge d\,e_1 \wedge \cdots \wedge d\,e_r &= \xi(e_1, \ldots, e_r)\cdot \\ &\cdot d\psi_1 \wedge \cdots \wedge d\psi_m \wedge \Sigma^N \prod_{1\le k\le l\le r} \wedge\, d\phi_{kl}, \qquad \phi_{kl} = (e_k, e_l), \end{aligned} \tag{5.23}$$

where $\Sigma^N$ is the canonical volume form on $\mathcal{O}_r$ and $\xi(e_1, \ldots, e_r)$ is a certain nonvanishing function. The latter can be found by inserting the polyvector

$$X_{\mathcal{M}_{12}} \wedge \cdots \wedge X_{\mathcal{M}_{n-1,n}} \wedge X_{e_{11}} \wedge \cdots \wedge X_{e_{nr}}$$

into the left and right hand sides of (5.23) and taking into account relations

$$\Sigma(X_{\mathcal{M}_{ij}}, X_{\mathcal{M}_{pq}}) = \{\mathcal{M}_{ij}, \mathcal{M}_{pq}\}, \quad \Sigma(X_{\mathcal{M}_{ij}}, X_{e_{pk}}) = \{\mathcal{M}_{ij}, e_{pk}\},$$
$$\Sigma(X_{e_{pk}}, X_{e_{ql}}) = \{e_{pk}, e_{ql}\} = 0.$$

One can always choose such a basis of functions $\psi_k(\mathcal{M}, e)$ that $\xi$ becomes a constant on the whole orbit $\mathcal{O}_r$. Moreover, for this basis, the time derivative with respect to the flow (5.6)–(5.8), has the form $\dot{\psi}_k = \sum_{s=1}^m \varkappa_s \psi_s$ with some functions $\varkappa_s$ such that $\varkappa_k = 0$ on $\mathcal{O}_r$.

Now let $\mathcal{L}_v = d\,i_v + i_v\,d$ denote the Lie derivative in the space $(\mathcal{M}, e_1, \ldots, e_r)$ with respect to this flow, $i_v$ being the interior product corresponding to the flow. Since $\phi_{kl}$ are its generic first integrals, one has

$$\mathcal{L}_v\, d\phi_{kl} \equiv d(\dot{\phi}_{kl}) = 0. \tag{5.24}$$

On the other hand, the chosen functions $\psi_k$ are particular integrals of the flow. Then, for any $k$

$$\mathcal{L}_v d\psi_k = d(\dot{\psi}_k) = \sum_{j=1}^m (\varkappa_j d\psi_j + \psi_j d\varkappa_j) \qquad (\varkappa_k = 0 \text{ on } \mathcal{O}_r).$$

Hence, at points of $\mathcal{O}_r$ we have

$$d\psi_1 \wedge \cdots \wedge (\mathcal{L}_v\, d\psi_k) \wedge \cdots \wedge d\psi_m \equiv 0. \tag{5.25}$$

As a result, since

$$\mathcal{L}_v\Theta \equiv \mathcal{L}_v\left(1/\sqrt{\det(\mathcal{I}|_{\mathcal{D}_r})}\, d\psi_1\wedge\cdots\wedge d\psi_m\wedge\varSigma^N \prod_{1\le k\le l\le r}\wedge d\phi_{kl}\right)=0\,,$$

relations (5.24), (5.25) imply $\mathcal{L}_v(1/\sqrt{\det(\mathcal{I}|_{\mathcal{D}_r})}\,\varSigma^N)=0$, and the volume form under the Lie derivative is an integral invariant. Replacing here $\varSigma^N$ by the volume form $\varOmega^N$ on $T^*V(r,n)$, we arrive at the statement of the theorem.

Reducibility of the system (5.22) to the Hamiltonian form via a change of time for an arbitrary rank $r$ and an arbitrary inertia tensor $\mathcal{I}$ is an open problem.

# 6 Rank 1 case and integrable geodesic flow on $S^{n-1}$

Now we concentrate on the case $r=1$ given by the original condition (5.2) and again assume that the inertia tensor has the form (5.11). The variety $V(1,n)$ can be realized as the unit sphere $S^{n-1}$ in $\mathbb{R}^n$,

$$S^{n-1}=\{q\in\ \mathbb{R}^{n-1},\ q_1^2+\cdots+q_n^2=1\},$$

where we set $q=e_1$, and the moment map (5.17) is reduced to

$$\Omega=\Phi(q,\dot q)=q\wedge\dot q. \tag{6.1}$$

Therefore, for solution $e_1(t)$, $\Omega(t)=e_1(t)\wedge\dot e_1(t)$ of (5.6) (5.8), $q(t)=e_1(t)$ is a motion of a reduced system on the sphere $S^{n-1}$.

For the analysis of the reduced system we can use Theorems 5.3, 5.4 of the previous section. However, for our future purposes we shall use the reduction procedure described in the Proposition 2.1.

Under the condition (5.11) and, in view of (6.1), the reduced Lagrangian $L(q,\dot q)$ and the right hand side of the Lagrange-d'Alambert equation (2.10) take the form

$$L=\frac{1}{2\det A}[(A\dot q,\dot q)(Aq,q)-(Aq,\dot q)^2]\,, \tag{6.2}$$

$$\begin{aligned}\langle\mathcal{I}\Phi(q,\dot q),\mathrm{pr}_{g^{-1}\mathfrak{h}g}[\Phi(q,\dot q),\Phi(q,\xi)]\rangle &= \frac{1}{\det A}\langle Aq\wedge A\dot q,\mathrm{pr}_{g^{-1}\mathfrak{h}g}\,\xi\wedge\dot q\rangle\\ &=\frac{1}{\det A}(\dot q,A\dot q)(Aq,\xi)-\frac{1}{\det A}(\dot q,Aq)(A\dot q,\xi)=\Psi(q,\dot q,\xi)\,.\end{aligned} \tag{6.3}$$

Here we used relation $\mathrm{pr}_{g^{-1}\mathfrak{h}g}\,\xi\wedge\dot q=\xi\wedge\dot q$ for any admissible vector $\xi=(\xi_1,\dots,\xi_n)^T\in T_qS^{n-1}$.

As in Section 4, the reduction of the LR system (5.4) onto $T^*S^{n-1}$ can explicitly be written in terms of local coordinates $q_1,\dots,q_{n-1}$ on $S^{n-1}$ and the corresponding momenta.

As an alternative, below we shall keep using the redundant coordinates $q_i$ and velocities $\dot q_i$, in which the Lagrange equations have the form

$$\frac{\partial L}{\partial q_i}-\frac{d}{dt}\frac{\partial L}{\partial\dot q_i}=\pi_i+\Lambda q_i\,,\qquad i=1,\dots,n, \tag{6.4}$$

$$\pi_i=\frac{\partial\Psi}{\partial\xi_i}=\frac{1}{\det A}(\dot q,A\dot q)A_iq_i-\frac{1}{\det A}(\dot q,Aq)A_i\dot q_i,$$

where $\Lambda$ is a Lagrange multiplier. We now want to represent the reduced LR system on $T^*S^{n-1}$ as a restriction of a system on the Euclidean space $\mathbb{R}^{2n} = \{q, p\}$. Note that $L(q, \dot{q})$ is degenerate in the redundant velocities $\dot{q}$, hence they cannot be expressed uniquely in terms of the redundant momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \equiv \frac{1}{\det A}(q, Aq)A_i\dot{q}_i - \frac{1}{\det A}(\dot{q}, Aq)A_i q_i. \tag{6.5}$$

In this case one can apply the Dirac formalism for Hamiltonian systems with constraints in the phase space (see, e.g., [19, 2, 35]). Namely, from (6.5) we find that $(q, p) = 0$, hence the cotangent bundle $T^*S^{n-1}$ is realized as a subvariety of $\mathbb{R}^{2n} = (q, p)$ defined by constraints

$$\phi_1 \equiv (q, q) = 1, \quad \phi_2 \equiv (q, p) = 0.$$

Under these conditions, relations (6.5) can be uniquely inverted to yield

$$\dot{q} = \frac{\det A}{(q, Aq)} \left[ A^{-1}p - (p, A^{-1}q)q \right] . \tag{6.6}$$

On the other hand, we note that $\partial L/\partial q_i = \pi_i$. Then, from (6.4) we obtain $\dot{p} = -\Lambda q$ and, from the condition $(\dot{q}, p) + (q, \dot{p}) = 0$,

$$\dot{p} = -\Lambda q, \qquad \Lambda = \det A \frac{(p, A^{-1}p) - (p, q)(q, A^{-1}p)}{(q, Aq)} . \tag{6.7}$$

The system (6.6), (6.7) on $T^*S^{n-1}$ coincides with the restriction of the following system on $\mathbb{R}^{2n} = \{q, p\}$

$$\dot{q}_i = \{q_i, \hat{H}\}_*, \quad \dot{p}_i = \{p_i, \hat{H}\}_* - \hat{\pi}_i,$$
$$\hat{\pi}_i(q, p) = \pi_i(q, \dot{q}(q, p)), \quad \hat{H} = \frac{1}{2} \det A \frac{(p, A^{-1}p)}{(q, Aq)},$$

which is quasi-Hamiltonian with respect to the Dirac bracket on $\mathbb{R}^{2n}$

$$\{F, G\}_* = \{F, G\} + \frac{\{F, \phi_1\}\{G, \phi_2\} - \{F, \phi_2\}\{G, \phi_1\}}{\{\phi_1, \phi_2\}},$$

$\{\cdot, \cdot\}$ being the standard Poisson bracket on $\mathbb{R}^{2n}$. The latter system has explicit vector form

$$\begin{aligned} \dot{q} &= \frac{\det A}{(q, Aq)} \left[ A^{-1}p - \frac{(p, A^{-1}q)}{(q, q)} q \right] , \\ \dot{p} &= -\det A \frac{(p, A^{-1}p)(q, q) - (p, q)(q, A^{-1}p)}{(q, Aq)(q, q)^2} \, q \, . \end{aligned} \tag{6.8}$$

The bracket $\{\cdot, \cdot\}_*$ is degenerate and possesses Casimir functions $\phi_1, \phi_2$ specified above.

Notice that from (6.1) and (6.6) we get

$$\Omega = q \wedge \frac{\det A}{(q, Aq)} \left[ A^{-1}p - \frac{(p, A^{-1}q)}{(q, q)} q \right] .$$

Then equations (6.8) can also be obtained directly from Theorem 5.3 by setting $r = 1$, $\mathcal{X} = q, \mathcal{P} = p$.

Finally, from Theorems 5.4 and 5.1 we get the following corollary.

**Corollary 6.1** *The reduced LR system (6.6), (6.7) on $T^*S^{n-1}$ possesses an invariant measure*

$$(Aq,q)^{-(n-2)/2}\,\sigma, \qquad \sigma = \Omega^{n-1}$$

*where $\sigma$ is the volume $2(n-1)$-form and $\Omega$ is the restriction of the canonical symplectic form $dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$ onto $T^*S^{n-1}$.*

In particular, for the reduction of the Veselova LR system (4.2) onto $T^*S^2$, the density of its invariant measure is proportional to $1/\sqrt{(q,Aq)}$, as was claimed in Section 4.

**Reducibility.** As follows from Corollary 6.1, item 1) of Theorem 3.5, and the fact that the dimension of the reduced configuration manifold equals $n-1$, if our reduced LR system on $T^*S^{n-1}$ were transformable to a Hamiltonian form by a change of time, then the corresponding reducing multiplier $\mathcal{N}$ should be proportional to $1/\sqrt{(q,Aq)}$.

Although Chaplygin's reducibility theorem does not admit a straightforward multidimensional generalization, i.e., item 1) of Theorem 3.5 cannot be inverted, remarkably, for the reduced LR system on $T^*S^{n-1}$ the inverse statement becomes applicable.

**Theorem 6.2** *1). Under change of time $d\tau = \sqrt{\det A/(Aq,q)}\,dt$ and appropriate change of momenta, the reduced LR system (6.4) or (6.6), (6.7) becomes a Hamiltonian system describing a geodesic flow on $S^{n-1}$ with the following Lagrangian obtained from (6.2)*

$$L^*(q,dq/d\tau) = \frac{1}{2}(q,Aq)^{-1}\left[\left(A\frac{d\,q}{d\tau},\frac{d\,q}{d\tau}\right)(Aq,q) - \left(Aq,\frac{d\,q}{d\tau}\right)^2\right]. \quad (6.9)$$

*2). The latter system is algebraic completely integrable for any dimension $n$. In the spheroconic coordinates $\lambda_1,\ldots,\lambda_{n-1}$ on $S^{n-1}$ such that*

$$q_i^2 = \frac{(I_i-\lambda_1)\cdots(I_i-\lambda_{n-1})}{\prod_{j\neq i}(I_i-I_j)}, \qquad I_i = A_i^{-1} \quad (6.10)$$

*the Lagrangian $L^*(q,dq/d\tau)$ takes the Stäckel form and the evolution of $\lambda_k$ is described by the Abel–Jacobi quadratures*

$$\frac{\lambda_1^{k-1}d\lambda_1}{2\sqrt{R(\lambda_1)}} + \cdots + \frac{\lambda_{n-1}^{k-1}d\lambda_{n-1}}{2\sqrt{R(\lambda_{n-1})}} = \delta_{k,n-1}\sqrt{2h}\,d\tau, \quad (6.11)$$
$$k = 1,\cdots,n-1,$$

*where*

$$R(\lambda) = -(\lambda-I_1)\cdots(\lambda-I_n)\lambda(\lambda-c_2)\cdots(\lambda-c_{n-1}), \quad (6.12)$$

*$h = L^*$ being the energy constant and $c_2,\cdots,c_{n-1}$ being other constants of motion (we set $c_1 = 0$). For generic values of these constants the corresponding invariant manifolds are $(n-1)$-dimensional tori.*

We start with the proof of item 2) of Theorem 6.2, which is quite standard. Namely, using the Jacobi identities

$$\text{for any distinct } \rho_1,\dots,\rho_N, \quad \sum_{s=1}^{N} \frac{\rho^m}{\prod_{l\neq s}(\rho_l-\rho_s)} = \begin{cases} 0 & \text{for } 0\le m< N-1,\\ 1 & \text{for } m=N-1,\end{cases}$$

in the spheroconic coordinates we have

$$\left(A\frac{d\,q}{d\tau},\frac{d\,q}{d\tau}\right)(Aq,q)-\left(Aq,\frac{d\,q}{d\tau}\right)^2 \tag{6.13}$$

$$=\frac{1}{4}\frac{\lambda_1\cdots\lambda_{n-1}}{I_1\cdots I_n}\sum_{k=1}^{n-1}\frac{\prod_{s\neq k}(\lambda_k-\lambda_s)}{(\lambda_k-I_1)\cdots(\lambda_k-I_n)\,\lambda_k}\left(\frac{d}{d\tau}\lambda_k\right)^2,$$

$$(Aq,q)\equiv(I^{-1}q,q)=\frac{\lambda_1\cdots\lambda_{n-1}}{I_1\cdots I_n}\,. \tag{6.14}$$

Then the reduced Lagrangian $L^*(q,dq/d\tau)$ in (6.9) takes form

$$L^*=\frac{1}{8}\sum_{k=1}^{n-1}\frac{\prod_{s\neq k}(\lambda_k-\lambda_s)}{(\lambda_k-I_1)\cdots(\lambda_k-I_n)\,\lambda_k}\left(\frac{d}{d\tau}\lambda_k\right)^2,$$

As a result, the corresponding Hamiltonian written in terms of

$$\lambda_k,\quad \mu_k=\frac{\partial L^*}{\partial(d\lambda_k/d\tau)}$$

is of Stäckel type (see e.g., [2]), which leads to the quadratures (6.11) and proves the integrability of the system.

The proof of item 1) of Theorem 6.2 is based on a relation between the reduced LR system and the celebrated Neumann system and will be given in the end of this section.

**Reduction to the Neumann system.** It appears that Theorem 4.1 relating the Veselova LR system and the classical Neumann system has the following multidimensional generalization. Namely, introduce another new time $\tau_1$ by formula

$$d\tau_1=\hat\mu^{-1}dt,\qquad \hat\mu^{-1}=\sqrt{\det A\frac{\langle q\wedge\dot q,\mathcal{I}(q\wedge\dot q)\rangle}{(Aq,q)}}\,dt, \tag{6.15}$$

and let $'$ denotes the derivation in the new time.

**Theorem 6.3** *Under the time change (6.15), the solutions $q(t)$ of the reduced multidimensional Veselova system on $S^{n-1}$ transforms to the solution of the integrable Neumann problem with the potential $U(q)=\frac{1}{2}(A^{-1}q,q)$,*

$$q''=-\frac{1}{A}q+\lambda q,\qquad q'=\frac{dq}{d\tau_1}. \tag{6.16}$$

*corresponding to zero value of the integral*

$$F_0(q,q')=\langle Aq',q'\rangle\langle Aq,q\rangle-\langle Aq,q'\rangle^2-\langle Aq,q\rangle \tag{6.17}$$

*and vise versa.*

**Remark 6.1** In the case $n = 3$ we have the mentioned Veselov–Veselova result with inertia tensor $I = A^{-1}$ [41]. The theorem is obtained recently by Fedorov and Kozlov by using special solutions of multidimensional Clebsch–Perelomov system. The proof we shall present here is similar as three-dimensional Veselov–Veselova proof.

*Proof of Theorem* 6.3. Let, as above, $\Omega = q \wedge \dot{q}$ and set $P = q \wedge q'$. Then the energy integral of the reduced Veselova system and the integral (6.17) of the Neumann system can be written as:

$$E(q, \dot{q}) = \frac{1}{2}\langle \mathcal{I}(\Omega), \Omega\rangle, \quad F(q, q') = \det A \langle \mathcal{I}P, P\rangle - (Aq, q).$$

The change of time (6.15) induces a bijection between invariant submanifolds $\mathcal{E}_h = \{E = h\} \subset TS^{n-1}\{q, \dot{q}\}$ and $\mathcal{F}_0 = \{F = 0\} \subset TS^{n-1}\{q, q'\}$. Indeed, on $\mathcal{E}_h$ we have

$$dt = \mu_h d\tau_1\,, \qquad \mu_h^{-1} = \sqrt{\frac{2h \det A}{(Aq, q)}}. \tag{6.18}$$

Then the point $(q, \dot{q}) \in \mathcal{E}_h$ corresponds to $(q, q')$, $q' = \mu_h \dot{q}$, and the equation $\langle I\Omega, \Omega\rangle/2 = h$ corresponds to relation

$$\frac{1}{2\mu_h^2}\langle \mathcal{I}P, P\rangle \equiv \frac{1}{2}\frac{2h \det A}{(Aq, q)}\langle P, \mathcal{I}P\rangle = h.$$

Therefore $F = \det A\langle \mathcal{I}P, P\rangle - (Aq, q) = 0$, and $(q, q') \in \mathcal{F}_0$.

Next, let us note that equations (5.7) with $\Omega = q \wedge \dot{q}$ are equivalent to the equations

$$(\mathcal{I}\dot{\Omega} \cdot q) \wedge q + (\mathcal{I}\Omega \cdot \dot{q}) \wedge q = 0. \tag{6.19}$$

After changing of time (6.18) we have that $P = \mu_h \Omega$ and

$$\frac{dP}{d\tau} = \frac{dP}{dt}\mu_h = \frac{d}{dt}(\mu_h \Omega)\mu_h = \mu_h^2 \frac{d\Omega}{dt} + \frac{1}{2}\frac{d}{dt}(\mu_h^2)\Omega. \tag{6.20}$$

Now we apply the inertia operator (5.11) to both sides of this relation, then multiply the result by the vector $q$, and finally take the wedge product with $q$. As a result, in view of (6.18), we get

$$(\mathcal{I}P' \cdot q) \wedge q = \frac{(Aq, q)}{2h \det A}(\mathcal{I}\dot{\Omega} \cdot q) \wedge q + \frac{(Aq, \dot{q})}{2h \det A}(\mathcal{I}\Omega \cdot q) \wedge q \tag{6.21}$$

Using (6.19), we transform (6.21) to

$$2h \det A\, (\mathcal{I}P' \cdot q) \wedge q = -(Aq, q)(\mathcal{I}\Omega \cdot \dot{q}) \wedge q + (Aq, \dot{q})(\mathcal{I}\Omega \cdot q) \wedge q. \tag{6.22}$$

The right hand side of (6.22) is of the form $\Xi \wedge q$, where

$$\begin{aligned} \Xi &= (Aq, \dot{q})\mathcal{I}\Omega \cdot q - (Aq, q)\mathcal{I}\Omega \cdot \dot{q} \\ &= \frac{1}{\det A}(Aq, \dot{q})(Aq \otimes A\dot{q} - A\dot{q} \otimes Aq) \cdot q \\ &\quad -\frac{1}{\det A}(Aq, \dot{q})(Aq \otimes A\dot{q} - A\dot{q} \otimes Aq) \cdot \dot{q} = -2hAq. \end{aligned} \tag{6.23}$$

For the last equality in (6.23) we used identity

$$2h = \langle \mathcal{I}(q \wedge \dot{q}), q \wedge \dot{q}\rangle = \frac{1}{\det A}\langle Aq \wedge A\dot{q}, q \wedge \dot{q}\rangle = \frac{1}{\det A}(Aq, q)(A\dot{q}, \dot{q}) - (Aq, \dot{q})^2.$$

Hence, (6.22), (6.23) yield

$$(\mathcal{I}P' \cdot q) \wedge q = \frac{1}{\det A} q \wedge Aq, \qquad P = q \wedge q'. \tag{6.24}$$

In view of the constraint $(q, q) = 1$, this is equivalent to equations (6.16).

Thus we proved that if $q(t)$ is a solution of reduced multidimensional Veselova system laying on $\mathcal{E}_h$, i.e., $q(t)$ satisfies (6.19), then $q(\tau_1)$ is a solution of the Neumann system (6.16) laying on $\mathcal{F}_0$.

Conversely, starting from (6.24) and repeating calculations in inverse direction, we arrive at (6.19). The theorem is proved.

It is known (see e.g., [32, 35, 37]) that the Neumann system on $S^{n-1}$ possesses the following family of quadratic first integrals

$$\mathcal{F}(\lambda) = \sum_{1\le i<j\le n} \frac{P_{ij}^2}{(\lambda - I_i)(\lambda - I_j)} + \sum_{i=1}^{n} \frac{q_i^2}{\lambda - I_i}, \tag{6.25}$$

and that the evolution of the spheroconic coordinates $\lambda_k$ defined by (6.10) is described by equations

$$\frac{\lambda_1^{k-1} d\lambda_1}{2\sqrt{\mathcal{R}(\lambda_1)}} + \cdots + \frac{\lambda_{n-1}^{k-1} d\lambda_{n-1}}{2\sqrt{\mathcal{R}(\lambda_{n-1})}} = \delta_{k,n-1}\, d\tau_1, \qquad k = 1, \cdots, n-1, \tag{6.26}$$

where $\mathcal{R}(\lambda)$ is a polynomial of degree $2n-1$,

$$\mathcal{R} = -\Phi^2(\lambda)\mathcal{F}(\lambda), \qquad \Phi(\lambda) = (\lambda - I_1)\cdots(\lambda - I_n)\,.$$

Next, as follows from (6.25), for the trajectories $q(\tau_1)$ corresponding to zero value of the integral (6.17), we have $\mathcal{F}(0) = 0$, hence, in this case, the polynomial $\mathcal{R}(\lambda)$ has the same form as (6.12), that is

$$\mathcal{R} = -(\lambda - I_1)\cdots(\lambda - I_n)\lambda(\lambda - c_2)\cdots(\lambda - c_{n-1}), \tag{6.27}$$

Now, comparing equations (6.26) with the quadratures (6.11), we arrive at the following

**Proposition 6.4** *Under the time change $d\tau_1 = \sqrt{2h}\, d\tau$ the solution $q(\tau_1)$ of the Neumann system (6.16) lying on $\mathcal{F}_0 = \{F_0 = 0\}$ transforms to a solution $q(\tau)$ of the geodesic flow on $S^{n-1}$ described by the Lagrangian $L^*$ in (6.9) and having the energy constant $h$, and vise versa.*

Now combining Theorem 6.3 and Proposition 6.4, we finally obtain the proof of item 1) of Theorem 6.2.

# 7 Reconstructed motion on the distribution $D$

Now we consider the integrability of the original (unreduced) LR system on the right-invariant distribution $D \subset TSO(n)$ of dimension $(n-1) + n(n-1)/2$, which is specified by constraints (5.2) and the left-invariant metric defined by (5.11).

In the Hamiltonian case, the integrability of the reduced system implies generally a non-commutative integrability of the original system, namely the phase space is foliated by invariant isotropic tori with quasi-periodic dynamic. In our nonholonomic case one has to solve the reconstruction problem: find all trajectories $(g(t), \dot g(t))$ in $D$ that under $SO(n-1)$–reduction $\pi : D \to TS^{n-1}$ are projected to the given trajectory $(q(t), \dot q(t))$ in $TS^{n-1}$. (In particular, for the Fedorov–Kozlov integrable case of the multidimensional nonholonomic Suslov problem, the reconstruction problem was studied in [7, 8].)

Since $SO(n-1)$ is a symmetry group of the LR system on $D$, and the reduced motion on $TS^{n-1}$ occurs on $(n-1)$–dimensional generic invariant tori with quasi-periodic dynamics, it is natural to expect that the reconstructed motion $(g(t), \dot g(t))$ is quasi-periodic over $(\rho + n - 1)$–dimensional tori, where $\rho$ does not exceed the dimension of the maximal commutative subgroup of $SO(n-1)$, that is $\rho \le \operatorname{rank}\ SO(n-1) = \left[\frac{n-1}{2}\right]$ (see [27]).

As we shall see below, for our case this is not quite true. In fact, the relation between the reduced LR system and the Neumann system described by Theorem 6.3 enables us to reconstruct the motion on $D$ exactly. For this purpose we also shall make use of the remarkable correspondence between the Neumann system and a geodesic flow on a quadric. Namely, consider a family of $(n-1)$-dimensional confocal quadrics in $\mathbb{R}^n = (X_1, \dots, X_n)$,

$$
Q(\alpha) = \left\{ \frac{X_1^2}{\alpha - A_1} + \cdots + \frac{X_n^2}{\alpha - A_n} = -1 \right\}, \quad \alpha \in \mathbb{R}. \tag{7.1}
$$

**Theorem 7.1** ([32]). *Let $X(s)$ be a geodesic on the quadric $Q(0)$, $s$ being a natural parameter. Then under the change of time*

$$
ds = \sqrt{\frac{(dX/ds, A^{-1} dX/ds)}{(X, A^{-2}X)}}\, d\tau_1 \tag{7.2}
$$

*the unit normal vector $q(\tau_1) = A^{-1}X/|A^{-1}X|$ is a solution to the Neumann system (6.16) corresponding to zero value of the integral $F_0(q, q')$ in (6.16) and vise versa*

It is well known that the problem of geodesics on a quadric $Q(0)$ is completely integrable, and qualitative behavior of the geodesics is described by the remarkable *Chasles theorem* (see e.g., [32, 35]): the tangent line

$$
\ell_s = \{X(s) + \sigma\, dX/ds \mid \sigma \in \mathbb{R}\}
$$

of a geodesic $X(s)$ on $Q(0)$ is also tangent to a fixed set of confocal quadrics $Q(\alpha_2), \dots, Q(\alpha_{n-1}) \subset \mathbb{R}^n$, where $\alpha_2, \dots, \alpha_{n-1}$ are parameters playing the role of constants of motion (we set $\alpha_1 = 0$). Now let $\mathfrak{n}_k$ be the normal vector of the quadric $Q(\alpha_k)$ at the touching point $\mathfrak{p}_k = \ell \cap Q(\alpha_k)$. Then another classical theorem of geometry says that the normal vectors $\mathfrak{n}_1, \dots, \mathfrak{n}_{n-1}$, together with the unit tangent vector $\gamma = dX/ds$, form an orthogonal basis in $\mathbb{R}^n$.

On the other hand, in [35], Moser proved the following

**Proposition 7.2** *1). Let $x$ be the position vector of a point on the line $\ell_s$, which is tangent to geodesic $X(s)$. Then in the new parametrization $s_1$ such that $ds = -(X, A^{-2}X)\, ds_1$ the evolution of the line is described by the Lax equations in $n \times n$ matrix form*

$$
\frac{d}{ds_1}\mathcal{L} = [\mathcal{L}, \mathcal{B}], \qquad \mathcal{L} = \Pi_\gamma (A - x \otimes x) \Pi_\gamma, \tag{7.3}
$$

$$
\mathcal{B} = A^{-1}x \otimes A^{-1}\gamma - A^{-1}\gamma \otimes A^{-1}x\,, \tag{7.4}
$$

*where* $\Pi_\gamma = Id - (\gamma,\gamma)^{-1}\gamma\otimes\gamma$ *is the projection onto the orthogonal complement of* $\gamma$ *in* $\mathbb{R}^n$.

*2). The conserved eigenvalues of* $\mathcal{L}$ *are given by the parameters* $\alpha_1 = 0, \alpha_2, \dots, \alpha_{n-1}$ *of the confocal quadrics and by an extra zero. The corresponding eigenvectors are parallel to the normal vectors* $\mathfrak{n}_1 = q, \dots, \mathfrak{n}_{n-1}$, *and to* $\gamma$.

Now we are ready to describe generic solutions of the original LR system on $D \subset TSO(n)$. Let $q(\tau_1)$ be the solution of the Neumann system (6.16) with $F_0(q,q') = 0$, which is associated to a solution $(q(t), p(t))$ of the reduced LR system as described by Theorem 6.3. Let

$$X = (q, Aq)^{-1/2} Aq(s), \quad \mathfrak{n}_1 = q(s), \dots, \mathfrak{n}_{n-1}(s), \quad \gamma(s) = \frac{dX}{ds} \tag{7.5}$$

be the corresponding geodesic on $Q(0)$ in the new parametrization $s$ given by (7.2) and the unit eigenvectors of $\mathcal{L}$. (According to (6.15) and (7.2), we can treat $s$ as a known function of the original time $t$.) Then we have the following reconstruction theorem.

**Theorem 7.3** *A solution* $(g(t), \dot g(t))$ *of the original LR system on the distribution* $D \subset so(n)\times SO(n)$ *is given by the momentum map* $\Omega(t) = q\wedge \dot q$ *and the orthogonal frame formed by the unit vectors*

$$e_1 = q(t),\ e_2 = \mathfrak{n}_2(t),\ \dots,\ e_{n-1} = \mathfrak{n}_{n-1}(t), \quad e_n = \gamma(t).$$

*The other solutions* $(g(t), \dot g(t))$ *that are projected onto the same trajectory* $(q(t), p(t))$ *have the same* $\Omega, e_1$, *while the rest of the frame is obtained by the orthogonal transformations,*

$$(e_2(t)\cdots e_n(t)) = (\mathfrak{n}_2(t)\cdots \mathfrak{n}_{n-1}(t)\,\gamma(t))\,\mathfrak{R}, \tag{7.6}$$

*where* $\mathfrak{R}$ *is a constant matrix ranging over the group* $SO(n-1)$.

From Theorems 7.3, 6.3 and the integrability properties of the Neumann system on $T^*S^{n-1}$ we conclude that the phase space $D \subset T\,SO(n)$ of the multidimensional Veselova LR system with the left-invariant metric defined by (5.11) is almost everywhere foliated by $(n-1)$-dimensional invariant tori, on which the motion is straight-line but not uniform. This also implies that, apart from the pull-back of the $n-1$ integrals of the Neumann system, the LR system possesses $(n-1)(n-2)/2$ generally independent integrals on $D$. Indeed, according to the nonholonomic momentum theorem, the system has linear integrals $l_k = \langle \mathcal{M}, e_1\wedge e_k\rangle$, $k = 2, \dots, n$, of which $n-2$ ones are independent, since $l_2^2 + \cdots + l_n^2 = p^2$. Further, as we shall see below (relations (7.7)), for the special reconstructed solution $q(t), \mathfrak{n}_k(t), \gamma(t)$, the vector $A\dot q$ belongs to 2-plane spanned by $e_n = \gamma$ and $Aq$. Hence $\mathcal{M}\wedge e_n \equiv \frac{1}{\det A}(Aq\wedge A\dot q)\wedge e_n = 0$, which yields a set of scalar conditions, which are linear in $\mathcal{M}$.

*Proof of Theorem* 7.3. As follows from Proposition 7.2, for the geodesic motion on $Q(0)$, the unit normal vectors $\mathfrak{n}_1, \dots, \mathfrak{n}_{n-1}$ and $\gamma$ satisfy the kinematic (Poisson) equations with the "angular velocity" matrix $\mathcal{B}$, i.e.,

$$\frac{d}{ds_1}\mathfrak{n}_k = -\mathcal{B}\,\mathfrak{n}_k, \quad \frac{d}{ds_1}\gamma = -\mathcal{B}\,\gamma, \qquad k = 1, \dots, n-1,$$
$$ds = -\nu\, ds_1, \qquad \nu = (X, A^{-2}X).$$

Let us choose $x = X(s)$ in the expression for $\mathcal{B}$. From Theorem 7.1 we have

$$X = \sqrt{\nu}\, Aq, \quad \text{and} \quad \gamma \equiv \frac{dX}{ds} = \sqrt{\nu}\, A\frac{dq}{ds} + \frac{d\sqrt{\nu}}{ds}Aq. \tag{7.7}$$

Substituting this into (7.4), we find $\mathcal{B} = \nu\, q \wedge dq/ds$. Then the above Poisson equations take the simple form $d\mathfrak{n}_k/ds = -(q \wedge dq/ds)\mathfrak{n}_k$, $d\gamma/ds = -(q \wedge dq/ds)\gamma$. Changing here the time parameter $s$ to $t$ and taking into account relation (6.1), we finally obtain

$$\dot{\mathfrak{n}}_k = -\Omega(t)\, \mathfrak{n}_k, \quad k = 1, \dots, n-1, \quad \dot{\gamma} = -\Omega(t)\gamma, \tag{7.8}$$

where $\Omega \in \mathcal{D} \subset so(n)$ is the admissible angular velocity of the $n$-dimensional body. This implies that the orthogonal frame $\{\mathfrak{n}_1(t), \dots, \mathfrak{n}_{n-1}(t), \gamma(t)\}$ gives a solution of the LR system on $D$.

Clearly, the vectors of the frames that are obtained by the orthogonal transformations (7.6) also satisfy the Poisson equations (7.8) and therefore also give such solutions. Since the fiber of the map $\pi : D \to TS^{n-1}$ is the group $SO(n-1)$, there are no other solutions on $D$ that are projected onto the same trajectory $(q(t), \dot{q}(t))$. The theorem is proved.

In order to find explicit expressions for the components of $\mathfrak{n}_k$ and $\gamma$, following Jacobi [29], we first introduce ellipsoidal coordinates $\nu_1, \dots, \nu_{n-1}$ on $Q(0)$ according to the formulas

$$X_i^2 = \frac{A_i(A_i - \nu_1)\cdots(A_i - \nu_{n-1})}{\prod_{j\neq i}(A_i - A_j)}, \qquad i = 1, \dots, n.$$

Matching these with the expressions (6.10) for $q_i$ in terms of the spheroconic coordinates $\lambda_1, \dots, \lambda_{n-1}$ on $S^{n-1}$ and taking into account (7.5), (6.14), we find that, up to permutation of indices,

$$\nu_k = \lambda_k^{-1}, \qquad k = 1, \dots, n-1.$$

Using this property one can also prove that the *nonzero* parameters $\alpha_2, \dots, \alpha_{n-1}$ of the confocal quadrics in the Chasles theorem are just inverse of the constants $c_2, \dots, c_{n-1}$ in the invariant polynomial (6.27). As a result, making use of definition of the vectors $\mathfrak{n}_k$, $\gamma$, one can express their components in terms of $\lambda_1, \dots, \lambda_{n-1}$ and $c_2, \dots, c_{n-1}$. The evolution of $\lambda$-coordinates in the time $\tau$ is described by the quadratures (6.11), (6.12).

Finally, by using the classical algebraic geometrical methods ([3, 18]), the components of $q, \mathfrak{n}_2, \dots, \mathfrak{n}_{n-1}$, $\gamma$, as well as the function $\sqrt{\lambda_1 \cdots \lambda_{n-1}}$ can be represented as quotients of theta-functions with half-integer theta-characteristics associated to the hyperelliptic curve $\{w^2 = R(\lambda)\}$ of genus $n-1$, whose arguments depend linearly on $\tau_1$. The dependence of $t$ in $\tau_1$ is obtained by the integration of (6.15), which, in view of (6.14), leads to the simple quadrature

$$t = \frac{1}{\sqrt{2h}} \int \sqrt{\lambda_1(\tau_1)\cdots\lambda_{n-1}(\tau_1)}\, d\tau_1 + \text{const.}$$

# Conclusion

In this paper we considered LR systems on compact Lie groups and showed that their reductions to homogeneous spaces always possess an invariant measure. We calculated it explicitly in case of the Stiefel varieties $V(r,n) =$

$SO(n)/SO(n-r)$. It appeared that for $r=1$ and the special inertia tensor on $so(n)$, the reduced flow is transformed to an integrable geodesic flow on $S^{n-1}$ via the change of time prescribed by the density of the invariant measure and Theorem 3.5. Moreover, in this case the unreduced flow on the right-invariant distribution $D \in T\,SO(n)$ is also integrable.

Such a behavior of a multidimensional nonholonomic system is exceptional and may be explained by the existence of a rich underlying geometry coming from the Chasles theorem and the Jacobi problem on geodesics on an ellipsoid. (The latter is known to be responsible for integrability of various problems of mechanics due to their close relations to it.)

In this connection the following questions arise: are there other inertia tensors of LR systems on $SO(n)$, for which the above properties hold and how wide is the class of such tensors? Can reduced flows on $V(r,n)$, $r>1$ be transformed to the Hamiltonian form (with respect to the canonical symplectic structure $\Omega$) in the same manner? Are the reduction to a Hamiltonian form and integrability possible in case of nonhomogeneous right-invariant constraints on $SO(n)$ (similarly to what takes place for the classical case $n=3$) ?

A part of our analysis can be extended to LR systems on noncompact Lie groups and their reductions. It would be interesting to study meaningful examples of such systems.

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