

## Research Article

# Nonhomogeneous Boundary Value Problem for One-Dimensional Compressible Viscous Micropolar Fluid Model: Regularity of the Solution

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An initial-boundary value problem for 1D flow of a compressible viscous heat-conducting micropolar fluid is considered; the fluid is thermodynamically perfect and polytropic. Assuming that the initial data are Hölder continuous on  $]0, 1[$  and transforming the original problem into homogeneous one, we prove that the state function is Hölder continuous on  $]0, 1[ \times ]0, T[$ , for each  $T > 0$ . The proof is based on a global-in-time existence theorem obtained in the previous research paper and on a theory of parabolic equations.

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## 1. Introduction

In this paper, we consider a nonstationary 1D flow of a compressible viscous and heat-conducting micropolar fluid, being in a thermodynamical sense perfect and polytropic. In [1–3], we considered the problem with homogeneous boundary conditions.

Here we study, as in [4, 5], the case of nonhomogeneous boundary conditions for velocity and microrotation which is called in gas dynamics “problem on piston” (see [6]). Assuming that the initial data are Hölder continuous on  $]0, 1[$  and transforming the original problem into homogeneous one, we prove that, for each  $T > 0$ , the mass density, velocity, microrotation velocity, and temperature are Hölder continuous on  $]0, 1[ \times ]0, T[$ . The proof is based on a global-in-time existence theorem [5] and on a theory of parabolic equations [7]. We use some ideas of Antontsev et al. [8] applied to the case of classical fluid with homogeneous boundary conditions, results from [3] as well and some inequalities for Hölder norms obtained by the Nirenberg-Gagliardo inequality.

## 2. Statement of the problem and its equivalent setting

Let  $\rho, v, \omega$ , and  $\theta$  denote, respectively, the mass density, velocity, microrotation velocity, and temperature of the fluid in the Lagrangean description. Then the problem which we consider

has the formulation as follows [1]:

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial v}{\partial x} = 0, \quad (2.1)$$

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x} \left( \rho \frac{\partial v}{\partial x} \right) - K \frac{\partial}{\partial x} (\rho \theta), \quad (2.2)$$

$$\rho \frac{\partial \omega}{\partial t} = A \left[ \rho \frac{\partial}{\partial x} \left( \rho \frac{\partial \omega}{\partial x} \right) - \omega \right], \quad (2.3)$$

$$\rho \frac{\partial \theta}{\partial t} = -K \rho^2 \theta \frac{\partial v}{\partial x} + \rho^2 \left( \frac{\partial v}{\partial x} \right)^2 + \rho^2 \left( \frac{\partial \omega}{\partial x} \right)^2 + \omega^2 + D \rho \frac{\partial}{\partial x} \left( \rho \frac{\partial \theta}{\partial x} \right) \quad (2.4)$$

in  $]0, 1[ \times ]0, T[$ ,  $T > 0$ , where  $K, A$ , and  $D$  are positive constants. Equations (2.1)–(2.4) are, respectively, local forms of the conservations laws for the mass, momentum, momentum moment, and energy. We take the following nonhomogeneous initial and boundary conditions:

$$\rho(x, 0) = \rho_0(x), \quad (2.5)$$

$$v(x, 0) = v_0(x), \quad (2.6)$$

$$\omega(x, 0) = \omega_0(x), \quad (2.7)$$

$$\theta(x, 0) = \theta_0(x), \quad (2.8)$$

$$v(0, t) = \mu_0(t), \quad v(1, t) = \mu_1(t), \quad (2.9)$$

$$\omega(0, t) = \nu_0(t), \quad \omega(1, t) = \nu_1(t), \quad (2.10)$$

$$\frac{\partial \theta}{\partial x}(0, t) = \frac{\partial \theta}{\partial x}(1, t) = 0, \quad (2.11)$$

for  $x \in \Omega = ]0, 1[$ ,  $t \in ]0, T[$ . Here  $\rho_0, v_0, \omega_0, \theta_0, \mu_0, \mu_1, \nu_0$ , and  $\nu_1$  are given functions. We assume the compatibility conditions

$$v_0(0) = \mu_0(0), \quad v_0(1) = \mu_1(0), \quad (2.12)$$

$$\omega_0(0) = \nu_0(0), \quad \omega_0(1) = \nu_1(0), \quad (2.13)$$

$$\frac{\partial \theta_0}{\partial x}(0) = \frac{\partial \theta_0}{\partial x}(1) = 0, \quad (2.14)$$

and the inequalities

$$0 < m \leq \rho_0(x) \leq M, \quad m \leq \theta_0(x) \leq M \quad \text{for } x \in \Omega, \quad (2.15)$$

where  $m, M \in \mathbb{R}^+$ . We assume also that there exists a constant  $\delta > 0$  such that

$$l(t) = \int_0^1 \frac{1}{\rho_0(x)} dx + \int_0^t [\mu_1(\tau) - \mu_0(\tau)] d\tau \geq \delta, \quad t \in ]0, T[. \quad (2.16)$$

In the previous work [5] we proved that for

$$\begin{aligned} \mu_0, \mu_1, \nu_0, \nu_1 &\in H^2(]0, T[), \\ \rho_0, \nu_0, \omega_0, \theta_0 &\in H^1(\Omega), \end{aligned} \quad (2.17)$$

the problems (2.1)–(2.4) have a unique generalized solution

$$(x, t) \longrightarrow (\rho, v, \omega, \theta)(x, t), \quad (x, t) \in Q_T = \Omega \times ]0, T[, \quad (2.18)$$

$$\rho \in L^\infty(0, T; H^1(\Omega)) \cap H^1(Q_T), \quad \inf_{Q_T} \rho > 0, \quad (2.19)$$

$$v, \omega, \theta \in L^\infty(0, T; H^1(\Omega)) \cap H^1(Q_T) \cap L^2(0, T; H^2(\Omega)), \quad (2.20)$$

that satisfies (2.1)–(2.4) a.e. in  $Q_T$  and conditions (2.5)–(2.11) in the sense of traces. Moreover,

$$\theta > 0 \text{ in } \overline{Q_T}. \quad (2.21)$$

From embedding and interpolation theorems (e.g., [9]) one can conclude that from (2.19) and (2.20) it follows:

$$\rho \in L^\infty(0, T; C(\overline{\Omega})) \cap C([0, T], L^2(\Omega)), \quad (2.22)$$

$$v, \omega, \theta \in L^2(0, T; C^1(\overline{\Omega})) \cap C([0, T], H^1(\Omega)), \quad (2.23)$$

$$v, \omega, \theta \in C(\overline{Q_T}). \quad (2.24)$$

Now, instead of the velocity  $v$  and microrotation  $\omega$  we introduce new functions  $V$  and  $W$  in order to obtain a problem with the homogeneous boundary conditions.

Notice that using (2.9) from (2.1) we get

$$\int_0^1 \frac{dx}{\rho(x, t)} = l(t), \quad t \in ]0, T[, \quad (2.25)$$

where the function  $l$  is defined by (2.16). We introduce the functions

$$v_1(x, t) = \frac{\mu(t)}{l(t)} \int_0^x \frac{d\xi}{\rho(\xi, t)} + \mu_0(t), \quad (2.26)$$

$$\omega_1(x, t) = \frac{\nu(t)}{l(t)} \int_0^x \frac{d\xi}{\rho(\xi, t)} + \nu_0(t) \text{ on } Q_T, \quad (2.27)$$

where  $\mu(t) = \mu_1(t) - \mu_0(t)$  and  $\nu(t) = \nu_1(t) - \nu_0(t)$ . It is evident that

$$\begin{aligned} v_1(0, t) &= \mu_0(t), & v_1(1, t) &= \mu_1(t), \\ \omega_1(0, t) &= \nu_0(t), & \omega_1(1, t) &= \nu_1(t), \quad t \in ]0, T[. \end{aligned} \quad (2.28)$$

Inserting

$$V(x, t) = v(x, t) - v_1(x, t), \quad W(x, t) = \omega(x, t) - \omega_1(x, t) \quad (2.29)$$

into (2.1)–(2.4) we get the following equivalent system:

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial V}{\partial x} + \frac{\mu}{l} \rho = 0, \quad (2.30)$$

$$\frac{\partial V}{\partial t} = \frac{\partial}{\partial x} \left( \rho \frac{\partial V}{\partial x} \right) - K \frac{\partial}{\partial x} (\rho \theta) - \frac{\partial v_1}{\partial t}, \quad (2.31)$$

$$\rho \frac{\partial W}{\partial t} = A \left[ \rho \frac{\partial}{\partial x} \left( \rho \frac{\partial W}{\partial x} \right) - \omega_1 - W \right] - \rho \frac{\partial \omega_1}{\partial t}, \quad (2.32)$$

$$\begin{aligned} \rho \frac{\partial \theta}{\partial t} = & -K \rho^2 \theta \frac{\partial V}{\partial x} - K \rho \theta \frac{\mu}{l} + \rho^2 \left( \frac{\partial V}{\partial x} \right)^2 + 2\rho \frac{\partial V}{\partial x} \frac{\mu}{l} + \left( \frac{\mu}{l} \right)^2 \\ & + \rho^2 \left( \frac{\partial W}{\partial x} \right)^2 + 2\rho \frac{\partial W}{\partial x} \frac{\nu}{l} + \left( \frac{\nu}{l} \right)^2 + (W + \omega_1)^2 + D\rho \frac{\partial}{\partial x} \left( \rho \frac{\partial \theta}{\partial x} \right), \end{aligned} \quad (2.33)$$

with the homogeneous boundary conditions

$$V(0, t) = V(1, t) = 0, \quad W(0, t) = W(1, t) = 0, \quad (2.34)$$

$$\frac{\partial \theta}{\partial x}(0, t) = \frac{\partial \theta}{\partial x}(1, t) = 0 \quad (2.35)$$

for  $t \in ]0, T[$  and initial conditions

$$\rho(x, 0) = \rho_0(x), \quad V(x, 0) = V_0(x), \quad (2.36)$$

$$W(x, 0) = W_0(x), \quad \theta(x, 0) = \theta_0(x), \quad (2.37)$$

for  $x \in \Omega$ , where

$$\begin{aligned} V_0(x) &= v_0(x) - \frac{\mu(0)}{l(0)} \int_0^x \frac{1}{\rho_0(\xi)} d\xi - \mu_0(0), \\ W_0(x) &= \omega_0(x) - \frac{\nu(0)}{l(0)} \int_0^x \frac{1}{\rho_0(\xi)} d\xi - \nu_0(0) \end{aligned} \quad (2.38)$$

are known functions. In the article [5], we proved that the problems (2.30)–(2.37) have a unique generalized solution  $(\rho, V, W, \theta)$  in the domain  $Q_T$  with property (2.21) as well. Moreover, we obtained that

$$v_1, \omega_1 \in L^\infty(0, T; H^2(\Omega)), \quad \frac{\partial v_1}{\partial t}, \frac{\partial \omega_1}{\partial t} \in L^\infty(0, T; L^2(\Omega)). \quad (2.39)$$

In the following  $C^{k+\alpha}(\overline{Q_T})$  ( $k \in \mathbb{N} \cup \{0\}, 0 < \alpha < 1$ ) is the Banach space of functions of the class  $C^k(\overline{Q_T})$ , having  $k$ th derivatives Hölder continuous with the exponent  $\alpha$  on  $\overline{Q_T}$ ; the norm is defined by

$$|f|_{k+\alpha, Q_T} = \sum_{m+j=0}^k |D_x^m D_t^j f|_{0, Q_T} + \sum_{m+j=k} H^\alpha(D_x^m D_t^j f), \quad (2.40)$$

where

$$H^\alpha(f) = \sup_{y, z \in Q_T} \frac{|f(y) - f(z)|}{|y - z|^\alpha}, \quad (2.41)$$

and  $|\cdot|_{0, Q_T}$  is the norm on  $C(\overline{Q_T})$ ;  $D_x^m$  and  $D_t^j$  are, respectively, the  $m$ th derivatives with respect to  $x$  and the  $j$ th derivatives with respect to  $t$ .  $C^{k+\alpha, m+\beta}(\overline{Q_T})$  ( $k, m \in \mathbb{N} \cup \{0\}, 0 < \alpha, \beta < 1$ ) is the Banach space of functions which have  $k$ th derivatives with respect to  $x$  and  $m$ th derivatives with respect to  $t$  Hölder continuous on  $\overline{Q_T}$ . The norm is defined by

$$|f|_{k+\alpha, m+\beta, Q_T} = \sum_{l=0}^k |D_x^l f|_{0, Q_T} + \sum_{j=1}^m |D_t^j f|_{0, Q_T} + H_x^\alpha(D_x^k f) + H_t^\beta(D_x^k f) + H_x^\alpha(D_t^m f) + H_t^\beta(D_t^m f), \quad (2.42)$$

where

$$H_x^\alpha(f) = \sup_{(x_1, t), (x_2, t) \in Q_T} \frac{|f(x_1, t) - f(x_2, t)|}{|x_1 - x_2|^\alpha}, \quad (2.43)$$

$$H_t^\beta(f) = \sup_{(x, t_1), (x, t_2) \in Q_T} \frac{|f(x, t_1) - f(x, t_2)|}{|t_1 - t_2|^\beta}.$$

By  $C \in \mathbf{R}^+$  we denote a generic constant, having possibly different values at different places. Also we use some inequalities for Hölder norms obtained by the following Nirenberg-Gagliardo interpolation inequality

$$|f|_{1/\mu} \leq |f|_{1/\lambda}^{(\nu-\mu)/(\nu-\lambda)} |f|_{1/\nu}^{(\mu-\lambda)/(\nu-\lambda)}, \quad (2.44)$$

where  $\mu, \nu, \lambda \in \mathbf{R}$  and  $\lambda \leq \mu \leq \nu$ . Here, for bounded domain  $D \subset \mathbf{R}^n$  and  $f : D \rightarrow \mathbf{R}$  the norm  $|f|_q$  is defined by

$$|f|_q = \begin{cases} \|f\|_{L^q(D)}, & q > 0, \\ |f|_{k+\beta, D}, & q < 0, \end{cases} \quad (2.45)$$

where  $k = [-n/q]$  and  $\beta = -n/q - k$  (e.g., [8, page 27]). Some of our considerations are very similar or identical to that of [8] or [3]. In these cases we omit proofs or details of proofs,

making reference to correspondent pages of the book [8] or article [3]; we use the notation  $\|\cdot\| = \|\cdot\|_{L^2}$ .

### 3. The main results

The aim of this paper is to prove the following regularity result.

**Theorem 3.1.** *Let the functions*

$$\mu_0, \mu_1, \nu_0, \nu_1 \in C^2([0, T]), \quad (3.1)$$

$$\rho_0 \in C^{1+\alpha}(\overline{\Omega}), \quad v_0, \omega_0, \theta_0 \in C^{2+\alpha}(\overline{\Omega}), \quad 0 < \alpha < 1 \quad (3.2)$$

satisfy the compatibility conditions

$$\frac{d}{dx} \left( \rho_0 \frac{dv_0}{dx} \right) - K \frac{d}{dx} (\rho_0 \theta_0) = \begin{cases} \frac{d\mu_0(0)}{dt}, & \text{for } x = 0 \\ \frac{d\mu_1(0)}{dt}, & \text{for } x = 1, \end{cases} \quad (3.3)$$

$$A \left[ \frac{d}{dx} \left( \rho_0 \frac{d\omega_0}{dx} \right) - \frac{\omega_0}{\rho_0} \right] = \begin{cases} \frac{d\nu_0(0)}{dt}, & \text{for } x = 0 \\ \frac{d\nu_1(0)}{dt}, & \text{for } x = 1, \end{cases} \quad (3.4)$$

and (2.12)–(2.16). Then the generalized solution of the problems (2.1)–(2.11) has the properties

$$\rho \in C^{1+\alpha}(\overline{Q_T}), \quad v, \omega, \theta \in C^{2+\alpha, 1+\alpha/2}(\overline{Q_T}), \quad 0 < \alpha < 1. \quad (3.5)$$

Notice that because of (3.1) and (3.2) we have

$$V_0, W_0 \in C^{2+\alpha}(\overline{\Omega}), \quad l \in C^3([0, T]), \quad (3.6)$$

and for  $t = 0$  we can easily conclude that

$$v_1|_{t=0}, \omega_1|_{t=0}, \frac{\partial v_1}{\partial t} \Big|_{t=0}, \frac{\partial \omega_1}{\partial t} \Big|_{t=0} \in C^{2+\alpha}(\overline{\Omega}). \quad (3.7)$$

Now, (2.12), (2.13), (3.3), and (3.4) become the following compatibility conditions for the problems (2.30)–(2.37)

$$V_0(0) = V_0(1) = 0, \quad W_0(0) = W_0(1) = 0, \quad (3.8)$$

$$\frac{d}{dx} \left( \rho_0 \frac{dV_0}{dx} \right) - K \frac{d}{dx} (\rho_0 \theta_0) = \begin{cases} \frac{d\mu_0(0)}{dt}, & \text{for } x = 0 \\ \frac{d\mu_1(0)}{dt}, & \text{for } x = 1, \end{cases} \quad (3.9)$$

$$\frac{d}{dx} \left( \rho_0 \frac{dW_0}{dx} \right) - \frac{W_0}{\rho_0} = \begin{cases} \frac{\nu_0(0)}{\rho_0} + A^{-1} \frac{d\nu_0(0)}{dt}, & \text{for } x = 0 \\ \frac{\nu_1(0)}{\rho_0} + A^{-1} \frac{d\nu_1(0)}{dt}, & \text{for } x = 1. \end{cases} \quad (3.10)$$

In this paper, we will prove the following result first.

**Theorem 3.2.** *Under the assumptions of Theorem 3.1 the problems (2.30)–(2.37) have a generalized solution  $(\rho, V, W, \theta)$  in  $Q_T$  with the properties*

$$\rho \in C^{1+\alpha}(\overline{Q}_T), \quad V, W, \theta \in C^{2+\alpha, 1+\alpha/2}(\overline{Q}_T), \quad 0 < \alpha < 1. \quad (3.11)$$

Moreover,

$$v_1, \omega_1 \in C^{2+\alpha, 1+\alpha/2}(\overline{Q}_T). \quad (3.12)$$

Theorem 3.1 is an immediate consequence of this result. In the proof of Theorem 3.2 we apply, as in [3], the method of the book [8], where Theorem 3.1 was proved for the classical fluid ( $\omega = 0$ ) with homogeneous boundary conditions.

In that what follows, we assume that the conditions (2.12)–(2.16) and (3.1)–(3.4) are fulfilled.

#### 4. Some properties of the solution $(\rho, V, W, \theta)$ and functions $v_1$ and $\omega_1$

**Lemma 4.1.** *It holds*

$$\frac{\partial^2 v_1}{\partial t^2}, \frac{\partial^2 \omega_1}{\partial t^2} \in L^2(0, T; L^2(\Omega)). \quad (4.1)$$

*Proof.* Using (2.30) from (2.26) and (2.27) we get

$$\frac{\partial v_1}{\partial t} = \left[ \left( \frac{\mu}{l} \right)' + \left( \frac{\mu}{l} \right)^2 \right] \int_0^x \frac{1}{\rho} d\xi + \frac{\mu}{l} V + \mu_0', \quad (4.2)$$

$$\frac{\partial \omega_1}{\partial t} = \left[ \left( \frac{\nu}{l} \right)' + \frac{\mu\nu}{l^2} \right] \int_0^x \frac{1}{\rho} d\xi + \frac{\nu}{l} V + \nu_0'. \quad (4.3)$$

After differentiating (4.2) with respect to  $t$ , squaring, integrating over  $\Omega$  and taking into account (2.25), (2.30), (3.1), and (3.6) we get

$$\left\| \frac{\partial^2 v_1}{\partial t^2}(t) \right\|^2 \leq C \left( 1 + \|V(t)\|^2 + \left\| \frac{\partial V}{\partial t}(t) \right\|^2 \right). \quad (4.4)$$

With the help of (2.20) for  $V$  we conclude that

$$\int_0^T \left\| \frac{\partial^2 v_1}{\partial t^2}(\tau) \right\|^2 d\tau \leq C \left( 1 + \int_0^T \|V(\tau)\|^2 d\tau + \int_0^T \left\| \frac{\partial V}{\partial t}(\tau) \right\|^2 d\tau \right) \leq C. \quad (4.5)$$

From (4.3) follows the same estimation for  $\partial^2 \omega_1 / \partial t^2$ .  $\square$

**Lemma 4.2.** *The inclusions*

$$\frac{\partial \rho}{\partial t}, \frac{\partial V}{\partial t}, \frac{\partial W}{\partial t}, \frac{\partial \theta}{\partial t} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \quad (4.6)$$

hold true.

*Proof.* Using (2.22) for  $\rho$  and (2.20) for  $V$  from (2.30) we get immediately that  $\partial \rho / \partial t \in L^\infty(0, T; L^2(\Omega))$ . Differentiating (2.30) with respect to  $x$ , using the inequalities

$$\begin{aligned} |f|^2 &\leq C \|f\| \|f'\| \leq C \|f'\|^2, \\ |f'|^2 &\leq C \|f'\| \|f''\| \leq C \|f''\|^2 \end{aligned} \quad (4.7)$$

(valid for a function  $f$  or its derivative vanishing at  $x = 0$  and  $x = 1$ ), (2.22) and (3.1) we obtain

$$\begin{aligned} \left\| \frac{\partial^2 \rho}{\partial x \partial t}(t) \right\|^2 &\leq C \left( \int_0^1 \left| \frac{\partial \rho}{\partial x}(t) \right|^2 \left| \frac{\partial V}{\partial x} \right|^2 dx + \int_0^1 \left| \frac{\partial^2 V}{\partial x^2} \right|^2 dx + \left| \frac{\mu}{l} \right|^2 \int_0^1 \left| \frac{\partial \rho}{\partial x} \right|^2 dx \right) \\ &\leq C \left( \left\| \frac{\partial \rho}{\partial x}(t) \right\|^2 \left\| \frac{\partial^2 V}{\partial x^2}(t) \right\|^2 + \left\| \frac{\partial^2 V}{\partial x^2}(t) \right\|^2 + \left\| \frac{\partial \rho}{\partial x}(t) \right\|^2 \right). \end{aligned} \quad (4.8)$$

Taking into account (2.20) and (2.19) we get

$$\int_0^T \left\| \frac{\partial^2 \rho}{\partial x \partial t}(\tau) \right\|^2 d\tau \leq C. \quad (4.9)$$

After differentiating (2.31) with respect to the time variable, multiplying by  $\partial V / \partial t$  and integrating by parts over  $\Omega$  we obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left\| \frac{\partial V}{\partial t}(t) \right\|^2 + \int_0^1 \rho \left( \frac{\partial^2 V}{\partial x \partial t} \right)^2 dx \\ &= - \int_0^1 \frac{\partial \rho}{\partial t} \frac{\partial V}{\partial x} \frac{\partial^2 V}{\partial x \partial t} dx + K \int_0^1 \frac{\partial \rho}{\partial t} \theta \frac{\partial^2 V}{\partial x \partial t} dx + K \int_0^1 \rho \frac{\partial \theta}{\partial t} \frac{\partial^2 V}{\partial x \partial t} dx - \int_0^1 \frac{\partial^2 v_1}{\partial t^2} \frac{\partial V}{\partial t} dx. \end{aligned} \quad (4.10)$$



Applying (4.6) and (2.22) for  $\rho$ , (2.24) for  $\theta$ , (4.7) for  $\partial V/\partial x$  and the Young inequality with a parameter  $\varepsilon > 0$  we obtain

$$\left| \int_0^1 \frac{\partial \rho}{\partial t} \frac{\partial V}{\partial x} \frac{\partial^2 V}{\partial x \partial t} dx \right| \leq \varepsilon \int_0^1 \rho \left( \frac{\partial^2 V}{\partial x \partial t} \right)^2 dx + C \left\| \frac{\partial^2 V}{\partial x^2}(t) \right\|^2, \quad (4.11)$$

$$\left| K \int_0^1 \frac{\partial \rho}{\partial t} \theta \frac{\partial^2 V}{\partial x \partial t} dx \right| \leq C + \varepsilon \int_0^1 \rho \left( \frac{\partial^2 V}{\partial x \partial t} \right)^2 dx, \quad (4.12)$$

$$\left| K \int_0^1 \rho \frac{\partial \theta}{\partial t} \frac{\partial^2 V}{\partial x \partial t} dx \right| \leq \varepsilon \int_0^1 \rho \left( \frac{\partial^2 V}{\partial x \partial t} \right)^2 dx + C \left\| \frac{\partial \theta}{\partial t}(t) \right\|^2, \quad (4.13)$$

$$\left| \int_0^1 \frac{\partial^2 v_1}{\partial t^2} \frac{\partial V}{\partial t} dx \right| \leq C \left( \left\| \frac{\partial^2 v_1}{\partial t^2}(t) \right\|^2 + \left\| \frac{\partial V}{\partial t}(t) \right\|^2 \right). \quad (4.14)$$

For sufficiently small  $\varepsilon > 0$  from (4.10) and (4.11)–(4.14) it follows that for  $t \in ]0, T[$  we have

$$\begin{aligned} \left\| \frac{\partial V}{\partial t}(t) \right\|^2 + \int_0^t \left\| \frac{\partial^2 V}{\partial x \partial t}(\tau) \right\|^2 d\tau &\leq C \left( 1 + \left\| \frac{\partial V}{\partial t}(0) \right\|^2 + \int_0^t \left\| \frac{\partial \theta}{\partial t}(\tau) \right\|^2 d\tau + \int_0^t \left\| \frac{\partial^2 V}{\partial x^2}(\tau) \right\|^2 d\tau \right. \\ &\quad \left. + \int_0^t \left\| \frac{\partial^2 v_1}{\partial t^2}(\tau) \right\|^2 d\tau + \int_0^t \left\| \frac{\partial V}{\partial t}(\tau) \right\|^2 d\tau \right). \end{aligned} \quad (4.15)$$

Taking into account (3.2), (3.6), and (3.7) from (2.31) we can easily conclude that

$$\left\| \frac{\partial V}{\partial t}(0) \right\| = \left\| \rho'_0 V'_0 + \rho_0 V''_0 - K \rho'_0 \theta_0 - K \rho_0 \theta'_0 - \frac{\partial v_1}{\partial t}(0) \right\| \leq C, \quad (4.16)$$

and using (2.20) and (4.1) we get that inclusion (4.6) is satisfied for the function  $V$ . In the similar way from (2.32) and (2.33) we obtain (4.6) for  $W$  and  $\theta$ .  $\square$

Now, taking into account (4.6) and (2.20), we can introduce the following inequalities for

$$\eta \in \{V, W, \theta\} \quad (4.17)$$

derived in [3] by the Nirenberg-Gagliardo inequality (2.44).

**Lemma 4.3** (see [3, Lemmas 2.2–2.4]). For  $0 < \alpha < 1$  and  $\varepsilon > 0$ , the function  $\eta$  satisfies the inequalities

$$\left| \frac{\partial \eta}{\partial x} \right|_{0, Q_T} \leq C |\eta|_{2+\alpha, 1+\alpha/2, Q_T}^a \quad (4.18)$$

$$\left| \frac{\partial \eta}{\partial x} \right|_{\alpha, \alpha/2, Q_T} \leq C \left( \varepsilon |\eta|_{2+\alpha, 1+\alpha/2, Q_T} + \sup_{Q_T} \left| \frac{\partial \eta}{\partial t} \right| + 1 \right), \quad (4.19)$$

$$\left| \frac{\partial \eta}{\partial x} \right|_{\alpha, Q_T} \leq C |\eta|_{2+\alpha, 1+\alpha/2, Q_T}^d (|\eta|_{2+\alpha, 1+\alpha/2, Q_T} + 1)^{1-d}, \quad (4.20)$$

where  $a = 1/(3 + 2\alpha)$  and  $d = \alpha/(2 - \alpha)$ . For  $0 < \alpha \leq 1/2$  it holds

$$\left| \frac{\partial \eta}{\partial x} \right|_{\alpha, \alpha/2, Q_T} \leq C |\eta|_{2+\alpha, 1+\alpha/2, Q_T}^b \quad (4.21)$$

where  $b = (1 + 2\alpha)/(3 + 2\alpha)$ .

## 5. The proofs of Theorems 3.1 and 3.2

The conclusions of Theorems 3.1 and 3.2 are immediate consequences of the following lemmas.

**Lemma 5.1.** *It holds*

$$\rho, V, W, \theta \in C^{1/2, 1/2}(\overline{Q_T}). \quad (5.1)$$

Moreover,

$$v_1, \omega_1 \in C^{1+1/2, 1+1/2}(\overline{Q_T}). \quad (5.2)$$

*Proof.* Taking into account (4.6) we get inclusion (5.1) for the functions  $V, W$ , and  $\theta$  in the same way as for  $\rho$  in (see [8, pages 54–55]). Using (3.1), (3.6), and (5.1) from (2.26), (2.27), (4.2), and (4.3) we get (5.2) immediately.  $\square$

**Lemma 5.2.** For  $0 < \alpha < 1$  and  $\gamma = \min\{1/2, \alpha\}$  it holds

$$\frac{\partial \rho}{\partial x} \in C^{\gamma, \gamma}(\dot{Q}_T). \quad (5.3)$$

*Proof.* With the help of (5.1), (5.2), and (3.2) we obtain (5.3) in the similar way as in (see [8, pages 57–58]).  $\square$

**Lemma 5.3.** For  $0 < \alpha < 1$ ,  $\gamma = \min\{1/2, \alpha\}$ ,  $a = 1/(3 + 2\alpha)$ , and  $b = (1 + 2\gamma)/(3 + 2\gamma)$  the inequalities

$$|V|_{2+\gamma, 1+\gamma/2, Q_T} \leq C(1 + |\theta|_{1+\gamma, \gamma/2, Q_T}), \quad (5.4)$$

$$|W|_{2+\gamma, 1+\gamma/2, Q_T} \leq C, \quad (5.5)$$

$$|\theta|_{2+\gamma, 1+\gamma/2, Q_T} \leq C \left( |V|_{2+\gamma, 1+\gamma/2, Q_T}^a + |V|_{2+\gamma, 1+\gamma/2, Q_T}^b + |V|_{2+\gamma, 1+\gamma/2, Q_T}^{a+b} + |V|_{2+\gamma, 1+\gamma/2, Q_T}^{2a} + 1 \right) \quad (5.6)$$

hold true.

*Proof.* We write (2.31), (2.32), and (2.33) in the form

$$\begin{aligned} \frac{\partial V}{\partial t} - \rho \frac{\partial^2 V}{\partial x^2} - \frac{\partial \rho}{\partial x} \frac{\partial V}{\partial x} &= -K \frac{\partial \rho}{\partial x} \theta - K \rho \frac{\partial \theta}{\partial x} - \frac{\partial v_1}{\partial t}, \\ \frac{\partial W}{\partial t} - A \rho \frac{\partial^2 W}{\partial x^2} - A \frac{\partial \rho}{\partial x} \frac{\partial W}{\partial x} + A \frac{W}{\rho} &= -A \frac{\omega_1}{\rho} - \frac{\partial \omega_1}{\partial t}, \\ \frac{\partial \theta}{\partial t} - D \rho \frac{\partial^2 \theta}{\partial x^2} - D \frac{\partial \rho}{\partial x} \frac{\partial \theta}{\partial x} &= -K \rho \theta \frac{\partial V}{\partial x} - K \theta \frac{\mu}{l} + \rho \left( \frac{\partial V}{\partial x} \right)^2 + 2 \frac{\partial V}{\partial x} \frac{\mu}{l} + \frac{1}{\rho} \left( \frac{\mu}{l} \right)^2 \\ &\quad + \rho \left( \frac{\partial W}{\partial x} \right)^2 + 2 \frac{\partial W}{\partial x} \frac{\nu}{l} + \frac{1}{\rho} \left( \frac{\nu}{l} \right)^2 + \frac{1}{\rho} (W + \omega_1)^2, \end{aligned} \quad (5.7)$$

and we consider them as parabolic equations for  $V, W$ , and  $\theta$ , respectively, with Hölder continuous coefficients with exponent  $\gamma = \min\{1/2, \alpha\}$ . Taking into account the compatibility conditions (3.8)–(3.10), (2.14), and  $|fg|_{\alpha, \alpha/2} \leq |f|_0 |g|_{\alpha, \alpha/2} + |f|_{\alpha, \alpha/2} |g|_0$ , from a parabolic theory (see [7, Theorems 5.2 and 5.3]) we conclude that the solutions  $V, W$ , and  $\theta$  satisfy the following inequalities

$$\begin{aligned} |V|_{2+\gamma, 1+\gamma/2, Q_T} &\leq C \left( \left| \frac{\partial \rho}{\partial x} \right|_{\gamma, \gamma/2, Q_T} |\theta|_{0, Q_T} + \left| \frac{\partial \rho}{\partial x} \right|_{0, Q_T} |\theta|_{\gamma, \gamma/2, Q_T} + |\rho|_{\gamma, \gamma/2, Q_T} \left| \frac{\partial \theta}{\partial x} \right|_{0, Q_T} \right. \\ &\quad \left. + |\rho|_{0, Q_T} \left| \frac{\partial \theta}{\partial x} \right|_{\gamma, \gamma/2, Q_T} + \left| \frac{\partial v_1}{\partial t} \right|_{\gamma, \gamma/2, Q_T} + |V_0|_{2+\gamma, \Omega} \right), \end{aligned} \quad (5.8)$$

$$|W|_{2+\gamma, 1+\gamma/2, Q_T} \leq \left( \left| \frac{1}{\rho} \right|_{0, Q_T} |\omega_1|_{\gamma, \gamma/2, Q_T} + \left| \frac{1}{\rho} \right|_{\gamma, \gamma/2, Q_T} |\omega_1|_{0, Q_T} + \left| \frac{\partial \omega_1}{\partial t} \right|_{\gamma, \gamma/2, Q_T} + |W_0|_{2+\gamma, \Omega} \right), \quad (5.9)$$

$$\begin{aligned}
|\theta|_{2+\gamma,1+\gamma/2,Q_T} \leq C & \left( |\rho\theta|_{0,Q_T} \left| \frac{\partial V}{\partial x} \right|_{\gamma,\gamma/2,Q_T} + |\rho\theta|_{\gamma,\gamma/2,Q_T} \left| \frac{\partial V}{\partial x} \right|_{0,Q_T} + |\theta|_{\gamma,\gamma/2,Q_T} \right. \\
& + \left| \rho \frac{\partial V}{\partial x} \right|_{0,Q_T} \left| \frac{\partial V}{\partial x} \right|_{\gamma,\gamma/2,Q_T} + \left| \rho \frac{\partial V}{\partial x} \right|_{\gamma,\gamma/2,Q_T} \left| \frac{\partial V}{\partial x} \right|_{0,Q_T} + \left| \frac{\partial V}{\partial x} \right|_{\gamma,\gamma/2,Q_T} \\
& + \left| \frac{1}{\rho} \right|_{\gamma,\gamma/2,Q_T} + \left| \rho \frac{\partial W}{\partial x} \right|_{0,Q_T} \left| \frac{\partial W}{\partial x} \right|_{\gamma,\gamma/2,Q_T} + \left| \frac{\partial W}{\partial x} \right|_{\gamma,\gamma/2,Q_T} \\
& + \left| \rho \frac{\partial W}{\partial x} \right|_{\gamma,\gamma/2,Q_T} \left| \frac{\partial W}{\partial x} \right|_{0,Q_T} + \left| \frac{W}{\rho} \right|_{0,Q_T} |W|_{\gamma,\gamma/2,Q_T} \\
& \left. + \left| \frac{W}{\rho} \right|_{\gamma,\gamma/2,Q_T} |W|_{0,Q_T} + \left| \frac{\omega_1^2}{\rho} \right|_{\gamma,\gamma/2,Q_T} + |\theta_0|_{2+\gamma,\Omega} \right). \tag{5.10}
\end{aligned}$$

Using the inequalities

$$|f|_{0,Q_T} \leq |f|_{\gamma,\gamma/2,Q_T} \leq |f|_{\gamma,\gamma,Q_T}, \tag{5.11}$$

and (3.6), (3.2), and (5.1)–(5.3), from (5.8)–(5.9) we get easily (5.4) and (5.5). With the help of (4.18), (4.21) for  $\eta = W$  and (5.5) from (5.10) it follows

$$|\theta|_{2+\gamma,1+\gamma/2,Q_T} \leq C \left( \left| \frac{\partial V}{\partial x} \right|_{\gamma,\gamma/2,Q_T} + \left| \frac{\partial V}{\partial x} \right|_{0,Q_T} + \left| \frac{\partial V}{\partial x} \right|_{\gamma,\gamma/2,Q_T} \left| \frac{\partial V}{\partial x} \right|_{0,Q_T} + \left| \frac{\partial V}{\partial x} \right|_{0,Q_T}^2 + 1 \right). \tag{5.12}$$

Using (4.18) and (4.21) for  $\eta = V$  we get (5.6) immediately.  $\square$

**Lemma 5.4.** For  $\gamma$  from Lemma 5.2 the estimations

$$|V|_{2+\gamma,1+\gamma/2,Q_T} \leq C, \tag{5.13}$$

$$|\theta|_{2+\gamma,1+\gamma/2,Q_T} \leq C, \tag{5.14}$$

$$|\rho|_{1+\gamma,Q_T} \leq C, \tag{5.15}$$

$$|v_1|_{2+\gamma,1+\gamma/2,Q_T} \leq C, \quad |\omega_1|_{2+\gamma,1+\gamma/2,Q_T} \leq C \tag{5.16}$$

hold true.

*Proof.* For  $\eta = \theta$  and  $0 < \alpha < 1$  from (4.19) we can conclude that

$$|\theta|_{1+\alpha,\alpha/2,Q_T} \leq C(|\theta|_{2+\alpha,1+\alpha/2,Q_T} + 1), \tag{5.17}$$

and from (5.4) it follows

$$|V|_{2+\gamma,1+\gamma/2,Q_T} \leq C(|\theta|_{2+\gamma,1+\gamma/2,Q_T} + 1). \tag{5.18}$$

Inserting (5.6) on the right-hand side of (5.18) we obtain

$$|V|_{2+\gamma,1+\gamma/2,Q_T} \leq C(|V|_{2+\gamma,1+\gamma/2,Q_T}^a + |V|_{2+\gamma,1+\gamma/2,Q_T}^b + |V|_{2+\gamma,1+\gamma/2,Q_T}^{a+b} + |V|_{2+\gamma,1+\gamma/2,Q_T}^{2a} + 1), \quad (5.19)$$

where  $a, b, a + b, 2a \in ]0, 1[$ . Applying the Young inequality with a parameter  $\varepsilon > 0$  we obtain

$$|V|_{2+\gamma,1+\gamma/2,Q_T} \leq C(\varepsilon|V|_{2+\gamma,1+\gamma/2,Q_T} + 1), \quad (5.20)$$

and hence (5.13). Using this result from (5.6) follows (5.14). From (2.30) we get

$$\left| \frac{\partial \rho}{\partial t} \right|_{\gamma, Q_T} \leq |\rho^2|_{0, Q_T} \left| \frac{\partial V}{\partial x} \right|_{\gamma, Q_T} + |\rho^2|_{\gamma, Q_T} \left| \frac{\partial V}{\partial x} \right|_{0, Q_T} + \left| \frac{\mu}{l} \rho \right|_{\gamma, Q_T}. \quad (5.21)$$

Using (4.20) and (5.13) for  $V$ , the inequality  $|\rho|_{\gamma, Q_T} \leq |\rho|_{\gamma, \gamma, Q_T}$ , (5.1) and (3.1) we obtain

$$\left| \frac{\partial \rho}{\partial t} \right|_{\gamma, Q_T} \leq C, \quad (5.22)$$

and with the help of (5.3) we get (5.15). Notice that from (5.15) follows

$$\int_0^x \rho^{-1} d\xi \in C^{\gamma/2}(\overline{Q_T}), \quad (5.23)$$

and using (5.13) and (3.1) from (4.2) and (2.26) we obtain

$$\begin{aligned} \left| \frac{\partial v_1}{\partial t} \right|_{\gamma, \gamma/2, Q_T} &\leq \left| \left( \frac{\mu}{l} \right)' + \left( \frac{\mu}{l} \right)^2 \int_0^x \frac{1}{\rho} d\xi \right|_{\gamma, \gamma/2, Q_T} + \left| \frac{\mu}{l} V \right|_{\gamma, \gamma/2, Q_T} + |\mu'_0|_{\gamma/2, ]0, T[} \leq C, \\ \left| \frac{\partial^2 v_1}{\partial x^2} \right|_{\gamma, \gamma/2, Q_T} &\leq \left| \frac{\mu}{l\rho} \right|_{\gamma, \gamma/2, Q_T} \left| \frac{\partial \rho}{\partial x} \right|_{0, Q_T} + \left| \frac{\mu}{l\rho} \right|_{0, Q_T} \left| \frac{\partial \rho}{\partial x} \right|_{\gamma, \gamma/2, Q_T} + \left| \frac{\mu}{l} \right|_{\gamma, ]0, T[} \leq C. \end{aligned} \quad (5.24)$$

Taking into account (5.2) it is evident that the inequality

$$|v_1|_{2+\gamma,1+\gamma/2,Q_T} \leq C \quad (5.25)$$

is satisfied. From (2.27) and (4.3) follows the same estimation for the function  $\omega_1$ .  $\square$

Now, from the above estimations we derive the conclusion that if  $\alpha \leq 1/2$  then  $\alpha = \gamma$  and Lemmas 5.1–5.4 are the proofs of Theorems 3.2 and 3.1. If  $\alpha > 1/2$  we have

$$V, W, \theta, v_1, \omega_1 \in C^{2+1/2, 1+1/4}(\overline{Q_T}), \quad \rho \in C^{1+1/2}(\overline{Q_T}). \quad (5.26)$$

**Lemma 5.5.** For  $1/2 < \alpha < 1$  we have

$$\rho, V, W, \theta \in C^{\alpha, \alpha}(\overline{Q_T}), \quad (5.27)$$

$$v_1, \omega_1 \in C^{1+\alpha, 1+\alpha}(\overline{Q_T}), \quad (5.28)$$

$$\frac{\partial \rho}{\partial x} \in C^{\alpha, \alpha}(Q_T). \quad (5.29)$$

*Proof.* Inclusions (5.27) follows directly from (5.26). Using this result from (2.26), (2.27), (4.2), and (4.3) we get (5.28). Estimation (5.29) is proved in (see [8, pages 57-58]).  $\square$

**Lemma 5.6.** For  $1/2 < \alpha < 1$  the estimations

$$|V|_{2+\alpha, 1+\alpha/2, Q_T} \leq C, \quad (5.30)$$

$$|W|_{2+\alpha, 1+\alpha/2, Q_T} \leq C, \quad (5.31)$$

$$|\theta|_{2+\alpha, 1+\alpha/2, Q_T} \leq C, \quad (5.32)$$

$$|\rho|_{1+\alpha, Q_T} \leq C. \quad (5.33)$$

$$|v_1|_{2+\alpha, 1+\alpha/2, Q_T} \leq C, \quad |\omega_1|_{2+\alpha, 1+\alpha/2, Q_T} \leq C \quad (5.34)$$

are true.

*Proof.* We consider (2.31)–(2.33) again as parabolic equations for  $V, W$ , and  $\theta$ , respectively, with Hölder continuous coefficients with exponent  $\alpha$ . In the same way as before from (5.8) and (5.9) we get

$$|V|_{2+\alpha, 1+\alpha/2, Q_T} \leq C(1 + |\theta|_{2+\alpha, 1+\alpha/2, Q_T}), \quad (5.35)$$

$$|W|_{2+\alpha, 1+\alpha/2, Q_T} \leq C, \quad (5.36)$$

and with the help of (5.26) from (5.10) we obtain

$$|\theta|_{2+\alpha, 1+\alpha/2, Q_T} \leq C \left( 1 + \left| \frac{\partial V}{\partial x} \right|_{\alpha, \alpha/2, Q_T} \right). \quad (5.37)$$

Inserting (5.37) in (5.35), using (4.19) and (5.26) for the function  $V$ , we obtain (5.30). With the help of (4.19) and (5.30) from (5.37) it follows (5.32). In the same way as before we get

$$\left| \frac{\partial \rho}{\partial t} \right|_{\alpha, Q_T} \leq |\rho^2|_{\alpha, Q_T} \left| \frac{\partial V}{\partial x} \right|_{0, Q_T} + |\rho^2|_{0, Q_T} \left| \frac{\partial V}{\partial x} \right|_{\alpha, Q_T} + \left| \frac{\mu}{l} \rho \right|_{\alpha, Q_T}. \quad (5.38)$$

Because of (5.27), (5.30), and (4.20) for  $\eta = V$  we obtain

$$\left| \frac{\partial \rho}{\partial t} \right|_{\alpha, Q_T} \leq C, \quad (5.39)$$

and using (5.29) we have (5.33). Taking into account (5.33) in the same way as in Lemma 5.4 we get (5.34).  $\square$

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