NONINJECTIVE CYCLIC MODULES

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In [3], it is shown that a ring R such that every cyclic right R-module is injective must be semisimple Artin. In this note, that proof is greatly simplified, and it is shown that a hereditary ring cannot contain an infinite direct product of subrings.

R will denote a ring with 1, all modules will be unital right R-modules, and all homomorphisms R-homomorphisms. For a module M, $E(M)$ will denote its injective hull (see [2]).

THEOREM. *Let $\{e_i \mid i \in \mathcal{I}\}$ be an infinite set of orthogonal idempotents of R. Assume for each $A \subseteq \mathcal{I}$, there exists $m_A \in R$ such that $m_A e_i = e_i$ for all $i \in A$, and $e_j m_A = 0$ for all $j \in \mathcal{I} - A$. Then for all $M_R \supseteq R_R$, $M/(\sum_{i \in \mathcal{I}} e_i R + \ker \pi)$ is not injective, where $\pi\colon R \to \prod_{i \in \mathcal{I}} e_i R$, $\pi(m) = \langle e_i m \rangle$.*

PROOF. Let $\mathcal{I} = \bigcup_{A \in \mathfrak{A}} A$, where $\mathfrak{A}$ is infinite and for all $A, B \in \mathfrak{A}$, A is infinite and $A \cap B \neq \emptyset \Leftrightarrow A = B$. By Zorn's lemma, $\mathfrak{A}$ can be enlarged to a set $\mathfrak{B} \subseteq$ the power set of $\mathcal{I}$ maximal with respect to the properties

(i) for all $A \in \mathfrak{B}$, A is infinite, and

(ii) for all A and B in $\mathfrak{B}$, $A \neq B \Rightarrow A \cap B$ is finite.

Let $\Sigma = \sum_{i \in \mathcal{I}} e_i R + \ker \pi$. Then Σ is precisely the set of elements of R annihilated by almost all e_i. Let $A \in \mathfrak{B}$, $r \in R$, and assume $m_A r \notin \Sigma$. Then there exist an infinite number of i (all in A) such that $e_i m_A r \neq 0$. For any set $\{A_j \mid 1 \leq j \leq n\} \subseteq \mathfrak{B} - \{A\}$, $A \cap \bigcup_{j=1}^{n} A_j$ is finite. Thus for all but a finite number of $i \in A$, $e_i m_{A_j} = 0$ for all j, $1 \leq j \leq n$. Then $m_A r \notin \sum_{j=1}^{n} m_{A_j} R + \Sigma$, so $\sum_{A \in \mathfrak{B}} (m_A + \Sigma) R$ is direct in M/Σ.

Define $\phi\colon \sum_{A \in \mathfrak{B}} (m_A R + \Sigma)/\Sigma \to M/\Sigma$ by

$$\begin{aligned} \phi(m_A) &= m_A \quad A \in \mathfrak{A}, \\ &= 0 \qquad A \in \mathfrak{B} - \mathfrak{A}. \end{aligned}$$

Assume ϕ extends to a homomorphism $\bar{\phi}$ from $R/\Sigma \to M/\Sigma$. Let $\bar{\phi}(1 + \Sigma) = m + \Sigma$. Then for all $A \in \mathfrak{A}$, $m m_A = m_A + \sum_{l=1}^{n} e_{i_l} r_l + k$, so $A' = \{a \in A \mid e_a m e_a = e_a\} \supseteq A - \{i_l \mid 1 \leq l \leq n\}$ is infinite.

Let C be a choice set for $\{A' \mid A \in \mathfrak{A}\}$. By the maximality of $\mathfrak{B}$, $C \cap D$ is infinite for some $D \in \mathfrak{B}$, and D cannot belong to $\mathfrak{A}$. But then

Received by the editors July 10, 1967.

[1] The author gratefully acknowledges partial support from the National Science Foundation under grant GP 7162.

$mm_D \in \Sigma$, so for all but a finite number of $i \in \mathcal{I}$, $e_i mm_D = 0$. Hence for all but a finite number of $d \in C \cap D$, $0 = e_d mm_D$, but for all $d \in C \cap D$, $e_d = e_d mm_D e_d$, a contradiction.

COROLLARY. *Let R contain an infinite ring direct product $\prod_{i \in \mathcal{I}} R_i$, where R_i is a ring with identity e_i. Then R is not hereditary.*

PROOF. By [1, p. 14], a ring R is hereditary if and only if every quotient of an injective module is injective. $\{e_i \mid i \in \mathcal{I}\}$ are orthogonal idempotents, and the characteristic function of A will serve as m_A in the theorem. Then $E(R)/\Sigma$ is not injective.

COROLLARY. *Let R be a ring such that every cyclic R-module is injective. Then R is semisimple Artin.*

PROOF. R is von Neumann regular and self injective. For any set of orthogonal idempotents $\{e_i \mid i \in \mathcal{I}\} \subseteq R$ and $A \subseteq \mathcal{I}$, let m_A be the projection of 1 on $E(\sum_{i \in A} e_i R) \subseteq R$. Clearly $m_A e_i = e_i$ for all $i \in A$. Let $j \in \mathcal{I} - A$. Then $Re_j m_A = Re$ for some $e = e^2$. If $x = m_A er \in \sum_{i \in A} e_i R$, then $e_j x = 0$ so $e_j m_A er = 0$, $er = 0$, and finally $x = 0$. Since $m_A R$ is an essential extension of $\sum_{i \in A} e_i R$, $m_A eR = 0$. Then $e_j m_A = e_j m_A e = 0$. The theorem then shows that $\mathcal{I}$ cannot be infinite, so R is semisimple Artin (see [4]).

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