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NONLINEAR AND DIRECTION CONNECTIONS

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ABSTRACT. Nonlinear connections and direction connections are two types of connections arising in Finsler geometry. In his work on generalized sprays, P. Dazord showed that there is a relationship between these two types (nonlinear connections were called sections by him). This relationship has also been used by J. Grifone in a work on prolongation of direction connections. In this paper we examine this relationship in a general setting. In particular, we show that E. Cartan's condition "D" is necessary and sufficient for a direction connection to define a nonlinear one. Also, we prove a nonuniqueness result for direction connections associated to a given nonlinear one.

1. **Connections on vector-bundles.** We first recall our definition of a connection on a vector bundle [5], [6]. For a smooth (C^∞) vector bundle $p:E\to M$, set $E_0=E-0$ and $p_0=p|E_0$. The bundle $p^{-1}E$ over E is canonically isomorphic to the bundle VE of vertical vectors in the tangent bundle TE of E. Hence we have the exact sequence

$$(1)\qquad 0\to p^{-1}E\xrightarrow{J}TE\xrightarrow{p'}p^{-1}TM\to 0$$

of vector bundles over E, where J corresponds to the inclusion map $VE\subset TE$ and p' is essentially the tangent map $p_*:TE\to TM$.

A smooth *nonlinear connection* on the vector bundle $p:E\to M$ is a smooth splitting of (1) over E_0. Since $TE|E_0=TE_0$ and $p^{-1}E|E_0=p_0^{-1}E$, such a splitting is given by a smooth linear map $V:TE_0\to p_0^{-1}E$ (i.e. continuous linear on the fibres), satisfying the equation $VJ=\mathrm{id}$. The splitting can be conveniently described by its connection map $D:TE_0\to E$ defined as $D=r\circ V$, where $r:p^{-1}E\to E$ is the canonical surjection over p. D is continuous linear on the fibres and is smooth.

The connection on $p:E\to M$ is *homogeneous* (resp. *linear*) if the map D is homogeneous of degree 1 (resp. linear) on the p_* fibres of TE. For a linear connection, the splitting of (1) automatically extends to all of E; in fact, a linear connection can be defined as a splitting of (1) which is smooth over all of E.

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Let us now introduce local coordinates. Suppose locally TM and E are isomorphic to product bundles $U\times\boldsymbol{B}$ and $U\times\boldsymbol{E}$, respectively, where $\boldsymbol{B}$ and $\boldsymbol{E}$ are real Banach spaces (the reader should substitute $\boldsymbol{B}=\boldsymbol{R}^n$, $\boldsymbol{E}=\boldsymbol{R}^m$ if he is not familiar with calculus in Banach spaces) and $U\subset\boldsymbol{B}$ is an open set. Then TE_0 is locally isomorphic to the product bundle $(U\times\boldsymbol{E}_0)\times\boldsymbol{B}\times\boldsymbol{E}$ (where $\boldsymbol{E}_0=\boldsymbol{E}-0$), and fiberwise for each $(x, a)\in U\times\boldsymbol{E}$, D is defined by a map $D_{(x,a)}:\boldsymbol{B}\times\boldsymbol{E}\to\boldsymbol{E}$ of the form

$$D_{(x,a)}(\lambda, b) = b + \omega(x, a)\lambda. \tag{2}$$

Here $\omega: U\times\boldsymbol{E}_0\to L(\boldsymbol{B}, \boldsymbol{E})$ is a smooth map (L denotes a space of continuous linear maps). ω is called a *local component* of the connection.

The connection is homogeneous (resp. linear) iff each ω is homogeneous of degree 1 (resp. continuous linear) in the second variable, a. (Hence for a linear connection we can define $\omega(x, 0)=0$ and obtain a smooth map on $U\times\boldsymbol{E}$.)

2. **Linear connections on $p_0^{-1}E\to E_0$.** Next we consider a linear connection on the vector bundle $p_0^{-1}E\to E_0$ and define its torsion.

Since $p_0^{-1}E$ is locally isomorphic to $(U\times\boldsymbol{E}_0)\times\boldsymbol{E}$, and $T(p_0^{-1}E)$ to $((U\times\boldsymbol{E}_0)\times\boldsymbol{E})\times(\boldsymbol{B}\times\boldsymbol{E})\times\boldsymbol{E}$, the connection map $\mathfrak{D}: T(p_0^{-1}E)\to p_0^{-1}E$ of such a connection is, fiberwise for each $((x, a), b)\in(U\times\boldsymbol{E}_0)\times\boldsymbol{E}$, given by a map $\mathfrak{D}_{((x,a),b)}:(\boldsymbol{B}\times\boldsymbol{E})\times\boldsymbol{E}\to\boldsymbol{E}$ of the form

$$\mathfrak{D}_{((x,a),b)}((\lambda, c), d) = d + \Omega((x, a), b)(\lambda, c).$$

Ω is continuous linear in (λ, c) and, the connection being linear, also in b. Hence $\mathfrak{D}$ must actually be locally of the form

$$\mathfrak{D}_{((x,a),b)}((\lambda, c), d) = d + \Gamma_{(x,a)}(b, \lambda) + C_{(x,a)}(b, c), \tag{3}$$

where Γ, C are smooth maps of $U\times\boldsymbol{E}_0$ into $L^2(\boldsymbol{E}, \boldsymbol{B}; \boldsymbol{E})$ and $L^2(\boldsymbol{E}, \boldsymbol{E}; \boldsymbol{E})$, respectively ($L^2$ denotes a space of continuous bilinear maps).

Although the two parts Γ and C of the local component of $\mathfrak{D}$ do not, of course, transform together as a tensor under a change of coordinates, it happens that the C part does define a tensor, which corresponds to the Finsler torsion tensor of [1]. (Since $p^{-1}E\approx VE$, one can transcribe the coordinate change equation of [6, p. 1127] to the present notation and set $\mu=0$ to see this fact.) Thus there is a smooth section $\mathfrak{I}$ of the bundle of bilinear maps from $p_0^{-1}E$ to $p_0^{-1}E$, which is locally given by maps $\mathfrak{I}_{(x,a)}\in L^2(\boldsymbol{E}, \boldsymbol{E}; \boldsymbol{E})$ with $\mathfrak{I}_{(x,a)}(b, c)=C_{(x,a)}(b, c)$. We call $\mathfrak{I}$ the *torsion tensor* of the given linear connection on $p_0^{-1}E\to E_0$.

3. **Direction connections.** We now define direction connections for vector bundles. Namely, a *direction connection* for the vector bundle

$p: E \to M$ is a linear connection on the vector bundle $p_0^{-1}E \to E_0$ which is invariant under the diffeomorphisms of the manifold $p_0^{-1}E$ defined by radial expansions in the fibres of $E_0 \to M$.

Let us describe this invariance condition precisely. Radial expansions of the fibres of $E_0 \to M$ are diffeomorphisms of type $e \mapsto f(p_0 e)e$, where f is a smooth nonvanishing real-valued function on M. Tensoring such a map on the right with the identity yields a diffeomorphism of the total space of the bundle $E_0 \otimes E \to M$. Since this total space is diffeomorphic with $p_0^{-1}E$, we get a diffeomorphism of $p_0^{-1}E$ denoted by $\hat{f}$. The condition of invariance is that the connection map $\mathfrak{D}: T(p_0^{-1}E) \to p_0^{-1}E$ of the given linear connection satisfies

$$\mathfrak{D} \circ (\hat{f})_* = \hat{f} \circ \mathfrak{D} \tag{4}$$

for all such diffeomorphisms $\hat{f}$.

Next we derive the local coordinate description of direction connections. Namely, consider a local representation $\hat{f}(x, a, b) = (x, \phi(x)a, b)$ of a diffeomorphism $\hat{f}$ defined above, where $\phi: U \to \boldsymbol{R}$ is a smooth nonvanishing function. An easy calculation shows (4) to mean

$$\begin{aligned} \Gamma_{(x,a)}(b, \lambda) + C_{(x,a)}(b, c) = \Gamma_{(x,\phi(x)a)}(b, \lambda) &+ C_{(x,\phi(x)a)}(b, \phi(x)c) \\ &+ C_{(x,\phi(x)a)}(b, \phi'(x)(\lambda)a). \end{aligned} \tag{5}$$

Putting $c = 0$ and then setting ϕ constant, we see that (5) is equivalent to the two conditions that

$$\Gamma_{(x,a)} \text{ is homogeneous of degree 0 in } a, \text{ and} \tag{6}$$

$$C_{(x,a)}(b, c) = C_{(x,\phi(x)a)}(b, \phi(x)c) + C_{(x,\phi(x)a)}(b, \phi'(x)(\lambda)a). \tag{7}$$

Taking ϕ constant, we see (7) implies

$$C_{(x,a)} \text{ is homogeneous of degree } -1 \text{ in } a. \tag{8}$$

Setting $\phi(x) = \exp(-f(x))$, where f is a continuous linear functional on $\boldsymbol{B}$ such that $f(\lambda) = 1$ (Hahn-Banach Theorem), we get $C_{(x,a)}(-, a) = 0$, i.e.

$$\mathfrak{J}_e(-, e) = 0, \tag{9}$$

where we identify $(p_0^{-1}E)_e$ with E_{pe}. As a summary, we have

PROPOSITION 1. *A linear connection on $p_0^{-1}E \to E_0$ is a direction connection iff all its local components* (Γ, C) *satisfy* (6) *and* (7). *Moreover, a direction connection satisfies* (8) *and* (9).

COROLLARY. *A linear connection on $p_0^{-1}E \to E_0$ with torsion zero is a direction connection.*

REMARK. In one of his earlier papers on Finsler geometry, M. Matsumoto [4] studied a more general class of linear connections on $p_0^{-1}E \to E_0$, namely, those invariant under diffeomorphisms of $p_0^{-1}E$ defined by *uniform* radial expansions in the fibres of $E_0 \to M$, i.e. (4) holds only for constant f. Locally these connections are characterized by (6) and (8) (with only constant ϕ, (5) is equivalent to these two equations); (9) need not be satisfied. Let us call these connections *weak direction connections* (in [4] they were called Finsler connections). Note that they differ from ordinary direction connections only by the behavior of the torsion tensor.

4. **From $\mathfrak{D}$ to D.** Next we turn to the relationship between linear connections on $p_0^{-1}E \to E_0$ and homogeneous nonlinear connections on $p: E \to M$.

In this section we derive a necessary and sufficient condition that a linear connection $\mathfrak{D}$ on $p_0^{-1}E \to E_0$ defines a nonlinear connection D on $p: E \to M$ according to the prescription in [1, §5].

The vector bundle $p_0^{-1}E \to E$ has a canonical section $\mathfrak{V}: E_0 \to p_0^{-1}E$ which is defined as $\mathfrak{V}(e) = e$ (using the identification $(p_0^{-1}E)_e = E_{pe}$). Letting $\mathfrak{D}$ denote the connection map, define the map $D: TE_0 \to E$ as $D = r \circ \mathfrak{D} \circ \mathfrak{V}_*$. (This is just the covariant derivative map $Z \mapsto \mathfrak{D}_Z \mathfrak{V}$, see [5, §2].)

PROPOSITION 2. *For a linear connection $\mathfrak{D}$ on $p_0^{-1}E \to E_0$, the map $D = r \circ \mathfrak{D} \circ \mathfrak{V}_*$ defines a nonlinear connection on $E \to M$ iff the torsion of $\mathfrak{D}$ satisfies*

$$\mathfrak{J}_e(e, -) = 0. \tag{10}$$

Moreover, if $\mathfrak{D}$ is a (weak) direction connection then the connection D is homogeneous.

PROOF. We work locally. TE_0 is locally $U \times \boldsymbol{E}_0 \times \boldsymbol{B} \times \boldsymbol{E}$ and $\mathfrak{V}(x, a) = (x, a, a)$. Calculating $\mathfrak{V}_*$ and using (3) we get $D(x, a, \lambda, b) = (x, b + \Gamma_{(x,a)}(a, \lambda) + C_{(x,a)}(a, b))$. By [5, Lemma 1, p. 239] this defines a nonlinear connection iff $\Gamma_{(x,a)}(a, \lambda) + C_{(x,a)}(a, b)$ is linear in λ and independent of b. But this clearly occurs iff $C_{(x,a)}(a, b) = 0$ for all b, which is precisely (10).

The local component of D is thus given by

$$\omega(x, a)\lambda = \Gamma_{(x,a)}(a, \lambda). \tag{11}$$

In case $\mathfrak{D}$ is a weak direction connection, $\Gamma_{(x,a)}$ is homogeneous of degree 0 in a. Since $\Gamma_{(x,a)}(b, \lambda)$ is linear in b, we see $\omega(x, a)$ is homogeneous of degree 1 in a, which means D is homogeneous. Q.E.D.

Let us say that in the setting of Proposition 2 the connection $\mathfrak{D}$ is *associated* to the connection D. This means that (11) holds for local components.

REMARK. Equation (10) is also known as E. Cartan's condition "D" (see [**4**]). This condition must be added to the results of Grifone in [**3**], since the step $\mathfrak{D}$ to D is used there. In Dazord [**1**] only direction connections with symmetric torsion ("regular") are treated, so that (9) implies (10).

5. **From** D **to** $\mathfrak{D}$. We now consider the reverse of the situation just discussed. Our results are as follows.

PROPOSITION 3. *For each homogeneous nonlinear connection D on $E \to M$, there exists an associated direction connection $\mathfrak{D}$ with torsion zero. If D is linear, the pullback $r^{-1}D$ is such a $\mathfrak{D}$.*

$\mathfrak{D}$ is not unique (even if it is assumed that dimensions are finite, $E = TM$, D is linear and comes from a spray, and $\mathfrak{D}$ is symmetric).

PROOF. Using $VE \approx p^{-1}E$ we apply the proposition of [**6**, §2], which assigns to D the linear Berwald connection $\mathfrak{D}$ on $p_0^{-1}E \to E_0$. If ω denotes the local component of D, then that of $\mathfrak{D}$ is by definition

$$\Gamma_{(x,a)}(b, \lambda) = \partial_a \omega(x, a)(b)\lambda, \qquad C_{(x,a)} = 0. \tag{12}$$

Hence clearly $\mathfrak{D}$ has torsion zero. If D is homogeneous, then ω is homogeneous of degree 1 in a, whence its derivative with respect to this variable is homogeneous of one less degree, namely 0. This means (6) holds for Γ. Since also (7) holds trivially for C, $\mathfrak{D}$ is a direction connection by Proposition 1. Now the homogeneity of degree 1 of ω in a implies by Euler's theorem that

$$\partial_a \omega(x, a)(a)\lambda = \omega(x, a)\lambda. \tag{13}$$

But (12) and (13) give (11), which means $\mathfrak{D}$ is associated to D. For ω linear in a, $\Gamma_{(x,a)}(b, \lambda) = \omega(x, b)\lambda$. This together with $C = 0$ means $\mathfrak{D}$ is the pullback connection $r^{-1}D$.

To prove the nonuniqueness assertions let $M = \boldsymbol{R}^2$ and $E \to M$ be the tangent bundle $\boldsymbol{R}^2 \times \boldsymbol{R}^2 \to \boldsymbol{R}^2$ of M. Let D be the linear connection defined by $\omega(x, a)\lambda = \langle a, \lambda \rangle x$, where $\langle\ ,\ \rangle$ denotes the inner product in $\boldsymbol{R}^2$. The Berwald connection $\mathfrak{D}$ of D is given by $C = 0$ and $\Gamma_{(x,a)}(b, \lambda) = \partial_a \omega(x, a)(b)\lambda = \langle b, \lambda \rangle x$. Now define a direction connection $\mathfrak{D}^0$ by $C^0 = 0$ and

$$\Gamma^0_{(x,a)}(b, \lambda) = \left[(a_1)^2 b_1 \lambda_1 + a_1 a_2 (b_1 \lambda_2 + b_2 \lambda_1) + (a_2)^2 b_2 \lambda_2\right](x / |a|^2),$$

where $a = (a_1, a_2)$. Then $\Gamma^0_{(x,a)}(a, \lambda) = \langle a, \lambda \rangle x = \omega(x, a)\lambda$, which means $\mathfrak{D}^0$ is associated to D. But clearly $\mathfrak{D} \neq \mathfrak{D}^0$.

Both $\mathfrak{D}$ and $\mathfrak{D}^0$ are symmetric, because it can be shown in a straightforward way from the definition of symmetry given in [1] that if torsion of $\mathfrak{D}$ is zero, then $\mathfrak{D}$ is symmetric iff the local component Γ is symmetric.

To see that D arises from a spray (see [1], [2]) on M, consider the spray defined by $G(x, a) = -\langle a, a\rangle x$. Then with ω as above, we see that $\omega(x, a)\lambda = -\frac{1}{2}\partial_a G(x, a)(\lambda)$, which means D comes from the spray G. Q.E.D.

REMARK. Proposition 3 shows that the uniqueness statement in Theorem 1 [2] cannot be interpreted to mean one-to-one correspondence between symmetric torsion zero direction connections and their associated sprays. Therefore, it is not clear what sense this uniqueness statement could make.

If we add the assumption that $\mathfrak{D}$ is continuously extendible to a connection on the bundle $p^{-1}E\to E$, and that D is linear, then we do get uniqueness for torsion zero associated direction connections. For (as noted in [1]), the extendibility condition means locally that $\Gamma_{(x,a)}$ does not depend on a. Differentiation of (11) then yields $\Gamma_{(x,a)}(b, \lambda) = \omega(x, b)\lambda$, which together with $C=0$ means that $\mathfrak{D}$ is equal to the pullback connection $r^{-1}D$. (See also Matsumoto [4], where such connections are called simple connections.) However, the extendibility assumption is too restrictive for Finsler geometry.

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