1172. Nonlinear backstepping design for the underactuated TORA system

Diantong Liu[1], Weiping Guo[2]
Institute of Computer Science and Technology, Yantai University, Yantai City, People's Republic of China
[1]Corresponding author
E-mail: [1]*diantong.liu@163.com*, [2]*weiping.guo@163.com*

Abstract. The nonlinear feedback cascade model of the underactuated translational oscillators with rotating actuator is obtained through a collocated partial feedback linearization and a global change of coordinates. A nonlinear controller is designed with the backsteping technology, which treats the state variables as virtual control inputs to design the virtual controllers step by step. The system stability is proved with the Lyapunov stability theorem. The simulation results show the system under any initial states can be asymptotically stabilized to the origin and the controller has a good control performance.

Keywords: underactuated mechanical systems, TORA, backstepping, vibration control.

1. Introduction

The TORA (translational oscillators with rotating actuator) is a classical underactuated mechanical system, which is composed of a non-actuated translational oscillators and an actuated pendulum. It was developed as a simplified model of a dual spin spacecraft [1]. One research motivation to control the TORA comes from the possible use to suppress the translational vibration with a rotational actuation. The other is an independent interest to control the TORA as a benchmark problem in the nonlinear control design.

The control problem on the TORA has attracted a considerable amount of attentions from many researchers. Some focus on the intelligent design methods [2-4] and some focus on the nonlinear design methods [5-11]. Among the intelligent design methods, a self-tuning fuzzy sliding mode control is proposed [2-3] and there are some fuzzy rules and parameters to be preset. A neural network controller is proposed with SPSA Algorithm [4] and a time-consuming learning process is needed. Among the nonlinear design methods, the energy-based design [5-7], the backstepping design [8-9] and sliding mode based design [10], the design processes and the obtained controllers are relatively cumbersome. A comparison study of nonlinear control techniques for the TORA is performed [5].

In this paper, the TORA system is transformed into a nonlinear feedback cascade system through a collocated partial feedback linearization and a global change of coordinates. Then a nonlinear backstepping design method is proposed. The design procedure treats the state variables as virtual control inputs to design the virtual controllers step by step, and the system stability is proved via the Lyapunov stability theorem.

2. The TORA system model

The TORA is shown schematically in Fig. 1. The oscillator consists of a cart of mass m_1 connected to a fixed wall by a linear spring of stiffness k. The cart is constrained to have one-dimensional travel and its position is q_1. The rotational actuator with the output torque is attached to the cart and it has a pendulum with the equivalent mass m_2, the rotate radius r, the moment of inertia I and the rotate angle q_2. Since the rotating torque can not drive the cart directly, it is a benchmark example of the underactuated mechanical system, which has one control input τ and two configuration variables (q_1, q_2), and its Euler-Lagrange equations of motion can be obtained as:

$$\begin{cases} (m_1 + m_2)\ddot{q}_1 + m_2 r\cos(q_2)\ddot{q}_2 - m_2 r\sin(q_2)\dot{q}_2^2 + kq_1 = 0, \\ m^2 r\cos(q_2)\ddot{q}_1 + (m_2 r^2 + I)\ddot{q}_2 + m_2 gr\sin(q_2) = \tau. \end{cases} \tag{1}$$
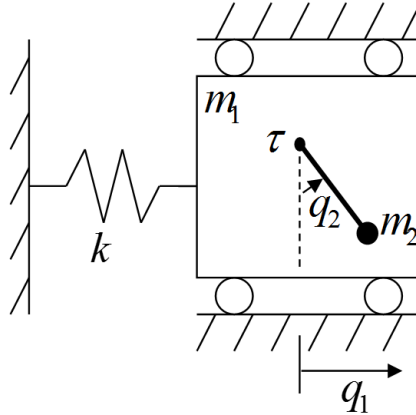

Fig. 1. TORA system configuration

Since the partial feedback linearization technology proposed by Spong [12] can simplify the system dynamics. The following collocated partial feedback linearization is adopted:

$$\tau = \left((m_2 r^2 + I) - \frac{(m_2 r\cos(q_2))^2}{m_1 + m_2} \right) u + \frac{(m_2 r)^2 \sin(q_2)\cos(q_2)}{m_1 + m_2} \dot{q}_1^2 - km_2 rq_1\cos(q_2) \tag{2}$$
$$+ m_2 gr\sin(q_2),$$

and it can simplify the dynamics of the shape variable q_2 as:

$$\ddot{q}_2 = u.$$

Olfati-Saber proposed a global change of coordinates [13] such as:

$$\begin{cases} x_1 = q_1 + \dfrac{m_2 r\sin(q_2)}{m_1 + m_2}, \\ x_2 = q_2, \\ x_3 = (m_1 + m_2)\dot{q}_1 + m_2 r\cos(q_2)\dot{q}_2, \\ x_4 = \dot{q}_2. \end{cases} \tag{3}$$

It can transform the system dynamics into a nonlinear system as:

$$\begin{cases} \dot{x}_1 = \dfrac{x_3}{m_1 + m_2}, \\ \dot{x}_2 = x_4, \\ \dot{x}_3 = -kx_1 + \dfrac{km_2 r\sin x_2}{m_1 + m_2}, \\ \dot{x}_4 = u. \end{cases} \tag{4}$$

Therefore, the system model of TORA can be described with a nonlinear feedback cascade model [14] as:

$$\begin{cases} \ddot{x}_1 = -\dfrac{kx_1}{m_1 + m_2} + \dfrac{km_2r}{(m_1 + m_2)^2}\sin x_2, \\ \ddot{x}_2 = u. \end{cases} \tag{5}$$

3. The nonlinear controller design through backstepping

Since the model of TORA can be transformed into a cascade nonlinear system through the collocated partial feedback linearization (2) and the global change of coordinates (3), a nonlinear controller can be designed with the backstepping technology [14]. The design process is:

Step 1.

From the dynamic equation of state x_1 in the TORA system model (5):

$$\ddot{x}_1 = -\frac{k}{m_1 + m_2}x_1 + \frac{km_2r}{(m_1 + m_2)^2}\sin x_2. \tag{6}$$

It can be seen that only x_2 is in the dynamic equation (6) of state x_1. So x_2 can be looked as the virtual control input to Equation (6) to control the system state x_1 and $\dot{x}_1$. A reference trajectory x_{2r} is defined for x_2 to track as:

$$x_{2r} = \frac{k}{m_1 + m_2}x_1 - \frac{km_2r}{(m_1 + m_2)^2}\sin x_2 + x_2 - k_1x_1 - k_2\dot{x}_1,$$

which leads to an error e_2 defined as:

$$e_2 = x_2 - x_{2r} = -\frac{k}{m_1 + m_2}x_1 + \frac{km_2r}{(m_1 + m_2)^2}\sin x_2 + k_1x_1 + k_2\dot{x}_1, \tag{7}$$

where, k_1 and k_2 are positive constants that are chosen such that:

$$\ddot{x}_1 + k_2\dot{x}_1 + k_1x_1 = 0,$$

is asymptotically stable at $(x_1, \dot{x}_1) = (0,0)$.

The dynamic Equation (6) of state x_1 can be described as:

$$\begin{bmatrix} \dot{x}_1 \\ \ddot{x}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ \dot{x}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}e_2. \tag{8}$$

Define $X_1 = \begin{bmatrix} x_1 \\ \dot{x}_1 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So the Equation (8) is written as:

$$\dot{X}_1 = AX_1 + Be_2.$$

According to the control theory for the linear time-invariant system [15], the choose of k_1 and k_2 can implies the existence of a positive definite symmetry matrix P and Q so that:

$$A^TP + PA = -Q.$$

Consider a scalar positive definite Lyapunov function given by:

$$V_1 = \frac{1}{2}(X_1^TPX_1 + e_2^2) \geq 0.$$

Its time derivative $\dot{V}_1$ is given by:

$$
\begin{aligned}
\dot{V}_1 &= -\frac{1}{2}X_1^T Q X_1 + e_2(\dot{e}_2 + X_1^T PB) \\
&= -\frac{1}{2}X_1^T Q X_1 + e_2\left(-\frac{k}{m_1+m_2}\dot{x}_1 + \frac{km_2 r\dot{x}_2}{(m_1+m_2)^2}\cos x_2 + k_1\dot{x}_1 + k_2\ddot{x}_1 + X_1^T PB\right),
\end{aligned}
\tag{9}
$$

We note that the variable $\dot{x}_2$ enters the right hand side of Equation (9). We now proceed to look $\dot{x}_2$ as the control variable and design a reference trajectory $\dot{x}_{2r}$ for it to make the second term of right hand in Equation (9) be non-positive.

Step 2.

In step 1, the time derivative of the Lyapunov function V_1 is obtained in Equation (9). In order to make the $\dot{V}_1$ be negative definite, the state $\dot{x}_2$ is looked as the virtual control input in Equation (9). A reference trajectory $\dot{x}_{2r}$ is defined as:

$$
\dot{x}_{2r} = \frac{k}{m_1+m_2}\dot{x}_1 - \frac{km_2 r\dot{x}_2}{(m_1+m_2)^2}\cos x_2 - k_1\dot{x}_1 - k_2\ddot{x}_1 - X_1^T PB + \dot{x}_2 - k_3 e_2.
$$

The tracking error $\hat{e}_2$ for $\dot{x}_2$ can be defined as:

$$
\hat{e}_2 = \dot{x}_2 - \dot{x}_{2r} = -\frac{k}{m_1+m_2}\dot{x}_1 + \frac{km_2 r\dot{x}_2}{(m_1+m_2)^2}\cos x_2 + k_1\dot{x}_1 + k_2\ddot{x}_1 + X_1^T PB + k_3 e_2.
\tag{10}
$$

So:

$$
\dot{V}_1 = -\frac{1}{2}X_1^T Q X_1 - k_3 e_2^2 + e_2\hat{e}_2.
$$

We modify the scalar positive Lyapunov function $V_1 \geq 0$ as:

$$
V_2 = V_1 + \frac{1}{2}\hat{e}_2^2 \geq 0.
$$

Differentiating V_2:

$$
\begin{aligned}
\dot{V}_2 &= -\frac{1}{2}X_1^T Q X_1 - k_3 e_2^2 + \hat{e}_2(e_2 + \dot{\hat{e}}_2) = -\frac{1}{2}X_1^T Q X_1 - k_3 e_2^2 \\
&+ \hat{e}_2\left(e_2 - \frac{k}{m_1+m_2}\ddot{x}_1 + \frac{km_2 r\ddot{x}_2}{(m_1+m_2)^2}\cos x_2 - \frac{km_2 r\dot{x}_2^2}{(m_1+m_2)^2}\sin x_2 + k_1\ddot{x}_1 \right. \\
&\left. - \frac{kk_2}{m_1+m_2}\dot{x}_1 + \frac{k_2 km_2 r\dot{x}_2}{(m_1+m_2)^2}\cos x_2 + X_1^T PB + k_3\dot{e}_2\right).
\end{aligned}
\tag{11}
$$

Step 3.

The system control variable $u = \ddot{x}_2$ enters in the right hand of the Equation (11). So, In order to make the $\dot{V}_2$ be negative definite, we can make the following equation hold:

$$
\begin{aligned}
& e_2 - \frac{k}{m_1+m_2}\ddot{x}_1 + \frac{km_2 r\ddot{x}_2}{(m_1+m_2)^2}\cos x_2 - \frac{km_2 r\dot{x}_2^2}{(m_1+m_2)^2}\sin x_2 + k_1\ddot{x}_1 - \frac{kk_2}{m_1+m_2}\dot{x}_1 \\
& + \frac{k_2 km_2 r\dot{x}_2}{(m_1+m_2)^2}\cos x_2 + X_1^T PB + k_3\dot{e}_2 = -k_4\hat{e}_2.
\end{aligned}
\tag{12}
$$

Such that:

$$\dot{V}_2 = -\frac{1}{2}X_1^T Q X_1 - k_3 e_2^2 - k_4 \hat{e}_2^2 \le 0. \tag{13}$$

Therefore, the control law can be obtained from (12) as:

$$u = -\frac{(m_1+m_2)^2}{km_2 r \cos x_2}\left\{e_2 - \frac{k}{m_1+m_2}\ddot{x}_1 - \frac{km_2 r \dot{x}_2^2}{(m_1+m_2)^2}\sin x_2 + k_1\ddot{x}_1 + \frac{k_2 km_2 r \dot{x}_2}{(m_1+m_2)^2}\cos x_2 \right.$$
$$\left. -\frac{kk_2}{m_1+m_2}\dot{x}_1 + \dot{X}_1^T PB + k_3\dot{e}_2 + k_4\hat{e}_2\right\}. \tag{14}$$

Theorem 1: The TORA system described by Equation (1) is asymptotically stable under the control input (2) and the nonlinear controller (14) with the feedback cascade model (5) and the global change of coordinates (3).

Proof:

The backstepping design process has proved: the time derivative of the chosen positive definite Lyapunov function V_2 is negative definite. That is to say, the three terms of the right hand in Equation (13) is asymptotically approach to 0. Since that the first term $-\frac{1}{2}\bar{x}^T Q\bar{x}$ approaches to 0, $(x_1,\dot{x}_1)$ must asymptotically approach to $(0,0)$. From the second term $-k_3 e_2^2$ approaches to 0, it is known from Equation (7) that x_2 must asymptotically approach to 0. The third term $-k_4\hat{e}_2^2$ approaches to 0 implies that $\dot{x}_2$ asymptotically approach to 0. Therefore, the system states $(x_1,\dot{x}_1,x_2,\dot{x}_2)$ of the TORA described by Equation (5) asymptotically approach to (0,0,0,0).

The global change of coordinates (3) is an invertible transformation, which is:

$$\begin{cases} q_1 = x_1 - \dfrac{m_2 r \sin x_2}{m_1+m_2}, \\ \dot{q}_1 = \dot{x}_1 - \dfrac{m_2 r \cos x_2}{m_1+m_2}\dot{x}_2, \\ q_2 = x_2, \\ \dot{q}_2 = \dot{x}_2. \end{cases} \tag{15}$$

It can be seen from Equation (15) that $(x_1,\dot{x}_1,x_2,\dot{x}_2)$ asymptotically approach to (0,0,0,0) implies that $(q_1,\dot{q}_1,q_2,\dot{q}_2)$ approach to (0,0,0,0). That is to say, the control input τ that is calculated with Equation (2), (14) and (4) can asymptotically stabilize the TORA system.

Remark 1: There is a singularity when $x_2 = \pi/2$ in the controller (14). The method to deal with the singularity is: a linear controller is designed through the LQR method [15] with the linearized system around $x_2 \in (\pi/2 - \Delta, \pi/2 + \Delta)$ for the TORA system:

$$u = Kx. \tag{16}$$

The controller switch between the nonlinear controller (14) and the linear controller (16) according to the system state x_2.
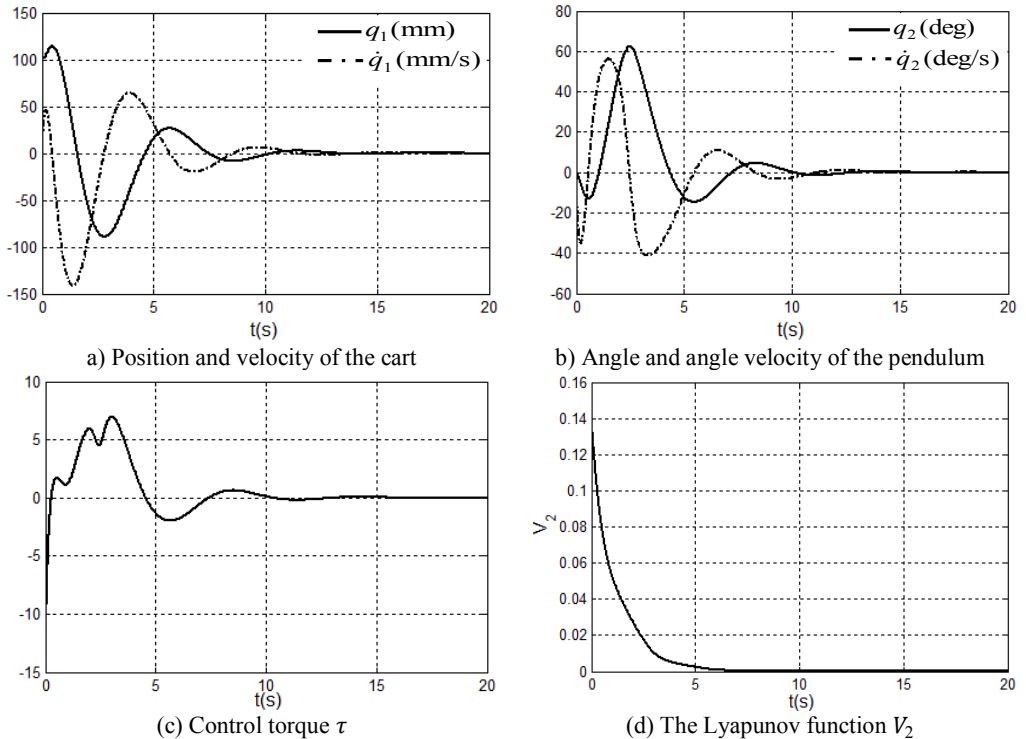
Remark 2: Only three steps are needed to obtain the nonlinear controller with the backstepping technology. The first step is designed for a one-input and two-state system, the second step is designed for a one-input and on state system, and the third step is to obtain the controller. The design process and the proposed controller are simple compared with some the literature [5-6].

4. Simulation studies

In order to test the proposed nonlinear control algorithm, the following system parameters [16]

are used: $m_1 = 10$, $m_2 = 1$, $k = 1$, $r = 1$, $I = 1$. The parameters of the nonlinear controller are chosen as $P = [6.5 \quad 1; 1 \quad 0.3]$, $Q = [10 \quad 0; 0 \quad 1]$, $K = [-2 \quad -2 \quad 8 \quad 4]$, $k_1 = 5$, $k_2 = 5$, $k_3 = 0.5$, $k_4 = 0.5$.
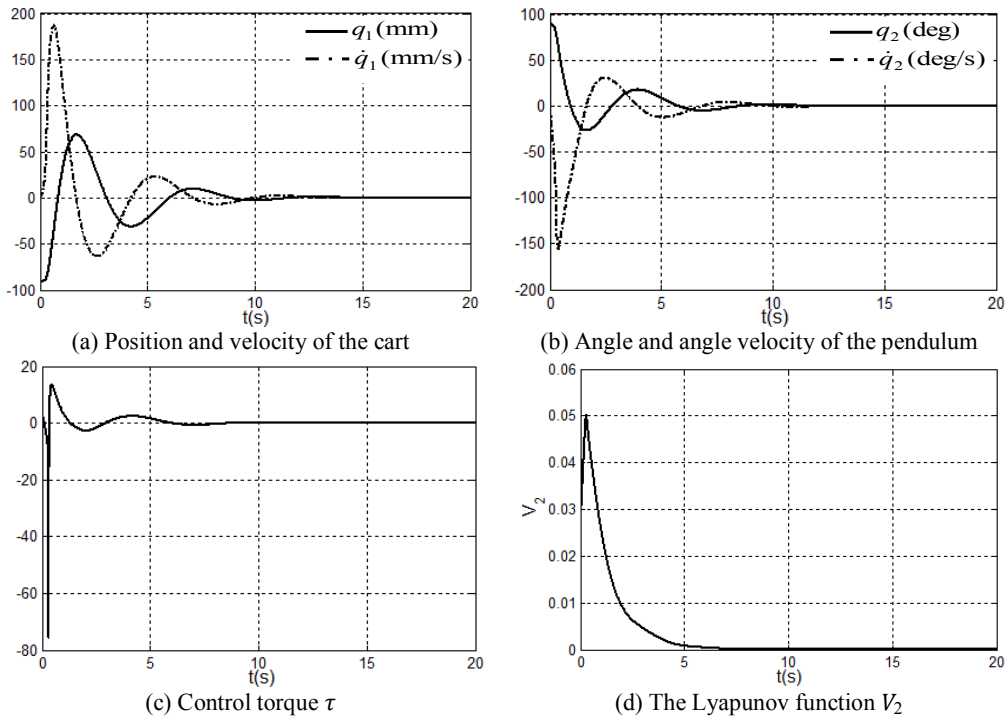
The simulation results are shown in Fig. 2-4, where the position unit and angle unit respectively are meter and radian to calculate the V_2. The Fig. 2 is obtained under the initial state $(x_1, \dot{x}_1, x_2, \dot{x}_2) = (100,0,0,0)$ i.e. $(q_1, \dot{q}_1, q_2, \dot{q}_2) = (100,0,0,0)$, Fig. 3 is obtained under the initial state $(x_1, \dot{x}_1, x_2, \dot{x}_2) = (0,0,90°,0)$ i.e. $(q_1, \dot{q}_1, q_2, \dot{q}_2) = (-90.9,0,90°,0)$ and Fig. 4 are obtained under the initial state $(x_1, \dot{x}_1, x_2, \dot{x}_2) = (100,0,90°,0)$ i.e. $(q_1, \dot{q}_1, q_2, \dot{q}_2) = (9.09, 90°,0,0)$.

a) Position and velocity of the cart

b) Angle and angle velocity of the pendulum
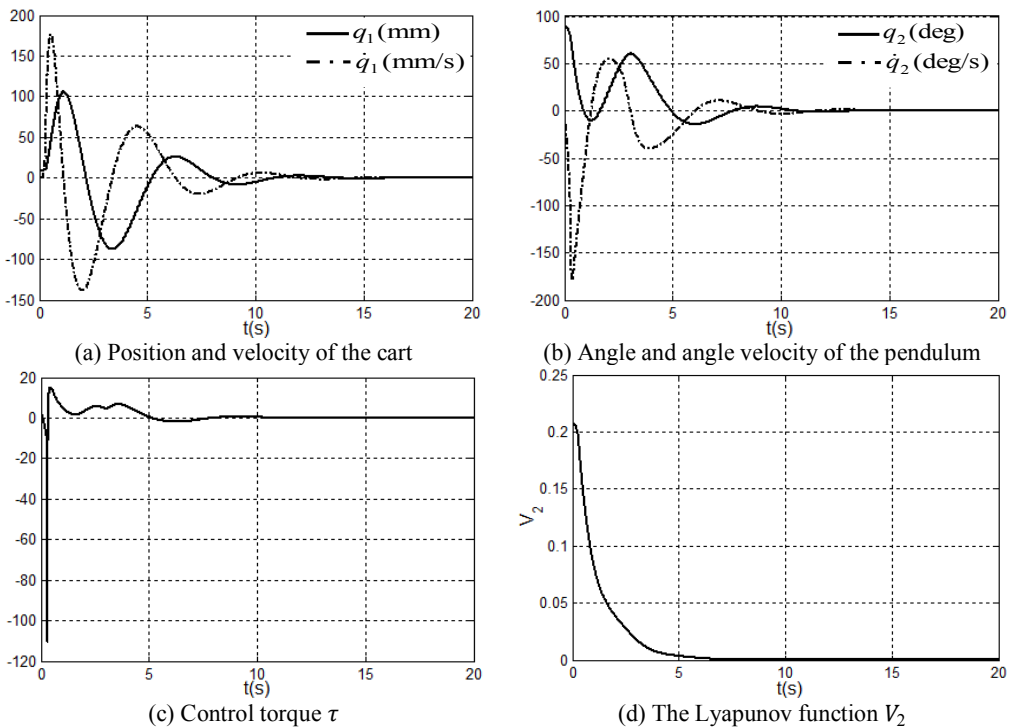
(c) Control torque τ

(d) The Lyapunov function V_2

Fig. 2. Simulation results under system state $(q_1, \dot{q}_1, q_2, \dot{q}_2) = (100,0,0,0)$ i.e. $(x_1, \dot{x}_1, x_2, \dot{x}_2) = (100,0,0,0)$

It can be seen form the simulation results that the TORA system is asymptotically stable under any initial states with the proposed control algorithm. At the same time, it is shown that it is possible to suppress the translational vibration with a rotational actuation. On the other hand, the control performance can be improved through adjusting the parameters of the proposed controller.

From Fig. 3(d), the Lyapunov function is not a monotonically decreasing function. The reason is the controller switches between the nonlinear controller (14) and the linear controller (16). When the linear controller is in use, the Lyapunov function may be not a monotonically decreasing function. Lots of simulation experiments show that the parameters k_1, k_2, k_3, k_4 respectively correspond to the system states x_1, $\dot{x}_1$, x_2, $\dot{x}_2$ therefore it is easy to adjust the parameters for a improved system performance.

(a) Position and velocity of the cart

(b) Angle and angle velocity of the pendulum

(c) Control torque τ

(d) The Lyapunov function V_2

Fig. 3. Simulation results under system state $(q_1, \dot{q}_1, q_2, \dot{q}_2) = (-90.9, 90°, 0, 0)$
i.e. $(x_1, \dot{x}_1, x_2, \dot{x}_2) = (0, 0, 90°, 0)$

(a) Position and velocity of the cart

(b) Angle and angle velocity of the pendulum

(c) Control torque τ

(d) The Lyapunov function V_2

Fig. 4. Simulation results under system state $(q_1, \dot{q}_1, q_2, \dot{q}_2) = (9.09, 90°, 0, 0)$
i.e. $(x_1, \dot{x}_1, x_2, \dot{x}_2) = (100, 0, 90°, 0)$

5. Conclusions

The underactuated TORA is transformed to a nonlinear feedback cascade system with a collocated partial feedback linearization and a global change of coordinates. A nonlinear control algorithm is proposed from the backstepping technology. The design procedure treats the state variables as virtual control inputs to design the virtual controllers step by step. The first step is designed for a one-input and two-state system, the second step is designed for a one-input and one-state system, and the third step is to obtain the nonlinear controller. The system stability and the control performance are proved via the Lyapunov stability theorem or simulation results.

Acknowledgements

References

[1] **Robert T. B., Bernstein D. S., Coppola V. T.** A benchmark problem for nonlinear control design. International Journal of Robust and Nonlinear Control, Vol. 8, Issue 4-5, 1998, p. 307-310.

[2] **Hung L. C., Lin H. P., Chung H. Y.** Design of self-tuning fuzzy sliding mode control for TORA system. Expert Systems with Applications, Vol. 32, Issue 1, 2007, p. 201-212.

[3] **Li C. D., Yi J. Q., Zhao D. B.** Control of the TORA system using SIRMs based type-2 fuzzy logic. Proceedings of IEEE International Conference on Fuzzy System, 2009, p. 694-699.

[4] **Lee C. H., Yu T. M., Chien J. C.** Adaptive neural network controller design for a class of nonlinear systems using SPSA algorithm. Proceedings of the International Multi Conference on Engineers and Computer Scientists, Vol. 2, 2011, p. 967-974.

[5] **Avis J. M., Nersesov S. G., Nathanb R., et al.** A comparison study of nonlinear control techniques for the RTAC system. Nonlinear Analysis: Real World Applications, Vol. 11, Issue 4, 2010, p. 2647-2658.

[6] **Jankovic M., Fontaine D., Kokotovic P. V.** TORA example: cascade and passivity based control designs. IEEE Trans. Control Syst. Technol., Vol. 4, Issue 3, 1996, p. 292-297.

[7] **Gao B. T., Zhang X. H., Chen H. J., et al.** Energy-based control design of an underactuated 2-dimensional TORA system. Proceeding of IROS'09 Proceedings of the 2009 IEEE/RSJ International Conference on Intelligent Robots and Systems, 2009, p. 1296-1301.

[8] **Bupp R., Bernstein D. S., Coppola V. T.** Experimental implementation of integrator backstepping and passive nonlinear controllers on the RTAC testbed. International Journal of Robust and Nonlinear Control, Vol. 8, Issue 4-5, 1998, p. 435-457.

[9] **Wang Y. Q., Li S., Chen Q. W.** A Recursive design method of controller for the stabilization of the translational oscillator with a rotational actuator system. Proceedings of the 2011 IEEE International Conference on Computer Science and Automation Engineering, Shanghai, USA, p. 558-561.

[10] **Chen Y. F., Huang A. C.** Controller design for a class of underactuated mechanical systems. IET Control Theory Application, Vol. 6, Issue 1, 2012, p. 103-110.

[11] **Celani F.** Output regulation for the TORA benchmark via rotational position feedback. Automatica, Vol. 47, Issue 3, 2011, p. 584-590.

[12] **Spong M. W.** Underactuated mechanical systems. Control Problems in Robotics and Automation, Springer Verlag, London, UK, 1997.

[13] **Olfati-Saber R.** Normal forms for underactuated mechanical systems with symmetry. IEEE Transactions on Atomatic Control, Vol. 47, Issue 2, 2002, p. 305-308.

[14] **Khalil H. K.** Nonlinear systems. 3rd Edition. Prentise Hall, USA, 2002.

[15] **Katsuhiko Ogata** Modern control engineering. Prentice Hall, 2009.

[16] **Olfati-Saber R.** Nonlinear control of underactuated mechanical systems with application to robotics and aerospace vehicles. Ph. D. thesis, Massachusetts Institute of Technology, 2001.