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Nonlinear Characteristics of Pulse-Duration Modulation

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Abstract

A comparison of pulse-duration demodulation with pulse-code demodulation indicates that there is a pulse-code modulation (PCM) system that will outperform any pulse-duration modulation (PDM) system by about 2½ db input power at 20 db output signal-to-noise ratio, 4½ db at 30 db, and 7 db at 40 db. However, the PDM systems outperform the PCM systems at a sufficiently high input signal-to-noise density ratio. Pulse-duration modulation thus tends to make better use of high signal strengths, while pulse-code modulation makes better use of low signal strengths.

Nonlinear Characteristics of Pulse-Duration Modulation

I. Introduction

The most widely known analyses of pulse-duration demodulation are the linear analysis of A. J. Viterbi (Ref. 1) and the nonlinear analysis of M. H. Nichols and A. T. Bublitz (Ref. 2). The present report expands on these studies by producing new results on the nonlinear behavior of pulse-duration demodulation both above and below threshold. Although the nonlinear demodulator of Nichols and Bublitz exhibits better performance than the linear demodulator of Viterbi, its threshold behavior is much poorer than that assumed by Nichols and Bublitz.

A comparison of pulse-duration with pulse-code demodulation indicates that for any given output signal-to-noise ratio there is a PCM system that will outperform any PDM system by about 2½ db input power at 20 db output signal-to-noise ratio, 4½ db at 30 db, and 7 db at at 40 db. Considerably more bandwidth is required for a PDM system than for an equivalent PCM system. On the other hand, every PDM system will outperform every PCM system at a sufficiently high ratio of input signal energy to noise density; the PDM system excels in performance over a comparable PCM system at almost all power levels except the lower ones. Pulse-duration modulation thus tends to make better use of high signal strengths, while pulse-code modulation makes better use of low signal strengths.

II. Pulse-Duration Modulation

Figure 1 shows a method of representing the sample a by pulse-duration modulation. This type of modulation has the capability of continuously representing samples

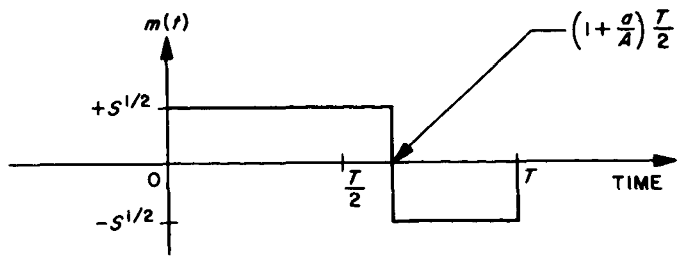

Fig. 1. Pulse-duration modulation representing the sample a by the reversal of polarity at the time $(1 + a/A)(T/2)$

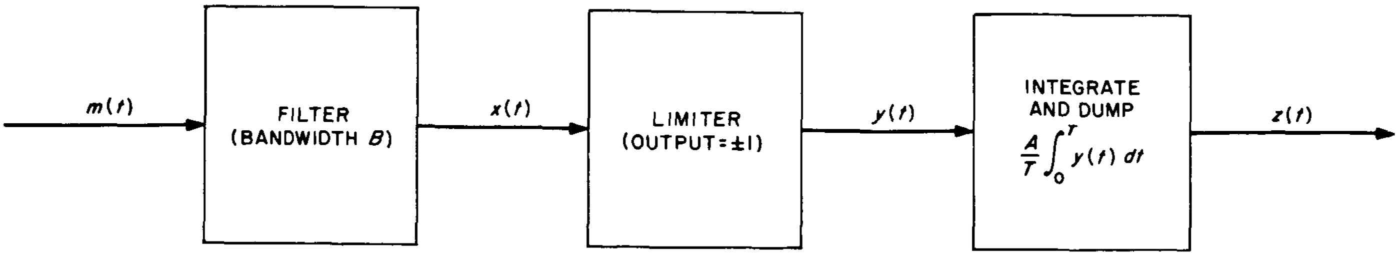

Fig. 2. The demodulation of a PDM waveform, using a filter and a limiter to reduce the effects of noise

ranging in value from $-A$ to $+A$. The duration of the positive pulse is directly proportional to the value of the sample a. Demodulation of pulse-duration modulation thus requires synchronization to establish the zero time reference and isolation to separate the waveforms representing different samples. The various methods available for providing the synchronization and isolation will not be discussed in this report.

The demodulation of the waveform of Fig. 1 is performed by filtering with a bandwidth B (cycles per second) equal to twice the low-frequency bandwidth B_L, symmetrically limiting, and integrating the resultant waveform as in Fig. 2.

Under noiseless conditions, the output of the integrate-and-dump circuit would be the integral of the pulse-duration waveform scaled to produce the sample a:

$$z(T) = \frac{A}{T}\int_0^T y(t)\,dt$$

$$= \frac{A}{T}\left\{+\left[\left(1+\frac{a}{A}\right)\frac{T}{2}\right] - \left[T - \left(1+\frac{a}{A}\right)\frac{T}{2}\right]\right\}$$

$$= a$$

Where the sample values are uniformly distributed between $+A$ and $-A$, the output power is

$$S_o = \int_{-A}^{+A} \frac{1}{2A} a^2\,da = \frac{A^2}{3}$$

III. Noise Reduction

A limited amount of noise can be removed from the pulse-duration waveform by first filtering the waveform and then limiting it to recover the transition point. Viterbi's analysis did not use the limiter that had been previously proposed by Nichols and Bublitz.

A spectral density of noise power N/B for frequencies from $-B/2$ to $+B/2$ corresponds to a high-frequency noise power N in a bandwidth B that has been incoherently demodulated. Coherent demodulation (as used by Viterbi) would lead to a spectral density of $N/(2B)$.

The filtering of the pulse-duration waveform distorts the transition so that when noise $n(t)$ is added to the waveform at the transition time, the transition moves by a value of t_n approximately equal to the value of the noise divided by the slope $2S^{1/2}/\tau$ of the transition as in Fig. 3.

The fall time τ is related to the double-sided bandwidth B in cycles per second by the relation (Ref. 3)

$$\tau = \frac{1}{B}$$

which also corresponds to the sampling theorem (Ref. 4).

The variation t_n of the transition time, owing to noise, produces an additive noise z_n at the output of the integration:

$$z_n = 2A\,\frac{t_n}{T}$$

$$= \frac{An(t)}{S^{1/2}(TB)}$$

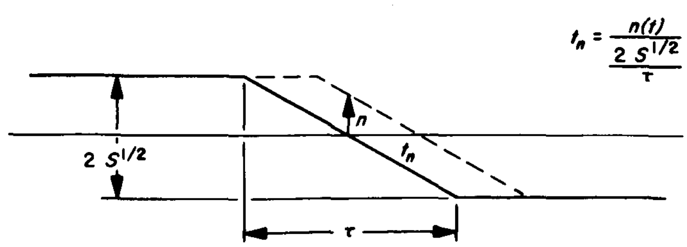

Fig. 3. Zero crossing of a PDM waveform that has been filtered with a bandwidth $B = 1/\tau$

The average noise power, because of variations in the transition time, is thus

$$\overline{z_n^2} = \frac{A^2 N}{S(TB)^2}$$

which produces an output signal-to-noise power ratio

$$\frac{S_o}{N_o} = \frac{S(TB)^2}{3N} = \frac{1}{3}\frac{ST}{\frac{N}{B}}k = \frac{1}{3}\frac{S}{N}k^2$$

where $k = TB = T/\tau$ is the number of fall times τ in the interval T. This relation is valid only for increasingly higher input signal-to-noise ratios as the index k is made larger (7¾ db for $k=4$; 9¼ db for $k=16$; and 10¾ db for $k=64$). Two other noise components become important as the input signal-to-noise ratio deteriorates. The first noise component is due to the breaking through of the limiter at times other than the transition time. The second noise component, which is found at still lower signal-to-noise levels, is bias distortion in the integration of the pulse-duration waveform and is also caused by the breakthrough of the limiter.

IV. Sampling

The use of the sampling theorem (Ref. 4) provides a means of characterizing the waveform $x(t)$ at the output of the filter of bandwidth B in Fig. 2. The waveform $x(t)$ can be sufficiently described by $k = TB$ equally spaced samples in the time interval T. Each sample is plus or minus the signal amplitude $S^{1/2}$ plus noise of variance N resulting from a spectral density of N/B being passed through the filter of bandwidth B.

The output of the limiter is then a sequence of equally spaced samples of $+1$ and -1, depending on the polarity and magnitude of the signal and noise.

In order to perform a number of summations, the sample values of $x(t)$ are assumed to occur on the odd integers between $+2m$ and $-2m$. The actual transition time is assumed to occur at a time h belonging to the even integers. Thus, in the noiseless case, the limited waveform $y(t)$ would be $+1$ at sample times $t < h$ and -1 at sample times $t > h$:

$$y(t) = \begin{cases} +1, & t < h \quad \text{at } m + \frac{h}{2} \text{ samples} \\ -1, & t > h \quad \text{at } m - \frac{h}{2} \text{ samples} \end{cases}$$

Assuming a fixed transition time h, integration of the limited waveform $y(t)$ is then the summation

$$\begin{aligned} z(t) &= \frac{A}{2m}\sum_{t=-2m}^{+2m} y(t) \\ &= \frac{A}{2m}\left[\left(m + \frac{h}{2}\right) - \left(m - \frac{h}{2}\right)\right] \\ &= \frac{Ah}{2m} \end{aligned}$$

which is the transition time h scaled by $A/(2m)$.

The variance of the output of the integrator in the noiseless case is then

$$\begin{aligned} S &= \frac{1}{2m+1}\sum_{\substack{h=-2m \\ \text{even}}}^{+2m}\left(\frac{Ah}{2m}\right)^2 \\ &= \frac{A^2}{3}\frac{m+1}{m} = \frac{A^2}{3}\frac{k+2}{k} \approx \frac{A^2}{3} \end{aligned}$$

where the transition times are assumed to be equally distributed over the $2m+1$ possible transition times, $k = TB = 2m$, and

$$\sum_{\substack{h=-2m \\ \text{even}}}^{+2m}\left(\frac{h}{2}\right)^2 = \frac{m(m+1)(2m+1)}{3}$$

V. Limiter Noise

For Gaussian noise, the limiter in Fig. 2 reverses its polarity because of noise with probability p_1:

$$p_1 = \frac{1}{(2\pi)^{1/2}}\int_{(S/N)^{1/2}}^{\infty}\exp\left(-\frac{u^2}{2}\right)du$$

The polarity remains unchanged with probability $p_0 = 1 - p_1$.

The sample output of the limiter $y(t)$ for times before the transition time h is then $+1$ with probability p_0 and -1 with probability p_1. For times after the transition time h, these probabilities are reversed:

$$P\{y(t) = +1\} = \begin{cases} p_0, & t < h \\ p_1, & t > h \end{cases}$$

$$P\{y(t) = -1\} = \begin{cases} p_0, & t < h \\ p_1, & t > h \end{cases}$$

VI. Summation Noise

The noise at the output of the summation with zero modulation ($h = 0$) due to breaking through the limiter is then

$$N_s = \overline{\left[\frac{A}{2m} \sum_{t=2m}^{2m} x(t)\right]^2}$$

$$= \frac{A^2}{4m^2} \sum_{t=-2m}^{+2m} \sum_{s=-2m}^{+2m} \overline{x(t)\, x(s)}$$

$$= \frac{A^2}{4m^2}\left[2m + 2(m^2 - m)(p_0 - p_1)^2 - 2m^2(p_0 - p_1)^2\right]$$

$$= \frac{2A^2 p_1 p_0}{m} = \frac{4A^2 p_1 p_0}{k}$$

The first term is the sum of the diagonal terms; the second term is the sum of the two principal minors, less the diagonal; and the third term is the sum of the two off-diagonal minors.

VII. Bias Errors

The error in the summation

$$\frac{A}{2m} \sum_{t=-2m}^{+2m} y(t)$$

is the difference

$$\text{error} = \left[\frac{A}{2m} \sum_{t=-2m}^{+2m} y(t)\right] - \frac{Ah}{2m}$$

which has the nonzero mean value

$$\overline{\text{error}} = \left[\frac{A}{2m} \sum_{-2m}^{+2m} \overline{y(t)}\right] - \frac{Ah}{2m}$$

$$= \frac{A}{2m}\left[(p_0 - p_1)\left(m + \frac{h}{2}\right) - (p_0 - p_1)\left(m - \frac{h}{2}\right)\right] - \frac{Ah}{2m}$$

$$= -\frac{Ah\, p_1}{m}$$

Thus for any particular transition time h, the integration produces a bias error of $(-Ahp_1)/m$. The average squared value of the bias error is then

$$N_b = \frac{1}{2m+1} \sum_{\substack{h=-2m \\ \text{even}}}^{+2m} \left(\frac{Ahp_1}{m}\right)^2 = \frac{4}{3} A^2 p_1^2 \frac{m+1}{m} = \frac{4}{3} A^2 p_1^2 \frac{k+2}{k} \approx \frac{4}{3} a^2 p_1^2$$

VIII. Error Noise

The average squared error is the sum of the summation noise N_s and the bias error noise N_b. This can be obtained directly by finding the average squared error N_h for a fixed transition time h and then averaging over the possible transition times:

$$N_h = \overline{\left[\frac{1}{2m}\sum_{-2m}^{+2m} y(t) - \frac{h}{2m}\right]^2} = \frac{1}{4m^2}\left[\sum_{t=-2m}^{+2m}\sum_{s=-2m}^{+2m}\overline{y(t)\,y(s)} - h\sum_{-2m}^{+2m}\overline{y(t)} + h^2\right]$$

After the averages $\overline{y(t)y(s)}$ and $\overline{y(t)}$ are evaluated, the average squared error is

$$N_h = \frac{2A^2p_1p_0}{m} + \left(\frac{Ahp_1}{m}\right)^2$$

where the first term is the summation noise N_s and the second term is the square of the bias error. Further averaging of the transition times h produces the result

$$N = N_s + N_b$$

where

$$N_s = \frac{2A^2p_1p_2}{m} = \frac{4A^2p_1p_0}{k}$$

$$N_b = \frac{4}{3}A^2p_1\frac{2m+1}{m} \approx \frac{4}{3}A^2p_1^2$$

IX. Thresholds

Above threshold, the only noise in a PDM system is the noise $(A^2N)/(Sk^2)$ due to a slight displacement in the transit time. However, at threshold, the noise begins to change the polarity of the output of the limiter and thus produces an increase in the output noise resulting from the noisy summation. This term varies essentially with the first power of the breakthrough probability p_1 and has a maximum value of A^2/k. Further, below threshold, the bias error predominates by adding in the term $(4/3)\,A^2p_1^2$ varying with the square of the breakthrough probability p_1 and having a maximum value of $A^2(4/3)$.

The total average squared noise is then essentially

$$N = \frac{A^2N}{Sk^2} + \frac{4A^2p_1p_0}{k} + \frac{4}{3}A^2p_1^2$$

as compared with the average squared signal

$$S = \frac{A^2}{3}$$

The output signal-to-noise ratio is then

$$\frac{S_o}{N_o} = \frac{1}{\frac{3N}{Sk^2} + \frac{12p_1p_0}{k} + 4p_1^2}$$

Figure 4 is a plot of the output signal-to-noise ratio as a function of

$$u = \frac{ST}{\frac{N}{B}} = \frac{S}{N}k$$

for various values of k. Viterbi's linear result (using non-coherent demodulation to baseband)

$$\frac{S_o}{N_o} = \frac{1}{3}\frac{ST}{\frac{N}{B}}$$

is obtained for $k = 1$ and is considerably less than that obtained with nonlinear demodulation.

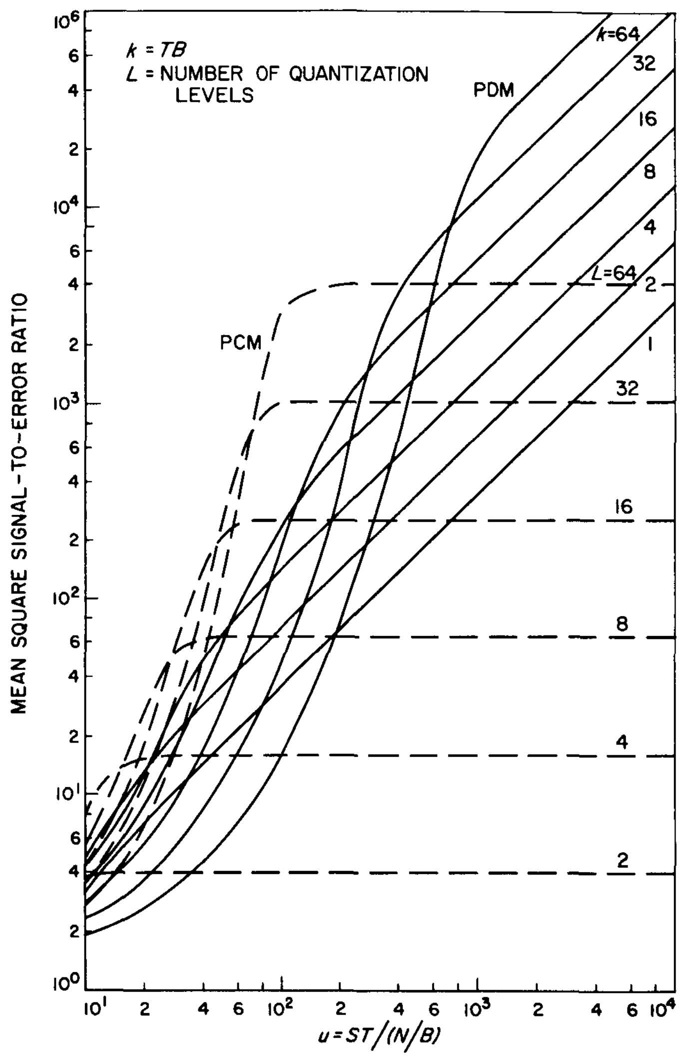

Fig. 4. A comparison of pulse-duration modulation and pulse-code modulation

The 3-db deterioration points (sometimes defined as threshold) occur at higher and higher input signal-to-noise ratios for large time–bandwidth products (8½ db for $k=8$, 9¼ db for $k=16$, 10 db for $k=32$, and 10¾ for $k=64$). For a fixed time interval T and a coefficient k, the bandwidth required is k/t:

$$B_{PDM} = \frac{k}{T}$$

X. Pulse-Code Modulation Comparison

As a means of evaluating the effectiveness of PDM, it is compared with PCM, which is known to have an output signal-to-noise ratio of

$$\frac{S_o}{N_o} = \frac{L^2}{1 + 4L^2 p}$$

where L is the number of quantization levels and p is the probability of binary error:

$$p = \frac{1}{(2\pi)^{1/2}} \int^{\infty} \exp\left(\frac{u^2}{2}\right) du$$

where the lower integration limit is

$$\left(\frac{ST}{\frac{N}{B}\log_2 L}\right)$$

This result is obtained by noting that the average squared signal uniformly distributed over $\pm A$ is

$$S_o = \frac{A^2}{3}$$

The average squared quantization error N_q resulting from L quantization intervals is similarly

$$N_q = \frac{A^2}{3L^2}$$

False detection of the most significant binary digit would result in an error of magnitude A. False detection of the next most significant digit would result in an error of magnitude $A/2$, etc. The average squared false detection error is then

$$N_d = p(A)^2 + p\left(\frac{A}{2}\right)^2 + p\left(\frac{A}{4}\right)^2 + \cdots$$

$$= pA^2 \sum_{i=0} \frac{1}{2^{2i}} = \frac{4}{3} pA^2$$

where p is the probability of false detection of a single binary digit.

The output signal-to-noise ratio then follows immediately when the output noise is set equal to the sum of the quantization noise plus the false detection noise.

The integrate-and-dump circuit used in the detection of the binary digits has an effective bandwidth B_e of $1/t$, where t is the integration time:

$$B_e = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin \omega \frac{t}{2}}{\omega \frac{t}{2}} \right)^2 d\omega = \frac{1}{t}$$

where the linear transfer function of the integrate-and-dump circuit is

$$H(\omega) = \exp\left(-j\omega \frac{t}{2} \right) \frac{\sin \omega \frac{t}{2}}{\omega \frac{t}{2}}$$

The variance of the detection noise is then

$$N_e = \frac{1}{t} \frac{N}{B}$$

where the baseband spectral density is N/B.

False detection occurs where the noise changes the polarity of the output digit. For Gaussian noise, this occurs with probability

$$p = \frac{1}{(2\pi)^{1/2}} \int^{\infty} \exp\left(\frac{u^2}{2}\right) du$$

where the lower integration limit is

$$\left(\frac{St}{\frac{N}{B}} \right)^{1/2}$$

assuming that each binary digit uses an equal portion of the interval T for detection,

$$t = \frac{T}{\log_2 L}$$

The output signal-to-noise ratio for PCM is plotted in Fig. 4 for various quantization levels. The double-sided bandwidth in cycles per second required for a PCM system is

$$B_{PCM} = \frac{\log_2 L}{T}$$

as compared with the bandwidth for the PDM system

$$B_{PDM} = \frac{k}{T}$$

The wider bandwidth of PDM results in an increasing probability of error, which is partly offset by the PDM system's reducing the noise by averaging. However, the smaller noise bandwidth of the PCM demodulation predominates in an exponential manner to produce a better performance at any fixed output signal-to-noise ratio.

Over wide ranges of input signal to noise, the PDM system makes better use of the higher signal-to-noise ratios and thus can outperform any fixed PCM system.

XI. Optimum Demodulation of Pulse-Duration Modulation

A more sophisticated approach to the demodulation of pulse-duration modulation is the use of the optimum solution for minimum mean-square error. This solution is highly nonlinear and at present is not in a form that can be constructed readily.

The optimum mean-square filter requires that the output $z(T)$ of the filter be the conditional average

$$z(T) = \int_{-A}^{+A} ap(a \mid m)\, da$$

where m is the input waveform, a is the data sample, and $p(a \mid m)$ is the conditional probability of the data sample for the given input waveform m. Where the data sample is uniformly distributed, this expression can be written in the more computable form

$$z(T) = \frac{\int_{-A}^{+A} ap(m \mid a)\, da}{\int_{-A}^{+A} p(m \mid a)\, da}$$

where $p(m \mid a)$ is the conditional probability of input waveform m, given data sample a. The denominator of this expression normalizes the averaging process. The averaging process of the numerator is based on the probability of the waveform m resulting from a particular data sample a. The data associated with small values of $p(m \mid a)$ are ignored, while the other data are averaged in relation to their probability of occurrence. A close approximation to this type of demodulation is the use of a limiter followed by an integrate-and-dump circuit.

Another approximation can be obtained by assuming a discrete model where the input waveform is sampled on an odd integer base between $+2m$ and $-2m$, and where the transition time h is assumed to occur at an even integer. The pulse-duration signal $S(t)$ has the representation

$$S_h(t) = \begin{cases} S^{1/2}, & t < h \\ S^{1/2}, & t > h \end{cases}$$

On the assumption of additive Gaussian noise with distribution

$$p(n) = \frac{1}{(2\pi)^m \mid \phi \mid^{1/2}} \exp\left[\frac{1}{2 \mid \phi \mid} \sum_{t=-2m}^{+2m} \sum_{u=-2m}^{+2m} \left| \phi \right|_{tu} n(t)n(u)\right]$$

the conditional distributions become

$$p(m \mid h) = \frac{1}{(2\pi)^m \mid \phi \mid^{1/2}} \exp\left\{\frac{1}{2 \mid \phi \mid} \sum_{t=-2m}^{2m} \sum_{u=-2m}^{+2m} \left| \phi \right|_{tu} [m(t) - S_h(t)]\,[m(u) - S_h(u)]\right\}$$

where ϕ is the correlation matrix and $\left| \phi \right|_{tu}$ is the tu cofactor.

The optimum processor is then the mechanization of the formula

$$z(T) = \frac{A}{2m} \frac{\sum_{h=-2m}^{+2m} h \exp\left\{\frac{1}{2 \mid \phi \mid} \sum_{t=-2m}^{+2m} \sum_{u=-2m}^{+2m} \left| \phi \right|_{tu} [m(t) - S_h(t)]\,[m(u) - S_h(u)]\right\}}{\sum_{h=-2m}^{+2m} \exp\left\{\frac{1}{2 \mid \phi \mid} \sum_{t=-2m}^{+2m} \sum_{u=-2m}^{+2m} \left| \phi \right|_{tu} [m(t) - S_h(t)]\,[m(u) - S_h(u)]\right\}}$$

On the assumption that the correlation matrix ϕ is diagonal ($\phi = NI$), the formula reduces to

$$z(T) = \frac{A}{2m} \frac{\sum_{h=-2m}^{+2m} h \prod_{t=-2m}^{+2m} \exp\left\{\frac{[m(t) - S_h(t)]^2}{2N}\right\}}{\sum_{h=-2m}^{+2m} \prod_{t=-2m}^{+2m} \exp\left\{\frac{[m(t) - S_h(t)]^2}{2N}\right\}}$$

which still represents a rather formidable computing task. The computation could, however, be feasibly performed on a high-speed computing system.

Nomenclature

A sampling interval

a sample

B bandwidth

B_e effective bandwidth

B_L low-frequency bandwidth

k number of fall times in the interval T

L number of quantization levels

$m(t)$ input waveform

N high-frequency noise power

N_b bias error noise power

N_d average squared false detection error power

N_e variance of detection noise power

N_h average squared error power for fixed transition time

N_q average squared quantization error power

N_s summation noise power

N/B spectral density of noise

$n(t)$ noise waveform

$S(t)$ pulse-duration signal

t_n variation of the transition time

$x(t)$ waveform at output of filter

$y(t)$ waveform at output of limiter

$z(t)$ output waveform

τ fall time

ϕ correlation matrix

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