Nonlinear Compensator Synthesis via Linear Parameter-Varying Control

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***Abstract*— For the nonlinear output regulation, or servomechanism, problem a nonlinear compensator synthesis framework is presented that merges techniques for linear parameter-varying (LPV) systems with ideas derived from linearization-based gain scheduling. Plant linearizations about zero-error trajectories have an LPV structure upon which the synthesis of an LPV compensator is based. A key issue is whether, loosely speaking, the linearization process can be reversed wherein a nonlinear compensator is sought that linearizes to the LPV compensator about every zero-error trajectory. Necessary and sufficient existence conditions are derived for the existence of compensators satisfying this linearization requirement. Based on this, it is shown that error feedback compensators that contain an internal model of the exosystem are guaranteed to exist under mild hypotheses. A nonlinear compensator is designed for the ball and beam apparatus to illustrate the technique.**

I. Introduction

Linear parameter-varying (LPV) control has, over the past decade, emerged as an effective methodology to accommodate plants exhibiting parameter-dependent dynamics that preclude the application of linear time-invariant techniques [1], [2], [3], [5], [11], [14]. LPV plant models are often derived from a nonlinear plant whose dynamics over a specified operating regime vary significantly but admit a parameterization by a subset of the system variables. For example, in flight control applications a vehicle's linearized aerodynamics are commonly parameterized by variables such as angle-of-attack, dynamics pressure, altitude, and mach number that specify the required flight envelope.

In the case where LPV plant models arise from the linearization of a nonlinear plant about nominal trajectories, the feedback interconnection of the nonlinear plant and an LPV controller is not guaranteed to linearize to the feedback interconnection of the LPV plant and LPV controller precisely due to the parameter-dependence in the controller. This can produce unexpected and undesirable consequences. It is appropriate instead to consider more a general nonlinear compensator structure and impose the requirement that the nonlinear compensator should linearize to the LPV compensator about the designated nominal trajectories. This linearization requirement leads to the controller existence conditions derived in [10] for the case of plant linearizations about a family of equilibria and a novel compensator architecture that satisfies these conditions is presented [9]. The paper [12] surveys the interplay between LPV methods and the linearization-related issues raised above under the general heading of gain scheduling.

In this paper, we demonstrate the applicability of LPV methods in the context of the nonlinear output regulation, or servomechanism, problem. Namely, LPV plant models naturally arise from the linearization of a nonlinear plant about a parameterized collection of *zero-error trajectories* specified by a certain *zero-error submanifold*. An LPV compensator can then be constructed to meet additional performance objectives beyond error regulation, directly taking into account the time-varying nature of the parameters as generated by an associated exosystem. As in [10], we derive conditions for the existence of nonlinear compensators that meet an appropriate linearization requirement. Moreover, we show that a general class of error feedback compensators containing an internal model of the exosystem is guaranteed to satisfy this requirement. This approach on the one hand rigorously ties LPV methods to nonlinear systems and on the other hand extends prior work on linearization-based gain scheduling to the case of time-varying trajectories rather than just equilibria. It is reasonable to expect that this marriage of LPV-based techniques and linearization-based gain scheduling will allow for the design of nonlinear compensators that yield improved performance over those designed on the basis of a single plant linearization about a nominal equilibrium [8] or on the basis of plant linearizations about zero-error trajectories but for which LTI design methods are applied point-wise along the trajectory [7].

II. Problem Formulation

We consider a nonlinear continuous-time plant of the form

$$\begin{aligned} \dot{x} &= f(x,w,d,u) \\ z &= h_z(x,w,d,u) \\ y &= h_y(x,w,d) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $w(t) \in \mathbb{R}^{m_w}$ is generated by a known exosystem,

$$\dot{w} = s(w). \tag{2}$$

In addition, $d(t) \in \mathbb{R}^{m_d}$ is an unmeasured disturbance, $u(t) \in \mathbb{R}^{m_u}$ is the control input, $z(t) \in \mathbb{R}^q$ is the regulated output, and $y(t) \in \mathbb{R}^p$ is the measured output available for control purposes. For convenience, we assume that the plant functions, along with all other functions to appear, are smooth. Next, we assume that output regulation is achievable for the undisturbed plant ($d = 0$).

Assumption 2.1: For the nonlinear plant (1) and exosystem (2) we assume that there exists an open set $0 \in \mathcal{W} \subset \mathbb{R}^{m_w}$ and functions

$$x^\circ : \mathcal{W} \longrightarrow \mathbb{R}^n, \quad u^\circ : \mathcal{W} \longrightarrow \mathbb{R}^{m_u}, \quad y^\circ : \mathcal{W} \longrightarrow \mathbb{R}^p,$$

all zero at the origin, such that

$$\begin{aligned} \frac{\partial x^\circ(w)}{\partial w} s(w) &= f(x^\circ(w), w, 0, u^\circ(w)) \\ 0 &= h_z(x^\circ(w), w, 0, u^\circ(w)) \end{aligned} \qquad (3)$$

and

$$y^\circ(w) = h_y(x^\circ(w), w, 0)$$

for $w \in \mathcal{W}$. □

The regulator equations (3) define a *zero-error submanifold* via

$$\mathcal{M} = \{x \in \mathbb{R}^n \mid x = x^\circ(w),\ w \in \mathcal{W}\}.$$

Linearizing the nonlinear plant (1) about any zero-error trajectory lying in $\mathcal{M}$ leads to the family of linearizations having the linear parameter-varying (LPV) form

$$\begin{aligned} \dot{x}_\delta &= A(w)x_\delta + B_d(w)d_\delta + B_u(w)u_\delta \\ z_\delta &= C_z(w)x_\delta + D_{zd}(w)d_\delta + D_{zu}(w)u_\delta \\ y_\delta &= C_y(w)x_\delta + D_{yd}(w)d_\delta \end{aligned} \qquad (4)$$

where δ–subscripts indicate deviations from nominal values along a zero-error trajectory,

$$x_\delta = x - x^\circ(w), \quad d_\delta = d, \quad u_\delta = u - u^\circ(w),$$

$$z_\delta = z, \quad y_\delta = y - y^\circ(w)$$

and the coefficient matrices are given by, for example,

$$A(w) = \frac{\partial f}{\partial x}(x^\circ(w), w, 0, u^\circ(w))$$

with the others defined analogously.

Suppose an LPV compensator of the form

$$\begin{aligned} \dot{x}_{\mathcal{C}\delta} &= A_{\mathcal{C}}(w)x_{\mathcal{C}\delta} + B_{\mathcal{C}}(w)y_\delta \\ u_\delta &= C_{\mathcal{C}}(w)x_{\mathcal{C}\delta} + D_{\mathcal{C}}(w)y_\delta \end{aligned} \qquad (5)$$

has been constructed such that performance objectives are met by the feedback interconnection of (4) and (5) in an LPV sense for all exosystem trajectories generated by (2). It is important to note that the LPV compensator is also described in terms of deviation variables but in this case the compensator state function $x^\circ_{\mathcal{C}}(w)$ associated with $x_{\mathcal{C}\delta}$ has yet to be specified.

Of interest is the existence of a nonlinear compensator

$$\begin{aligned} \dot{x}_{\mathcal{C}} &= a(x_{\mathcal{C}}, y, w) \\ u &= c(x_{\mathcal{C}}, y, w) \end{aligned} \qquad (6)$$

that meets the following important linearization requirement with respect to the LPV compensator (5).

Requirement 2.2: For the nonlinear compensator (6) there must exist a smooth function $x^\circ_{\mathcal{C}}(\cdot)$ satisfying

$$\begin{aligned} \frac{\partial x^\circ_{\mathcal{C}}(w)}{\partial w} s(w) &= a(x^\circ_{\mathcal{C}}(w), y^\circ(w), w) \\ u^\circ(w) &= c(x^\circ_{\mathcal{C}}(w), y^\circ(w), w). \end{aligned} \qquad (7)$$

In addition, the following partial derivative identities must hold

$$\begin{aligned} \frac{\partial a}{\partial x_{\mathcal{C}}}(x^\circ_{\mathcal{C}}(w), y^\circ(w), w) &= A_{\mathcal{C}}(w), \\ \frac{\partial a}{\partial y}(x^\circ_{\mathcal{C}}(w), y^\circ(w), w) &= B_{\mathcal{C}}(w), \\ \frac{\partial a}{\partial w}(x^\circ_{\mathcal{C}}(w), y^\circ(w), w) &= 0, \\ \frac{\partial c}{\partial x_{\mathcal{C}}}(x^\circ_{\mathcal{C}}(w), y^\circ(w), w) &= C_{\mathcal{C}}(w), \\ \frac{\partial c}{\partial y}(x^\circ_{\mathcal{C}}(w), y^\circ(w), w) &= D_{\mathcal{C}}(w), \\ \frac{\partial c}{\partial w}(x^\circ_{\mathcal{C}}(w), y^\circ(w), w) &= 0. \end{aligned} \qquad (8)$$

□

The first part of Requirement 2.2 ensures that the the undisturbed ($d = 0$) closed-loop system possesses a zero-error submanifold. The second part guarantees that linearizations of the nonlinear compensator about closed-loop zero-error trajectories agree with the LPV compensator. Consequently, linearizations of the nonlinear closed-loop system about zero-error trajectories exactly match the feedback interconnection of (4) and (5). Note that terms in (8) involving partial derivatives with respect to w are required to vanish since they have no counterpart in the LPV compensator (5). The following theorem gives a necessary and sufficient existence condition for nonlinear compensators satisfying this linearization requirement.

Theorem 2.3: Given LPV compensator (5), there exists a nonlinear compensator (6) satisfying Requirement 2.2 if and only if there exists a smooth function $x^\circ_{\mathcal{C}}(\cdot)$ satisfying the partial differential equation

$$\begin{aligned} \frac{\partial}{\partial w}\left[\frac{\partial x^\circ_{\mathcal{C}}(w)}{\partial w} s(w)\right] &= A_{\mathcal{C}}(w)\frac{\partial x^\circ_{\mathcal{C}}}{\partial w} + B_{\mathcal{C}}(w)\frac{\partial y^\circ}{\partial w} \\ \frac{\partial u^\circ(w)}{\partial w} &= C_{\mathcal{C}}(w)\frac{\partial x^\circ_{\mathcal{C}}}{\partial w} + D_{\mathcal{C}}(w)\frac{\partial y^\circ}{\partial w} \end{aligned} \qquad (9)$$

Proof: For necessity, if a nonlinear compensator satisfying Requirement 2.2 exists, then differentiating the identities in (7) with respect to w and substituting the identities in (8) yields the identities in (9). Conversely, if there exists a smooth function $x^\circ_{\mathcal{C}}(\cdot)$ satisfying (9), then it is straightforward to verify that the nonlinear compensator specified by

$$\begin{aligned} a(x_{\mathcal{C}}, y, w) = {} & A_{\mathcal{C}}(w)\left[x_{\mathcal{C}} - x^\circ_{\mathcal{C}}(w)\right] + B_{\mathcal{C}}(w)\left[y - y^\circ(w)\right] \\ & + \frac{\partial x^\circ_{\mathcal{C}}(w)}{\partial w} s(w) \\ c(x_{\mathcal{C}}, y, w) = {} & C_{\mathcal{C}}(w)\left[x_{\mathcal{C}} - x^\circ_{\mathcal{C}}(w)\right] + D_{\mathcal{C}}(w)\left[y - y^\circ(w)\right] \\ & + u^\circ(w) \end{aligned}$$

satisfies Requirement 2.2 □

At first glance, the existence condition of Theorem 2.3 appears to be restrictive as it involves a type of partial differential equation that may not have a solution. On the other hand, if the condition above is not satisfied then linearizations of any compensator of the form (6) about a zero-error trajectory will not completely agree with the LPV compensator (5). Resulting mismatches constitute so-called hidden coupling terms that can potentially impact system performance. Compensators arising from a direct LPV implementation of (5) are likely to introduce hidden coupling terms corresponding to the exogenous variable w because partial derivative terms with respect to w that arise in the linearization process are typically nonzero. It is therefore of interest to identify compensator architectures that automatically satisfy the existence condition thereby decoupling the LPV design process from the existence issue. The next section presents such a situation.

III. Error Feedback Compensators

For the case of error feedback ($y = z = h_z(x, w, d)$), a nonlinear compensator satisfying Requirement 2.2 necessarily incorporates an internal model of the exosystem [8]. For such error feedback compensators, we first show that there exist local coordinates in which the LPV compensator has a particular structure. For this, define

$$\mathcal{M}_{\mathcal{C}} = \{x_{\mathcal{C}} \in \mathbb{R}^{n_C} \mid x_{\mathcal{C}} = x_{\mathcal{C}}^{\circ}(w),\, w \in \mathcal{W}\}$$

Assuming $\mathcal{M}_{\mathcal{C}}$ is an embedded submanifold, there exist about each $x_{\mathcal{C}}^{\circ} \in \mathcal{M}_{\mathcal{C}}$ local coordinates $(v_{\mathcal{C}}, w_{\mathcal{C}})$ for which

$$\mathcal{M}_{\mathcal{C}} = \{x_{\mathcal{C}} \in \mathbb{R}^{n_C} \mid v_{\mathcal{C}}(x_{\mathcal{C}}) = 0\}$$

and $w_{\mathcal{C}}(x_{\mathcal{C}}^{\circ}(w)) = w$.

Lemma 3.1: A nonlinear error feedback compensator satisfying Requirement 2.2 when expressed in the local coordinates $(v_{\mathcal{C}}, w_{\mathcal{C}})$ has linearizations about zero-error trajectories characterized by parameterized coefficient matrices

$$\begin{aligned} A_{\mathcal{C}}(w) &= \begin{bmatrix} A_{11}(w) & 0 \\ A_{21}(w) & A_{22}(w) \end{bmatrix}, \quad B_{\mathcal{C}}(w) = \begin{bmatrix} B_1(w) \\ B_2(w) \end{bmatrix} \\ C_{\mathcal{C}}(w) &= \begin{bmatrix} C_1(w) & C_2(w) \end{bmatrix}, \quad D_{\mathcal{C}}(w) = D_C(w) \end{aligned} \qquad (10)$$

in which $A_{22}(w) = \frac{\partial s(w)}{\partial w}$ and $C_2(w) = \frac{\partial u^{\circ}(w)}{\partial w}$.

Proof: Given the function $x_{\mathcal{C}}^{\circ}(w)$ associated with a nonlinear controller satisfying Requirement 2.2, we have by definition $v_{\mathcal{C}}^{\circ}(w) := v_{\mathcal{C}}(x_{\mathcal{C}}^{\circ}(w)) = 0$ and $w_{\mathcal{C}}^{\circ}(w) := w_{\mathcal{C}}(x_{\mathcal{C}}^{\circ}(w)) = w$. Since the existence conditions of Theorem 2.3 must also hold in the $(v_{\mathcal{C}}, w_{\mathcal{C}})-$coordinates and for the error feedback case $y^{\circ}(w) = z^{\circ}(w) = 0$, we have

$$\begin{aligned} \frac{\partial}{\partial w}\left[\begin{bmatrix} 0 \\ I \end{bmatrix} s(w)\right] &= \begin{bmatrix} A_{11}(w) & A_{12}(w) \\ A_{21}(w) & A_{22}(w) \end{bmatrix}\begin{bmatrix} 0 \\ I \end{bmatrix} \\ \frac{\partial u^{\circ}(w)}{\partial w} &= \begin{bmatrix} C_1(w) & C_2(w) \end{bmatrix}\begin{bmatrix} 0 \\ I \end{bmatrix} \end{aligned}$$

which forces $A_{12}(w) = 0$, $A_{22}(w) = \frac{\partial s(w)}{\partial w}$, and $C_2(w) = \frac{\partial u^{\circ}(w)}{\partial w}$ as required. □

Loosely speaking, Lemma 3.1 establishes that a nonlinear error feedback compensator satisfying Requirement 2.2 has linearizations about zero-error trajectories that specify an LPV compensator containing an internal model of the linearized exosystem. Moreover, since a coordinate transformation on the compensator state does not affect closed-loop linearizations, we can assume without loss of generality that the LPV compensator has the structure presented in the lemma. This allows us to establish the following converse result which guarantees the existence of a nonlinear error feedback compensator satisfying Requirement 2.2 that, as a result of our choice of coordinates, clearly exhibits the internal model of the exosystem.

Lemma 3.2: For the LPV compensator characterized in Lemma 3.1, there exists a nonlinear error feedback compensator satisfying Requirement 2.2. Moreover, one such compensator is specified by

$$\begin{aligned} a(v_{\mathcal{C}}, w_{\mathcal{C}}, z, w) &= \begin{bmatrix} A_{11}(w)v_{\mathcal{C}} + B_1(w)z \\ A_{21}(w)v_{\mathcal{C}} + B_2(w)z + s(w_{\mathcal{C}}) \end{bmatrix} \\ c(v_{\mathcal{C}}, w_{\mathcal{C}}, z, w) &= C_1(w)v_{\mathcal{C}} + D_C(w)z + u^{\circ}(w_{\mathcal{C}}) \end{aligned} \qquad (11)$$

for which Requirement 2.2 is satisfied with $v_{\mathcal{C}}^{\circ}(w) = 0$ and $w_{\mathcal{C}}^{\circ}(w) = w$.

Proof: For the error feedback controller specified by (11) and $v_{\mathcal{C}}^{\circ}(w) = 0$ and $w_{\mathcal{C}}^{\circ}(w) = w$, we have

$$\begin{aligned} a(v_{\mathcal{C}}^{\circ}(w), w_{\mathcal{C}}^{\circ}(w), 0, w) &= \begin{bmatrix} 0 \\ s(w_{\mathcal{C}}^{\circ}(w)) \end{bmatrix} = \begin{bmatrix} \frac{\partial v_{\mathcal{C}}^{\circ}(w)}{\partial w} \\ \frac{\partial w_{\mathcal{C}}^{\circ}(w)}{\partial w} \end{bmatrix} s(w) \\ c(v_{\mathcal{C}}^{\circ}(w), w_{\mathcal{C}}^{\circ}(w), 0, w) &= u^{\circ}(w_{\mathcal{C}}^{\circ}(w)) = u^{\circ}(w) \end{aligned}$$

so the first part of Requirement 2.2 is satisfied. Next, the partial derivative identities with respect to $v_{\mathcal{C}}$ and z clearly hold. The partial derivative identities with respect to $w_{\mathcal{C}}$ hold by definition of $A_{22}(w)$ and $C_2(w)$. Finally, the partial derivative terms with respect w vanish along zero-error trajectories since $v_{\mathcal{C}}^{\circ}(w) = 0$ and $z^{\circ}(w) = 0$. □

It is interesting to note that, as a consequence of the fact $w_{\mathcal{C}}^{\circ}(w) = w$, the nonlinear controller specified by

$$\begin{aligned} a(v_{\mathcal{C}}, w_{\mathcal{C}}, z) &= \begin{bmatrix} A_{11}(w_{\mathcal{C}})v_{\mathcal{C}} + B_1(w_{\mathcal{C}})z \\ A_{21}(w_{\mathcal{C}})v_{\mathcal{C}} + B_2(w_{\mathcal{C}})z + s(w_{\mathcal{C}}) \end{bmatrix} \\ c(v_{\mathcal{C}}, w_{\mathcal{C}}, z) &= C_1(w_{\mathcal{C}})v_{\mathcal{C}} + D_C(w_{\mathcal{C}})z + u^{\circ}(w_{\mathcal{C}}) \end{aligned} \qquad (12)$$

also satisfies Requirement 2.2 but does not require $w(t)$ as an input. Although the linearization requirement ensures that for sufficiently small deviations from zero-error trajectories the two controllers will deliver comparable performance, this is not necessarily the case for large deviations.

To summarize, the LPV compensator structure characterized in Lemma 3.1 necessarily must result from a nonlinear error feedback compensator satisfying Requirement 2.2 when expressed in suitable local coordinates. Conversely, a nonlinear error feedback compensator satisfying Requirement 2.2 always exists with respect to an LPV compensator possessing the structure described in Lemma 3.1. In this sense, the LPV compensator design process is not further constrained by the nonlinear compensator existence issue beyond the natural structural requirement that the LPV compensator contain an internal model of the linearized exosystem.

IV. LPV Compensator Synthesis

Having resolved the existence issue for nonlinear error feedback compensators, we now turn our attention to the synthesis of LPV compensators that possess the structure described in Lemma 3.1 and achieve stability and disturbance attenuation for the closed-loop LPV system represented by (omitting $w-$arguments):

$$G_{cl} = \left(\begin{array}{ccc|c} A + B_u D_C C_z & B_u C_1 & B_u C_2 & B_d + B_u D_C D_{zd} \\ B_1 C_z & A_{11} & 0 & B_1 D_{zd} \\ B_2 C_z & A_{21} & A_{22} & B_2 D_{zd} \\ \hline C_z & 0 & 0 & D_{zd} \end{array}\right) \tag{13}$$

Note that in principle the problem above is not a standard LPV design problem since the controller is subject to structural constraints, namely $A_{12} = 0$, A_{22} and C_2 given. Nevertheless, as we show in the sequel, the problem can be reduced to that of stabilizing an auxiliary plant, whose state space realization can be obtained from the problem data. This constitutes an LPV version of the development in [13] for LTI systems.

Lemma 4.1: Given the LPV system:

$$G = \left(\begin{array}{c|cc} A & B_d & B_u \\ \hline C_z & D_{zd} & 0 \\ C_z & D_{zd} & 0 \end{array}\right) \tag{14}$$

consider the following LPV auxiliary plant

$$G_{aux} = \left(\begin{array}{cc|ccc} A & B_u C_2 & B_d & B_u & 0 \\ 0 & A_{22} & 0 & 0 & I \\ \hline C_z & 0 & D_{zd} & 0 & 0 \\ C_z & 0 & D_{zd} & 0 & 0 \end{array}\right) \tag{15}$$

Assume that there exists an LPV controller K_{aux}

$$K_{aux} = \left(\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D_{\mathcal{C}} \\ A_{21} & B_2 \end{array}\right) \tag{16}$$

that internally stabilizes G_{aux} and achieves a closed–loop performance level $\gamma = \sup_{\rho \in \mathcal{P}} \|\mathcal{F}_\ell(G_{aux}, K_{aux})\|_*$, where $\mathcal{P}$ and $\|.\|_*$ denote the set of admissible parameter trajectories and a suitable norm, such as ℓ^2 induced, respectively. Then the controller

$$K = \left(\begin{array}{cc|c} A_{11} & 0 & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D_{\mathcal{C}} \end{array}\right) \tag{17}$$

has the following properties:

i.- It satisfies the structural constraints (10).
ii.- It internally stabilizes the original plant (14), and
iii.- The closed loop system achieves the same performance level obtained for the auxiliary plant, i.e. $\sup_{\rho \in \mathcal{P}} \|\mathcal{F}_\ell(G, K)\|_* = \gamma$.

Proof: The first property holds by construction. Properties (ii) and (iii) follows by noting that combining the auxiliary plant and controller equations (15)–(16) leads to the following closed-loop LPV system:

$$\left(\begin{array}{ccc|c} A + B_u D_C C_z & B_u C_2 & B_u C_1 & B_d + B_u D_C D_{zd} \\ B_2 C_z & A_{22} & A_{21} & B_2 D_{zd} \\ B_1 C_z & 0 & A_{11} & B_1 D_{zd} \\ \hline C_z & 0 & 0 & D_{zd} \end{array}\right)$$

Interchanging now the second and third partitions of the underlying state vector yields (13), which are precisely the equations that one obtains closing the loop around the plant (14) using the controller (17). Finally, internal stability of the auxiliary closed–loop system guarantees that all internal closed–loop subsystems are stable. □

V. Example: The Ball and Beam

We apply the synthesis methodology of the previous sections to the well-known ball and beam experiment whose nonlinear equations of motion are given by ([6])

$$\left(\frac{J_b}{R^2} + M\right)\ddot{r} + MG\sin(\theta) - Mr\dot{\theta}^2 = 0$$
$$\left(Mr^2 + J + J_b\right)\ddot{\theta} + 2Mr\dot{r}\dot{\theta} + MGr\cos(\theta) = \tau$$

in which r is the ball position, θ is the beam angle , τ is the applied torque, and the remaining system parameters are listed in Table 4.1. The applied torque is assumed to have the form

$$\tau = \tau_d + \tau_m$$

where τ_d is a disturbance torque and τ_m is the torque produced by a servomotor connected to the beam, modelled by

$$\dot{\tau}_m = \alpha\left(\tau_c - \tau_m\right)$$

in which τ_c is the commanded torque.

We define state variables, disturbance input, and control input according to

$$x_1 = r, \quad x_2 = \dot{r}, \quad x_3 = \theta, \quad x_4 = \dot{\theta}, \quad x_5 = \tau_m$$

along with disturbance input $d = \tau_d$ and control input $u = \tau_c$ yielding the nonlinear state equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= B\left(x_1 x_4^2 - G\sin(x_3)\right) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{-2Mx_1x_2x_4 - MG\,x_1\cos(x_3) + x_5 + d}{Mx_1^2 + J + J_b} \\ \dot{x}_5 &= \alpha\left(u - x_5\right) \end{aligned}$$

where $B := M/(J_b/R^2 + M)$.

The exosystem generating constant-velocity ball position commands is given by

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} w_2 \\ 0 \end{bmatrix}.$$

Table 4.1: Ball and Beam Parameters

Parameter	Description	Value
M	ball mass	0.05 kg
R	ball radius	0.01 m
J	beam inertia	0.02 kg m^2
J_b	ball inertia	2×10^{-6} kg m^2
G	acceleration due to gravity	9.81 m/s^2

The regulated and measured outputs are taken to be ball position error

$$z = y = x_1 - w_1.$$

It is straightforward to verify that Assumption 2.1 is satisfied for

$$x^\circ(w) = \begin{bmatrix} w_1 \\ w_2 \\ 0 \\ 0 \\ MG\,w_1 \end{bmatrix}, \quad u^\circ(w) = MG\,(w_1 + w_2/\alpha)$$

and the associated zero-error submanifold is given by

$$\mathcal{M} = \{x \in \mathbb{R}^5 \mid x_3 = 0, x_4 = 0, x_5 = MGx_1\}.$$

Plant linearizations about zero-error trajectories are specified by the coefficient matrices

$$A(w) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -BG & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{MG}{Mw_1^2+J+J_b} & 0 & 0 & -\frac{2Mw_1\,w_2}{Mw_1^2+J+J_b} & \frac{1}{Mw_1^2+J+J_b} \\ 0 & 0 & 0 & 0 & -\alpha \end{bmatrix}$$

$$B_d(w) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{Mw_1^2+J+J_b} \\ 0 \end{bmatrix}, \qquad B_u(w) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \alpha \end{bmatrix},$$

$$C_z(w) = C_y(w) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

The auxiliary plant (15) can be formed by using in addition

$$A_{22}(w) = \frac{\partial s(w)}{\partial w} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$C_2(w) = \frac{\partial u^\circ(w)}{\partial w} = \begin{bmatrix} MG & MG/\alpha \end{bmatrix}$$

which, defining $\rho = (\rho_1, \rho_2)$ according to

$$\rho_1 = \frac{1}{Mw_1^2 + J + J_b}, \quad \rho_2 = -\frac{2Mw_1\,w_2}{Mw_1^2 + J + J_b},$$

can be cast in the affine parameter-dependent form

$$\left(\begin{array}{c|c} A_{\rm aux}(\rho) & B_{\rm aux}(\rho) \\ \hline C_{\rm aux}(\rho) & D_{\rm aux}(\rho) \end{array}\right) = \left(\begin{array}{c|c} A^0_{\rm aux} & B^0_{\rm aux} \\ \hline C^0_{\rm aux} & D^0_{\rm aux} \end{array}\right) + \rho_1 \left(\begin{array}{c|c} A^1_{\rm aux} & B^1_{\rm aux} \\ \hline C^1_{\rm aux} & D^1_{\rm aux} \end{array}\right) + \rho_2 \left(\begin{array}{c|c} A^2_{\rm aux} & B^2_{\rm aux} \\ \hline C^2_{\rm aux} & D^2_{\rm aux} \end{array}\right)$$

Even though this description is affine in ρ, since ρ_1 and ρ_2 are not independent, designing an LPV controller for this plant entails finding a solution to a set of *functional* matrix inequalities, or, equivalently an infinite set of LMIs (see for instance [4] or [3]). An approximate solution to this problem can be obtained by gridding the parameter space and enforcing these LMIs at a finite number of points [1], but pursuing this approach still requires both expanding the solution to the original set of functional inequalities in terms of some basis and solving a large number of LMIs. To circumvent this difficulty, in this example we will take advantage of the fact that both C_y and B_u are independent of ρ to recast the problem as the synthesis of an $\mathcal{H}_\infty$ controller for a polytopic plant. To this effect note that, upon independently restricting $|w_1(t)| \le 1$ m and $|w_2(t)| \le 1$ m/s, the parameter vector $\rho(t)$ is guaranteed to lie in the box $[\underline{\rho}_1, \overline{\rho}_1] \times [\underline{\rho}_2, \overline{\rho}_2]$ with

$$\underline{\rho}_1 = \tfrac{1}{M+J+J_b}, \qquad \overline{\rho}_1 = \tfrac{1}{J+J_b},$$
$$\underline{\rho}_2 = -\sqrt{M/(J+J_b)}, \qquad \overline{\rho}_2 = \sqrt{M/(J+J_b)}.$$

Finally, neglecting the correlation between ρ_1 and ρ_2 leads to a standard quadratic stability problem for polytopic parameter dependent plants [3] that can be solved by synthesizing an $\mathcal{H}_\infty$ controller for each of the vertices, for instance using the `hinfgs` command in Matlab's LMI Control Toolbox. The LPV controller is then implemented by interpolating these four vertex controllers. Note that in this case, since $s(w)$ and $u^\circ(w)$ are linear in w, the nonlinear controller given by (11) allows for a direct implementation of the LPV controller having constant A_{22} and C_2.

Nonlinear simulations were conducted to assess the disturbance rejection performance of the LPV controller. A nominal trajectory corresponding to an initial ball position of -1 m and a constant velocity of 0.25 m/s was commanded. The plant and controller were initialized to yield identically zero tracking error for the disturbance-free case. A bandlimited (500 Hz) disturbance torque, plotted in Figure 1, was then applied. Note that this disturbance has large amplitude compared to the nominal torque command which has $|\tau_c(t)| = MG|w_1(t)| \le 0.49$ N-m. The ball position, ball position tracking error, and beam angle responses are shown in Figures 2 – 4. These plots indicate that the controller substantially attenuates the influence that the disturbance torque has on the position tracking error without causing excessive beam activity. As another figure of merit, we numerically calculate

$$\frac{\|z\|_{\ell_2[0,8]}}{\|d\|_{\ell_2[0,8]}} = 0.156$$

which compares favorably to the worst-case value of 8.2 achieved in the LPV design process.

VI. Concluding Remarks

The nonlinear output regulation problem is a natural setting in which to apply LPV synthesis techniques based

[1]Usually a coarser mesh is used for synthesis, followed by validation using a finer grid.

on an LPV plant model derived from nonlinear plant linearizations about a manifold of zero-error trajectories. The LPV formalism allows for other performance objectives to be addressed, such as the rejection of disturbances not generated by the exosystem, in addition to traditional error regulation.

In this paper, we have presented a framework for the synthesis of nonlinear compensators that meet an important linearization requirement with respect to LPV controllers designed on the basis of LPV plant linearizations. We have derived necessary and sufficient existence conditions that are automatically satisfied for a class of error feedback compensators satisfying the internal model principle. Moreover, we have shown that the associated structural constraints imposed on the underlying LPV compensator can be easily accommodated in the LPV design process. Finally, we have illustrated these ideas by designing a compensator for the ball and beam apparatus for which simulation results indicate excellent disturbance rejection performance along time-varying zero-error trajectories.

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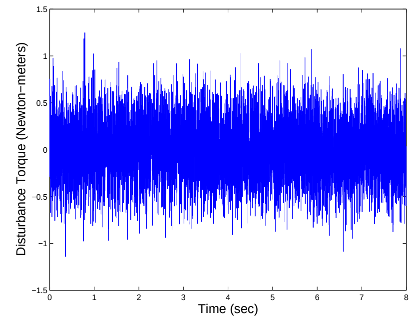

Fig. 1 Disturbance torque.

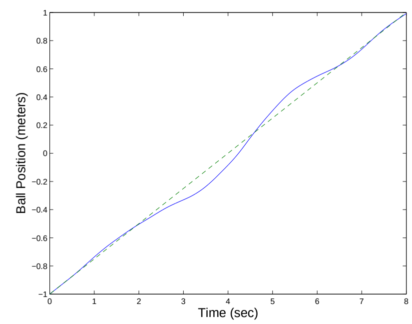

Fig. 2 Commanded ball position (dashed) and ball position response (solid).

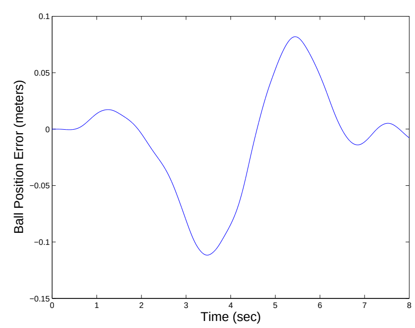

Fig. 3 Ball position tracking error response.

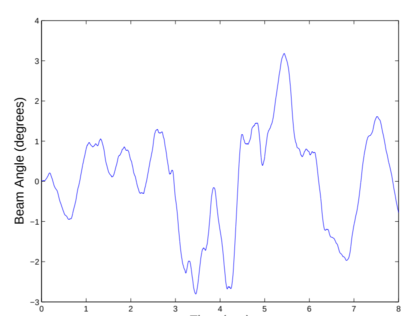

Fig. 4 Beam angle response.