Nonlinear Decision Weights in Choice under Uncertainty

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In most real-world decisions, consequences are tied explicitly to the outcome of events. Previous studies of decision making under uncertainty have indicated that the psychological weight attached to an event, called a decision weight, usually differs from the probability of that event. We investigate two sources of nonlinearity of decision weights: subadditivity of probability judgments, and the overweighting of small probabilities and underweighting of medium and large probabilities. These two sources of nonlinearity are combined into a two-stage model of choice under uncertainty. In the first stage, events are taken into subjective probability judgments, and the second stage takes probability judgments into decision weights. We then characterize the curvature of the decision weights by extending a condition employed by Wu and Gonzalez (1996) in the domain of risk to the domain of uncertainty and show that the nonlinearity of decision weights can be decomposed into subadditivity of probability judgments and the curvature of the probability weighting function. Empirical tests support the proposed two-stage model and indicate that decision weights are concave then convex. More specifically, our results lend support for a new property of subjective probability judgments, interior additivity (subadditive at the boundaries, but additive away from the boundaries), and show that the probability weighting function is inverse S-shaped as in Wu and Gonzalez (1996).
(*Decision Making under Uncertainty*; *Prospect Theory*; *Decision Weights*; *Support Theory*)

1. Introduction

Making decisions in the face of uncertainty is an important part of a manager's life. For instance, decisions whether to add capacity to an existing facility, introduce a new product, and hire a new employee are made difficult largely because future costs, demand, and productivity are not known with certainty. Subjective expected utility (SEU; Savage 1954) is regarded by most decision analysts to be the normative model for how individuals should make decisions under uncertainty (Howard 1992). Individuals, however, violate SEU, as was first and most famously illustrated by the Ellsberg Paradox (Ellsberg 1961; see Camerer and Weber 1992, Heath and Tversky 1991, and Fox and Tversky 1995). Beyond Ellsberg, a more general empirical characterization of decision under uncertainty has been lacking. Most empirical research has considered decision under risk. Research on risk has in turn been criticized: textbook gambles do not resemble real world decision alternatives, and hence gambles are often viewed as insufficiently rich to provide a complete picture of decision making.

While this view has some merit, we propose that decision under uncertainty resembles decision under risk with one important qualification. We illustrate our point with a new violation of the SEU model that uses a device we call a *preference ladder*. The prospects were presented to University of Chicago undergraduates as a between-subjects test. Each prospect is

tied to the high temperature (degrees Fahrenheit) in Chicago on Thanksgiving Day, 1996 (we denote this temperature by T). For example, R_2 offers \$140 if $25 < T \le 30$, \$100 if $30 < T \le 40$, and \$0 otherwise. For each question i, 75 respondents chose between R_i and S_i.

Question		% Choice	z-Statistic	$T \le 25$	$25 < T \le 30$	$30 < T \le 40$	$40 < T \le 50$	$50 < T \le 60$	$60 < T$
1	R_1	**19**		0	140	0	0	0	0
	S_1	81		100	100	0	0	0	0
			4.73						
2	R_2	**56**		0	140	100	0	0	0
	S_2	44		100	100	100	0	0	0
			0.50						
3	R_3	**60**		0	140	100	100	0	0
	S_3	40		100	100	100	100	0	0
			-1.47						
4	R_4	**48**		0	140	100	100	100	0
	S_4	52		100	100	100	100	100	0
			-3.25						
5	R_5	**23**		0	140	100	100	100	100
	S_5	77		100	100	100	100	100	100

Note that the prospects in one question are created from the prospects in the previous question by replacing \$0 in a particular state with \$100. Moving down the "ladder" from question 1 to question 5, the percentage of subjects choosing the risky (R) option increases and then decreases, creating an inverted U-shaped pattern of responses. This pattern is inconsistent with SEU maximization and the Sure Thing Principle, which require that the percentage of subjects choosing the risky option remain constant across the ladder.[1] In contrast, our respondents made different choices in questions 1 and 2 ($p < 0.0001$), questions 3 and 4 ($p = 0.07$), and questions 4 and 5 ($p < 0.001$).

Recall that question 2 is created from question 1 by replacing \$0 in the event "$30 < T \le 40$" with \$100. Although this shift in probability mass improves both R_2 and S_2 equally in objective terms, the choice data suggest that R_2 benefits more from the probability shift in subjective terms. Consider two possible explanations for this effect. First, the subjective probability

assigned to the event "$30 < T \le 40$", $\rho(30 < T \le 40)$, might exceed the increase in probability from adding the same event to "$T \le 30$", $\rho(T \le 40) - \rho(T \le 30)$. This explanation is consistent with recent demonstrations of subadditive probability judgments: $\rho(A) \ge \rho(A \cup B) - \rho(B)$ (Tversky and Koehler 1994, Tversky and Fox 1995, Fox et al. 1996, Rottenstreich and Tversky 1997). Second, an individual may assign additive probabilities but fail to weight these probabilities linearly. For example, an individual may assess the following additive probabilities, $\rho(25 < T \le 30) = \rho(30 < T \le 40) = 0.1$, $\rho(T \le 30) = 0.2$, and $\rho(T \le 40) = 0.3$, but could weight the 0.1 probability of the event "$30 < T \le 40$" more when it is added to the 0.1 probability of "$30 < T \le 40$" (in R_2) than when it is added to the 0.2 probability of "$T \le 30$" (in S_2). This story is suggested by several studies on decision under risk that show a tendency to overweight small objective probabilities and underweight medium and large objective probabilities (Tversky and Kahneman 1992, Camerer and Ho 1994, Tversky and Fox 1995, Wu and Gonzalez 1996, Gonzalez and Wu in press; for an exception, see Birnbaum and McIntosh 1996). Such a pattern can be captured with

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[1] For related violations of SEU, see Tversky and Kahneman (1992), Tversky and Fox (1995), and Fox and Tversky (1998), as well as Camerer's (1995) review of decision making.

an inverse S-shaped probability weighting function as originally proposed by Tversky and Kahneman (1992).

The two factors, subadditive probability judgments and an inverse S-shaped probability weighting function, can be accommodated within prospect theory (Kahneman and Tversky 1979, Tversky and Kahneman 1992) by a nonlinear *decision weighting function*. In this paper, we characterize the decision weighting function as a composition of two functions: a function that maps events into subjective probability judgments and a probability weighting function that maps the judged probabilities to decision weights. We present a condition that characterizes nonlinearity of the decision weighting function. In empirical tests of the condition, we observe an inverted U-shaped pattern (as in the Chicago temperature example), a pattern predicted by a decision weighting function that is concave for low probability events and convex for medium to high probability events.

The condition also allows us to tease apart the possible sources of nonlinearity of the decision weighting function, i.e., whether the nonlinearity is due to subadditivity of probability judgments or nonlinearity of the probability weighting function. It is important for decision analysts to understand which of the two factors is producing the pattern exhibited in the example above. If failure to obey the axioms of probability is the culprit, then the decision analyst should concentrate on ensuring additivity of probability judgments. On the other hand, if the failure is attributable to the curvature of the probability weighting function, then eliciting utilities using standard procedures (e.g., the certainty equivalent method) will be problematic (Hershey and Schoemaker 1985). In this case, the decision analyst might then wish to consider alternative methods for eliciting utilities such as the tradeoff method (Wakker and Deneffe 1996).

The paper is structured as follows. In §2, we present a two-stage model for decision making under uncertainty. The two-stage model maps (1) events into judged probabilities through a nonadditive subjective probability measure, and (2) judged probabilities into decision weights through a probability weighting function. The composition of these two stages produces the decision weighting function. We present conditions for concavity and convexity of the decision weighting function and describe the implications of these conditions for the two-stage model. In §§3 and 4, we present two empirical studies (one choice and one cash equivalence) designed to test for nonlinearity of the decision weighting function. In §5, we make some concluding remarks.

2. Framework

2.1. Preliminaries

Tversky and Fox (1995) proposed a two-stage theory for decision making under uncertainty. Their model considered only simple prospects, prospects with at most two possible outcomes. For a reason that will become clear in §2.3, we must extend their model to three-outcome prospects. We characterize non-SEU behavior in terms of nonlinearity of decision weights, a feature of a family of nonexpected utility theories that includes rank-dependent expected utility (Quiggin 1982) and cumulative prospect theory (Tversky and Kahneman 1992, Starmer and Sugden 1989, Luce and Fishburn 1991).

We consider three-outcome prospects in which possible outcomes are $x > y > 0$. Let S be an event space endowed with an algebra A. Let E_i be a subset of S and $\varnothing$ be the null set. We denote $S - E_i$ as the disjoint complement of E_i, i.e., both $E_i \cup (S - E_i) = S$ and $E_i \cap (S - E_i) = \varnothing$. Furthermore, let $E_{ij} = E_i \cup E_j$, $E_{ijk} = E_i \cup E_j \cup E_k$, and $E_{i,\ldots,j} = \cup_{k=i}^{j} E_k$. We denote a prospect in which x is received if E_i obtains, y is received if E_j obtains, and 0 is received if $S - E_i - E_j$ obtains, $(x, E_i; y, E_j)$ where E_i and E_j are disjoint. Unless otherwise noted, we assume that E_i and E_j are disjoint.

We assume that a prospect $(x, E_i; y, E_j)$ is valued according to the following representation, which is separable in events and outcomes:

$$U(x, E_i; y, E_j) = \omega(E_i)v(x) + [\omega(E_{ij}) - \omega(E_i)]v(y).$$

$$(2.1)$$

The representation consists of two scales, $v(\cdot)$, a value function, and $\omega(\cdot)$, a *decision weighting function* or *capacity* (Gilboa 1987, Schmeidler 1989, Wakker 1989). (Since $v(\cdot)$ is unique up to an affine transformation, we

have set $v(0) = 0$.) The capacity $\omega(\cdot)$ satisfies monotonicity with respect to set inclusion ($\omega(E_i) \leq \omega(E_{ij})$ for all i, j), as well as the restrictions $\omega(\varnothing) = 0$ and $\omega(S) = 1$. It is important to note that $\omega(\cdot)$ does not necessarily satisfy additivity: $\omega(E_{ij}) - \omega(E_i) \neq \omega(E_j)$ for disjoint E_i, E_j. Finally, the decision impact of E_j when the event is added to E_i is given by $\omega(E_{ij}) - \omega(E_i)$.

2.2. The Two-Stage Model

As we noted in the introduction, nonlinearity of decision weights can be a result of subadditivity of probability judgments, curvature of the probability weighting function, or both. We describe a two-stage model in which (1) events are first mapped into judged probabilities; and (2) judged probabilities are mapped into decision weights. Stage (i) is captured by $\rho: A \rightarrow [0, 1]$, where ρ is a *nonadditive subjective probability measure*. We assume that $\rho(\cdot)$ is a *support function*, i.e., is consistent with Tversky and Koehler's (1994) support theory. More specifically, $\rho(\cdot)$ is *subadditive* ($\rho(E_i) + \rho(E_j) \geq \rho(E_{ij})$ for E_i, E_j disjoint) and satisfies *binary complementarity* ($\rho(E_i) + \rho(S - E_i) = 1$ for all E_i). A more restrictive condition on $\rho(\cdot)$ is *interior additivity*: $\rho(\cdot)$ is *interior additive* if $\rho(E_{ij}) - \rho(E_j) = \rho(E_{ijk}) - \rho(E_{jk})$ for all disjoint E_i, E_j, E_k such that $\rho(E_j) > 0$ and $\rho(E_{ijk}) < 1$. Interior additivity requires that subadditivity be driven exclusively by the categorical difference between shifting from "impossibility" to "possibility" and from "near certainty" to "certainty." Note that if $\rho(\cdot)$ is subadditive and interior additive, then the "direct" probability of E_i will typically exceed the "revealed" probability of E_i: $\rho(E_i) > \rho(E_{ij}) - \rho(E_j)$. Interior additivity will prove to be a crucial concept in teasing apart the sources of nonlinear decision weights.

Stage (ii) is modeled by $\pi: [0, 1] \rightarrow [0, 1]$, a *probability weighting function* that is nondecreasing, with $\pi(0) = 0$ and $\pi(1) = 1$.

The two-stage model follows below.

> *Two-stage Model*: A prospect, $(x, E_i; y, E_j)$, is evaluated by (2.1), where $\omega(E_i) = \pi(\rho(E_i))$, where $\rho(\cdot)$ is a support function and $\pi(\cdot)$ is a probability weighting function.

Our two-stage model is rank-dependent in the sense that the weight assigned to an outcome depends on where that outcome is ranked vis-a-vis the other outcomes. Note that our two-stage model extends Tversky and Fox's two-stage theory to multiple outcomes.[2]

2.3. Concavity and Convexity of the Decision Weighting Function

We begin by defining concavity and convexity of the decision weighting function $\omega(\cdot)$. The function $\omega(\cdot)$ is *concave* if $\omega(E_{ij}) - \omega(E_j) \geq \omega(E_{ijk}) - \omega(E_{jk})$ for all E_k. Likewise, $\omega(\cdot)$ is *convex* if $\omega(E_{ij}) - \omega(E_j) \leq \omega(E_{ijk}) - \omega(E_{jk})$ for all E_k. Concavity of $\omega(\cdot)$ captures the diminishing impact of E_i: E_i has a smaller and smaller impact as it is added to more inclusive events. Convexity, of course, has the opposite interpretation.

Our formulation of a concavity (convexity) condition for decision making under uncertainty extends the following risky choice condition tested in Wu and Gonzalez (1996):

$$\text{if } R = (x, p; y, q) \sim (y, p + q + \epsilon) = S,$$

$$\text{then } R' = (x, p; y, q + q') \succcurlyeq (\preccurlyeq)$$

$$(y, p + q + q' + \epsilon) = S'. \quad (2.2)$$

Note that the second pair of gambles is constructed by adding q' chance at y to both R and S. Under rank-dependent expected utility or cumulative prospect theory, (2.2) implies

$$\pi(p + q + q') - \pi(p + q)$$

$$\geq (\leq) \pi(p + q + q' + \epsilon) - \pi(p + q + \epsilon),$$

which holds if $\pi(\cdot)$ is concave (convex). In an empirical test of (2.2), Wu and Gonzalez (1996) found that the number of subjects who chose the risky option increased and then decreased as q' increased, a pattern consistent with an inverse S-shaped probability weighting function.[3]

There are three noteworthy features of conditions

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[2] Tversky and Fox suggested that the value for a simple prospect (x, E) is given by $\pi(\rho(E))v(x)$. The generalization of (2.1) to multiple-outcome prospects, $(x_1, E_1; \ldots; x_i, E_i; \ldots; x_n, E_n)$, where $x_i > x_{i+1}$, is $\omega(E_1)v(x_1) + \cdots + [\omega(E_{1\ldots i}) - \omega(E_{1\ldots i-1})]v(x_i) + \cdots + [1 - \omega(E_{1\ldots n-1})]v(x_n)$.

[3] For related "common consequence" conditions, see Segal (1987), Tversky and Wakker (1995), Wu and Gonzalez (1998). See also Prelec (1998).

(2.2). First, the conditions are nonparametric in the following sense: inferences about $\pi(\cdot)$ do not depend on assumptions about the value function. Second, the conditions involve two nonzero outcomes, x and y, and thus solve the indeterminacy problem: when simple prospects with one nonzero outcome are used, the value and weighting function are determined only to a power (if $\pi(p)v(x)$ represents preferences, so does $\pi(p)^\lambda v(x)^\lambda$). Finally, because the conditions are identical except for the sign of the inequality, we can test for both concavity and convexity simultaneously.

The following two conditions extend (2.2) to decision making under uncertainty, while also inheriting the three aforementioned features (see Wakker 1996 for a more comprehensive treatment):

Concavity Condition:

$$\text{If } R = (x, E_j; y, E_k) \sim (y, E_{ijk}) = S$$

$$\text{then } R' = (x, E_j; y, E_{kl}) \geqslant (y, E_{ijkl}) = S'.$$
(2.3)

Convexity Condition:

$$\text{If } R = (x, E_j; y, E_k) \sim (y, E_{ijk}) = S \quad \text{then } R'$$

$$= (x, E_j; y, E_{kl}) \leqslant (y, E_{ijkl}) = S'. \quad (2.4)$$

If changing the outcome attached to E_l from 0 to y improves the risky alternative (R) more than it improves the safe alternative (S), then preferences are consistent with the Concavity Condition. Preferences are consistent with the Convexity Condition if the same change improves S more than R. Note that the Sure Thing Principle of SEU implies that $R \geqslant S \Leftrightarrow R' \geqslant S'$.

3. Study 1: Binary Choice

In the next two sections, we describe two empirical studies. The first study is a direct between-subject test of the concavity/convexity conditions using choice questions. The second study tests the conditions using a within-subject cash equivalence design. We also test for properties of $\rho(\cdot)$ such as subadditivity, binary complementarity, and interior additivity.

3.1. Procedure
We recruited 420 University of Washington undergraduates to complete a questionnaire of approxi-

mately 25 choice questions. Subjects were paid $5 for completing the questionnaire. The questionnaires consisted mostly of risky choice questions. However, each questionnaire contained either two or four uncertainty questions. Table 1 lists the questions, labeling the events to match the convention of the concavity/convexity conditions ($i = 1$, $j = 2$, etc.). We refer to the group of related questions (e.g., the three Seattle temperature questions, 1.1, 1.2, and 1.3) as a *ladder* and questions within a given ladder as *rungs*. Note that questions within a ladder differ only by a common consequence. For example, $0 in the event "50 $< T \leq 60$" in question 1.1 is replaced by $100 in question 1.2.

The sources of uncertainty included the high temperature in Seattle on a specific day, the number of football victories by the University of Washington football team for the 1995–1996 season, the 1996 U.S. national election, and the close of the Dow Jones Industrial average on a particular day. Ladder 2 involved conjunctions and disjunctions of four primitive events, DP (RP), "A Democrat (Republican) wins the Presidential election" and DH (RH), "The Democratic (Republican) Party gains control of the House of Representatives." For example, E_2 was described as "A Democrat wins the Presidential election *and* The Democratic Party gains control of the House of Representatives," whereas E_{123} was described as "A Democrat wins the Presidential election *or* The Democratic Party gains control of the House of Representatives."

All critical comparisons were between-subject. A total of 70 subjects answered each question (1.1, 1.2, etc.). There were two groups of subjects. Each member of the first group was given one question from each of ladders 1 and 2, while members of the second group were given one question from each of ladders 3–6. All prospects were displayed in a simplified form. For example, R_2 in Question 1.2 was described as

$$A \quad \begin{array}{ll} \$200, & \text{if } 45 < T \leq 50 \\ \$100, & \text{if } 50 < T \leq 60 \\ \$0, & \text{otherwise.} \end{array}$$

Finally, the questions were counter-balanced so that the risky option (i.e., the R prospect) was the first choice on the sheet half of the time. In addition, the order of the events (e.g., Seattle temperature, election

Table 1   Choice Questions for Study 1

	Ladder 1					

Question		% Choice	z-Statistic	E_1 $40 < T \leq 45$	E_2 $45 < T \leq 50$	E_3 $50 < T \leq 60$	E_4 $T \leq 40 \cup 60 < T$
1.1	R_1	**54**		0	200	0	0
	S_1	46		100	100	0	0
			3.04*				
1.2	R_2	**79**		0	200	100	0
	S_2	21		100	100	100	0
			6.86*				
1.3	R_3	**21**		0	200	100	100
	S_3	79		100	100	100	100

T = high temperature of Seattle on December 25, 1994.

	Ladder 2					

Question		% Choice	z-Statistic	E_1 $DP \cap RH$	E_2 $DP \cap DH$	E_3 $ERP \cap DH$	E_4 $RP \cap RH$
2.1	R_1	**35**		0	350	0	0
	S_1	65		300	300	0	0
			3.68*				
2.2	R_2	**67**		0	350	300	0
	S_2	33		300	300	300	0
			6.49*				
2.3	R_3	**13**		0	350	300	300
	S_3	87		300	300	300	300

DP = A Democrat wins the Presidential election.
RP = A Republican wins the Presidential election.
DH = The Democratic Party gains control of the House of Representatives.
RH = The Republican Party gains control of the House of Representatives.

	Ladder 3					

Question		% Choice	z-Statistic	E_2 $9 < W$	E_1 $W = 8$	E_3 $6 \leq W \leq 7$	E_4 $W < 6$
3.1	R_1	**44**		200	0	0	0
	S_1	56		120	120	0	0
			0.94				
3.2	R_2	**52**		200	0	120	0
	S_2	48		120	120	120	0
			2.43*				
3.3	R_3	**32**		200	0	120	120
	S_3	68		120	120	120	120

W = number of University of Washington football victories, 1995–1996 season.

Table 1   *Continued*

				Ladder 4			
				E_1	E_2	E_3	E_4
Question		% Choice	z-Statistic	$84 \leq T$	$81 \leq T < 84$	$75 \leq T < 81$	$T < 75$
4.1	R_1	**54**		0	100	0	0
	S_1	46		60	60	0	0
			1.03				
4.2	R_2	**60**		0	100	60	0
	S_2	40		60	60	60	0
			4.47*				
4.3	R_3	**18**		0	100	60	60
	S_3	82		60	60	60	60

T = high temperature of Seattle on June 15, 1995.

				Ladder 5			
				E_1	E_2	E_3	E_4
Question		% Choice	z-Statistic	$J < 70 \cap A \geq 70$	$J < 70 \cap A < 70$	$J \geq 70 \cap A < 70$	$J \geq 70 \cap A \geq 70$
5.1	R_1	**59**		0	80	0	0
	S_1	41		40	40	0	0
			0.77				
5.2	R_2	**66**		0	80	40	0
	S_2	34		40	40	40	0
			4.11*				
5.3	R_3	**31**		0	80	40	40
	S_3	69		40	40	40	40

J = July 1, 1995 High in Seattle.
A = August 1, 1995 High in Seattle.

				Ladder 6				
				E_2	E_1	E_3	E_4	E_5
Question		% Choice	z-Statistic	$D < 4200$	$4200 \leq D < 4250$	$4250 \leq D < 4300$	$4300 \leq D < 4600$	$D > 4600$
6.1	R_1	**52**		300	0	150	0	0
	S_1	48		150	150	150	0	0
			1.56					
6.2	R_2	**65**		300	0	150	150	0
	S_2	35		150	150	150	150	0
			5.14*					
6.3	R_3	**28**		300	0	150	150	150
	S_3	72		150	150	150	150	150

D = Dow Jones Industrial Average close on June 30, 1995 (asked on May 19, 1995, close of 4341).

* Denotes significance at the 0.05 level.

outcome) was counter-balanced within the questionnaire.

3.2. Results

The percentage of subjects choosing the risky or safe prospect in each of the questions is given in Table 1. Note that in all ladders, we see an inverse U-shaped pattern, the percentage of risky choices (%R, for short) increases then decreases, a finding consistent with concave and then convex decision weights. Table 1 also shows the z-statistics for the differences between pairs of proportions (binomial test). For example, in ladder 1, $\%R_1$ (%R for rung 1) is significantly different from $\%R_2$ ($p < 0.001$), and $\%R_2$ is significantly different from $\%R_3$ ($p < 0.001$). We can thus reject SEU, which requires that %R remain constant across rungs. Whereas $\%R_1 < \%R_2$ in all ladders, only two of the six comparisons of $\%R_1$ and $\%R_2$ were statistically significant at $\alpha = 0.05$. It is noteworthy that $\%R_3$ is significantly lower than $\%R_2$ in all six ladders, indicating a very pronounced certainty effect. Note that the U-shaped pattern exhibited in this study mirrors the results presented in the introduction using Chicago temperature.

4. Study 2: Cash Equivalence

The second study tests the concavity/convexity condition using a within-subject cash equivalence procedure instead of a choice procedure. We also collected probability assessments of the relevant events to test for properties of $\rho(\cdot)$ such as subadditivity, interior additivity, and binary complementarity.

4.1. Procedure

We recruited 19 University of Washington advanced undergraduate and graduate students in psychology to complete a questionnaire. Subjects were paid $5 for participation and asked to give cash equivalents for each of 18 prospects (Table 2). Before proceeding, respondents received instruction (available upon request from the authors) and were given a practice problem. Ladder 7 involved the same six 1996 national election questions used in ladder 2. Ladder 8 contained 12 questions tied to the number of 1995–1996 University of Washington football team victories. In addition, subjects provided 23 probability judgments

(Table 3). Sixteen of the 19 subjects provided usable responses (the remaining 3 subjects failed to answer all the probability questions).

4.2. Choice Results

The results using a cash equivalence procedure are similar to those reported for study 1. The first test we conduct is an ordinal test. For each rung i, subjects gave a cash equivalent for the risky prospect, $CE(R_i)$, and for the safe prospect, $CE(S_i)$. If $CE(R_i) > CE(S_i)$, then we infer that $R_i > S_i$ for that subject. On the other hand, $CE(R_i) < CE(S_i)$ implies that $R_i < S_i$, while $CE(R_i) = CE(S_i)$ implies that $R_i \sim S_i$. Counting indifference as 0.5, we arrive at the choice percentages shown in Table 2 (%Choice). It is reassuring that the pattern for ladder 7 is virtually the same as for ladder 2, because the ladders involve the same questions.

An ordinal test is clearly inefficient. Thus, we also conduct a cardinal analysis using the cash equivalents directly. Table 2 provides $CE(R_i) - CE(S_i)$ for ladders 7 and 8. The ladders exhibit the same inverted U-shaped pattern as the ladders in study 1. Since all questions were run within-subject, we conducted a one-sample t-test on the paired second differences, $(CE(R_i) - CE(S_i)) - (CE(R_j) - CE(S_j))$. (This test is equivalent to the test of the interaction in a two-way analysis of variance with repeated measures.) In ladder 7, for example, the second difference between rungs 1 and 2 was statistically significant ($t = 3.11$, $df = 18$), as was the difference between rungs 2 and 3 using the same test ($t = 7.45$, $df = 18$). The second differences in ladder 8 between rungs 1 and 3 ($t = 2.85$, $df = 18$), rungs 2 and 3 ($t = 2.37$, $df = 18$), and rungs 3 and 5 ($t = 2.26$, $df = 18$) were statistically significant as well. Note that this analysis explicitly assumes a linear value function, $v(x) = x$. A sensitivity analysis indicates that the same results are statistically significant for a range of concave or convex power value functions ($0.5 \leq \alpha \leq 1.5$).

4.3. Probability Judgments

Mean and median judged probabilities are given in Table 3. If probability judgments are additive in ladder 7, then

$$\rho(DP \cap DH) + \rho(RP \cap DH) = \rho(DH).$$

Table 2   Choice Questions for Study 2

		Ladder 7								

		Cash Equivalents					E_1	E_2	E_3	E_4
Question		Median	Mean	Std. Dev.	%Choice	$CE(R_i) - CE(S_i)$	$DP \cap RH$	$DP \cap DH$	$RP \cap DH$	$RP \cap RH$
7.1	R_1	55	73	60	15%	−58	0	350	0	0
	S_1	105	131	82			300	300	0	0
7.2	R_2	100	123	90	55%	1	0	350	300	0
	S_2	100	122	85			300	300	300	0
7.3	R_3	150	173	77	3%	−126	0	350	300	300
	S_3	300	300	0			300	300	300	300

DP = A Democrat wins the Presidential election.
RP = A Republican wins the Presidential election.
DH = The Democratic Party gains control of the House of Representatives.
RH = The Republican Party gains control of the House of Representatives.

		Ladder 8											

		Cash Equivalents			$CE(R_i) -$	E_2	E_1	E_3	E_4	E_5	E_6	E_7	E_8
Question		Median	Mean	Std. Dev.	%Choice	$CE(S_i)$	$V \geq 9$	$V = 8$	$V = 7$	$V = 6$	$V = 5$	$V = 4$	$V = 3$	$V < 3$
8.1	R_1	81	88	44	38%	8.0	200	0	0	0	0	0	0	0
	S_1	79	78	28			120	120	0	0	0	0	0	0
8.2	R_2	104	110	34	56%	14.6	200	0	120	0	0	0	0	0
	S_2	96	105	25			120	120	120	0	0	0	0	0
8.3	R_3	120	120	35	88%	27.5	200	0	120	120	0	0	0	0
	S_3	99	100	22			120	120	120	120	0	0	0	0
8.4	R_4	114	110	39	66%	22.5	200	0	120	120	120	0	0	0
	S_4	98	105	27			120	120	120	120	120	0	0	0
8.5	R_5	117	120	38	66%	16.3	200	0	120	120	120	120	0	0
	S_5	106	115	21			120	120	120	120	120	120	0	0
8.6	R_6	123	120	40	66%	15.7	200	0	120	120	120	120	120	0
	S_6	114	120	14			120	120	120	120	120	120	120	0

V = number of University of Washington football victories in 1995–96 season.

To the contrary, the mean probability estimates for $DP \cap DH$, $RP \cap DH$, and DH are 0.22, 0.30, and 0.34, respectively. Figure 1 shows subadditivity at the level of individual subjects. Note that of 16 subjects, 2 are additive, and 14 are subadditive ($p < 0.001$, one sample t-test). Subadditivity is also exhibited in the following judgments (mean estimates):

$$\rho(DP) + \rho(RP \cap DH) = 0.44 + 0.30$$

$$= 0.74 > \rho(DP \cup DH) = 0.54.$$

In ladder 8, subadditivity of $\rho(\cdot)$ is exhibited on the following partition: $\rho(9^+) + \rho(7) > \rho(7 \cup 9^+)$ ($p < 0.001$, one sample t-test). As with ladder 7, there is subadditivity at the level of individual subjects. Figure 1 indicates that of 16 subjects, 13 are subadditive, 2 are additive, and 1 is superadditive. We also find subadditivity of $\rho(\cdot)$ on all other relevant partitions.

In ladder 7, we tested for binary complementarity on the binary partition "Republican wins Presidency" (RP) and "Democrat win Presidency" (DP). The mean

Table 3 Probability Estimates for Study 2

Ladder 7

Assessed Event	Probability		
	Median	Mean	Std. Dev.
$DP \cap DH$	0.20	0.22	0.14
DP	0.40	0.44	0.18
$DP \cup DH$	0.53	0.56	0.21
$RP \cap DH$	0.30	0.30	0.17
RP	0.55	0.54	0.10
DH	0.32	0.34	0.17

Ladder 8

Assessed Event	Probability		
	Median	Mean	Std. Dev.
9 or more	0.53	0.50	0.20
8 or more	0.60	0.65	0.19
7 or more	0.72	0.72	0.17
6 or more	0.78	0.78	0.17
5 or more	0.90	0.85	0.13
4 or more	0.95	0.90	0.10
3 or more	0.92	0.90	0.09
7	0.30	0.35	0.23
6 or 7	0.40	0.43	0.19
5 to 7	0.44	0.44	0.17
4 to 7	0.43	0.46	0.18
3 to 7	0.50	0.53	0.18
7 or 9 or more	0.59	0.58	0.16
6 or 7 or 9 or more	0.65	0.64	0.13
5 to 7 or 9 or more	0.63	0.64	0.15
4 to 7 or 9 or more	0.70	0.71	0.15

and median probability estimates for $\rho(RP) + \rho(DP)$ were 0.98 and 1.00, respectively, consistent with binary complementarity (n.s., $t = -0.78$, $p = 0.45$, one sample t-test).[4]

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[4] Strictly speaking, RP and DP do not constitute a binary partition. Both studies were conducted prior to the peak speculation about General Colin Powell's third party candidacy: Ladder 4 (December 1994) and Ladder 7 (August 1995). Starting in October 1994, the Iowa Electronic Markets offered a winner-take-all contract that paid $1 if neither the Democrat nor Republican nominee was elected President. The prices can be interpreted as "market probabilities." The "other candidate" contract peaked at $0.18 on September 26,

Finally, we observe a new property of $\rho(\cdot)$, interior additivity. The mean judged probability for the event "University of Washington wins nine or more games" (9^+) is 0.50. Probability estimates for the same event are also "revealed" from other estimates, e.g., $\rho(7 \cup 9^+) - \rho(7)$. Consistent with subadditivity, the mean of the revealed probability estimate is much lower, 0.23. Four other mean estimates [$\rho(6 \text{ to } 7 \cup 9+) - \rho(6 \text{ to } 7)$, $\rho(5 \text{ to } 7 \cup 9+) - \rho(5 \text{ to } 7)$, $\rho(4 \text{ to } 7 \cup 9+) - \rho(4 \text{ to } 7)$, and $\rho(3 \text{ to } 7 \cup 9+) - \rho(3 \text{ to } 7)$] obtained similarly are all between 0.20 and 0.25, and all significantly different from the mean estimate for rung 1.

4.4. Theoretical Interpretation
In both studies 1 and 2, we found that the decision weighting function $\omega(\cdot)$ is concave and then convex, as suggested by the consistent inverse U-shaped pattern in each of the ladders. Recall that these patterns are consistent with the example given in the Introduction, as well as Wu and Gonzalez's (1996) study of decision making under risk. In study 2, we also found that $\rho(\cdot)$ is a support function. In addition, interior additivity of $\rho(\cdot)$ held for events investigated in study 2, although whether interior additivity holds more broadly is an empirical question.[5]

We return to the problem of teasing apart the two sources of curvature of decision weights. Our findings could be due to some combination of concavity, subadditivity, or linearity of $\pi(\cdot)$ and of $\rho(\cdot)$. It turns out that the three empirical conclusions imply that the probability weighting function is S-shaped. Strict concavity of decision weights implies that

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1995, and was trading at $0.03 at the time of study 2 and at $0.05 during study 1.

[5] Note that interior additivity is a very strong condition. Denote $q_i = \rho(E_i)$, $r_i = \rho(E_{ij}) - \rho(E_j)$ for non-null E_j, and $\Delta_i = q_i - r_i$, the difference between direct and revealed estimates. Thus, Δ_i can be thought of as a residual or bias. Since $\rho(E_{ij}) = q_i + r_j = q_j + r_i$, $\Delta_i = \Delta_j$ for all i, j. Thus, one simple model that gives rise to interior additivity and constant Δ_i follows. Suppose that t_i is an individual's "true" subjective probability of E_i. Then the direct probability is given by $q_i = t_i + \alpha$, where $\alpha > 0$ is a constant that reflects a degree of overestimation and the revealed probability is given by $r_i = t_i - \beta$, where $\beta > 0$ is a constant that reflects a degree of underestimation.

Figure 1   Subadditivity of Probability Judgments
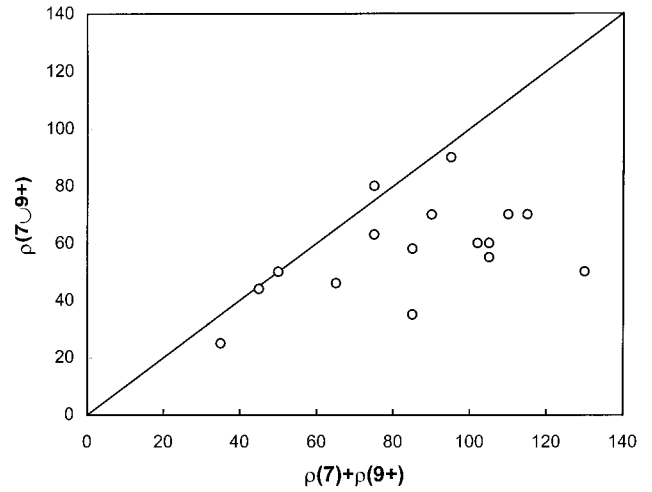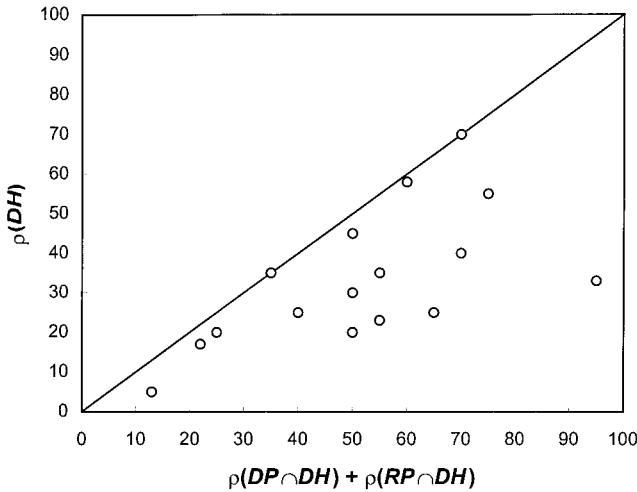
$$\omega(E_{12}) - \omega(E_2) > \omega(E_{123}) - \omega(E_{23})$$

$$> \omega(E_{1234}) - \omega(E_{234}),$$

or

$$\pi(\rho(E_{12})) - \pi(\rho(E_2)) > \pi(\rho(E_{123})) - \pi(\rho(E_{23}))$$

$$> \pi(\rho(E_{1234})) - \pi(\rho(E_{234})). \qquad (4.1)$$

Defining

$$r_1 = \rho(E_{12}) - \rho(E_2) = \rho(E_{123}) - \rho(E_{23})$$

$$= \rho(E_{1234}) - \rho(E_{234})$$

by interior additivity; and $r_3 = \rho(E_{23}) - \rho(E_2) > 0$ and $r_4 = \rho(E_{234}) - \rho(E_{23}) > 0$ by monotonicity of $\rho(\cdot)$, we can rewrite (4.1) as

$$\pi(\rho(E_2) + r_1) - \pi(\rho(E_2))$$

$$> \pi(\rho(E_2) + r_1 + r_3) - \pi(\rho(E_2) + r_3)$$

$$> \pi(\rho(E_2) + r_1 + r_3 + r_4) - \pi(\rho(E_2) + r_3 + r_4). \qquad (4.2)$$

Equation (4.2) is implied by concavity of $\pi(\cdot)$ below $\rho(E_{1234})$. Thus interior additivity proves to be a *crucial* concept: it allows us to attribute part of the inverse U-shaped pattern in studies 1 and 2 to curvature of $\pi(\cdot)$. We thus conclude that the probability weighting function is concave and then convex in the domain of uncertainty, as it is in risk.

5. Conclusion

We found that individuals violate SEU in a systematic fashion. The patterns observed in studies 1 and 2 resemble the findings of Wu and Gonzalez (1996) in the domain of risk. This resemblance suggests the possibility of general principles underlying behavior in both domains. We, like Tversky and Fox (1995) and Fox and Tversky (1998), are encouraged that the same basic concepts and principles seem to underlie both risk and uncertainty. In fact, one psychological principle, diminishing sensitivity, explains a great deal in both domains. The various functions, $\pi(\cdot)$, $\rho(\cdot)$, and $v(\cdot)$, all appear to exhibit diminishing sensitivity, a decrease in marginal effect as the distance from a reference point increases. With $\pi(\cdot)$, the reference points seem to be 0, impossibility, and 1, certainty; with $\rho(\cdot)$, the reference points seem to be ϕ, the null set, and S, the entire set; with $v(\cdot)$, the reference point seems to be the "0" point or status quo.

Thus, even though some have argued that gambles are too barren to have any implications for real world decision making, the present studies suggest that risk illuminates a basic fact about decision making. The decision weighting function is inverse S-shaped (concave/convex) for both risk and uncertainty. However, our analysis also adds substance to the critique about gambles. One reason gambles are barren is because they do not consider the first phase of the two-stage

model: the probability assessment phase. Gambles provide an adequate abstraction if one is interested in studying the probability weighting function but are insufficiently rich to permit a complete understanding of decision making. We proposed a two-stage model, presented a new property of judged probabilities, which we call interior additivity, and showed how interior additivity permits the contributions of the two stages to be disentangled.[6]

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