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NONLINEAR DEGENERATE EVOLUTION EQUATIONS AND PARTIAL DIFFERENTIAL EQUATIONS OF MIXED TYPE*

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Abstract. The Cauchy problem for the evolution equation $Mu'(t) + N(t, u(t)) = 0$ is studied, where M and $N(t, \cdot)$ are, respectively, possibly degenerate and nonlinear monotone operators from a vector space to its dual. Sufficient conditions for existence and for uniqueness of solutions are obtained by reducing the problem to an equivalent one in which M is the identity but each $N(t, \cdot)$ is multivalued and accretive in a Hilbert space. Applications include weak global solutions of boundary value problems with quasilinear partial differential equations of mixed Sobolev-parabolic-elliptic type, boundary conditions with mixed space-time derivatives, and those of the fourth or fifth type. Similar existence and uniqueness results are given for the semilinear and degenerate wave equation $Bu''(t) + F(t, u'(t)) + Au(t) = 0$, where each nonlinear $F(t, \cdot)$ is monotone and the nonnegative B and positive A are self-adjoint operators from a reflexive Banach space to its dual.

1. Introduction. Suppose we are given a nonnegative and symmetric linear operator $\mathcal{M}$ from a vector space E into its (algebraic) dual E^*. This is equivalent to specifying the nonnegative and symmetric bilinear form $m(x, y) = \langle \mathcal{M}x, y \rangle$ on E, where the brackets denote $E^* - E$ duality. Since m is a semiscalar-product on E, we have a (possibly non-Hausdorff) topological vector space (E, m) with seminorm "$x \to m(x, x)^{1/2}$", and its dual $(E, m)' = E'$ is a Hilbert space which contains the range of $\mathcal{M}$. We let $\mathcal{N}(t, \cdot)$ be a family of (possibly) nonlinear functions from E into E^*, $0 \leqq t \leqq T$, and consider the evolution equation

$$\frac{d}{dt}(\mathcal{M}u(t)) + \mathcal{N}(t, u(t)) = 0, \qquad 0 \leqq t \leqq T. \tag{1.1}$$

By a *solution* of (1.1) we mean a function $u:[0, T] \to E$ such that $\mathcal{M}u:[0, T] \to E'$ is absolutely continuous (hence, differentiable almost everywhere), with $\mathcal{N}(t, u(t)) \in E'$ for all t, and (1.1) is satisfied at almost every $t \in [0, T]$. The *Cauchy problem* is to find a solution u of (1.1) for which $\mathcal{M}u(0)$ is specified in E'.

The plan of the paper is as follows. In § 2 we use elementary linear algebra to show that (1.1) is equivalent to an evolution problem essentially of the form

$$-u'(t) \in \mathcal{M}^{-1} \circ \mathcal{N}(t, u(t)), \tag{1.2}$$

where $\mathcal{M}^{-1}$ denotes the (possibly) multivalued operator or relation that is inverse to $\mathcal{M}$. Our main results on the existence and uniqueness of solutions of (1.1) (or (1.2)) are stated and proved in § 3, and provide a natural application of nonlinear evolution problems with multivalued operators. Section 4 gives some applications of our results to various nonlinear boundary value problems which may contain derivatives in time of at most first order. Each such problem is reduced to (1.1) in an appropriate space. The examples include boundary value problems for

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equations of the form

$$\frac{\partial}{\partial t}\left(m_0(x)u(x,t) - \frac{\partial}{\partial x}\left(m(x)\frac{\partial u}{\partial x}\right)\right) - \frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right) = 0,$$

where $m_0(x) \geqq 0$, $m(x) \geqq 0$, and $p \geqq 2$. A first order time derivative may also appear in boundary conditions such as in boundary value problems of the fourth and fifth type. In § 5 we study the abstract wave equation

$$Bu''(t) + F(t, u'(t)) + Au(t) = 0, \tag{1.3}$$

where A and B are self-adjoint with A strictly positive and B nonnegative and each $F(t,\cdot)$ is monotone. When the operators in (1.3) are realizations of partial differential equations, we obtain results on the solvability of (e.g.)

$$\frac{\partial^2}{\partial t^2}\left(m_0(x)u(x,t) - \sum_{j=1}^{n}\frac{\partial}{\partial x_j}m_j(x)\frac{\partial u}{\partial x_j}\right) + \left|\frac{\partial u}{\partial t}\right|^{p-1}\frac{\partial u}{\partial t} - \Delta u = f,$$

wherein each m_j is nonnegative and bounded, and $p \geqq 2$, and boundary conditions may contain second order time derivatives.

Abstract equations of the form (1.1) have been considered by C. Bardos, H. Brezis, O. Grange and F. Mignot, H. Levine, J.-L. Lions, M. Visik and this writer. Our results of § 3 are closest to those in § 5 of [3] and those of [14], while in [4] it is assumed the leading operator in (1.1) is bounded from a Hilbert space into itself. The writers in [21, pp. 69–73] and [29] consider linear equations with time-dependent operators uniformly bounded from below by a positive quantity, hence, nondegenerate, and this last assumption was removed in [28]. Each of the preceding works has been directed toward the solution of boundary value problems, many of which have been studied by more direct methods. We refer the reader to the extensive bibliographies of [26], [27] for the theory and application of nondegenerate equations with mixed space and time derivatives (i.e., of Sobolev type) and to [7], [28] for additional references to (degenerate) mixed elliptic-parabolic type. See [24] for a treatment of (1.3) when B is the identity.

2. Two Cauchy problems. Let m denote the nonnegative and symmetric bilinear form given on the vector space E. Let K be the kernel of m, i.e., the subspace of those $x \in E$ with $m(x,x) = 0$, and denote the corresponding quotient space by E/K. Then the quotient map $q: E \to E/K$ given by

$$q(x) = \{y \in E : m(x-y, x-y) = 0\}$$

is a linear surjection, and it determines a scalar product $\mathbf{m}$ on E/K by

$$\mathbf{m}(q(x), q(y)) = m(x,y), \qquad x, y \in E. \tag{2.1}$$

The completion of $(E/K, \mathbf{m})$ is a Hilbert space W whose scalar product is the extension by continuity of $\mathbf{m}$, and we denote this extension also by $\mathbf{m}$.

Let E' denote the strong dual of the seminormed topological vector space (E, m). E' is a Hilbert space which is important in the discussion below, so we consider it briefly. Letting $(E/K)'$ and W' denote the duals of the indicated scalar product space and Hilbert space, respectively, and noting that E/K is dense in W, we have each $f \in W'$ uniquely determined by its restriction to E/K. This restriction gives a bijection of W' onto $(E/K)'$ and we hereafter identify these spaces. Regard q as a map from E into W. Its dual is the linear map $q^*: W' \to E'$ defined by

$$\langle q^*(f), x \rangle = \langle f, q(x) \rangle, \qquad f \in W', \quad x \in E. \tag{2.2}$$

Since $q(E) = E/K$ is dense in W, q^* is injective. Furthermore, each $g \in E'$ necessarily vanishes on K, so there is a unique element $f \in (E/K)'$ with $f \circ q = g$. That is, $g = q^*(f)$, so q^* is a bijection of W' onto E'. It follows from (2.1) and (2.2) that q^* is norm-preserving.

We easily relate the linear map $\mathcal{M}: E \to E'$ given to us by

$$\langle \mathcal{M}x, y \rangle = m(x, y), \qquad x, y \in E,$$

to the Hilbert space isomorphism $\mathcal{M}_0: W \to W'$ of F. Riesz defined by

$$\langle \mathcal{M}_0 x, y \rangle = \mathbf{m}(x, y), \qquad x, y \in W.$$

For any pair $x, y \in E$ we have $\langle q^*\mathcal{M}_0 qx, y \rangle = \langle \mathcal{M}_0 qx, qy \rangle = \mathbf{m}(qx, qy) = m(x, y) = \langle \mathcal{M}x, y \rangle$, and, hence,

$$\mathcal{M} = q^*\mathcal{M}_0 q. \tag{2.3}$$

The notion of a relation $\mathcal{R}$ on a Cartesian product $X \times Y$ of linear spaces will be essential. A *relation* $\mathcal{R}$ on $X \times Y$ is a subset of $X \times Y$. For each $x \in X$, the *image* of x by $\mathcal{R}$ is the set $\mathcal{R}(x) = \{y \in Y : [x, y] \in \mathcal{R}\}$, and the *domain* of $\mathcal{R}$ is the set of $x \in X$ for which $\mathcal{R}(x)$ is nonempty. The *range* of $\mathcal{R}$ is $\bigcup\{\mathcal{R}(x) : x \in X\}$. The graph of every function from a subset of X into Y is a relation on $X \times Y$, and we so identify functions as relations. The *inverse* of $\mathcal{R}$ is the relation $\mathcal{R}^{-1} = \{[y, x] : [x, y] \in \mathcal{R}\}$ on $Y \times X$. If a is a real number, we define

$$a\mathcal{R} = \{[x, ay] : [x, y] \in \mathcal{R}\}.$$

If $\mathcal{S}$ is a second relation on $X \times Y$, then

$$\mathcal{R} + \mathcal{S} = \{[x, y + z] : [x, y] \in \mathcal{R} \text{ and } [x, z] \in \mathcal{S}\}.$$

If $\mathcal{T}$ is a relation on $Y \times Z$, then the *composition* of $\mathcal{R}$ and $\mathcal{T}$ is

$$\mathcal{T} \circ \mathcal{R} = \{[x, z] : [x, y] \in \mathcal{R} \text{ and } [y, z] \in \mathcal{T} \text{ for some } y \in Y\}.$$

If $\mathcal{P}$ is a relation on $W \times X$, then composition is associative, i.e.,

$$(\mathcal{T} \circ \mathcal{R}) \circ \mathcal{P} = \mathcal{T} \circ (\mathcal{R} \circ \mathcal{P}).$$

Also, we identify the identity function I_Y on Y with its graph $\{[y, y] : y \in Y\}$, and easily obtain the inclusion $\mathcal{R} \circ \mathcal{R}^{-1} \supseteq I_Y$. These sets are equal if (and only if) $\mathcal{R}$ is a function, i.e., each $\mathcal{R}(x)$ is a singleton. Finally we note that

$$(\mathcal{T} \circ \mathcal{R})^{-1} = \mathcal{R}^{-1} \circ \mathcal{T}^{-1}.$$

Suppose that for each $t \in [0, T]$ we are given a (not necessarily linear) function $\mathcal{N}(t): E \to E^*$. Define a corresponding relation $\mathcal{N}_0(t)$ on $W \times W'$ as follows: $[w, f] \in \mathcal{N}_0(t)$ if and only if there is an $x \in E$ such that $q(x) = w$ and $\mathcal{N}(t, x) = q^*(f)$. Since q^* is onto E', it follows that the domain of $\mathcal{N}_0(t)$ is precisely the image $q(D(t))$, where we define $D(t) = \{x \in E : \mathcal{N}(t, x) \in E'\}$. Also, for each $t \in [0, T]$ and

$x \in D(t)$, there is exactly one $f \in W'$ with $\mathscr{N}(t,x) = q^*(f)$, so we have

$$\mathscr{N}(t,x) = q^* \circ \mathscr{N}_0(t) \circ q(x), \qquad 0 \leqq t \leqq T, \quad x \in D(t). \tag{2.4}$$

Finally, we define a family of composite relations on $W \times W$ by $\mathscr{A}(t) = \mathscr{M}_0^{-1} \circ \mathscr{N}_0(t)$, $t \in [0, T]$. That is, $[x, z] \in \mathscr{A}(t)$ if and only if there is a $y \in W'$ for which $[x, y] \in \mathscr{N}_0(t)$ and $y = \mathscr{M}_0 z$. Since $\mathscr{M}_0$ is a bijection, $\mathscr{A}(t)$ and $\mathscr{N}_0(t)$ have the same domain, $q(D(t))$.

Remark 1. Note that $\mathscr{N}_0(t)$ is a function (as is $\mathscr{A}(t)$) if and only if $\mathscr{N}(t,x) = \mathscr{N}(t,y)$ for every pair $x, y \in E$ such that $\mathscr{N}(t,x)$ and $\mathscr{N}(t,y)$ belong to E' and $\mathscr{M}x = \mathscr{M}y$. This is frequently (but not always) the case in applications, even where $\mathscr{M}$ is not injective.

We shall relate solutions of the evolution equation (1.1) to those of an evolution problem determined by the relations $\mathscr{A}(t)$, $0 \leqq t \leqq T$. A function $v:[0,T] \to W$ is called a *solution* of the evolution problem

$$v'(t) + \mathscr{A}(t, v(t)) \ni 0, \qquad 0 \leqq t \leqq T, \tag{2.5}$$

if it is (strongly) absolutely continuous (hence, differentiable a.e.), $v(t) \in q(D(t))$ for every t, and (2.5) is satisfied at a.e. t. Since the domain of each $\mathscr{A}(t)$ is contained in E/K and since the maps $\mathscr{M}_0: W \to W'$ and $q^*: W' \to E'$ are linear isometries, it follows that v is a solution of (2.5) if and only if $v:[0,T] \to E/K$, is absolutely continuous, (hence, $q^*\mathscr{M}_0 v:[0,T] \to E'$ is differentiable a.e.), $v(t) \in q(D(t))$ for every t, and

$$(q^*\mathscr{M}_0 v(t))' \in -q^*\mathscr{N}_0(t, v(t))$$

at a.e. t. Let v be such a solution, and for each $t \in [0,T]$ choose a representative $u(t) \in D(t)$ from the coset $v(t) \in E/K$. Then $q(u(t)) = v(t)$, $\mathscr{M}u(t) = q^*\mathscr{M}_0 v(t)$ and $\mathscr{N}(t, u(t)) = q^*\mathscr{N}_0(t, v(t))$ for each t, so u is a solution of (1.1). Conversely, if u is any solution of (1.1), then the function $v \equiv q \circ u$ is a solution of (2.5), so we have the following result.

PROPOSITION 1. *If v is a solution of* (2.5) *and for each $t \in [0,T]$, $u(t) \in D(t)$ belongs to the coset $v(t) \in E/K$, then u is a solution of* (1.1). *Conversely, if u is a solution of* (1.1), *then $v \equiv q \circ u$ is a solution of* (2.5).

COROLLARY 1. *Let $u_0 \in D(0)$. Then there exists a solution v of* (2.5) *with $v(0) = q(u_0)$ if and only if there exists a solution u of* (1.1) *with $\mathscr{M}u(0) = \mathscr{M}u_0$.*

COROLLARY 2. *Let $u_0 \in D(0)$. Then there is at most one solution v of* (2.5) *with $v(0) = q(u_0)$ if and only if for every pair of solutions u_1, u_2 of* (1.1) *with $\mathscr{M}u_1(0) = \mathscr{M}u_2(0) = \mathscr{M}u_0$, we have $\mathscr{M}u_1(t) = \mathscr{M}u_2(t)$ for all $t \in [0,T]$, hence*

$$\mathscr{N}(t, u_1(t)) = \mathscr{N}(t, u_2(t)).$$

Remark 2. In the situation of Corollary 2, uniqueness holds for solutions of the Cauchy problem for (1.1) if for each $t \in [0,T]$ and each pair $x, y \in E$, $\mathscr{M}x = \mathscr{M}y$ and $\mathscr{N}(t,x) = \mathscr{N}(t,y) \in E'$ imply that $x = y$.

Example. Take $E = \mathbb{R}^2 = E^*$ with $\langle [x_1, x_2], [y_1, y_2] \rangle = x_1 y_1 + x_2 y_2$. Let $\mathscr{M}([x_1, x_2]) = [x_1, 0]$ and $\mathscr{N}(t)[x_1, x_2] = [x_2, -x_1]$. Then the kernel of $\mathscr{M} + \mathscr{N}(t)$ is null, so uniqueness holds. Note, however, $\mathscr{N}_0(t)$ is *not* a function. This corresponds to the (trivial) evolution equation

$$u_1'(t) = 0, \quad u_2(t) = 0, \qquad t \geqq 0,$$

for $u(t) \equiv [u_1(t), u_2(t)]$.

3. Existence and uniqueness. Evolution problems of the form (2.5) have been considered by many writers, and we refer to the recent work of M. Crandall and A. Pazy [10] and J. Dorroh [12] for references in this direction. In particular, a sufficient condition for uniqueness of solutions of Cauchy problems associated with (2.5) is that each $\mathscr{A}(t)$ be accretive.

DEFINITION 1. A relation $\mathscr{A}$ on $W \times W$ is *accretive* if for every pair $[x_1, w_1]$ and $[x_2, w_2]$ in $\mathscr{A}$ we have $\mathbf{m}(w_1 - w_2, x_1 - x_2) \geqq 0$.

A related notion is that of monotonicity for functions (or relations) mapping a subset of a vector space into its dual. Such a condition holds for many operators associated with the variational formulation of (possibly nonlinear) elliptic boundary value problems [5], [7].

DEFINITION 2. Let D be a subset of the vector space E and denote by E^* the (algebraic) dual of E. A function $\mathscr{N}: D \to E^*$ is *D-monotone* if for each pair x_1, $x_2 \in D$ we have $\langle \mathscr{N}(x_1) - \mathscr{N}(x_2), x_1 - x_2 \rangle \geqq 0$.

Because of our intended applications, it is essential that D-monotonicity of each $\mathscr{N}(t)$ imply accretiveness of the corresponding $\mathscr{A}(t)$, where $\mathscr{N}(t)$ and $\mathscr{A}(t)$ are the functions and relations of § 2. This was our motivation for portions of the construction in § 2, and its success in this direction is reflected in the following central result.

PROPOSITION 2. *For each $t \in [0, T]$ let $\mathscr{N}(t)$ and $\mathscr{A}(t)$ be the respective function and relation of* § 2. *Then $\mathscr{A}(t)$ is accretive if and only if $\mathscr{N}(t)$ is $D(t)$-monotone.*

Proof. Let $[x_1, w_1]$ and $[x_2, w_2]$ belong to $\mathscr{A}(t)$. Then there are u_1, u_2 in $D(t)$ such that $x_j = q(u_j)$ and $\mathscr{N}(t, u_j) = q^*\mathscr{M}_0 w_j$, $j = 1, 2$. Thus we have

$$\begin{aligned}\mathbf{m}(w_1 - w_2, x_1 - x_2) &= \langle \mathscr{M}_0(w_1 - w_2), q(u_1 - u_2)\rangle \\ &= \langle q^*\mathscr{M}_0(w_1 - w_2), u_1 - u_2\rangle \\ &= \langle \mathscr{N}(t, u_1) - \mathscr{N}(t, u_2), u_1 - u_2\rangle.\end{aligned}$$

Hence, if $\mathscr{N}(t)$ is $D(t)$-monotone, then $\mathscr{A}(t)$ is accretive.

Conversely, if u_1, $u_2 \in D(t)$, there is a unique pair w_j $(j = 1, 2)$ in W with $\mathscr{N}(t, u_j) = q^*\mathscr{M}_0 w_j$. Then $[q(u_j), w_j] \in \mathscr{A}(t)$ and, as above, $\langle \mathscr{N}(t, u_1) - \mathscr{N}(t, u_2), u_1 - u_2\rangle = \mathbf{m}(w_1 - w_2, x_1 - x_2)$, so $\mathscr{A}(t)$ being accretive implies $\mathscr{N}(t)$ is $D(t)$-monotone.

DEFINITION 3. If in the definition of accretive (or D-monotone) the inequality is strict whenever $x_1 \neq x_2$, then we say that $\mathscr{A}$ is *strictly accretive* (respectively, $\mathscr{N}$ is *strictly D-monotone*).

If u_1, $u_2 \in D(t)$, then $\mathscr{N}(t, u_1) - \mathscr{N}(t, u_2) \in E'$, so there is a constant k such that

$$|\langle \mathscr{N}(t, u_1) - \mathscr{N}(t, u_2), e\rangle| \leqq km(e, e)^{1/2}, \qquad e \in E.$$

If $\mathscr{M}u_1 = \mathscr{M}u_2$, then setting $e = u_1 - u_2$ in the above identity shows that $\langle \mathscr{N}(t, u_1) - \mathscr{N}(t, u_2), u_1 - u_2\rangle = 0$. Thus, if $\mathscr{N}(t)$ is strictly $D(t)$-monotone then $u_1 = u_2$, and Remark 1 shows that $\mathscr{A}(t)$ is a function. The first part of the proof of Proposition 2 shows then that $\mathscr{A}(t)$ is strictly accretive. Conversely, if $\mathscr{A}(t)$ is a strictly accretive function, and if $\langle \mathscr{N}(t, u_1) - \mathscr{N}(t, u_2), u_1 - u_2\rangle = 0$ in the second part of the proof of Lemma 1, then $w_1 = w_2$, hence $\mathscr{N}(t, u_1) = \mathscr{N}(t, u_2)$. These remarks prove the following.

COROLLARY 1. *In the situation of Proposition* 2, $\mathcal{N}(t)$ *is strictly* $D(t)$*-monotone if and only if it is injective and* $\mathcal{A}(t)$ *is a strictly accretive function.*

From Remark 2 and the preceding argument applied to $\mathcal{M} + \mathcal{N}(t)$, we obtain a sufficient condition for uniqueness.

THEOREM 1. *Let* $\mathcal{N}(t)$ *be* $D(t)$*-monotone and let* $\mathcal{M} + \mathcal{N}(t)$ *be strictly* $D(t)$*-monotone for each* $t \in [0, T]$. *Then, for each* $u_0 \in D(0)$, *there is at most one solution* u *of* (1.1) *for which* $\mathcal{M}u(0) = \mathcal{M}u_0$.

We turn now to the considerably more difficult question of existence. Our results in this direction will be obtained from recent results of M. Crandall, T. Liggett and A. Pazy on the existence of solutions of evolution problems like (2.5) in general Banach space [9], [10]. We shall present the special case of their results as they apply to Hilbert space and obtain through Proposition 1 a corresponding set of sufficient conditions for the existence of a solution of (1.1).

To begin, we shall describe the existence results of [10] that are relevant for (2.5). We assume that $\mathcal{A}(t)$ is accretive and that the range of $I + \mathcal{A}(t)$ is all of W for every $t \in [0, T]$ (Each $\mathcal{A}(t)$ is *m-accretive* [16] or *hyper-accretive* [12]). It follows that the range of $I + \lambda\mathcal{A}(t)$ is W for every $\lambda > 0$, so its inverse

$$J_\lambda(t) \equiv (I + \lambda\mathcal{A}(t))^{-1}, \qquad \lambda > 0,$$

is a function defined on all of W. The dependence on t will be restricted in two ways. First, the domain of $\mathcal{A}(t)$ is independent of t, and we shall denote it by $q(D)$. Second, there is a monotone increasing function $L:[0, \infty) \to [0, \infty)$ such that

$$\|J_\lambda(t, x) - J_\lambda(\tau, x)\|_W \leqq \lambda|t - \tau|L(\|x\|_W)(1 + \inf\{\|y\|_W : [x, y] \in \mathcal{A}(\tau)\}), \tag{3.1}$$
$$t, \tau \geqq 0, \quad x \in W, \quad 0 < \lambda \leqq 1.$$

It is shown in [10], under hypotheses somewhat more general than those above, that for each $v_0 \in q(D)$ there exists a (unique) solution v of (2.5) with $v(0) = v_0$.

The preceding result will be used to prove the following.

THEOREM 2. *Let the nonnegative and symmetric linear operator* $\mathcal{M}$ *from the vector space* E *into its dual* E^* *be given. Let* E' *denote the dual of the topological vector space* E *with the seminorm* $\langle \mathcal{M}x, x\rangle^{1/2}$; E' *is a Hilbert space with norm*

$$\|f\|_{E'} = \sup\{|\langle f, x\rangle| : x \in E, \langle \mathcal{M}x, x\rangle \leqq 1\}.$$

For each $t \in [0, T]$ *let* $\mathcal{N}(t): E \to E^*$ *be a (possibly nonlinear) function, and define* $D(t) = \{x \in E : \mathcal{N}(t, x) \in E'\}$. *Assume the following: for each* t, $\mathcal{N}(t)$ *is* $D(t)$*-monotone and the range of* $\mathcal{M} + \mathcal{N}(t)$ *contains* E'; $\mathcal{M}(D(t))$ *is independent of* t; *and there is a monotone increasing function* $L:[0, \infty) \to [0, \infty)$ *such that*

$$\|\mathcal{N}(t, w) - \mathcal{N}(\tau, w)\|_{E'} \leqq |t - \tau|L(\langle \mathcal{M}w, w\rangle)(1 + \|\mathcal{N}(t, w)\|_{E'}), \tag{3.2}$$
$$t, \tau \geqq 0, \quad w \in D(t).$$

Then, for each $u_0 \in D(0)$, *there exists a solution* u *of* (1.1) *with* $\mathcal{M}u(0) = \mathcal{M}u_0$.

Proof. From Proposition 1 it follows that we need only to verify that the relations $\mathcal{A}(t)$ constructed in § 2 satisfy the conditions listed above. Proposition 2 shows that each $\mathcal{A}(t)$ is accretive, and $\mathcal{M}(D(t))$ being constant implies that the domain $q(D(t))$ of $\mathcal{A}(t)$ is constant. Since $q^*\mathcal{M}_0$ maps W *onto* E', the identity

$$(\mathcal{M} + \lambda\mathcal{N}(t))(x) = q^* \circ \mathcal{M}_0 \circ (I + \lambda\mathcal{A}(t)) \circ q(x), \qquad x \in E, \tag{3.3}$$

shows that $I + \lambda\mathscr{A}(t)$ maps $q(D(t))$ onto W if (and only if) $\mathscr{M} + \lambda\mathscr{N}(t)$ maps $D(t)$ onto E'. Thus, we need only to verify the estimate (3.1).

Before proceeding to the verification of (3.1), we obtain some identities and estimates. First, we recall $q^*\mathscr{M}_0$ is an isomorphism of W onto E', and

$$\|q^*\mathscr{M}_0 w\|_{E'} = \|w\|_W, \qquad w \in W. \tag{3.4}$$

Also, we have from (2.1) and (2.3),

$$\|\mathscr{M}x\|_{E'} = \langle \mathscr{M}x, x\rangle^{1/2}, \qquad x \in E. \tag{3.5}$$

From (2.3), (2.4) and (3.3) follow the identities

$$\mathscr{A}(t)q = (q^*\mathscr{M}_0)^{-1}\mathscr{N}(t), \tag{3.6}$$

$$I + \lambda\mathscr{A}(t) = (q^*\mathscr{M}_0)^{-1}q^*(\mathscr{M}_0 + \lambda\mathscr{N}_0(t)). \tag{3.7}$$

This last result with the properties of relations mentioned in § 2 (e.g., $q \circ q^{-1} = I_{E/K}$) gives us

$$\begin{aligned} J_\lambda(t) &= [q^*(\mathscr{M}_0 + \lambda\mathscr{N}_0(t))]^{-1}q^*\mathscr{M}_0 \\ &= q(\mathscr{M} + \lambda\mathscr{N}(t))^{-1}q^*\mathscr{M}_0 \\ &= (q^*\mathscr{M}_0)^{-1}\mathscr{M}(\mathscr{M} + \lambda\mathscr{N}(t))^{-1}(q^*\mathscr{M}_0). \end{aligned} \tag{3.8}$$

This shows that $\mathscr{M}(\mathscr{M} + \lambda\mathscr{N}(t))^{-1}$ is a function. Thus, if $x_1,\ x_2 \in D(t)$ and $(\mathscr{M} + \lambda\mathscr{N}(t))x_1 = (\mathscr{M} + \lambda\mathscr{N}(t))x_2$, then $\mathscr{M}x_1 = \mathscr{M}x_2$. Furthermore, this shows $\mathscr{N}(t)x_1 = \mathscr{N}(t)x_2$, so $\mathscr{N}(t)(\mathscr{M} + \lambda\mathscr{N}(t))^{-1}$ also is a function on E'. From (3.8) we now obtain

$$\lambda^{-1}(I - J_\lambda(t)) = (q^*\mathscr{M}_0)^{-1}\mathscr{N}(t)(\mathscr{M} + \lambda\mathscr{N}(t))^{-1}(q^*\mathscr{M}_0). \tag{3.9}$$

Since $\mathscr{M}(\mathscr{M} + \lambda\mathscr{N}(t))^{-1}$ is a function, we have

$$\mathscr{M}(\mathscr{M} + \lambda\mathscr{N}(t))^{-1}(\mathscr{M} + \lambda\mathscr{N}(t))x = \mathscr{M}x, \qquad x \in D(t). \tag{3.10}$$

Also, $\mathscr{M} + \lambda\mathscr{N}(t)$ is a function and so follows

$$(\mathscr{M} + \lambda\mathscr{N}(t))(\mathscr{M} + \lambda\mathscr{N}(t))^{-1} = I_{E'}. \tag{3.11}$$

We recall that each $J_\lambda(t)$ is a contraction on W, i.e.,

$$\|J_\lambda(t)w_1 - J_\lambda(t)w_2\|_W \leqq \|w_1 - w_2\|_W, \qquad w_1, w_2 \in W,$$

and this implies through (3.4) and (3.8),

$$\|\mathscr{M}(\mathscr{M} + \lambda\mathscr{N}(t))^{-1}x_1 - \mathscr{M}(\mathscr{M} + \lambda\mathscr{N}(t))^{-1}x_2\|_{E'} \leqq \|x_1 - x_2\|_{E'}, \qquad x_1, x_2 \in E'. \tag{3.12}$$

That is, $\mathscr{M}(\mathscr{M} + \lambda\mathscr{N}(t))^{-1}$ is a contraction on E'. Also, (3.6) shows that $\mathscr{A}(t)q$ is a function and

$$\|\mathscr{A}(t)q(x)\|_W = \|\mathscr{N}(t)x\|_{E'}, \qquad x \in D(t), \tag{3.13}$$

and the identity [10]

$$\|\lambda^{-1}(I - J_\lambda(t))w\|_W \leqq \inf\{\|y\|_W : [w, y] \in \mathscr{A}(t)\}, \qquad w \in q(D),$$

together with $w = q(x)$ and (3.9) gives

$$\|\mathcal{N}(t)(\mathcal{M} + \lambda\mathcal{N}(t))^{-1}\mathcal{M}x\|_{E'} \leqq \|\mathcal{N}(t)x\|_{E'}, \qquad x \in D(t),$$

so we have the estimate

$$\|\mathcal{N}(t)(\mathcal{M} + \lambda\mathcal{N}(t))^{-1}\mathcal{M}x\|_{E'} \leqq \inf\{\|\mathcal{N}(t)y\|_{E'} : q(x - y) = 0\}, \qquad x \in D(t). \tag{3.14}$$

After this lengthy preparation, we are ready to verify (3.1) and thus complete the proof of Theorem 2. Let $t, \tau \in [0, T]$, $0 < \lambda \leqq 1$, and $x \in E$. From (3.4) and (3.8) we have

$$\|J_\lambda(\tau)q(x) - J_\lambda(t)q(x)\|_W = \|\mathcal{M}(\mathcal{M} + \lambda\mathcal{N}(\tau))^{-1}\mathcal{M}x - \mathcal{M}(\mathcal{M} + \lambda\mathcal{N}(t))^{-1}\mathcal{M}x\|_{E'}.$$

Using (3.10), we have this quantity given by

$$\|\mathcal{M}(\mathcal{M} + \lambda\mathcal{N}(t))^{-1}(\mathcal{M} + \lambda\mathcal{N}(t))(\mathcal{M} + \lambda\mathcal{N}(\tau))^{-1}\mathcal{M}x - \mathcal{M}(\mathcal{M} + \lambda\mathcal{N}(t))^{-1}\mathcal{M}x\|_{E'}.$$

Let $w \in (\mathcal{M} + \lambda\mathcal{N}(\tau))^{-1}\mathcal{M}(x)$. Then from (3.12) it follows that the above is bounded by

$$\|(\mathcal{M} + \lambda\mathcal{N}(t))(w) - \mathcal{M}x\|_{E'}.$$

By (3.11), this is equal to

$$\begin{aligned}
&\|(\mathcal{M} + \lambda\mathcal{N}(t))w - (\mathcal{M} + \lambda\mathcal{N}(\tau))(\mathcal{M} + \lambda\mathcal{N}(\tau))^{-1}\mathcal{M}x\|_{E'} \\
&\quad = \|(\mathcal{M} + \lambda\mathcal{N}(t))w - (\mathcal{M} + \lambda\mathcal{N}(\tau))w\|_{E'} \\
&\quad = \lambda\|\mathcal{N}(t, w) - \mathcal{N}(\tau, w)\|_{E'}.
\end{aligned}$$

From our hypothesis (3.2), the estimate (3.14), and (3.5) we now obtain

$$\begin{aligned}
\|J_\lambda(\tau)q(x) - J_\lambda(t)q(x)\|_W &\leqq \lambda|t - \tau|L(\|\mathcal{M}w\|_{E'}^2) \\
&\qquad \cdot (1 + \|\mathcal{N}(\tau)(\mathcal{M} + \lambda\mathcal{N}(\tau))^{-1}\mathcal{M}x\|_{E'}) \\
&\leqq \lambda|t - \tau|L(\|\mathcal{M}(\mathcal{M} + \lambda\mathcal{N}(\tau))^{-1}\mathcal{M}x\|_{E'}^2) \\
&\qquad \cdot (1 + \inf\{\|\mathcal{N}(\tau)y\|_{E'} : q(x - y) = 0\})
\end{aligned} \tag{3.15}$$

if $q(x) \in q(D)$. In order to estimate the term involving L in the above, we pick x_0 with $q(x_0) \in q(D)$ and then use (3.10) and (3.12) to obtain

$$\begin{aligned}
&\|\mathcal{M}(\mathcal{M} + \lambda\mathcal{N}(\tau))^{-1}\mathcal{M}x - \mathcal{M}x_0\|_{E'} \\
&\quad = \|\mathcal{M}(\mathcal{M} + \lambda\mathcal{N}(\tau))^{-1}\mathcal{M}x - \mathcal{M}(\mathcal{M} + \lambda\mathcal{N}(\tau))^{-1}(\mathcal{M} + \lambda\mathcal{N}(\tau))x_0\|_{E'} \\
&\quad \leqq \|\mathcal{M}x - (\mathcal{M} + \lambda\mathcal{N}(\tau))x_0\|_{E'}.
\end{aligned}$$

From (3.2) it follows that

$$K_T \equiv \sup\{\|\mathcal{N}(t, x_0)\|_{E'} : 0 \leqq t \leqq T\} + 2\|\mathcal{M}x_0\|_{E'} < \infty,$$

and so we have

$$\|\mathcal{M}(\mathcal{M} + \lambda\mathcal{N}(\tau))^{-1}\mathcal{M}x\|_{E'} \leqq \|\mathcal{M}x\|_{E'} + K_T, \qquad q(x) \in q(D), \quad 0 < \lambda \leqq 1.$$

This estimate together with (3.13) and (3.5) show that (3.15) implies (3.1) with L replaced by the monotone increasing function

$$L_1(\xi) = L((\xi + K_T)^2), \qquad \xi \geqq 0.$$

Remark 3. If each $\mathscr{A}(t)$ is a function, then Theorem 2 follows from T. Kato's result [16]. This will be the case in many of the applications below.

4. Boundary value problems. We shall describe realizations of the abstract evolution equation (1.1) as initial and boundary value problems for some partial differential equations of mixed elliptic-parabolic-Sobolev type. Our intent is to indicate a variety of such problems to which our results imply existence or uniqueness of solutions, so we do not attempt to attain technically best results in any sense. In particular, we shall limit consideration here to autonomous equations with spatial derivatives of at most second order. After introducing the Banach spaces which we shall use, we construct a quasilinear elliptic partial differential operator following the technique of F. Browder [5], [7]. Then we deduce from the appropriate surjectivity results of [5], [7] the information necessary to apply Theorem 2 of § 3. We illustrate the application of our resulting Theorem 3 to boundary value problems through the methods of J.-L. Lions [20], [21], [23].

Let G be a bounded open set in Euclidean space R^n with G locally on one side of its smooth boundary ∂G. The space of (equivalence classes of) functions on G with Lebesgue summable pth powers is denoted by $L^p(G)$, and $W^p(G)$ is the Sobolev space of those $\phi \in L^p(G)$ for which each of the (distribution) partial derivatives $D_j\phi$ belongs to $L^p(G)$, $1 \leqq j \leqq n$. Letting D_0 be the identity on $L^p(G)$, we can express the norm on $W^p(G)$ by

$$\|\phi\|_{W^p} = \left(\sum_{j=0}^{n} (\|D_j\phi\|_{L^p})^p\right)^{1/p}.$$

There is a continuous and linear *trace* map $\gamma: W^p(G) \to L^p(\partial G)$ with dense range, and it coincides with "restriction to ∂G" on smooth functions on G. (When it is appropriate to mention the variable $s \in \partial G$, we shall suppress the trace map by writing, e.g., $\phi(s) \equiv (\gamma\phi)(s)$ for $\phi \in W^p(G)$.) Since ∂G is smooth, there is a unit (outward) normal $n(s) = [n_1(s), \cdots, n_n(s)]$ at each $s \in \partial G$ for which we have

$$\int_G D_j\phi(x)\,dx = \int_{\partial G} \phi(s)n_j(s)\,ds, \qquad 1 \leqq j \leqq n, \tag{4.1}$$

for functions $\phi \in W^1(G)$.

Let V be a Banach space continuously embedded in $W^p(G)$ and containing the space $C_0^\infty(G)$ of infinitely differentiable functions with compact support in G. Suppose we are given a family of functions $N_j: G \times R^{n+1} \to R$, $0 \leqq j \leqq n$, for which we assume the following:

Each $N_j(x, y)$ is measurable in x for fixed $y \in R^{n+1}$, continuous in y for fixed $x \in G$, and there are a $C > 0$ and $g \in L^q(G)$ with $q = p/(p-1)$ and $p \geqq 2$, such that

$$|N_j(x, y)| \leqq C \sum_{k=0}^{n} |y_k|^{p-1} + g(x), \qquad x \in G, \quad y \in R^{n+1}, \quad 0 \leqq j \leqq n, \tag{4.2}$$

$$\sum_{j=0}^{n} (N_j(x, y) - N_j(x, z))(y_j - z_j) \geqq 0, \qquad y, z \in R^{n+1}, \quad x \in G, \tag{4.3}$$

and there are a $c > 0$ and $h \in L^q(G)$ such that

$$\sum_{j=0}^{n} N_j(x, y)y_j + h(x) \geqq c|y|^p, \qquad y \in R^{n+1}, \quad x \in G. \tag{4.4}$$

Letting $D\phi \equiv \{D_j\phi : 0 \leqq j \leqq n\}$ for $\phi \in W^p(G)$, we find that each $N_j(x, D\phi(x))$ belongs to $L^q(G)$, so we can define $\mathcal{N} : V \to V'$ by

$$(4.5) \qquad \langle \mathcal{N}\phi, \psi \rangle \equiv \sum_{j=0}^{n} \int_G N_j(x, D\phi(x)) D_j\psi(x)\, dx, \qquad \phi, \psi \in V.$$

Note that the restriction of $\mathcal{N}\phi$ to $C_0^\infty(G)$ is the distribution on G given by

$$(4.6) \qquad N(\phi) \equiv -\sum_{j=1}^{n} (D_j N_j(\cdot, D\phi)) + N_0(\cdot, D\phi).$$

This defines our quasilinear elliptic partial differential operator $N : V \to \mathcal{D}'(G)$, the space of distributions on G. From (4.1) we obtain the (formal) Green's identity

$$(4.7) \qquad \langle \mathcal{N}\phi - N\phi, \psi \rangle = \int_{\partial G} \frac{\partial \phi(s)}{\partial N} \psi(s)\, ds, \qquad \psi \in V,$$

whenever $\mathcal{N}(\phi)$, and hence $N(\phi)$, belong to $L^q(G)$, and we have let

$$(4.8) \qquad \frac{\partial \phi(s)}{\partial N} \equiv \sum_{j=1}^{n} N_j(s, D\phi(s)) n_j(s), \qquad s \in \partial G,$$

denote the conormal derivative.

(Note that $L^q(G)$ is simultaneously identified by duality with subsets of V' and $\mathcal{D}'(G)$.)

One can use (4.2) and dominated convergence to show that $\mathcal{N}$ is hemicontinuous, i.e., continuous from line segments in V to V' with the weak topology. Also, (4.3) shows that $\mathcal{N}$ is V-monotone while (4.4) implies

$$\langle \mathcal{N}\phi, \phi \rangle \geqq c(\|\phi\|_{W^p})^p - |h|_{L^q} \|\phi\|_{L^p}, \qquad \phi \in V,$$

so $\mathcal{N}$ is coercive: $\langle \mathcal{N}\phi, \phi \rangle \to \infty$ as $\|\phi\|_{W^p} \to \infty$. These three properties are sufficient to make $\mathcal{N}$ surjective [5], [7].

Suppose we are given a continuous, linear and monotone $\mathcal{M} : V \to V'$. Then $\mathcal{M} + \mathcal{N}$ is hemicontinuous, coercive and monotone, hence maps onto V'. Assume also $\mathcal{M}$ is symmetric and let E denote the space V with the seminorm induced by $\mathcal{M}$. Then the injection $V \to E$ is continuous, and hence $E' \subset V'$, so the range of $\mathcal{M} + \mathcal{N}$ includes E'. From Theorems 1 and 2 we obtain the following result.

THEOREM 3. *Let V be a reflexive Banach space and $\mathcal{M} : V \to V'$ a symmetric, continuous, linear and monotone operator. Let $\mathcal{N} : V \to V'$ be hemicontinuous, monotone and coercive, and $u_0 \in V$ with $\mathcal{N}u_0 \in E'$, where E is the space V with the seminorm induced by $\mathcal{M}$. Then there exists an absolutely continuous $u : [0, T) \to E$, such that $\mathcal{N}u(t) \in E'$ for all $t \in [0, T]$,*

$$\frac{d}{dt}(\mathcal{M}u(t)) + \mathcal{N}u(t) = 0, \qquad a.e.\ t \in [0, t],$$

and $\mathcal{M}(u(0) - u_0) = 0$. The solution is unique if $\mathcal{M} + \mathcal{N}$ is strictly monotone.

Remark 4. By our choice of V, we may obtain *stable* boundary conditions from the inclusions $u(t) \in V$, $t \in [0, T]$, or *variational* boundary conditions from the identity

$$\left\langle \frac{d}{dt}(\mathcal{M}u(t)) + \mathcal{N}u(t), v \right\rangle = \int_G \left\{ \frac{d}{dt}(\mathcal{M}u(t)) + \mathcal{N}u(t) \right\} v(x)\,dx, \tag{4.9}$$
$$v \in V, \quad t \in [0, T].$$

We shall illustrate the application of Theorem 3 to boundary value problems by four examples. In the first two examples, we choose $V = W_0^p(G)$, the space of those $\phi \in W^p(G)$ for which $\gamma(\phi) = 0$.

Example 1. *Degenerate elliptic-parabolic equations.* Let $m_0 \in L^r(G)$ with $m_0(x) \geqq 0$, a.e. $x \in G$, $r = p/(p-2)$, and define

$$\langle \mathcal{M}\phi, \psi \rangle \equiv \int_G m_0(x)\phi(x)\psi(x)\,dx, \qquad \phi, \psi \in V.$$

Let $u_0 \in V = W_0^p(G)$ be given with $N(u_0) = m_0^{1/2}g$ for some $g \in L^2(G)$. Then Theorem 3 asserts the existence of a solution of the equation

$$\frac{\partial}{\partial t}(m_0(x)u(x,t)) + N(u(x,t)) = 0, \qquad x \in G, \quad t > 0, \tag{4.10}$$

with $u(s,t) = 0$ for $s \in \partial G$ and $t \geqq 0$, and $u(x,0) = u_0(x)$ for all $x \in G$ with $m_0(x) > 0$. Such problems arise, e.g., in classical models of heat propagation, and $m_0(x)$ then denotes a variable specific heat capacity of the material.

Example 2. *Degenerate parabolic-Sobolev equations.* Let m_0 be as above, but define

$$\langle \mathcal{M}\phi, \psi \rangle \equiv \int_G \left\{ \phi\psi + m_0 \sum_{j=1}^{n} (D_j\phi)(D_j\psi) \right\}, \qquad \phi, \psi \in V.$$

Since $\langle \mathcal{M}\phi, \phi \rangle \geqq (\|\phi\|_{L^2})^2$, we have $L^2(G) \subset E'$, and so Theorem 3 shows that for each $u_0 \in W_0^p(G)$ with $N(u_0) \in L^2(G)$, there is a unique solution of the equation

$$\frac{\partial}{\partial t}\left\{ u(x,t) - \sum_{j=1}^{n} D_j(m_0(x)D_ju(x,t)) \right\} + N(u(x,t)) = 0, \qquad x \in G,$$

with $u(s,t) = 0$ for $s \in \partial G$, $t \in [0,T]$ and $u(x,0) = u_0(x)$ for all $x \in G$. Such equations have been used to describe diffusion processes wherein m_0 is a material constant with the dimensions of viscosity [11], [26]. Also see [8], [18], [25].

Many variations on the preceding examples are immediate. For example, one can use Sobolev imbedding results to get a smaller choice of r in the first example, and other choices of V could replace the Dirichlet boundary condition (in part) by a condition on the conormal derivative (4.8). Such is the case in our next two examples which consider equation (4.10) with *evolutionary boundary conditions.*

Example 3. *Parabolic boundary conditions.* In order to simplify some computations below, assume that ∂G intersects the hyperplane $R^{n-1} \times \{0\}$ in a set with relative interior S. Let $a \in L^\infty(S)$ be given with $a(s) \geqq 0$, $s \in S$, and define the space

$$V \equiv \{\phi \in W^p(G) : \phi(s) = 0 \text{ if } s \in \partial G \sim S,\ a^{1/2}(s)D_j\phi(s) \in L^2(S) \text{ for } 1 \leqq j \leqq n-1\}$$

with the norm

$$\|\phi\|_V \equiv \|\phi\|_{W^p} + \left(\int_S a(s) \sum_{j=1}^{n-1} (D_j\phi(s))^2 \, ds\right)^{1/2}.$$

Let m_0 be given as in Example 1 and $n_0 \in L^r(S)$ with $n_0(s) \geqq 0$, a.e. $s \in S$. Define

$$\langle \mathcal{M}\phi, \psi \rangle \equiv \int_G m_0 \phi\psi + \int_S n_0(s)\phi(s)\psi(s)\, ds,$$

and

$$\langle \mathcal{N}\phi, \psi \rangle \equiv (4.5) + \int_S a(s)\left(\sum_{j=1}^{n-1} D_j\phi(s) D_j\psi(s)\right) ds.$$

For u_0 as in Example 1, Theorem 3 asserts the existence of a solution of equation (4.10) which satisfies the initial conditions

$$m_0(x)(u(x,0) - u_0(x)) = 0, \qquad x \in G,$$
$$n_0(s)(u(s,0) - u_0(s)) = 0, \qquad s \in S.$$

Since for $\phi \in V$ we have $\phi(s) = 0$ for $s \in \partial G \sim S$ we obtain from (4.7), (4.9) and (4.1) (applied to S) the variational boundary condition

$$\frac{\partial}{\partial t}(n_0(s)u(s,t)) + \frac{\partial}{\partial N}(u(s,t)) = \sum_{j=1}^{n-1} D_j(a(s)D_j u(s,t)), \qquad s \in S, \quad t \in [0,T].$$

Also, we have the stable condition $u(s,t) = 0$ for $s \in \partial G \sim S$ and $t \in [0,T]$. Boundary value problems of this form describe models of fluid flow wherein S is an approximation of a narrow fracture characterized by a very high permeability. Thus, most of the flow in S occurs in the tangential directions. See [6], [28] for applications and references.

Example 4. *The fourth boundary value problem.* (This terminology is not ours, but comes from [1].) Let $V \equiv \{\phi \in W^p(G) : \gamma(\phi) \text{ is constant on } \partial G\}$ with the norm of $W^p(G)$, and define $\mathcal{N}$ by (4.5). Let m_0 be given as in Example 1 and define

$$\langle \mathcal{M}\phi, \psi \rangle \equiv \int_G m_0\phi\psi + \gamma(\phi)\cdot\gamma(\psi), \qquad \phi, \psi \in V.$$

Then from Theorem 3 it follows that for each $u_0 \in V$, with $N(u_0) = m_0^{1/2} g$ for some $g \in L^2(G)$, there exists a solution of equation (4.10) which satisfies the boundary conditions of the fourth kind

$$u(s,t) = f(t), \qquad s \in \partial G, \quad t \in [0,T],$$
$$f'(t) + \int_{\partial G} \frac{\partial u(s,t)}{\partial N}\, ds = 0,$$

as well as the initial conditions

$$m_0(x)(u(x,0) - u_0(x)) = 0, \qquad x \in G,$$
$$f(0) = u_0(s), \quad \text{a constant}.$$

Such problems are used to describe, for example, heat conduction in a region G which is submerged in a highly conductive material of finite mass, so the heat flow from G affects the temperature $f(t)$ in the enclosing material. This problem was introduced in [1], together with a problem of the fifth kind (to which our results can be applied).

5. Two degenerate wave equations. We shall give results on existence and uniqueness of two second order evolution equations with (possibly degenerate) operator coefficients on the time derivatives and then indicate some applications. As before, we illustrate the variety of potential applications through the simplest examples.

THEOREM 4. *Let A and B be symmetric and continuous linear operators from a reflexive Banach space V into its dual V', where B is monotone and A is coercive: there is a $k > 0$ such that*

$$\langle A\phi, \phi \rangle \geqq k\|\phi\|_V^2, \qquad \phi \in V.$$

Denote by V_b the space V with the seminorm induced by B and let $F: V \to V'$ be a (possibly nonlinear) monotone and hemicontinuous function. Then, for each pair u_1, $u_2 \in V$ with $Au_1 + F(u_2) \in V_b'$, there exists a unique absolutely continuous $u:[0, T] \to V$ with $Bu':[0, T] \to V_b'$ absolutely continuous, $u(0) = u_1$, $Bu'(0) = Bu_2$,

$$F(u'(t)) + Au(t) \in V_b', \quad \text{a.e. } t \in [0, T], \tag{5.1}$$

and

$$(Bu'(t))' + F(u'(t)) + Au(t) = 0, \quad \text{a.e. } t \in [0, T]. \tag{5.2}$$

Proof. Define a pair of operators from the product space $E \equiv V \times V$ into $E^* = V^* \times V^*$ by

$$\mathcal{M}([\phi_1, \phi_2]) \equiv [A\phi_1, B\phi_2],$$

$$\mathcal{N}([\phi_1, \phi_2]) \equiv [-A\phi_2, A\phi_1 + F(\phi_2)].$$

The symmetric and nonnegative $\mathcal{M}$ gives a seminorm on E for which the dual is $E' = V' \times V_b'$. The operator $A: V \to V'$ is an isomorphism, so u is a solution of the Cauchy problem for (5.2) if and only if $[u, u']$ is a solution of the Cauchy problem for (1.1) with the operators above. Uniqueness follows from Remark 2 of § 2, and existence will follow from Theorem 2 if we can verify that the range of $\mathcal{M} + \mathcal{N}$ contains E'. Since A is surjective, an easy exercise shows we need only to verify that $A + B + F$ maps onto V'. This follows by [5], [7], since $A + B + F$ is hemicontinuous, monotone and coercive.

The Cauchy problem solved by Theorem 4 appears to ask for too much in two directions. First, our previous results suggest we should specify (essentially) $F(u(0)) = F(u_1)$ instead of $u(0) = u_1$, since, e.g., we may take $B \equiv 0$ in (5.2). The second point to be noticed in the Cauchy problem associated with Theorem 4 is the inclusion (5.1). In applications, (5.1) can actually imply that a differential equation is satisfied, so this Cauchy problem possibly contains a *pair* of differential equations.

In our next and final result, we obtain a considerably weaker solution of a single equation similar to (5.2) subject to initial conditions with data that need not satisfy the compatibility condition, $Au_1 + F(u_2) \in V_b'$.

THEOREM 5. *Let the operators A, B and F and spaces V and V_b be given as in Theorem* 4. *Then for each pair $u_0, u_1 \in V$, there exists a unique summable function $w:[0, T] \to V$ for which $Bw:[0, T] \to V_b'$ is absolutely continuous,*

$$(Bw)' + F(w):[0, T] \to V'$$

is (equal a.e. to a function which is) absolutely continuous,

$$(Bw)(0) = Bu_0, \qquad ((Bw') + F(w))(0) = Au_1,$$

and

$$((Bw)' + F(w))' + Aw = 0 \tag{5.3}$$

a.e. in $[0, T]$.

Proof. The Cauchy problem above for (5.3) is equivalent to that of Theorem 4 as well as to that of (1.1) with the operators given in the proof of Theorem 4. In short, if u is the solution of (5.2), then $w \equiv u'$ is the solution of (5.3) and $[u, w]$ is the solution of (1.1).

We continue our listing of examples with references to their applications and history.

Example 5. *Degenerate wave-parabolic-Sobolev-elliptic equations.* Take

$$V \equiv W_0^2(G),$$

the indicated Sobolev space introduced in § 4, and define the coercive form

$$\langle A\phi, \psi\rangle \equiv \int_G \sum_{j=1}^n D_j\phi D_j\psi, \qquad \phi, \psi \in V.$$

Let $m_j \in L^\infty(G)$ with $m_j(x) \geqq 0$, a.e. $x \in G$, for $0 \leqq j \leqq n$, and define the operator B by

$$\langle B\phi, \psi\rangle \equiv \int_G \sum_{j=0}^n m_j(x)D_j\phi(x)D_j\psi(x)\,dx, \qquad \phi, \psi \in V.$$

Finally, let $F \equiv \mathcal{N}$ be given by (4.5), where we assume (4.2), (4.3) and $1 < p \leqq 2$. (This last requirement is quite restrictive but is relevant here since it gives the continuous inclusions $L^2(G) \to L^p(G)$ and $L^q(G) \to L^2(G)$. Then, Theorem 5 shows there is a unique generalized solution w of the equation

$$\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial t}(m_0 w(x, t) - \sum_{j=1}^n D_j m_j(x) D_j w(x, t)) + N(w)\right\} - \sum_{j=1}^n D_j^2 w(x, t) = 0, \tag{5.4}$$
$$x \in G, \quad t \in [0, T],$$

where N is given by (4.6), and w satisfies the boundary conditions

$$w(s, t) = 0, \qquad s \in \partial G, \quad t \in [0, T],$$

and the initial conditions

$$B(w(\cdot, 0) - u_0) = 0, \qquad (Bw)' + N(w)|_{t=0} = w_1,$$

where $u_0 \in V$ and $w_1 \in V'$ are given.

Equation (5.4) includes the classical wave equation as well as the equation

$$\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial t} - \lambda \Delta w\right) - \Delta w = 0,$$

which arises in classical hydrodynamics and the theory of elasticity [15]. Applications in which B is a homogeneous differential operator of order two include the modeling of infinitesimal waves [22] by the equation

$$\frac{\partial^2}{\partial t^2}\left(\sum_{j=1}^{2} D_j^2 w(x,t)\right) + \sum_{j=1}^{3} D_j^2 w(x,t) = 0$$

and the Sobolev equation

$$\frac{\partial^2}{\partial t^2}\left(\sum_{j=1}^{3} D_j^2 w(x,t)\right) + D_3^2 w(x,t) = 0,$$

which describes the motion of a fluid in a rotating vessel [27], [29]. (An elementary change of variables will bring this last equation to the form of (5.4).)

Example 6. *A gas diffusion equation.* Taking the special case of (5.4) with $B \equiv 0$, we can solve problems for the equation

$$\frac{\partial}{\partial t} N(w) - \sum_{j=1}^{n} D_j^2 w = 0, \tag{5.5}$$

in which N is given by (4.6). Setting $N_j = 0$ for $1 \leqq j \leqq n$ and

$$N_0(x,s) \equiv m_0(x)|s|^{p-1} \operatorname{sgn}(s),$$

where $m_0 \in L^\infty(G)$, $m_0(x) \geqq 0$, and $1 < p \leqq 2$ gives us the degenerate and nonlinear

$$\frac{\partial}{\partial t}(m_0(x)|w|^{p-1} \operatorname{sgn}(w)) - \Delta w = 0.$$

The change of variable $u \equiv |w|^{p-1} \operatorname{sgn}(w)$ puts this in the form [3], [4]

$$\frac{\partial}{\partial t}(m_0(x)u(x,t)) - (q-1) \sum_{j=1}^{n} D_j(|u|^{q-2} D_j u(x,t)) = 0, \tag{5.6}$$

with $q - 2 = (2-p)/(p-1) \geqq 0$.

Note that (5.6) is not of the form suitable for the results of § 3, since the nonlinear part is not monotone, but it can be rewritten as

$$\frac{\partial}{\partial t}(A^{-1} m_0 u) - |u|^{q-2} u(x,t) = 0,$$

where A is given in Example 5. We also note that (5.5) includes one of the Stefan free-boundary problems [4], [17], and the nonlinear term can contain spatial derivatives.

Our final example illustrates an application of both Theorem 4 and Theorem 5 to a situation in which B acts only on the boundary ∂G and F is multiplication by a nonnegative function on G. Other combinations are possible and useful, but the following is typical of higher order evolutionary boundary conditions.

Example 7. Second order boundary conditions. Let $S \subset \partial G$ and define V to be the subspace of $W^2(G)$ consisting of those functions which vanish on $\partial G \sim S$. Let the operator A and the function m_0 be given as in Example 5, and define

$$\langle F\varphi, \psi \rangle = \int_G m_0(x)\varphi(x)\psi(x)\,dx,$$

$$\langle B\varphi, \psi \rangle = \int_S \varphi(s)\psi(s)\,ds, \qquad \varphi, \psi \in V.$$

Let $w_0 \in V$, $w_1 \in L^2(G)$ and $w_2 \in L^2(S)$. Since A is an isomorphism, there is a $u_1 \in V$ with

$$\langle Au_1, v \rangle = \int_G w_1(x)v(x)\,dx + \int_S w_2(s)v(s)\,ds.$$

From Theorem 5 it follows that there is a unique $w:[0, T] \to V$ which is a generalized solution of the partial differential equation

$$\frac{\partial}{\partial t}(m_0(x)w(x,t)) - \Delta w(x,t) = 0, \qquad x \in G, t > 0, \tag{5.7}$$

subject to the boundary conditions

$$w(s,t) = 0, \qquad s \in \partial G \sim S, \quad t > 0,$$

$$\frac{\partial^2 w(s,t)}{\partial t^2} + \frac{\partial w(s,t)}{\partial n} = 0, \qquad s \in S, \qquad t > 0,$$

and the initial conditions

$$w(s,0) = w_0(s), \qquad s \in S,$$

$$\frac{\partial w(s,0)}{\partial t} = w_2(s), \qquad s \in S,$$

$$w(x,0) = w_1(x), \qquad \text{where } m_0(x) > 0.$$

Since $V_b' = L^2(S)$, the pair $u_1, u_2 \in V$ satisfies $Au_1 + Fu_2 \in V_b'$ if and only if

$$-\Delta u_1 + m_0 u_2 = 0 \quad \text{in } G,$$

and

$$\frac{\partial u_1}{\partial n} \in L^2(S).$$

These conditions imply a regularity result for u_1 depending on the smoothness of m_0. If u_1 and u_2 are so given, and if w denotes the solution of the Cauchy problem of Theorem 4, then (5.1) implies that w is regular (depending on m_0) and satisfies (5.7) and the null boundary condition on $\partial G \sim S$. The equation (5.2) implies the second boundary condition above, and the initial conditions in Theorem 4 assert that w satisfies $w(x,0) = u_1(x)$, $x \in \bar{G}$, and $\partial w(s,0)/\partial t = u_2(s)$, $s \in S$. The data in this case are more restricted and the conditions stronger than above, but we obtain a correspondingly stronger solution. Problems of the above type (with $m_0(x) \equiv 0$) originate from the equations of water waves or gravity waves. See [13], [22] for additional results and references.

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