Nonlinear Design of Adaptive Controllers for Linear Systems

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***Abstract*—A new approach to adaptive control of linear systems abandons the traditional certainty-equivalence concept and treats the control of linear plants with unknown parameters as a nonlinear problem. A recursive design procedure introduces at each step new design parameters and incorporates them in a novel Lyapunov function. This function encompasses all the states of the adaptive system and forces them to converge to a manifold of smallest possible dimension. Only as many controller parameters are updated as there are unknown plant parameters, and the dynamic order of the resulting controllers is no higher (and in most cases is lower) than that of traditional adaptive schemes. A simulation comparison with a standard indirect linear scheme shows that the new nonlinear scheme significantly improves transient performance without an increase in control effort.**

I. Introduction and Problem Statement

In this paper we present a new approach to adaptive control of linear systems. We abandon the traditional certainty-equivalence design, in which a parameter update law tunes a would-be linear controller [1]–[3]. Instead, we treat the control of linear plants with unknown parameters as a nonlinear problem to which we apply tools of adaptive nonlinear control [7]–[11]. We develop a recursive procedure to design different types of adaptive controllers. At each step, this procedure introduces new design parameters and incorporates them in a novel Lyapunov function, which is also constructed in a step-by-step fashion. This function encompasses all the states of the adaptive system and forces them to converge to a manifold of smallest possible dimension.

Another advantage of the new design procedure is that it is uncertainty-specific, with respect to both the number of unknown parameters and their locations: the closer unknown parameters are to the control input, the simpler is the adaptive controller. Only as many controller parameters are updated as there are unknown plant parameters. As for additional filters, only two are employed, one at the control u and one at the output y. Each of these filters is of dimension equal to the plant order. This means that the dynamic order of the resulting controllers is no higher (and in most cases is lower) than in traditional adaptive schemes.

Manuscript received July 28, 1992; revised June 8, 1993. Recommended by Past Associate Editor R. H. Middleton.

This work was supported in part by the National Science Foundation under Grant ECS-91-96178 and in part by the Air Force Office of Scientific Research under Grant AFOSR 92-J-0004.

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IEEE Log Number 9215721.

Different choices of design parameters offer new possibilities for improvement of transient performance and reduction of control effort. Particularly important are the "nonlinear damping" terms, which, when compared to update law normalizations, lead to faster and better-damped transients without an increase in control effort.

Problem statement: The control objective is to asymptotically track a reference signal $y_r(t)$ with the output y of the plant

$$y(s)=\frac{B(s)}{A(s)}u(s)=\frac{b_ms^m+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}u(s). \tag{1.1}$$

An adaptive approach to this problem is required because some or all of the a_i's and b_i's are unknown. We make the following standard assumptions about the plant and the reference signal.

Assumption 1.1 : The plant is minimum phase, i.e., the polynomial $B(s)=b_ms^m+\cdots+b_1s+b_0$ is Hurwitz, and the plant order (n), relative degree ($\rho=n-m$), and sign of the high-frequency gain ($\operatorname{sgn}(b_m)$) are known.

Assumption 1.2 : The reference signal $y_r(t)$ and its first ρ derivatives are known and bounded, and, in addition, $y_r^{(\rho)}(t)$ is piecewise continuous. In particular, $y_r(t)$ may be the output of a reference model of relative degree $\rho_r\geq\rho$ with a piecewise continuous input $r(t)$.

The paper is organized as follows. We first select a state estimation scheme and then present our design procedure followed by the proof of stability and tracking. Then we illustrate our procedure on an unstable third-order plant, and, using simulation results for this plant, we compare the new nonlinear design with a traditional certainty-equivalence design. This comparison illustrates the improvements in the trade-off between transient performance and control effort.

II. State Estimation

Since only the output y is available for measurement, we design a Kreisselmeier observer [4] by first representing the plant (1.1) as

$$\begin{aligned}\dot{x}_1&=x_2-a_{n-1}y\\ \dot{x}_2&=x_3-a_{n-2}y\\ &\vdots\\ \dot{x}_{\rho-1}&=x_\rho-a_{m+1}y\\ \dot{x}_\rho&=x_{\rho+1}-a_my+b_mu\\ &\vdots\\ \dot{x}_n&=-a_0y+b_0u\\ y&=x_1\end{aligned} \tag{2.1}$$

and then rewriting it in the form

$$\dot{x} = A_0 x + (k - a)y + bu$$
$$y = e_1^T x \tag{2.2}$$

where e_i denotes the ith coordinate vector in $\mathbb{R}^n$ and

$$A_0 = \begin{bmatrix} -k_1 & & \\ & & I \\ \vdots & & \\ -k_n & & 0 \cdots 0 \end{bmatrix}, \qquad k = \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix},$$

$$a = \begin{bmatrix} a_{n-1} \\ \vdots \\ a_0 \end{bmatrix}, \qquad \bar{b} = \begin{bmatrix} b_m \\ \vdots \\ b_0 \end{bmatrix}, \qquad b = \begin{bmatrix} 0_{(\rho-1)\times 1} \\ \bar{b} \end{bmatrix}. \tag{2.3}$$

The observer matrix A_0 is Hurwitz due to the choice of k. It is easy to check that

$$A_0^i e_n = e_{n-i}, \qquad 0 \le i \le n-1$$
$$A_0^n e_n = -k, \tag{2.4}$$

so that the polynomial functions $A(\cdot)$ and $B(\cdot)$ from (1.1) satisfy

$$A(A_0)e_n = a - k$$
$$B(A_0)e_n = b. \tag{2.5}$$

By filtering u and y with two n-dimensional filters

$$\dot{\eta} = A_0 \eta + e_n y$$
$$\dot{\lambda} = A_0 \lambda + e_n u \tag{2.6}$$

the state estimate is formed as

$$\hat{x} \triangleq B(A_0)\lambda - A(A_0)\eta. \tag{2.7}$$

Using (2.2) and (2.5)–(2.7), it is easy to verify that the estimation error $\epsilon = x - \hat{x}$ satisfies $\dot{\epsilon} = A_0 \epsilon$. In the expression (2.7) for $\hat{x}$, the vector signals multiplying the unknown parameters $a_0, \cdots, a_{n-1}$ and $b_0, \cdots, b_m$ are $\xi_i = A_0^i \eta, 0 \le i \le n-1$, and $v_i = A_0^i \lambda, 0 \le i \le m$, respectively. For convenience we also define $\xi_n = -A_0^n \eta$ and rewrite (2.7) as

$$x = \xi_n - \sum_{i=0}^{n-1} a_i \xi_i + \sum_{i=0}^{m} b_i v_i + \epsilon, \qquad \dot{\epsilon} = A_0 \epsilon. \tag{2.8}$$

All the $\xi-$ and v-signals and their derivatives are explicitly available:

$$\xi_n = -A_0^n \quad \dot{\xi}_n = A_0 \xi_n + ky$$
$$\xi_i = A_0^i \eta \quad \dot{\xi}_i = A_0 \xi_i + e_{n-i} y, \quad 0 \le i \le n-1$$
$$v_i = A_0^i \lambda \quad \dot{v}_i = A_0 v_i + e_{n-i} u, \quad 0 \le i \le m. \tag{2.9}$$

It is important to point out that these expressions are implemented as algebraic identities, along with filter equations (2.6).

In the adaptive controller design the unknown parameters $a_0, \cdots, a_{n-1}$ and $b_0, \cdots, b_m$, appearing in the state estimation equation (2.8), will be replaced by their estimates.

III. An Introductory Example

Our recursive design procedure employs integrator backstepping in the observer [8] combined with tuning functions [10]. In that sense, it removes the overparameterization employed in [9], where the results of [7] and [8] were combined to design adaptive output-feedback controllers for nonlinear systems. For readers unfamiliar with these references, the procedure is introduced by designing an adaptive controller for the plant $y(s) = (1/s(s-a))u(s)$, represented as

$$\dot{x}_1 = x_2 + a x_1$$
$$\dot{x}_2 = u \tag{3.1}$$

where a is unknown, and $y = x_1$ is measured. For simplicity we let y_r=const.

The observer filters (2.6) and the corresponding ξ and v variables are implemented as

$$\dot{\eta} = A_0 \eta + e_2 y, \qquad \xi_1 = A_0 \eta, \qquad \xi_2 = -A_0^2 \eta \tag{3.2}$$
$$\dot{\lambda} = A_0 \lambda + e_2 u, \qquad v = \lambda, \qquad A_0 = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix}. \tag{3.3}$$

From (2.8), using $\hat{a}$ as an estimate of a, the unmeasured state x_2 is expressed as

$$x_2 = \xi_{2,2} + \hat{a}\xi_{1,2} + (a - \hat{a})\xi_{1,2} + v_2 + \epsilon_2. \tag{3.4}$$

The plant (3.1) is of relative degree $\rho = 2$, and the design is in two steps.

Step 1: The equation for the tracking error $z_1 = y - y_r = x_1 - y_r$ is (since $\dot{y}_r \equiv 0$)

$$\dot{z}_1 = x_2 + ay = \xi_{2,2} + v_2 + \hat{a}(\xi_{1,2} + y)$$
$$+ (a - \hat{a})(\xi_{1,2} + y) + \epsilon_2. \tag{3.5}$$

The backstepping idea is to treat the filter state v_2 as a *virtual control*, replace it by $v_2 = z_2 + \alpha_1$, and design $\alpha_1(y, \hat{a}, y_r, \eta)$ to stabilize the z_1-equation

$$\dot{z}_1 = z_2 + \alpha_1 + \xi_{2,2} + \epsilon_2 + \hat{a}\omega + (a - \hat{a})\omega,$$
$$\omega = \xi_{1,2} + y \tag{3.6}$$

in the absence of z_2 and $(a - \hat{a})\omega$. This is simply achieved by

$$\alpha_1 = -c_1 z_1 - d_1 z_1 - \xi_{2,2} - \hat{a}\omega. \tag{3.7}$$

With this stabilizing function, (3.6) reduces to

$$\dot{z}_1 = -(c_1 + d_1)z_1 + z_2 + (a - \hat{a})\omega + \epsilon_2. \tag{3.8}$$

The reason for using two separate positive coefficients c_1 and d_1 will become obvious at the next step, where d_2 is a nonlinear gain. If an update law for $\hat{a}$ were to be designed at this step, a simple choice would be $\dot{\hat{a}} = \tau_1 = \gamma z_1 \omega$, because then the derivative of $V_1 = (1/2)z_1^2 + (1/2\gamma)(a - \hat{a})^2 + (1/d_1)\epsilon^T P_0 \epsilon$ along (3.8) would be $\dot{V} \le -c_1 z_1^2 - (3/4d_1)\epsilon^T \epsilon + z_1 z_2$, which is nonpositive when $z_2 \equiv 0$. However, $\tau_1 = \gamma z_1 \omega$ will not be used to update $\hat{a}$. It only defines our first "tuning function."

Step 2: To step back through the second integrator, we differentiate $z_2 = v_2 - \alpha_1$ and get

$$\begin{aligned}\dot{z}_2 &= u - k_2 v_1 - \dot{\alpha}_1\\ &= u - k_2 v_1 - \frac{\partial \alpha_1}{\partial y}(\xi_{2,2} + a\omega + v_2 + \epsilon_2)\\ &\quad - \frac{\partial \alpha_1}{\partial \hat{a}}\dot{\hat{a}} - \frac{\partial \alpha_1}{\partial \eta}(A_0\eta + e_2 y).\end{aligned} \tag{3.9}$$

Now the actual control u is available and for it we design the control law

$$\begin{aligned}u = &-z_1 - c_2 z_2 - d_2\left(\frac{\partial \alpha_1}{\partial y}\right)^2 z_2 + k_2 v_1\\ &+ \frac{\partial \alpha_1}{\partial y}(\xi_{2,2} + \hat{a}\omega + v_2)\\ &+ \frac{\partial \alpha_1}{\partial \eta}(A_0\eta + e_2 y) + u_a\end{aligned} \tag{3.10}$$

where u_a is yet to be determined. The nonlinear damping term $-d_2(\partial\alpha_1/\partial y)^2 z_2$ was introduced in [8] to counteract the destablizing effect of $(\partial\alpha_1/\partial y)\epsilon_2$. To choose u_a and the update law for $\hat{a}$ we consider the Lyapunov function

$$\begin{aligned}V_2 &= V_1 + \frac{1}{2}z_2^2 + \frac{1}{d_2}\epsilon^T P_0 \epsilon\\ &= \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2 + \frac{1}{2\gamma}(a - \hat{a})^2 + \left(\frac{1}{d_1} + \frac{1}{d_2}\right)\epsilon^T P_0 \epsilon.\end{aligned}$$

Its derivative for the system (3.8), (3.9) with the control (3.10) is

$$\begin{aligned}\dot{V}_2 \le &-c_1 z_1^2 - c_2 z_2^2 - \frac{3}{4}\left(\frac{1}{d_1} + \frac{1}{d_2}\right)\epsilon^T\epsilon\\ &+ (\hat{a} - a)\left[\frac{1}{\gamma}\dot{\hat{a}} - \left(z_1\omega - z_2\frac{\partial \alpha_1}{\partial y}\omega\right)\right]\\ &+ z_2\left(u_a - \frac{\partial \alpha_1}{\partial \hat{a}}\dot{\hat{a}}\right).\end{aligned}$$

Observe that the last two terms in $\dot{V}_2$ are indefinite. To cancel them, we select the actual update law

$$\dot{\hat{a}} = \gamma\left(z_1\omega - z_2\frac{\partial \alpha_1}{\partial y}\omega\right) = \tau_1 - \gamma z_2 \frac{\partial \alpha_1}{\partial y}\omega \triangleq \tau_2, \tag{3.11}$$

and let $u_a = (\partial\alpha_1/\partial\hat{a})\tau_2$. The design is completed because $\dot{V}_2 \le -c_1 z_1^2 - c_2 z_2^2 - (3/4)((1/d_1) + (1/d_2))\epsilon^T\epsilon$ is nonpositive.

This guarantees stability of the origin $z_1 = z_2 = 0$, $\epsilon = 0$, $\hat{a} = a$, as well as the regulation of z_1, z_2, ϵ to zero, which, in particular, means that asymptotic tracking is achieved; $\lim_{t\to\infty}[y(t) - y_r] = 0$.

IV. The Design Procedure

The general recursive design procedure defines at each step a new *stabilizing function* and a new *tuning function*. In the final ρ-th step, these functions determine the *actual control law* and the *actual update law*.

The first two steps of the general procedure are as in the above relative-degree-two example, except that now the high frequency gain b_m is also unknown. Like in other adaptive schemes, an additional parameter $\hat{p}$ is introduced to estimate b_m^{-1}.

A more substantial change occurs beyond Step 2, that is, for plants with $\rho \ge 3$. As shown in [10], the stabilizing functions designed at Steps $3, \cdots, \rho$ must also compensate for the mismatch between the tuning functions and the actual update law, because only the last tuning function τ_ρ is the actual update law.

Introducing the positive constants γ, c_i, d_i, $i = 1, \cdots, \rho$ and the positive definite matrix Γ of dimension $(n + m + 1) \times (n + m + 1)$ as design parameters, we are now ready to present the design procedure.

Step 1: The derivative of the tracking error $z_1 = y - y_r$ is

$$\dot{z}_1 = x_2 - a_{n-1}y - \dot{y}_r. \tag{4.1}$$

If x_2 were measured, we would treat it as a "virtual control" and use it to stabilize (4.1). Since x_2 is not measured, we replace it by its expression from (2.8)

$$x_2 = \xi_{n,2} - \xi_{(2)}a + v_{(2)}\bar{b} + \epsilon_2 \tag{4.2}$$

where ϵ_2 is an exponentially decaying signal and

$$\xi_{(2)} = [\xi_{n-1,2}, \cdots, \xi_{0,2}], \qquad v_{(2)} = [v_{m,2}, \cdots, v_{0,2}]. \tag{4.3}$$

Substituting (4.2) into (4.1), we obtain

$$\dot{z}_1 = \xi_{n,2} - \xi_{(2)}a + v_{(2)}\bar{b} - a_{n-1}y - \dot{y}_r + \epsilon_2. \tag{4.4}$$

A choice is now to be made of a measured variable in (4.4) to replace x_2 as a virtual control. We choose $v_{m,2}$, because (2.9) shows that the actual control u appears after only $\rho - 1$ differentiations of $v_{m,2}$, earlier than for any other variable in (4.4). Then, we denote

$$\theta = \begin{bmatrix} -a \\ \bar{b} \end{bmatrix}, \qquad \omega^T = [\xi_{(2)} + e_1^T y,\ v_{(2)}] \tag{4.5}$$

$$\bar{v}_{(2)} = [0,\ v_{m-1,2}, \cdots, v_{0,2}],$$

$$\bar{\omega}^T = [\xi_{(2)} + e_1^T y,\ \bar{v}_{(2)}] = \omega^T - e_{n+1}^T v_{m,2} \tag{4.6}$$

where e_{n+1} is the $(n+1)$st basis vector in $\mathbb{R}^{n+m+1}$, and write (4.4) as

$$\begin{aligned}\dot{z}_1 &= b_m v_{m,2} + \xi_{n,2} + [\xi_{(2)} + e_1^T y,\ \bar{v}_{(2)}]\theta - \dot{y}_r + \epsilon_2\\ &= b_m v_{m,2} + \xi_{n,2} + \bar{\omega}^T\theta - \dot{y}_r + \epsilon_2.\end{aligned} \tag{4.7}$$

If $v_{m,2}$ were the actual control input we would design for it a control law α_1 to stabilize (4.7). Let z_2 be the error between the actual and desired value of $v_{m,2}$

$$z_2 = v_{m,2} - \alpha_1 \tag{4.8}$$

and rewrite (4.7) as

$$\dot{z}_1 = b_m z_2 + b_m \alpha_1 + \xi_{n,2} + \bar{\omega}^T\theta - \dot{y}_r + \epsilon_2. \tag{4.9}$$

Since the parameters are unknown, one would first think of replacing them with estimates. As α_1 is multiplied by the unknown parameter b_m, however, the estimate of b_m

would have to be bounded away from zero. To avoid this inconvenience, we introduce an additional parameter $p = b_m^{-1}$. Adding and subtracting $(c_1 + d_1)z_1$,[1] we rewrite (4.7) as

$$\begin{aligned}\dot{z}_1 &= -c_1 z_1 - d_1 z_1 + b_m z_2 \\ &\quad + b_m\{\alpha_1 + p(c_1 z_1 + d_1 z_1 + \xi_{n,2} - \dot{y}_r) \\ &\quad + p\overline{\omega}^T\theta\} + \epsilon_2. \end{aligned} \tag{4.10}$$

We now pause to examine this equation. Although more complicated than its predecessor (4.1), this equation is more convenient for backstepping, since z_2 is a measured variable. In the design of α_1 the unknown parameters p, θ will be replaced by their estimates $\hat{p}$, $\hat{\theta}$. The form of (4.10) suggests that we define

$$\varphi = c_1 z_1 + d_1 z_1 + \xi_{n,2} - \dot{y}_r + \overline{\omega}^T\hat{\theta}. \tag{4.11}$$

Then, adding and subtracting the term $b_m\{\hat{p}\varphi + p\overline{\omega}^T\hat{\theta}\}$ and keeping in mind that $b_m p = 1$, we rewrite (4.10) as

$$\begin{aligned}\dot{z}_1 = -c_1 z_1 + b_m z_2 + b_m(\alpha_1 + \hat{p}\varphi) + b_m(p - \hat{p})\varphi \\ + \overline{\omega}^T(\theta - \hat{\theta}) + \epsilon_2 - d_1 z_1. \end{aligned} \tag{4.12}$$

We can now view (4.12) as a first-order system to be stabilized by α_1 with respect to the Lyapunov function

$$\begin{aligned}V_1 = \frac{1}{2}z_1^2 + \frac{1}{2}(\theta - \hat{\theta})^T\Gamma^{-1}(\theta - \hat{\theta}) \\ + \frac{|b_m|}{2\gamma}(p - \hat{p})^2 + \frac{1}{d_1}\epsilon^T P_0 \epsilon \end{aligned} \tag{4.13}$$

where P_0 is the positive definite solution of $P_0 A_0 + A_0^T P_0 = -I$. To design α_1 we examine the derivative of V_1

$$\begin{aligned}\dot{V}_1 &= -c_1 z_1^2 + b_m z_1 z_2 + b_m z_1(\alpha_1 + \hat{p}\varphi) \\ &\quad + |b_m|(p - \hat{p})\frac{1}{\gamma}(\gamma\,\mathrm{sgn}\,(b_m)\varphi z_1 - \dot{\hat{p}}) \\ &\quad + (\theta - \hat{\theta})^T\Gamma^{-1}(\Gamma\overline{\omega}z_1 - \dot{\hat{\theta}}) \\ &\quad - d_1 z_1^2 + z_1\epsilon_2 - \frac{1}{d_1}\epsilon^T\epsilon. \end{aligned} \tag{4.14}$$

If $v_{m,2}$ were our actual control, we would have $z_2 \equiv 0$ and we would eliminate $p - \hat{p}$, $\theta - \hat{\theta}$ and b_m from (4.14) with the choices

$$\alpha_1 = -\hat{p}\varphi = -\hat{p}[c_1 z_1 + d_1 z_1 + \xi_{n,2} - \dot{y}_r + \overline{\omega}^T\hat{\theta}] \tag{4.15}$$

$$\dot{\hat{p}} = \gamma\,\mathrm{sgn}\,(b_m)\varphi z_1 \tag{4.16}$$

and $\dot{\hat{\theta}} = \tau_1$, where

$$\tau_1 = \Gamma\overline{\omega}z_1. \tag{4.17}$$

With $z_2 \equiv 0$, this would yield the following expression for the derivative of V_1:

$$\dot{V}_1 = -c_1 z_1^2 - d_1\left(z_1 - \frac{1}{2d_1}\epsilon_2\right)^2 - \frac{1}{d_1}\Omega(\epsilon)$$

[1]Two separate positive coefficients c_1 and d_1 appear here for uniformity with subsequent steps.

where

$$\Omega(\epsilon) = \epsilon_1^2 + \frac{3}{4}\epsilon_2^2 + \cdots + \epsilon_n^2. \tag{4.18}$$

Since $v_{m,2}$ is not our actual control, we have $z_2 \not\equiv 0$ and we do not use $\dot{\hat{\theta}} = \tau_1$ as the update law for $\hat{\theta}$, because θ will reappear in subsequent steps. However, p will not reappear, so we do use (4.16) as the actual update law for $\hat{p}$. We retain (4.17) as our first *tuning function* and (4.15) as our first *stabilizing function*. Substituting (4.15) and (4.17) into (4.12) and (4.14), we obtain

$$\begin{aligned}\dot{z}_1 &= -c_1 z_1 - d_1 z_1 + b_m z_2 + \overline{\omega}^T(\theta - \hat{\theta}) \\ &\quad + b_m\varphi(p - \hat{p}) + \epsilon_2 \\ \dot{\hat{p}} &= \gamma\,\mathrm{sgn}\,(b_m)\varphi z_1 \\ \dot{V}_1 &\le -c_1 z_1^2 + b_m z_1 z_2 + (\theta - \hat{\theta})^T\Gamma^{-1}(\tau_1 - \dot{\hat{\theta}}) - \frac{1}{d_1}\Omega(\epsilon). \end{aligned} \tag{4.19}$$

Step 2: Using (4.5), we express the derivative of $z_2 = v_{m,2} - \alpha_1$ as

$$\begin{aligned}\dot{z}_2 &= v_{m,3} - k_2 v_{m,1} - \frac{\partial\alpha_1}{\partial y}(\xi_{n,2} + \omega^T\theta + \epsilon_2) \\ &\quad - \frac{\partial\alpha_1}{\partial y_r}\dot{y}_r - \frac{\partial\alpha_1}{\partial\dot{y}_r}\ddot{y}_r - \frac{\partial\alpha_1}{\partial\xi_n}(A_0\xi_n + ky) \\ &\quad - \sum_{i=0}^{n-1}\frac{\partial\alpha_1}{\partial\xi_i}(A_0\xi_i + e_{n-i}y) \\ &\quad - \sum_{i=0}^{m-1}\frac{\partial\alpha_1}{\partial v_{i,2}}(v_{i,3} - k_2 v_{i,1}) \\ &\quad - \frac{\partial\alpha_1}{\partial\hat{p}}\gamma\,\mathrm{sgn}\,(b_m)\varphi z_1 - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}} \\ &= v_{m,3} + \beta_2 - \frac{\partial\alpha_1}{\partial y}\omega^T\theta \\ &\quad - \frac{\partial\alpha_1}{\partial y}\epsilon_2 - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}} \end{aligned} \tag{4.20}$$

where β_2 denotes all the known terms except for $v_{m,3}$. We now treat $v_{m,3}$ as a virtual control, that is, introducing the new variable $z_3 = v_{m,3} - \alpha_2$, we use α_2 to stabilize the $(z_1, z_2, \hat{p})$-system

$$\begin{aligned}\dot{z} &= -c_1 z_1 - d_1 z_1 + b_m z_2 + \overline{\omega}^T(\theta - \hat{\theta}) \\ &\quad + b_m\varphi(p - \hat{p}) + \epsilon_2 \\ \dot{z}_2 &= z_3 + \alpha_2 + \beta_2 - \frac{\partial\alpha_1}{\partial y}\omega^T\theta - (\theta - \hat{\theta})^T\frac{\partial\alpha_1}{\partial y}\omega \\ &\quad - \frac{\partial\alpha_1}{\partial y}\epsilon_2 - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}} \\ \dot{\hat{p}} &= \gamma\,\mathrm{sgn}\,(b_m)\varphi z_1 \end{aligned} \tag{4.21}$$

with respect to the Lyapunov function

$$V_2 = V_1 + \frac{1}{2}z_2^2 + \frac{1}{d_2}\epsilon^T P_0 \epsilon. \tag{4.22}$$

The derivative of V_2 for (4.21) is computed as

$$\begin{aligned}\dot{V}_2 \le& -c_1 z_1^2 + z_2\Big[b_m z_1 + z_3 + \alpha_2 + \beta_2 \\ &- \frac{\partial \alpha_1}{\partial y}\omega^T\hat{\theta} - \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}}\Big] \\ &+ (\theta - \hat{\theta})^T\Gamma^{-1}\left(\tau_1 - \Gamma\frac{\partial \alpha_1}{\partial y}\omega z_2 - \dot{\hat{\theta}}\right) \\ &- z_2\frac{\partial \alpha_1}{\partial y}\epsilon_2 - \frac{1}{d_2}\epsilon^T\epsilon - \frac{1}{d_1}\Omega(\epsilon). \qquad (4.23)\end{aligned}$$

Using $z_2 b_m z_1 = z_2\hat{b}_m z_1 + z_2(b_m - \hat{b}_m)z_1$, we rewrite (4.23) as

$$\begin{aligned}\dot{V}_2 \le& -c_1 z_1^2 + z_2\Big[z_3 + \alpha_2 + \beta_2 - \frac{\partial \alpha_1}{\partial y}\omega^T\hat{\theta} \\ &+ \hat{b}_m z_1 - \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}}\Big] \\ &+ (\theta - \hat{\theta})^T\Gamma^{-1}\Big[\tau_1 - \Gamma\Big(\frac{\partial \alpha_1}{\partial y}\omega \\ &+ [0, \cdots, 0, -z_1, 0, \cdots, 0]^T\Big)z_2 - \dot{\hat{\theta}}\Big] \\ &- z_2\frac{\partial \alpha_1}{\partial y}\epsilon_2 - \frac{1}{d_2}\epsilon^T\epsilon - \frac{1}{d_1}\Omega(\epsilon) \qquad (4.24)\end{aligned}$$

with $-z_1$ appearing in the $(n+1)$st entry of the row vector.

If $v_{m,3}$ were our control, we would have $z_3 \equiv 0$ and we would eliminate $\theta - \hat{\theta}$ from (4.24) with the update law $\dot{\hat{\theta}} = \tau_2$, where

$$\begin{aligned}\tau_2 &= \tau_1 - \Gamma\left(\frac{\partial \alpha_1}{\partial y}\omega + [0, \cdots, 0, -z_1, 0, \cdots, 0]^T\right)z_2 \\ &= \Gamma\overline{\omega}z_1 - \Gamma\frac{\partial \alpha_1}{\partial y}\omega z_2 + \Gamma e_{n+1}z_1 z_2 \\ &= \Gamma\overline{\omega}z_1 - \Gamma\frac{\partial \alpha_1}{\partial y}\omega z_2 + \Gamma e_{n+1}z_1(v_{m,2} - \alpha_1) \\ &= \Gamma\omega\left(z_1 - \frac{\partial \alpha_1}{\partial y}z_2\right) + \Gamma e_{n+1}\hat{p}\varphi z_1 \qquad (4.25)\end{aligned}$$

where we have used (4.6), (4.15), and (4.17). Then, examining the terms in the bracket multiplying z_2 in (4.24), we see that, if $z_3 \equiv 0$ and $\dot{\hat{\theta}} = \tau_2$, the choice

$$\alpha_2 = -c_2 z_2 - d_2\left(\frac{\partial \alpha_1}{\partial y}\right)^2 z_2 - \beta_2 + \frac{\partial \alpha_1}{\partial y}\omega^T\hat{\theta} - \hat{b}_m z_1 + \frac{\partial \alpha_1}{\partial \hat{\theta}}\tau_2 \qquad (4.26)$$

would yield

$$\dot{V}_2 \le -c_1 z_1^2 - c_2 z_2^2 - d_2\left(z_2\frac{\partial \alpha_1}{\partial y} + \frac{1}{2d_2}\epsilon_2\right)^2 - \left(\frac{1}{d_1} + \frac{1}{d_2}\right)\Omega(\epsilon).$$

However, since $z_3 \not\equiv 0$, we do not use $\dot{\hat{\theta}} = \tau_2$ as an update law. Instead, we retain τ_2 in (4.25) as our second tuning function and α_2 in (4.26) as our second stabilizing function. Upon the substitution into (4.21), and (4.24), we obtain

$$\begin{aligned}\dot{z}_1 =& -c_1 z_1 - d_1 z_1 + b_m z_2 + \overline{\omega}^T(\theta - \hat{\theta}) \\ &+ b_m\varphi(p - \hat{p}) + \epsilon_2 \\ \dot{z}_2 =& -c_2 z_2 - d_2\left(\frac{\partial \alpha_1}{\partial y}\right)^2 z_2 + z_3 - b_m z_1 \\ &- \frac{\partial \alpha_1}{\partial y}\omega^T(\theta - \hat{\theta}) + z_1(b_m - \hat{b}_m) \\ &+ \frac{\partial \alpha_1}{\partial \hat{\theta}}(\tau_2 - \dot{\hat{\theta}}) - \frac{\partial \alpha_1}{\partial y}\epsilon_2 \\ \dot{\hat{p}} =& \gamma\,\mathrm{sgn}\,(b_m)\varphi z_1 \\ \dot{V}_2 \le& -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3 + (\theta - \hat{\theta})^T\Gamma^{-1}(\tau_2 - \dot{\hat{\theta}}) \\ &+ z_2\frac{\partial \alpha_1}{\partial \hat{\theta}}(\tau_2 - \dot{\hat{\theta}}) - \left(\frac{1}{d_1} + \frac{1}{d_2}\right)\Omega(\epsilon). \qquad (4.27)\end{aligned}$$

The mismatch term $(\partial \alpha_1/\partial \theta)(\tau_2 - \dot{\hat{\theta}})$ will be dealt with in subsequent steps.

Step $i(3 \le i < \rho)$: We express the derivative of $z_i = v_{m,i} - \alpha_{i-1}$ as

$$\dot{z}_i = v_{m,i+1} + \beta_i - \frac{\partial \alpha_{i-1}}{\partial y}\omega^T\theta - \frac{\partial \alpha_{i-1}}{\partial y}\epsilon_2 - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} \qquad (4.28)$$

where β_i encompasses all the known terms except for $v_{m,i+1}$. We now treat $v_{m,i+1}$ as a virtual control, that is, introducing the new variable $z_{i+1} = v_{m,i+1} - \alpha_i$, we use α_i to stabilize the $(z_1, \cdots, z_i, \hat{p})$-system

$$\begin{aligned}\dot{z}_1 =& -c_1 z_1 - d_1 z_1 + b_m z_2 + \overline{\omega}^T(\theta - \hat{\theta}) \\ &+ b_m\varphi(p - \hat{p}) + \epsilon_2 \\ \dot{z}_2 =& -c_2 z_2 - d_2\left(\frac{\partial \alpha_1}{\partial y}\right)^2 z_2 + z_3 - b_m z_1 \\ &- \frac{\partial \alpha_1}{\partial y}\omega^T(\theta - \hat{\theta}) + z_1(b_m - \hat{b}_m) \\ &+ \frac{\partial \alpha_1}{\partial \hat{\theta}}(\tau_2 - \dot{\hat{\theta}}) - \frac{\partial \alpha_1}{\partial y}\epsilon_2 \\ \dot{z}_j =& -c_j z_j - d_j\left(\frac{\partial \alpha_{j-1}}{\partial y}\right)^2 z_j + z_{j+1} - z_{j-1} \\ &- \frac{\partial \alpha_{j-1}}{\partial y}\omega^T(\theta - \hat{\theta}) \\ &+ \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}}(\tau_j - \dot{\hat{\theta}}) - \sum_{k=2}^{j-1} z_k\frac{\partial \alpha_{k-1}}{\partial \hat{\theta}}\Gamma\frac{\partial \alpha_{j-1}}{\partial y}\omega \\ &- \frac{\partial \alpha_{j-1}}{\partial y}\epsilon_2, \qquad 3 \le j \le i-1 \\ \dot{z}_i =& z_{i+1} + \alpha_i + \beta_i - \frac{\partial \alpha_{i-1}}{\partial y}\omega^T\hat{\theta} - \frac{\partial \alpha_{i-1}}{\partial y}\omega^T(\theta - \hat{\theta}) \\ &- \frac{\partial \alpha_{i-1}}{\partial y}\epsilon_2 - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} \\ \dot{\hat{p}} =& \gamma\,\mathrm{sgn}\,(b_m)\varphi z_1 \qquad (4.29)\end{aligned}$$

with respect to the Lyapunov function

$$V_i = V_{i-1} + \frac{1}{2}z_i^2 + \frac{1}{d_i}\epsilon^T P_0\epsilon \qquad (4.30)$$

where $\dot{V}_{i-1}$ satisfies

$$\dot{V}_{i-1} \le -\sum_{j=1}^{i-1} c_j z_j^2 + z_i z_{i-1} + (\theta - \hat{\theta})^T \Gamma^{-1} (\tau_{i-1} - \dot{\hat{\theta}}) + \sum_{j=2}^{i-1} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} (\tau_j - \dot{\hat{\theta}}) - \sum_{j=3}^{i-1}\sum_{k=2}^{j-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \Gamma \frac{\partial \alpha_{j-1}}{\partial y} \omega z_j - \sum_{j=1}^{i-1} \frac{1}{d_j} \Omega(\epsilon). \tag{4.31}$$

Let us first analyze (4.29) and (4.31). The form of $\dot{z}_3, \cdots, \dot{z}_{i-1}$ is the same as the form of $\dot{z}_2$ except for the term $-\sum_{k=2}^{j-1} z_k (\partial \alpha_{k-1}/\partial \hat{\theta}) \Gamma (\partial \alpha_{j-1}/\partial y) \omega$. Likewise, the form $\dot{V}_{i-1}$ is the same as that of $\dot{V}_2$ except for the term $-\sum_{j=3}^{i-1} \sum_{k=2}^{j-1} z_k (\partial \alpha_{k-1}/\partial \hat{\theta}) \Gamma (\partial \alpha_{j-1}/\partial y) \omega z_j$. These terms, incorporated at steps $2, \cdots, i-1$ by the stabilizing functions $\alpha_3, \cdots, \alpha_{i-1}$, would have cancelled $\sum_{j=2}^{i-1} z_j (\partial \alpha_{j-1}/\partial \hat{\theta})(\tau_j - \dot{\hat{\theta}})$ in $\dot{V}_{i-1}$, if the relative degree were $i-1$, in which case we would have chosen $\dot{\hat{\theta}} = \tau_{i-1}$. We give below a detailed explanation of these cancellations at step i.

The derivative of V_i for (4.29) is computed as

$$\begin{aligned} \dot{V}_i \le & -\sum_{j=1}^{i-1} c_j z_j^2 + z_i \\ & \cdot \left[z_{i-1} + z_{i+1} + \alpha_i + \beta_i - \frac{\partial \alpha_{i-1}}{\partial y} \omega^T \hat{\theta} - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} \right] \\ & + (\theta - \hat{\theta})^T \Gamma^{-1} \left(\tau_{i-1} - \Gamma \frac{\partial \alpha_{i-1}}{\partial y} \omega z_i - \dot{\hat{\theta}} \right) \\ & + \sum_{j=2}^{i-1} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} (\tau_j - \dot{\hat{\theta}}) \\ & - \sum_{j=3}^{i-1}\sum_{k=2}^{j-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \Gamma \frac{\partial \alpha_{j-1}}{\partial y} \omega z_j \\ & - z_i \frac{\partial \alpha_{i-1}}{\partial y} \epsilon_2 - \frac{1}{d_i} \epsilon^T \epsilon - \sum_{j=1}^{i-1} \frac{1}{d_j} \Omega(\epsilon). \end{aligned} \tag{4.32}$$

If $v_{m,i+1}$ were our control, we would have $z_{i+1} \equiv 0$ and we would eliminate $\theta - \hat{\theta}$ from (4.32) with the update law $\dot{\hat{\theta}} = \tau_i$, where

$$\tau_i = \tau_{i-1} - \Gamma \frac{\partial \alpha_{i-1}}{\partial y} \omega z_i. \tag{4.33}$$

Then, noting that

$$\tau_j - \dot{\hat{\theta}} = \tau_j - \tau_i = \Gamma \sum_{k=j+1}^{i} \frac{\partial \alpha_{k-1}}{\partial y} \omega z_k, \qquad 2 \le j \le i-1 \tag{4.34}$$

and making use of the algebraic identities

$$\begin{aligned} & \sum_{j=3}^{i-1}\sum_{k=2}^{j-1} z_k \frac{\partial \alpha_{k-1}}{\partial y} \Gamma \frac{\partial \alpha_{j-1}}{\partial y} \omega z_j \\ & = \sum_{k=2}^{i-2} \sum_{j=k+1}^{i-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \Gamma \frac{\partial \alpha_{j-1}}{\partial y} \omega z_j \\ & = \sum_{k=2}^{i-1} \sum_{j=k+1}^{i} z_k \frac{\partial \alpha_{k-1}}{\partial y} \Gamma \frac{\partial \alpha_{j-1}}{\partial y} \omega z_j \\ & \quad - \sum_{k=2}^{i-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \Gamma \frac{\partial \alpha_{i-1}}{\partial y} \omega z_i \end{aligned} \tag{4.35}$$

we would rewrite $\dot{V}_i$ as

$$\begin{aligned} \dot{V}_i < & -\sum_{j=1}^{i-1} c_j z_j^2 + z_i \\ & \cdot \left[z_{i-1} + \alpha_i + \beta_i - \frac{\partial \alpha_{i-1}}{\partial y} \omega^T \hat{\theta} - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \tau_i \right] \\ & + \sum_{j=2}^{i-1} \sum_{k=j+1}^{i} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \Gamma \frac{\partial \alpha_{k-1}}{\partial y} \omega z_k \\ & - \sum_{j=3}^{i-1}\sum_{k=2}^{j-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \Gamma \frac{\partial \alpha_{j-1}}{\partial y} \omega z_j - z_i \frac{\partial \alpha_{i-1}}{\partial y} \epsilon_2 \\ & - \frac{1}{d_i} \epsilon^T \epsilon - \sum_{j=1}^{i-1} \frac{1}{d_j} \Omega(\epsilon) \\ = & -\sum_{j=1}^{i-1} c_j z_j^2 + z_i \\ & \cdot \left[z_{i-1} + \alpha_i + \beta_i - \frac{\partial \alpha_{i-1}}{\partial y} \omega^T \hat{\theta} - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \tau_i \right] \\ & + \sum_{j=2}^{i-1} \sum_{k=j+1}^{i} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \Gamma \frac{\partial \alpha_{k-1}}{\partial y} \omega z_k \\ & - \sum_{k=2}^{i-2} \sum_{j=k+1}^{i-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \Gamma \frac{\partial \alpha_{j-1}}{\partial y} \omega z_j - z_i \frac{\partial \alpha_{i-1}}{\partial y} \epsilon_2 \\ & - \frac{1}{d_i} \epsilon^T \epsilon - \sum_{j=1}^{i-1} \frac{1}{d_j} \Omega(\epsilon) \\ = & -\sum_{j=1}^{i-1} c_j z_j^2 + z_i \left[z_{i-1} + \alpha_i + \beta_i - \frac{\partial \alpha_{i-1}}{\partial y} \omega^T \hat{\theta} \right. \\ & \left. - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \tau_i + \sum_{k=2}^{i-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \Gamma \frac{\partial \alpha_{i-1}}{\partial y} \omega \right] \\ & - z_i \frac{\partial \alpha_{i-1}}{\partial y} \epsilon_2 - \frac{1}{d_i} \epsilon^T \epsilon - \sum_{j=1}^{i-1} \Omega(\epsilon). \end{aligned} \tag{4.36}$$

The last term in the bracket in (4.36) is due to the mismatch between τ_i and $\tau_2, \cdots, \tau_{i-1}$. Along with the other terms in the bracket, this term will be cancelled by the stabilizing function

$$\alpha_i = -c_i z_i - d_i \left(\frac{\partial \alpha_{i-1}}{\partial y} \right)^2 z_i - z_{i-1} - \beta_i + \frac{\partial \alpha_{i-1}}{\partial y} \omega^T \hat{\theta} + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \tau_i - \sum_{k=2}^{i-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \Gamma \frac{\partial \alpha_{i-1}}{\partial y} \omega. \tag{4.37}$$

For $z_{i+1} \equiv 0$ and $\dot{\hat{\theta}} = \tau_i$, this α_i would yield

$$\dot{V}_i \leq -\sum_{j=1}^{i} c_j z_j^2 - d_i\left(z_i \frac{\partial \alpha_{i-1}}{\partial y} + \frac{1}{2d_i}\epsilon_2\right)^2 - \sum_{j=1}^{i} \frac{1}{d_j}\Omega(\epsilon).$$

Since $z_{i+1} \not\equiv 0$, however, we do not use $\dot{\hat{\theta}} = \tau_i$ as an update law. Instead, we retain (4.33) as our ith tuning function and (4.37) as our ith stabilizing function. Substituting them into (4.29) and (4.32), we obtain

$$\begin{aligned}
\dot{z}_1 &= -c_1 z_1 - d_1 z_1 + b_m z_2 + \overline{\omega}^T(\theta - \hat{\theta}) \\
&\quad + b_m \varphi(p - \hat{p}) + \epsilon_2 \\
\dot{z}_2 &= -c_2 z_2 - d_2\left(\frac{\partial \alpha_1}{\partial y}\right)^2 z_2 + z_3 - b_m z_1 \\
&\quad - \frac{\partial \alpha_1}{\partial y}\omega^T(\theta - \hat{\theta}) + z_1(b_m - \hat{b}_m) \\
&\quad + \frac{\partial \alpha_1}{\partial \hat{\theta}}(\tau_2 - \dot{\hat{\theta}}) - \frac{\partial \alpha_1}{\partial y}\epsilon_2 \\
\dot{z}_j &= -c_j z_j - d_j\left(\frac{\partial \alpha_{j-1}}{\partial y}\right)^2 z_j + z_{j+1} \\
&\quad - z_{j-1} - \frac{\partial \alpha_{j-1}}{\partial y}\omega^T(\theta - \hat{\theta}) \\
&\quad + \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}}(\tau_j - \dot{\hat{\theta}}) - \sum_{k=2}^{j-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}}\Gamma\frac{\partial \alpha_{j-1}}{\partial y}\omega \\
&\quad - \frac{\partial \alpha_{j-1}}{\partial y}\epsilon_2, \qquad 3 \leq j \leq i
\end{aligned} \tag{4.38}$$

$$\begin{aligned}
\dot{\hat{p}} &= \gamma \operatorname{sgn}(b_m)\varphi z_1 \\
\dot{V}_i &\leq -\sum_{j=1}^{i} c_j z_j^2 + z_i z_{i+1} + (\theta - \hat{\theta})^T \Gamma^{-1}(\tau_i - \dot{\hat{\theta}}) \\
&\quad + \sum_{j=2}^{i} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}}(\tau_j - \dot{\hat{\theta}}) \\
&\quad - \sum_{j=3}^{i}\sum_{k=2}^{j-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}}\Gamma\frac{\partial \alpha_{j-1}}{\partial y}\omega z_j - \sum_{j=1}^{i}\frac{1}{d_j}\Omega(\epsilon).
\end{aligned}$$

Step ρ: We express the derivative of $z_\rho = v_{m,\rho} - \alpha_{\rho-1}$ as

$$\dot{z}_\rho = u + \beta_\rho - \frac{\partial \alpha_{\rho-1}}{\partial y}\omega^T\theta - \frac{\partial \alpha_{\rho-1}}{\partial y}\epsilon_2 - \frac{\partial \alpha_{\rho-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} \tag{4.39}$$

where β_ρ encompasses all the known terms (including $v_{m,\rho+1}$) except u. Finally the *actual control* u appears in (4.39). We can now design u and and the *actual update law* $\dot{\hat{\theta}}$ to stabilize the $(z_1, \cdots, z_\rho, \hat{p})$-system with respect to the Lyapunov function

$$\begin{aligned}
V_\rho &= V_{\rho-1} + \frac{1}{2}z_\rho^2 + \frac{1}{d_\rho}\epsilon^T P_0 \epsilon \\
&= \sum_{j=1}^{\rho}\left(\frac{1}{2}z_j^2 + \frac{1}{d_j}\epsilon^T P_0 \epsilon\right) + \frac{|b_m|}{2\gamma}(p - \hat{p})^2 \\
&\quad + \frac{1}{2}(\theta - \hat{\theta})^T \Gamma^{-1}(\theta - \hat{\theta}).
\end{aligned} \tag{4.40}$$

Using (4.38) with $i = \rho - 1$ and (4.39), the derivative of V_ρ is computed as

$$\begin{aligned}
\dot{V}_\rho &\leq -\sum_{j=1}^{\rho-1} c_j z_j^2 + z_\rho \\
&\quad \cdot \left[z_{\rho-1} + u + \beta_\rho - \frac{\partial \alpha_{\rho-1}}{\partial y}\omega^T\hat{\theta} - \frac{\partial \alpha_{\rho-1}}{\partial \hat{\theta}}\dot{\hat{\theta}}\right] \\
&\quad + (\theta - \hat{\theta})^T\Gamma^{-1}\left(\tau_{\rho-1} - \Gamma\frac{\partial \alpha_{\rho-1}}{\partial y}\omega z_\rho - \dot{\hat{\theta}}\right) \\
&\quad + \sum_{j=2}^{i-1} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}}(\tau_j - \dot{\hat{\theta}}) \\
&\quad - \sum_{j=3}^{\rho-1}\sum_{k=2}^{j-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}}\Gamma\frac{\partial \alpha_{j-1}}{\partial y}\omega z_j \\
&\quad - z_\rho \frac{\partial \alpha_{\rho-1}}{\partial y}\epsilon_2 - \frac{1}{d_\rho}\epsilon^T\epsilon - \sum_{j=1}^{\rho-1}\frac{1}{d_j}\Omega(\epsilon).
\end{aligned} \tag{4.41}$$

To eliminate $\theta - \hat{\theta}$ from (4.41), we choose the update law

$$\dot{\hat{\theta}} = \tau_\rho = \tau_{\rho-1} - \Gamma\frac{\partial \alpha_{\rho-1}}{\partial y}\omega z_\rho. \tag{4.42}$$

Then, noting that

$$\tau_j - \dot{\hat{\theta}} = \tau_j - \tau_\rho = \Gamma\sum_{k=j+1}^{\rho}\frac{\partial \alpha_{k-1}}{\partial y}\omega z_k, \qquad 2 \leq j \leq \rho - 1 \tag{4.43}$$

and using the identities (4.35) with $i = \rho$, we rewrite $\dot{V}_\rho$ as

$$\begin{aligned}
\dot{V}_\rho &\leq -\sum_{j=1}^{\rho-1} c_j z_j^2 + z_\rho\left[z_{\rho-1} + u + \beta_\rho - \frac{\partial \alpha_{\rho-1}}{\partial y}\omega^T\hat{\theta}\right. \\
&\quad \left. - \frac{\partial \alpha_{\rho-1}}{\partial \hat{\theta}}\tau_\rho + \sum_{k=2}^{\rho-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}}\Gamma\frac{\partial \alpha_{\rho-1}}{\partial y}\omega\right] \\
&\quad - z_\rho\frac{\partial \alpha_{\rho-1}}{\partial y}\epsilon_2 - \frac{1}{d_\rho}\epsilon^T\epsilon - \sum_{j=1}^{\rho-1}\frac{1}{d_j}\Omega(\epsilon).
\end{aligned} \tag{4.44}$$

Finally, we choose the control u as

$$\begin{aligned}
u &= -c_\rho z_\rho - d_\rho\left(\frac{\partial \alpha_{\rho-1}}{\partial y}\right)^2 z_\rho - z_{\rho-1} - \beta_\rho \\
&\quad + \frac{\partial \alpha_{\rho-1}}{\partial y}\omega^T\hat{\theta} + \frac{\partial \alpha_{\rho-1}}{\partial \hat{\theta}}\tau_\rho - \sum_{k=2}^{\rho-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}}\Gamma\frac{\partial \alpha_{\rho-1}}{\partial y}\omega
\end{aligned} \tag{4.45}$$

which yields[2]

$$\dot{V}_\rho = -\sum_{j=1}^{\rho}\left\{c_j z_j^2 + d_j\left(z_j\frac{\partial \alpha_{j-1}}{\partial y} + \frac{1}{2d_j}\epsilon_2\right)^2 + \frac{1}{d_j}\Omega(\epsilon)\right\}. \tag{4.46}$$

[2]For notational convenience we introduce $\alpha_0 \triangleq 0$.

Our design is now complete. The resulting $(z_1, \cdots, z_\rho, \hat{p}, \hat{\theta})$-system is

$$\begin{aligned}
\dot{z}_1 &= -c_1 z_1 - d_1 z_1 + b_m z_2 + \bar{\omega}^T(\theta - \hat{\theta}) \\
&\quad + b_m \varphi(p - \hat{p}) + \epsilon_2 \\
\dot{z}_2 &= -c_2 z_2 - d_2\left(\frac{\partial \alpha_1}{\partial y}\right)^2 z_2 + z_3 - b_m z_1 \\
&\quad - \frac{\partial \alpha_1}{\partial y}\omega^T(\theta - \hat{\theta}) + z_1(b_m - \hat{b}_m) \\
&\quad + \frac{\partial \alpha_1}{\partial \hat{\theta}}(\tau_2 - \dot{\hat{\theta}}) - \frac{\partial \alpha_1}{\partial y}\epsilon_2 \\
\dot{z}_j &= -c_j z_j - d_j\left(\frac{\partial \alpha_{j-1}}{v}\right)^2 z_j + z_{j+1} \\
&\quad - z_{j-1} - \frac{\partial \alpha_{j-1}}{\partial y}\omega^T(\theta - \hat{\theta}) \\
&\quad + \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}}(\tau_j - \dot{\hat{\theta}}) - \sum_{k=2}^{j-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}}\Gamma\frac{\partial \alpha_{j-1}}{\partial y}\omega \\
&\quad - \frac{\partial \alpha_{j-1}}{\partial y}\epsilon_2, \qquad 3 \le j \le \rho - 1 \qquad (4.47) \\
\dot{z}_\rho &= -c_\rho z_\rho - d_\rho\left(\frac{\partial \alpha_{\rho-1}}{\partial y}\right)^2 z_\rho - z_{\rho-1} \\
&\quad - \frac{\partial \alpha_{\rho-1}}{\partial y}\omega^T(\theta - \hat{\theta}) \\
&\quad - \sum_{k=2}^{\rho-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}}\Gamma\frac{\partial \alpha_{\rho-1}}{\partial \hat{\theta}}\omega - \frac{\partial \alpha_{\rho-1}}{\partial y}\epsilon_2 \\
\dot{\hat{p}} &= \gamma\, \mathrm{sgn}\,(b_m)\varphi z_1 \\
\dot{\hat{\theta}} &= \Gamma\omega\left\{z_1 - \sum_{j=2}^{\rho}\frac{\partial \alpha_{j-1}}{\partial y} z_j\right\} + \Gamma e_{n+1}\hat{p}\varphi z_1.
\end{aligned}$$

V. Direct Stability Analysis

While the design procedure is intricate, its result—the structure of the z-system—is remarkably simple. Substituting (4.43) into (4.47), denoting $\sigma_{ki} = (\partial\alpha_{k-1}/\partial\hat{\theta})\Gamma(\partial\alpha_{i-1}/\partial y)$, and using (4.6) and (4.15) to derive

$$\begin{aligned}
&b_m z_2 + \bar{\omega}^T(\theta - \hat{\theta}) \\
&\quad = b_m z_2 + \omega^T(\theta - \hat{\theta}) - e_{n+1}^T v_{m,2}(\theta - \hat{\theta}) \\
&\quad = b_m z_2 + \omega^T(\theta - \hat{\theta}) - (b_m - \hat{b}_m)(z_2 + \alpha_1) \\
&\quad = \hat{b}_m z_2 + \omega^T(\theta - \hat{\theta}) + (b_m - \hat{b}_m)\hat{p}\varphi
\end{aligned}$$

the z-system is more compactly rewritten as (5.1), shown at the bottom of the page.

The stability properties of this system can be deduced by inspection. Thanks to the skew-symmetry of the off-diagonal terms, the stabilizing effects of the diagonal terms $-c_i - d_i((\partial\alpha_{i-1}/\partial y))^2$ dominate. The nonlinear damping terms $-d_i((\partial\alpha_{i-1}/\partial y))^2 z_i$ are significant when $(\partial\alpha_{i-1}/\partial y)$ is large, which is crucial, because $(\partial\alpha_{i-1}/\partial y)$ are the "nonlinear gains" for the estimation errors $\theta - \hat{\theta}$ and ϵ_2. We shall see how the nonlinear damping terms are used to improve transients caused by large initial estimation errors.

The skew-symmetry in (5.1) is achieved by incorporating the terms $(\partial\alpha_{i-1}/\partial\hat{\theta})\tau_i - \sum_{k=2}^{i-1} z_k\sigma_{ki}\omega$ in the stabilizing functions α_i and the control law (4.45). These terms directly compensate for the effect of $\dot{\hat{\theta}}$ and thus eliminate the need for complicated stability arguments dealing with "swapping terms."

The stability and tracking properties of the designed adaptive system will now be established through a direct Lyapunov analysis which will demonstrate that the closed-loop states converge to a manifold of the smallest possible dimension. The only variables not guaranteed to converge in our stability proof are the parameter errors $\theta - \hat{\theta}$ and $p - \hat{p}$.

From the previous section we know that the partial Lyapunov function V_ρ, defined in (4.40), is nonincreasing because of (4.46). This does not immediately establish the boundedness of all closed-loop signals, since V_ρ encompasses only $3n + 2$ of the $4n + m + 2$ states of the closed-loop adaptive system, which consists of the n-dimensional plant (2.1), the two n-dimensional filters (2.6), and the $n+m+2$ parameter estimates.

One of the advantages of this approach is that we can readily augment V_ρ by the remaining $n + m$ states, those of the zero dynamics of the plant (2.1) and of the η-filter (2.6). Using a

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\\ \dot{z}_4\\ \vdots\\ \dot{z}_\rho\end{bmatrix} = \begin{bmatrix}
-c_1 - d_1 & \hat{b}_m & 0 & 0 & \cdots & 0\\
-\hat{b}_m & -c_2 - d_2\left(\frac{\partial\alpha_1}{\partial y}\right)^2 & 1 + \sigma_{23}\omega & \sigma_{24}\omega & \cdots & \sigma_{2\rho}\omega\\
0 & -1 - \sigma_{23}\omega & -c_3 - d_3\left(\frac{\partial\alpha_2}{\partial y}\right)^2 & 1 + \sigma_{34}\omega & \cdots & \sigma_{3\rho}\omega\\
0 & -\sigma_{24}\omega & -1 - \sigma_{34}\omega & \ddots & \ddots & \vdots\\
\vdots & \vdots & \vdots & \ddots & \ddots & 1 + \sigma_{\rho-1,\rho}\omega\\
0 & -\sigma_{2\rho}\omega & -\sigma_{3\rho}\omega & \cdots & -1 - \sigma_{\rho-1,\rho}\omega & -c_\rho - d_\rho\left(\frac{\partial\alpha_{\rho-1}}{\partial y}\right)^2
\end{bmatrix}\cdot\begin{bmatrix}z_1\\ z_2\\ z_3\\ z_4\\ \vdots\\ z_\rho\end{bmatrix}$$
$$+ \begin{bmatrix}1\\ -\frac{\partial\alpha_1}{\partial y}\\ \vdots\\ -\frac{\partial\alpha_{\rho-1}}{\partial y}\end{bmatrix}[\omega^T(\theta - \hat{\theta}) + \epsilon_2] + \begin{bmatrix}\hat{p}\varphi\\ 0\\ \vdots\\ 0\end{bmatrix}(b_m - \hat{b}_m) + \begin{bmatrix}b_m\varphi\\ 0\\ \vdots\\ 0\end{bmatrix}(p - \hat{p}). \qquad (5.1)$$

similarity transformation we represent (2.1) as

$$\begin{aligned}\dot{x}_1 &= x_2 - a_{n-1}y\\ \dot{x}_2 &= x_3 - a_{n-2}y\\ &\vdots\\ \dot{x}_{\rho-1} &= x_\rho - a_{m+1}y\\ \dot{x}_\rho &= c_b^T\overline{x} - a_m y + b_m u\\ \dot{\zeta} &= A_b\zeta + b_b y\\ y &= x_1\end{aligned} \tag{5.2}$$

where $\overline{x} = [x_1, x_2, \cdots, x_\rho, \zeta^T]^T, c_b \in \mathbb{R}^n, b_b \in \mathbb{R}^m$. The eigenvalues of the $m \times m$ matrix A_b are the zeros of the polynomial $B(s)$, which is Hurwitz by Assumption 1.1. Although the transformation bringing (2.1) into (5.2) depends on the unknown plant parameters, this is not important because for our stability analysis we only need to know that such a transformation exists. Our task is to investigate the stability of the trajectories along which the tracking error is zero. The zero dynamics along these trajectories are governed by

$$\dot{\zeta}_r = A_b\zeta_r + b_b y_r, \qquad \zeta_r(0) = \zeta(0). \tag{5.3}$$

For stability analysis we are interested in the deviations $\tilde{\zeta} = \zeta - \zeta_r$ which are governed by

$$\dot{\tilde{\zeta}} = A_b\tilde{\zeta} + b_b z_1, \qquad \tilde{\zeta}(0) = 0. \tag{5.4}$$

For the η-variables in (2.6), we analogously define

$$\dot{\eta}_r = A_0\eta_r + e_n y_r, \qquad \eta_r(0) = \eta(0) \tag{5.5}$$

so that the deviations $\tilde{\eta} = \eta - \eta_r$ are governed by

$$\dot{\tilde{\eta}} = A_0\tilde{\eta} + e_n z_1, \qquad \tilde{\eta}(0) = 0. \tag{5.6}$$

In the coordinates z, ϵ, $\theta - \hat{\theta}$, $p - \hat{p}$, $\tilde{\eta}$, $\tilde{\zeta}$, the adaptive system, described by (2.8), (4.47), (5.4), and (5.6), has an equilibrium at the origin. Its stability will now be investigated using the augmented Lyapunov function

$$\begin{aligned}V &= V_\rho + \frac{1}{k_\eta}\tilde{\eta}^T P_0\tilde{\eta} + \frac{1}{k_\zeta}\tilde{\zeta}^T P_b\tilde{\zeta}\\ &= \sum_{j=1}^{\rho}\left(\frac{1}{2}z_j^2 + \frac{1}{d_j}\epsilon^T P_0\epsilon\right) + \frac{|b_m|}{2\gamma}(p-\hat{p})^2\\ &\quad + \frac{1}{2}(\theta-\hat{\theta})^T\Gamma^{-1}(\theta-\hat{\theta}) + \frac{1}{k_\eta}\tilde{\eta}^T P_0\tilde{\eta} + \frac{1}{k_\zeta}\tilde{\zeta}^T P_b\tilde{\zeta}\end{aligned} \tag{5.7}$$

where P_b satisfies $P_bA_b + A_b^TP_b = -I$, and k_η, k_ζ are positive constants to be chosen. Since this Lyapnov function is time-invariant, it will allow us to establish uniform stability properties. Using (4.46), (5.4), and (5.6), we obtain

$$\begin{aligned}\dot{V} &= -\sum_{j=1}^{\rho}\left\{c_jz_j^2 + d_j\left(z_j\frac{\partial\alpha_{j-1}}{\partial y} + \frac{1}{2d_j}\epsilon_2\right)^2 + \frac{1}{d_j}\Omega(\epsilon)\right\}\\ &\quad - \frac{1}{k_\eta}|\tilde{\eta}\|^2 - \frac{1}{k\zeta}\|\tilde{\zeta}\|^2 + \frac{2}{k_\eta}\tilde{\eta}^TP_0e_nz_1 + \frac{2}{k_\zeta}\tilde{\zeta}^TP_bb_bz_1\\ &= -\frac{c_1}{2}z_1^2 - \sum_{j=2}^{\rho}c_jz_j^2 - \sum_{j=1}^{\rho}\\ &\quad\times\left\{d_j\left(z_j\frac{\partial\alpha_{j-1}}{\partial y} + \frac{1}{2d_j}\epsilon_2\right)^2 + \frac{1}{d_j}\Omega(\epsilon)\right\}\\ &\quad - \frac{1}{2k_\eta}\|\tilde{\eta}\|^2 - \frac{1}{2k_\zeta}\|\tilde{\zeta}\|^2\\ &\quad - \frac{1}{4}\left[c_1z_1^2 + \frac{2}{k_\eta}\|\tilde{\eta}\|^2 + \frac{8}{k_\eta}\tilde{\eta}^TP_0e_nz_1\right]\\ &\quad - \frac{1}{4}\left[c_1z_1^2 + \frac{2}{k_\zeta}\|\tilde{\zeta}\|^2 + \frac{8}{k_\zeta}\tilde{\zeta}^TP_bb_bz_1\right]\\ &\le -\frac{c_1}{2}z_1^2 - \sum_{j=2}^{\rho}c_jz_j^2 - \sum_{j=1}^{\rho}\\ &\quad\times\left\{d_j\left(z_j\frac{\partial\alpha_{j-1}}{\partial y} + \frac{1}{2d_j}\epsilon_2\right)^2 + \frac{1}{d_j}\Omega(\epsilon)\right\}\\ &\quad - \frac{1}{2k_\eta}\|\tilde{\eta}\|^2 - \frac{1}{2k_\zeta}\|\tilde{\zeta}\|^2\\ &\quad - \frac{c_1}{4}\left(|z_1| - \frac{4\|P_0e_n\|}{c_1k_\eta}\|\tilde{\eta}\|\right)^2\\ &\quad - \frac{1}{2k_\eta^2}\left(k_\eta - \frac{8\|P_0e_n\|^2}{c_1}\right)\|\tilde{\eta}\|^2\\ &\quad - \frac{c_1}{4}\left(|z_1| - \frac{4\|P_bb_b\|}{c_1k_\zeta}\|\tilde{\zeta}\|\right)^2\\ &\quad - \frac{1}{2k_\zeta^2}\left(k_\zeta - \frac{8\|P_bb_b\|^2}{c_1}\right)\|\tilde{\zeta}\|^2.\end{aligned} \tag{5.8}$$

Thus, if we choose k_η and k_ζ such that

$$k_\eta \ge \frac{8\|P_0e_n\|^2}{c_1}, \qquad k_\zeta \ge \frac{8\|P_bb_b\|^2}{c_1} \tag{5.9}$$

the derivative of V will be nonpositive:

$$\dot{V} \le -\frac{c_1}{2}z_1^2 - \sum_{j=2}^{\rho}c_jz_j^2 - \sum_{j=1}^{\rho}\frac{3}{4d_j}\|\epsilon\|^2 - \frac{1}{2k_\eta}\|\tilde{\eta}\|^2 - \frac{1}{2k_\zeta}\|\tilde{\zeta}\|^2. \tag{5.10}$$

The nonpositivity of $\dot{V}$ proves the uniform stability of the origin. This fact, together with the boundedness of the reference signals $y_r, \dot{y}_r, \cdots, y_r^{(\rho)}$ implies that $z_1, \cdots, z_\rho$, $\hat{p}$, $\hat{\theta}$, ϵ, $\tilde{\eta}$, $\tilde{\zeta}$ are uniformly bounded.

Next we establish the boundedness of λ and u. The boundedness of $\lambda_1, \cdots, \lambda_{m+1}$ follows from the boundedness of y and (2.6), which gives[3]

$$\lambda_i = \frac{s^{i-1} + k_1s^{i-2} + \cdots + k_{i-1}}{K(s)}\frac{A(s)}{B(s)}y, \qquad 1 \le i \le n \tag{5.11}$$

where the polynomials $K(s) = \det(sI - A_0) = s^n + k_1s^{n-1} + \cdots + k_n$ and $B(s)$ are Hurwitz. To establish boundedness of $\lambda_{m+2}, \cdots, \lambda_n$, we note that since $v_m = A_0^m\lambda$, we have

$$v_{m,i} = \lambda_{m+i} + g_{m,i}^T\begin{bmatrix}\lambda_1\\ \vdots\\ \lambda_{m+i-1}\end{bmatrix}, \qquad 2 \le i \le \rho \tag{5.12}$$

[3] For notational convenience we define $k_0 \triangleq 1$.

for some $g_{m,i} \in \mathbb{R}^{m+i-1}$. By substituting into

$$z_i = v_{m,i} + \alpha_{i-1}(y, y_r, \cdots, y_r^{(i-1)}, \hat{p}, \hat{\theta}, \eta, \lambda_1, \cdots, \lambda_{m+i-1}), \quad 2 \le i \le \rho \quad (5.13)$$

we get

$$\lambda_{m+i} = -g_{m,i}^T \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{m+i-1} \end{bmatrix} + z_i - \alpha_{i-1}(y, y_r, \cdots, y_r^{(i-1)}, \hat{p}, \hat{\theta}, \eta, \lambda_1, \cdots, \lambda_{m+i-1}), \quad 2 \le i \le \rho. \quad (5.14)$$

This recursively proves that $\lambda_{m+2}, \cdots, \lambda_n$ are bounded. Therefore, the control u is bounded, and it follows from (2.8) that x is bounded.

To prove the convergence of the tracking error to zero, we note that the boundedness of x, η, λ, $\hat{p}$, $\hat{\theta}$, and u, together with (2.6), (2.8), (4.47), (5.4), (5.6), (5.7), and (5.8) implies that $\dot{V}$ is bounded and integrable on $[0, \infty)$, and moreover, $\ddot{V}$ is bounded. Hence, $\dot{V} \to 0$ as $t \to \infty$, which proves that $z_1, \cdots, z_\rho \to 0$ and ϵ, $\tilde{\eta}$, $\tilde{\zeta} \to 0$ as $t \to \infty$. Since $z_1 = y - y_r$, asymptotic tracking is achieved.

The above results are now summarized as follows.

Theorem 5.1: The closed-loop adaptive system, which consists of (2.8), (4.47), (5.4), and (5.6), has a globally uniformly stable equilibrium at the origin. All the solutions of this $(4n+m+2)$-dimensional dynamical system converge to the $(n + m + 2)$-dimensional equilibrium manifold $M = \{z_1 = z_2 = \cdots = z_\rho = 0, \epsilon = 0, \tilde{\eta} = 0, \tilde{\zeta} = 0\}$. This means, in particular, that global asymptotic tracking is achieved

$$\lim_{t \to \infty} [y(t) - y_r(t)] = 0. \quad (5.15)$$

VI. A Design Example

This section is a continuation of Section III. We illustrate the new design procedure on an unstable relative-degree-three plant

$$y(s) = \frac{1}{s^2(s-a)} u(s) \quad (6.1)$$

where $a = 3$ is considered to be unknown. The relative-degree-three design contains all the features of the general design procedure. The control objective is to asymptotically track the output of the reference model

$$y_r(s) = \frac{1}{(s+1)^3} r(s). \quad (6.2)$$

To derive the adaptive controller resulting from our nonlinear design, the plant (6.1) is first rewritten in the state-space form (2.1)

$$\begin{aligned} \dot{x}_1 &= x_2 + a x_1 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u \\ y &= x_1. \end{aligned} \quad (6.3)$$

The filters (2.6) and the corresponding ξ and v variables are implemented as

$$\dot{\eta} = A_0 \eta + e_3 y, \qquad \xi_2 = A_0^2 \eta, \qquad \xi_3 = -A_0^3 \eta \quad (6.4)$$

$$\dot{\lambda} = A_0 \lambda + e_3 u, \qquad v = \lambda, \quad (6.5)$$

$$A_0 = \begin{bmatrix} -k_1 & 1 & 0 \\ -k_2 & 0 & 1 \\ -k_3 & 0 & 0 \end{bmatrix}.$$

The signals y_r, $\dot{y}_r$, $\ddot{y}_r$, $y_r^{(3)}$ are implemented from the reference model (6.2) as follows:

$$y_r = r_1,\ \dot{y}_r = r_2,\ \ddot{y}_r = r_3,\ y_r^{(3)} = -3r_3 - 3r_2 - r_1 + r \quad (6.6)$$

where $\dot{r}_1 = r_2$, $\dot{r}_2 = r_3$, $\dot{r}_3 = -3r_3 - 3r_2 - r_1 + r$.

Since in this example the high-frequency gain is known, in the first step we can directly treat v_2 as a virtual control and do not need the additional parameter p. The virtual estimate (2.8) is $\xi_3 + a\xi_2 + v$, and by defining $\omega = \xi_{2,2} + y$ the results of the three steps of our design procedure are as follows.

Step 1:

$$z_1 = y - y_r \quad (6.7)$$

$$\tau_1 = \gamma \omega z_1 \quad (6.8)$$

$$\alpha_1 = -c_1 z_1 - d_1 z_1 - \xi_{3,2} + \dot{y}_r - \omega \hat{a}. \quad (6.9)$$

Step 2:

$$z_2 = v_2 - \alpha_1 \quad (6.10)$$

$$\tau_2 = \tau_1 - \gamma \frac{\partial \alpha_1}{\partial y} \omega z_2 \quad (6.11)$$

$$\begin{aligned} \alpha_2 = &-c_2 z_2 - d_2 \left(\frac{\partial \alpha_1}{\partial y} \right)^2 z_2 - z_1 + k_2 v_1 \\ &+ \frac{\partial \alpha_1}{\partial y}(v_2 + \xi_{3,2}) + \frac{\partial \alpha_1}{\partial y_r} \dot{y}_r + \frac{\partial \alpha_1}{\partial \dot{y}_r} \ddot{y}_r \\ &+ \frac{\partial \alpha_1}{\partial \xi_3}(A_0 \xi_3 + k y) + \frac{\partial \alpha_1}{\partial \xi_2}(A_0 \xi_2 + e_1 y) \\ &+ \frac{\partial \alpha_1}{\partial y} \omega \hat{a} + \frac{\partial \alpha_1}{\partial \hat{a}} \tau_2. \end{aligned} \quad (6.12)$$

Step 3:

$$z_3 = v_3 - \alpha_2 \quad (6.13)$$

$$\tau_3 = \tau_2 - \gamma \frac{\partial \alpha_2}{\partial y} \omega z_3 \quad (6.14)$$

$$\begin{aligned} u = &-c_3 z_3 - d_3\left(\frac{\partial \alpha_2}{\partial y}\right)^2 z_3 - z_2 + k_3 v_1 \\ &+ \frac{\partial \alpha_2}{\partial y}(v_2 + \xi_{3,2}) + \frac{\partial \alpha_2}{\partial y_r}\dot{y}_r \\ &+ \frac{\partial \alpha_2}{\partial \dot{y}_r}\ddot{y}_r + \frac{\partial \alpha_2}{\partial \ddot{y}_r} y_r^{(3)} \\ &+ \frac{\partial \alpha_2}{\partial \xi_3}(A_0\xi_3 + ky) + \frac{\partial \alpha_2}{\partial \xi_2}(A_0 \xi_2 + e_1 y) \\ &+ \frac{\partial \alpha_2}{\partial v_1}(v_2 - k_1 v_1) + \frac{\partial \alpha_2}{\partial v_2}(v_3 - k_2 v_1) \\ &+ \frac{\partial \alpha_2}{\partial y}\omega\hat{a} + \frac{\partial \alpha_2}{\partial \hat{a}}\tau_3 - \gamma z_2 \frac{\partial \alpha_1}{\partial \hat{a}}\frac{\partial \alpha_2}{\partial y}\omega. \end{aligned} \tag{6.15}$$

Because in this example the high-frequency gain $b_m = 1$ is known, the matrix form of the $(z_1, z_2, z_3, \hat{a})$-system is simpler than (5.1)

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\end{bmatrix} = \begin{bmatrix} -c_1 - d_1 & 1 & 0 \\ -1 & -c_2 - d_2\left(\frac{\partial \alpha_1}{\partial y}\right)^2 & 1 + \sigma\omega \\ 0 & -1 - \sigma\omega & -c_3 - d_3\left(\frac{\partial \alpha_2}{\partial y}\right)^2 \end{bmatrix} \times \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + [(a - \hat{a})\omega + \epsilon_2]\begin{bmatrix} 1 \\ -\frac{\partial \alpha_1}{\partial y} \\ -\frac{\partial \alpha_2}{\partial y}\end{bmatrix}$$

$$\dot{\hat{a}} = \gamma\left[1, -\frac{\partial \alpha_1}{\partial y}, -\frac{\partial \alpha_2}{\partial y}\right]\omega z \tag{6.16}$$

where $\sigma = \gamma(\partial\alpha_1/\partial\hat{a})(\partial\alpha_2/\partial y)$. Note again the skew-symmetry of the off-diagonal entries and the stabilizing role of the diagonal entries.

The block diagram in Fig. 1 shows that the overall structure of the new adaptive system has the familiar form of the input and output filters feeding into an estimator/controller block. The fundamental difference, is however, that this block is now a nonlinear controller. Whereas in traditional schemes this block would be a "certainty-equivalence" linear controller, the new three-step procedure produces the control law (6.15) in which both parameter estimates and filter signals enter nonlinearly.

VII. Improvement of Transient Performance

The new adaptive scheme is now compared with a standard certainty-equivalence scheme on the basis of transient performance and control effort. The comparison with a direct MRAC scheme is not pursued because such a scheme updates at *least three* parameters. This is clear from its control law

$$u = r + \left[\frac{\theta_0 s^2 + \theta_1 s + \theta_2}{s^2 + m_1 s + m_2}\right] y + \left[\frac{\theta_3 s + \theta_4}{s^2 + m_1 s + m_2}\right] u \tag{7.1}$$

where $s^2 + m_1 s + m_2$ is a Hurwitz polynomial. A calculation using the Bzout identity gives

$$\begin{aligned} s^5 &+ s^4[m_1 - \theta_3 - a] + s^3[m_2 - \theta_4 - a(m_1 - \theta_3)] \\ &- s^2[\theta_0 + (m_2 - \theta_4)a] \\ &= (s+1)^3(s^2 + m_1 s + m_2) + \theta_1 s + \theta_2 \end{aligned} \tag{7.2}$$

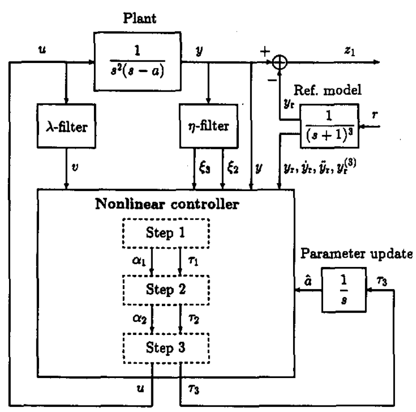

Fig. 1. The distinguishing feature of the new adaptive system is the "nonlinear controller" block. In contrast to certainty-equivalence designs which result in linear-like control laws, the new three-step procedure produces a control law in which both parameter estimates and filter signals enter nonlinearly.

which shows that θ_0, θ_3 and θ_4 have to be updated, while θ_1 and θ_2 can be fixed at $\theta_1 = -m_1 - 3m_2$, $\theta_2 = -m_2$. Simulations showed that the update of three parameters results in transient performance inferior to indirect linear schemes which update only one parameter estimate. Therefore, we compare our new controller to a standard indirect scheme [5], [6], in which the plant equation $s^2(s-a)y(s) = u(s)$ is filtered by a Hurwitz observer polynomial $s^3 + k_1 s^2 + k_2 s + k_3$ to obtain the estimation equation

$$\phi = \psi a$$

$$\phi = \frac{s^3}{s^3 + k_1 s^2 + k_2 s + k_3} y(s) - \frac{1}{s^3 + k_1 s^2 + k_2 s + k_3} u(s)$$

$$\psi = \frac{s^2}{s^3 + k_1 s^2 + k_2 s + k_3} y(s) \tag{7.3}$$

and the parameter update law is a normalized gradient[4]

$$\dot{\hat{a}} = \gamma \frac{\psi e}{1 + \psi^2}, \qquad e = \phi - \psi\hat{a}. \tag{7.4}$$

The control law (7.1) is implemented by replacing a with $\hat{a}$ in (7.2) and then solving it for the controller parameters: $\theta_3 = -(3 + \hat{a})$, $\theta_4 = -[3 + 3m_1 + \hat{a}(m_1 - \theta_3)]$, $\theta_0 = -[1 + 3m_1 + 3m_2 + (m_2 - \theta_4)\hat{a}]$, $\theta_1 = -m_1 - 3m_2$, $\theta_2 = -m_2$.

The above indirect adaptive linear scheme and our new nonlinear scheme were applied to the plant (6.1) with the true parameter $a = 3$. In all tests the initial parameter estimate was $\hat{a}(0) = 0$, so that, with the adaptation switched off, both closed-loop systems were unstable. The reference input was $r(t) = \sin t$.

[4]The simulation results with a least-squares update law were virtually identical and are therefore omitted.

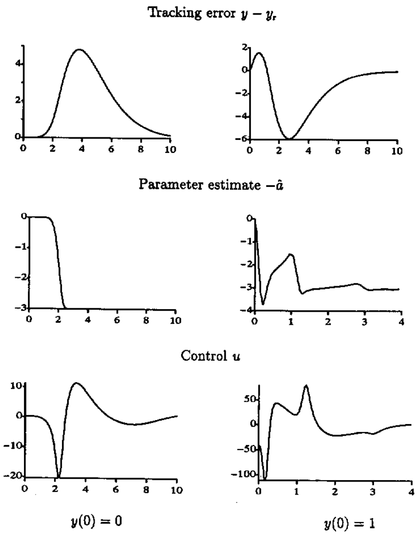

Fig. 2. Simulation results for the indirect adaptive linear scheme with $\gamma = 500$ for $y(0) = 0$ (left) and $y(0) = 1$ (right). The nonzero initial condition is misinterpreted as a parameter error by the parameter estimator, leading to the deterioration of both transient performance and control effort.

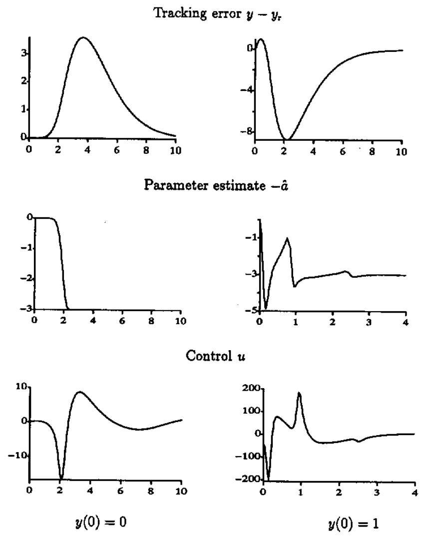

Fig. 3. The adaptation gain here is increased to $\gamma = 1000$. For $y(0) = 0$, this leads to faster parameter convergence, which results in better transient performance and smaller control compared to Fig. 2. This beneficial effect, however, is reversed for $y(0) = 1$, where both performance and control deteriorate further.

Adjustment of the indirect linear scheme : For a fair comparison, our first task was to adjust the design parameters of the indirect scheme to achieve the best transient performance with a prescribed control effort. The trade-off between transient performance and control effort was examined for various initial conditions. To reduce the transients due to the mismatch of initial conditions, the initial condition of the reference model output was set in all tests to be equal to the initial value of the plant output. In spite of numerous attempts, no conclusive guidelines were found for initialization of filters in the indirect linear scheme. The simulation results shown in Figs. 2–4 for $y(0) = 0$ and $y(0) = 1$ are representative. The available design constants were the adaptation gain γ and the coefficients of the observer polynomial $s^3 + k_1 s^2 + k_2 s + k_3$ and of the controller polynomial $s^2 + m_1 s + m_2$. After several attempts, all the roots of the observer polynomial were placed at $s = -2$ with $k_1 = 6$, $k_2 = 12$, $k_3 = 8$, while the roots of the controller polynomial were placed in a Butterworth configuration of radius 3 with $m_1 = 4.2426$, $m_2 = 9$. These were judged to yield the best trade-off between transient performance and control effort for different initial conditions. A final choice to be made was that of the adaptation gain. Figs. 2–4 present simulations results for three different values of that gain: $\gamma = 500$ (Fig. 2), $\gamma = 1000$ (Fig. 3), and $\gamma = 2000$ (Fig. 4). In each of these figures we show two simulation runs, one with $y(0) = r_1(0) = 0$ (left) and one with $y(0) = r_1(0) = 1$ (right), and for each run we show the tracking error $y - y_r$ (top), the parameter estimate $\hat{a}$ (middle), and the control effort u (bottom). From these three figures, it is evident that the effect of the adaptation gain on the transient performance and the control effort is very different for the two sets of initial conditions. For $y(0) = r_1(0) = 0$, both the performance and the control effort improve with increasing adaptation gain, while for $y(0) = r_1(0) = 1$ they both deteriorate. This difference can be explained by examining the behavior of the parameter estimate $\hat{a}$. The parameter estimator interprets the nonzero initial condition $y(0)$ as a parameter error and tries to adjust the parameter estimate to reduce it. This results in the simultaneous deterioration of both transient performance and control effort. An increase in the adaptation gain, while improving parameter convergence for $y(0) = 0$, increases the sensitivity of the estimator to

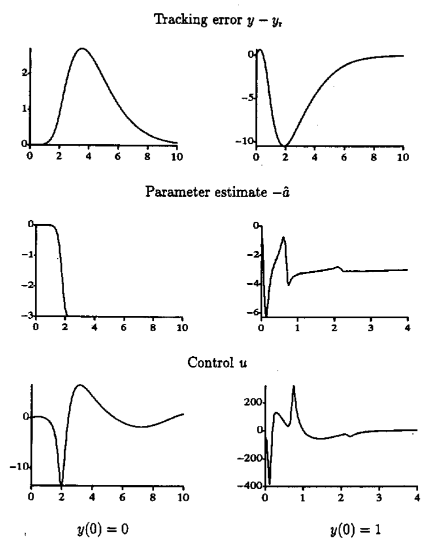

Fig. 4. A further increase of the adaptation gain to $\gamma = 2000$ confirms the trend detected in Figs. 1 and 2. The improvements obtained for $y(0) = 0$ are more than offset by the deterioration observed when $y(0) = 1$. The best compromise was judged to be achieved for $\gamma = 1000$ (Fig. 3).

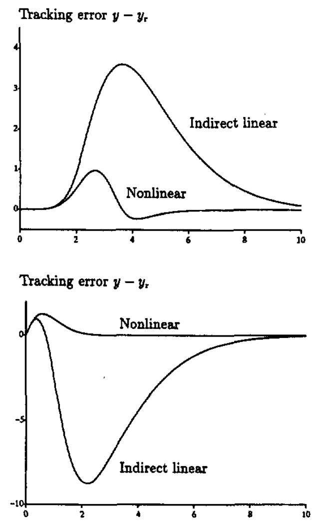

Fig. 5. Transient performance comparison of indirect linear scheme with new nonlinear scheme when $y(0) = 0$ (top) and when $y(0) = 1$ (bottom). In both cases, the nonlinear scheme achieves a dramatic performance improvement without any increase in control effort (see Figs. 6 and 7).

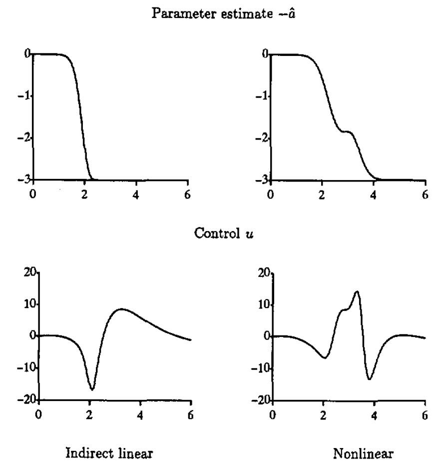

Fig. 6. The nonlinear damping terms provide better transient performance with a lower "effective adaptation gain." Here we see that for $y(0) = 0$, the improvement shown in Fig. 5 is achieved in spite of the slower parameter convergence of the nonlinear scheme.

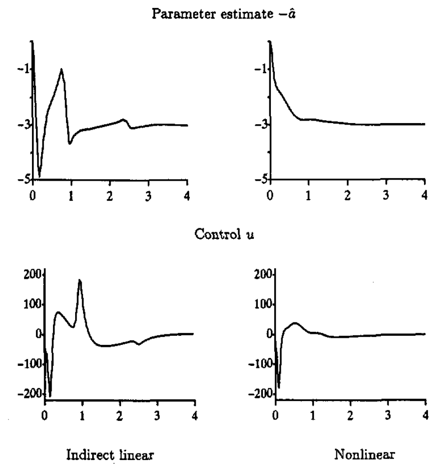

Fig. 7. The lower effective adaptation gain of the nonlinear scheme makes its parameter estimator less erratic when $y(0) = 1$. The benefits are evident not only in the transient performance (Fig. 5), but also in the control effort.

initial conditions, thus causing both transient performance and control to deteriorate even more for $y(0) = 1$. The best compromise was judged to be $\gamma = 1000$ (Fig. 3). The results in this figure were used for our comparison with the new nonlinear scheme.

Comparison with the nonlinear scheme : For a comparison of transient performance, the nonlinear scheme was adjusted to employ about the same control effort as that of the indirect linear scheme in Fig. 3. This was achieved with $k_1 = 6$, $k_2 = 12$, $k_3 = 8$, $c_1 = c_2 = c_3 = 1$, $d_1 = d_2 = d_3 = 0.1$, and the adaptation gain $\gamma = 0.5$. The plots in Fig. 5 show that the transient performance of the nonlinear scheme was far superior for both sets of initial conditions. Measured by any norm, the tracking error with the nonlinear scheme is only a fraction of the indirect linear scheme error.

We now proceed to discuss the three most important factors which contributed to the superior performance of our nonlinear scheme: nonlinear damping, incorporation of $\dot{\hat{a}}$ in the control law, and filter initialization.

Nonlinear damping: The nonlinear damping terms $-d_i((\partial\alpha_{i-1}/\partial y))^2 z_i$ contributed to a significant reduction of the effect of initial conditions on the new adaptive system. Their role is interpreted using Figs. 6 and 7. In Fig. 6, for $y(0) = 0$, the parameter convergence is slower in the nonlinear scheme. Fig. 5 shows that, in spite of this, the transient performance is much better without an increase in control effort. This was achieved by nonlinear damping, which attenuated the effect of initial parameter errors, so that fast parameter convergence was not required for good transient performance. The main benefit is the reduced sensitivity to initial conditions, as illustrated in Fig. 7. In contrast to the multiswing transient of the indirect linear scheme, the transient of the nonlinear scheme is nonoscillatory. The attenuating effect of nonlinear damping is particularly evident in Fig. 8. If the damping is increased over an optimum rate, the tracking error continues to decrease, but the control effort increases.

Incorporation of $\dot{\hat{a}}$ in u : Another important property of the nonlinear scheme is that the update law $\dot{\hat{a}} = \tau_3$ is contained *explicitly* in the control (6.15). This is not the case with certainty-equivalence schemes. The presence of $\dot{\hat{a}}$ in u indicates that some form of differential action is employed. The effect of this additional information about $\hat{a}$ is that the settling time of the tracking error is much shorter for the nonlinear scheme. Figs. 5–7 show that the settling time of the tracking error is closely coupled to that of the parameter error. In contrast, the tracking error of the indirect linear scheme continues to grow even after the parameter estimate has converged to its true value.

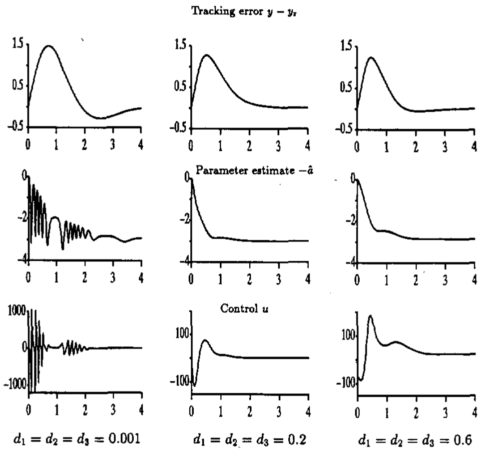

Fig. 8. The effect of nonlinear damping for $y(0) = 1$.

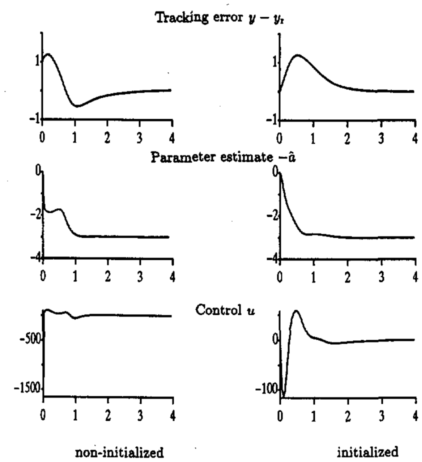

Fig. 9. Filter initialization for $y(0) = 1$.

Filter initialization : In contrast to the indirect scheme, the new nonlinear scheme provides clear guidelines for filter and reference model initialization, which follow from the design objective of driving the z-variables to zero. According to (6.7), (6.10), and (6.13), the initial values of z-variables are set to zero by choosing $r_1(0) = y(0)$, $v_2(0) = \alpha_1(0)$, $v_3(0) = \alpha_2(0)$. In general, it is always possible to set $z_1(0) = z_2(0) = \cdots = z_\rho(0) = 0$ by either one of the two methods:

1) *Setting* $z(0) = 0$ *by initializing the* λ*-filter.* Choosing $y_r(0) = y(0)$, we set $z_1(0) = 0$. Now, we perform the initialization using (5.14). For given $y(0), y_r(0), \dot{y}_r(0), \hat{p}(0), \hat{\theta}(0), \eta(0), \lambda_1(0), \cdots, \lambda_{m+i-1}(0)$ we set $z_i(0) = 0$ by choosing $\lambda_{m+i}(0) = -g_{m,i}^T[\lambda_1(0), \cdots, \lambda_{m+i-1}(0)]^T + \alpha_{i-1}|_{t=0}$, for $i = 2, \cdots, \rho$.
2) *Setting* $z(0) = 0$ *by initializing the reference model.* Again, by choosing $y_r(0) = y(0)$, we set $z_1(0) = 0$. Now, examining the expressions for α_i, we note that $(\partial\alpha_1/\partial\dot{y}_r) = \cdots = (\partial\alpha_{\rho-1}/\partial y_r^{(\rho-1)}) = \hat{p}$ and $z_i = v_{m,i} + \alpha_{i-1} = \bar{g}_i(y, y_r, \cdots, y_r^{(i-2)}, \hat{p}, \hat{\theta}, \eta, \lambda) + \hat{p}y_r^{(i-1)}$. It is reasonable to take $\hat{p}(0) \not\equiv 0$, because it reflects the fact that the high-frequency gain b_m is finite. Now, for a given $y(0)$, $y_r(0)$, $\hat{p}(0)$, $\hat{\theta}(0)$, $\eta(0)$, $\lambda(0)$, we set $z_i(0) = 0$ by choosing $y_r^{(i-1)}(0) = -(1/\hat{p}(0))\bar{g}_i|_{t=0}$, for $i = 2, \cdots, \rho$.

One simple way of setting $z(0) = 0$, which fits both cases a) and b) above, is to set $y_r(0) = y(0)$ and all remaining filter initial conditions to zero: $y_r^{(i)}(0) = 0$, $i = 1, \cdots, \rho - 1$, $\lambda(0) = \eta(0) = 0$.

In all tests, filter initialization was found to significantly improve both the transisent performance and the control effort. A typical example is Fig. 9, where $y_r(0) = y(0)$, $\dot{y}_r(0) = \ddot{y}_r(0) = 0$, $\lambda(0) = \eta(0) = 0$ have set $z(0) = 0$.

An explanation for the improvement of performances due to initialization is that by setting $z_1(0) = z_2(0) = \cdots = z_\rho(0) = 0$, we reduce the initial value of the Lyapunov function (5.7), that is, we reduce the initial deviation from the globally attractive manifold M. Recall from (5.4)–(5.6) that $\tilde{\eta}(0) = 0$ and $\tilde{\zeta}(0) = 0$. Thus, the only variables in the Lyapunov function which are not initialized to zero are $\hat{\theta} - \theta$, $\hat{p} - p$ and ϵ, that is, only those which are not known or not measured.

VIII. Concluding Remarks

The results of this paper show that recently developed tools for adaptive nonlinear control [7]–[11] can be used to design adaptive controllers for linear systems, which promise to outperform existing schemes. A particularly significant new property is the possibility to improve transient performance without an increase in control effort. This is achieved by nonlinear damping, which attenuates the effect of initial parameter errors so that fast parameter convergence is not required for good transient performance. The proof of stability is direct and reveals that the states of the adaptive system converge to a manifold whose dimension is the smallest possible. The robustness of the new class of adaptive controllers is a topic of current research.

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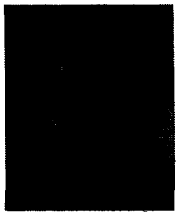

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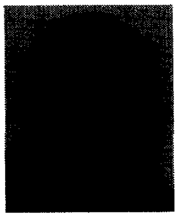

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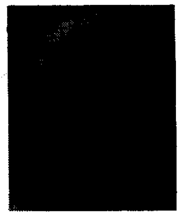

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Professor Kokotovic co-authored eight books and numerous articles contributing to sensitivity analysis, singular pertubation methods, and robust adaptive and nonlinear control. He received the 1990 Quazza Medal, the 1983 and 1993 Outstanding IEEE Transactions Paper Awards, and presented the 1991 Bode Prize Lecture.