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Abstract. The dynamics of Gaussian states for systems with Hamiltonians, quadratic in the position and momentum operators, gives rise to the definition of a system of nonlinear coupled differential equations for the density matrix parameters of the system states. In this work, we show that, using the Gaussian-state covariance-matrix evolution, it is possible to solve the system of these nonlinear equations. Some examples of applications of the method developed are given for one- and two-dimensional quadratic-system Gaussian states. It is noted that this formalism can also be used to find new constants of motion related to the covariance matrices of quadratic systems.

INTRODUCTION

In quantum mechanics, the physical system states are identified either with wave functions [1] (pure states) or with density matrices [2, 3] (mixed states). For given system Hamiltonians, the wave functions satisfy the linear Schrödinger evolution equation [4], and the density matrices satisfy the linear von Neumann equation [5]; for open systems, the density matrices satisfy the linear Gorini–Kossakowski–Sudarshan–Lindblad equation [6, 7]. Thus, the quantum dynamics is associated with linear quantum evolution equations, and the solutions of these equations are functions depending, in chosen representations, on quantum states, e.g., in the position representation on the position – coordinates, and extra depending-on-time parameters related to the properties of the functions and determining the physical characteristics of the solutions. Thus, the linear equations for the wave functions and density matrices, if one chooses different forms of the wave functions or density matrices depending on the evolving parameters, dictate the evolution equations for the time-dependent parameters, and these equations are nonlinear ones.

Thus, the linear quantum dynamics of the wave functions and density matrices determines the nonlinear dynamics of classical-like parameters describing the characteristics of the functions – solutions to the quantum evolution equations. This connection of the linear dynamics in quantum mechanics with nonlinear classical-like dynamics in the set of parameters characterizing the solutions of the Schrödinger evolution equation was employed in [8, 9, 10, 11, 12, 13, 14].

The important and useful ingredients of using the above-described connection of quantum linear dynamics with classical-like nonlinear dynamics is the possibility to use extra information on the nonlinear classical-like equations for the parameters characterizing the studied solutions of linear quantum evolution equations in order to obtain and solve the system of nonlinear equations. The important class of quantum systems is the systems described by Hamiltonians

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for which the integrals of motion are known. Such systems are, e.g., the sets of parametric oscillators. The linear in the position-and-momentum integrals of motion for such systems were found in [15, 16]. They were used to obtain the coherent-state wave functions for multimode parametric oscillators [17].

The quadratic in the position-and-momentum integrals of motion for quantum parametric oscillator were obtained in [18], where the quantum generalization of Ermakov invariant for classical parametric oscillator [19] was used to solve the Schrödinger equation. In this approach, the relation of the linear equation for the classical motion of parametric oscillator with nonlinear Riccati equation is elaborated [9, 14]; this method was also used in [8]. A particular example of the wave functions and density matrices considered in [10] are Gaussian functions.

The study of Gaussian states has lead to the many theoretical and practical advancements in science. These states are used in different subjects of quantum mechanics. A way to convert Gaussian states into non-Gaussian ones has been explored in [20]. In [21], it was shown that Gaussian states minimize the von Neumann entropy in the output of a Gaussian channel, which models the attenuation of an electromagnetic wave in the quantum case. The extendibility of bipartite system in a bosonic Gaussian state was studied in [22]. In [23], the symmetric logarithmic derivative for fermionic Gaussian states was used to obtain the quantum Fisher information of this type of states. The bounds for the negativity as a measure of entanglement in a fermionic Gaussian state has been computed in [24]. The entropy of formation of the Gaussian states was found in [25]. A way to simulate different quantum channels through the teleportation protocol has been given in [26]. Optimum quantum fidelity measures for Gaussian states have been reported in [27]. The Gaussian states have been considered as minimum uncertainty states in a information theory redefinition of the uncertainty relations. Also the study of Gaussian states through the solution of nonlinear equations, as the one defined by Riccati, has been studied in [12, 13, 14].

We established the nonlinear formalism to study the dynamics of Gaussian states in our previous work [10]. This type of formalism was used to determine the invariant states for such systems with quadratic Hamiltonians as the frequency converter and quasi-invariant states for the parametric amplifier. In this paper, we present an extension of our previous work; in particular, we study the solutions to quantum evolution equations within the framework of the nonlinear differential formalism in a more detailed way. To illustrate the application of this formalism in one- and two-dimensional quantum systems, we present some explicit examples.

This paper is organized as follows.

In section 2, a review of the differential formalism for the dynamics of the density matrix parameters of a Gaussian state is given. The solutions for the one-dimensional case are explored in section 3. Later in section 4, some examples for the evolution of a bipartite state are presented. Finally, some conclusions are given in section 5.

NONLINEAR DIFFERENTIAL FORMALISM FOR THE EVOLUTION OF GAUSSIAN STATES

Here, we construct the nonlinear differential formalism for an arbitrary Gaussian density matrix $\langle \mathbf{x}|\hat{\rho}|\mathbf{x}'\rangle$ in N dimensions; for this, we study the dynamics of the system determined by the Hamiltonian $\hat{H}(t)$, i.e., we obtain the solution to the von Neumann equation

$$\frac{d}{dt}\langle x|\hat{\rho}(t)|x'\rangle = \frac{i}{\hbar}\langle x|[\hat{H}(t),\hat{\rho}(t)]|x'\rangle\,, \tag{1}$$

where $x=(x_1,x_2,\ldots,x_N)$. Assume a Gaussian density matrix in the coordinate representation, which reads

$$\rho(x,x',t) = \exp\left(G(\mathbf{x},\mathbf{x}',t)\right), \tag{2}$$

where the exponent is given by the following quadratic function:

$$\begin{aligned} G(\mathbf{x},\mathbf{x}',t) = {}& \frac{1}{2}\sum_{j=1}^{N}\sum_{k\neq j}^{N}(a_{j,k}x_jx_k + a^*_{j,k}x'_jx'_k + a_{j,k+N}x_jx'_k + a^*_{j,k+N}x'_jx_k) \\ & - \sum_{j=1}^{N}(a_{j,j}x_j^2 + a^*_{j,j}x'^2_j - a_{j,j+N}\,x_jx'_j) + \sum_{j=1}^{N}(b_jx_j + b^*_jx'_j). \end{aligned} \tag{3}$$

If the system dynamics is determined by a quadratic Hamiltonian

$$\hat{H} = \hat{\xi}\,\boldsymbol{\Omega}(t)\,\hat{\xi} + c(t)\hat{\xi}\,, \tag{4}$$

where $\hat{\xi} = (\hat{p}_1, \hat{x}_1, \hat{p}_2, \hat{x}_2, \ldots, \hat{p}_N, \hat{x}_N)$, $c(t) = (c_1(t), c_2(t), \ldots, c_N(t))$, and

$$\mathbf{\Omega}(t) = \begin{pmatrix} \omega_{1,1}(t) & \omega_{1,2}(t) & \cdots & \omega_{1,2N}(t) \\ \omega_{1,2}(t) & \omega_{2,2}(t) & \cdots & \omega_{2,2N}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{1,2N}(t) & \omega_{2,2N}(t) & \cdots & \omega_{2N,2N}(t) \end{pmatrix}, \quad (5)$$

then von Neumann equation can be rewritten as follows:

$$\dot{G}(\mathbf{x}, \mathbf{x}', t)\rho(x, x', t) = (H(x,t) - H(x',t))\rho(x, x', t). \quad (6)$$

Here, $H(x,t) = \langle x|\hat{H}(t)$ and $H(x',t) = \hat{H}(t)|x'\rangle$ are the expressions of the Hamiltonian in differential form. By representing the momentum operators in the position representation $\hat{p}_j = -i\partial/\partial x_j$, one can obtain a set of nonlinear equations for the density matrix parameters a_{kl}. It is possible to solve this set of nonlinear differential equations, in view of the connection between the density matrix parameters and the covariance matrix.

As examples, we present the general one-dimensional Hamiltonian and some particular two-dimensional Hamiltonians, along with the nonlinear equations which they define.

ONE-DIMENSIONAL CASE

Let us consider an arbitrary Hamiltonian given by the following expression:

$$\hat{H} = \nu_1(t)\hat{p}^2 + \nu_2(t)\hat{q}^2 + \nu_3(t)(\hat{p}\hat{q} + \hat{q}\hat{p}) + c_1(t)\hat{p} + c_2(t)\hat{q}.$$

This Hamiltonian dictates the dynamics of the following initial Gaussian state in the coordinate representation

$$\langle x'|\hat{\rho}(0)|x\rangle = N \exp\left\{-ax^2 + a_{12}xx' - a^* x'^2 + bx + b^* x'\right\} \quad (7)$$

where the normalization constant N is given by

$$N = \left(\frac{a_1 + a_1^* - a_{12}}{\pi}\right)^{1/2} \exp\left\{-\frac{(b_1 + b_1^*)^2}{4(a_1 + a_1^* - a_{12})}\right\}.$$

The dynamics of such system leads us to the definition of the following nonlinear differential equations:

$$\begin{aligned} \dot{a}_1(t) &= i(a_{12}^2(t) - 4a_1^2(t))\nu_1(t) - 4a_1(t)\nu_2(t) + i\nu_3(t), \\ \dot{a}_1^*(t) &= -i(a_{12}^2(t) - 4a_1^{*2}(t))\nu_1(t) - 4a_1^*(t)\nu_2(t) - i\nu_3(t), \\ \dot{a}_{12}(t) &= 4a_{12}(t)(i(a_1^*(t) - a_1(t))\nu_1(t) - \nu_2(t)), \end{aligned} \quad (8)$$

On the other hand, the covariance matrix of the state (7) can be written as

$$\boldsymbol{\sigma}(0) = \begin{pmatrix} \sigma_{pp} & \sigma_{pq} \\ \sigma_{pq} & \sigma_{qq} \end{pmatrix} = \frac{1}{2\,(2a_{1R} - a_{12})} \begin{pmatrix} 4|a_1|^2 - a_{12}^2 & -2a_{1I} \\ -2a_{1I} & 1 \end{pmatrix}. \quad (9)$$

This covariance matrix satisfies the differential equation $\dot{\boldsymbol{\sigma}}(t) = 2\left[\boldsymbol{\sigma}(t)\mathbf{\Omega}(t)\mathbf{\Sigma} - \mathbf{\Sigma}\mathbf{\Omega}(t)\boldsymbol{\sigma}(t)\right]$, where $\mathbf{\Omega}(t)$ is defined by the Hamiltonian parameters, and $\mathbf{\Sigma}$ is a symplectic matrix

$$\mathbf{\Omega} = \frac{1}{2}\begin{pmatrix} \nu_1(t) & \nu_2(t) \\ \nu_2(t) & \nu_3(t) \end{pmatrix}, \quad \mathbf{\Sigma} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The explicit differential equations, which the covariances must satisfy, are

$$\dot{\sigma}_{pp}(t) = -4(\nu_2(t)\sigma_{pp}(t) + \nu_3(t)\sigma_{pq}(t)), \qquad \dot{\sigma}_{pq}(t) = 2(\nu_1(t)\sigma_{pp}(t) - \nu_3(t)\sigma_{qq}(t)),$$
$$\dot{\sigma}_{qq}(t) = 4(\nu_1(t)\sigma_{pq}(t) + \nu_2(t)\sigma_{qq}(t)). \quad (10)$$

The decoupling of these differential equations is not trivial, unless specific functions for the Hamiltonian parameters are taken into account. However, if this system is solvable then, using the inverse relation of Eq. (9), one can obtain the solutions to Eq. (8). To show this, we can think of the parameters $\nu_i,\ i=1,\ldots,3$ as time independent; in this case, Eq. (10) can be written in matrix form

$$\dot{\mathbf{s}}(t)=\mathbf{M}\mathbf{s}(t),\quad \mathbf{s}^T(t)=(\sigma_{pp}(t),\sigma_{pq}(t),\sigma_{qq}(t)),\quad \mathbf{M}=2\begin{pmatrix}-2\nu_2 & -2\nu_3 & 0\\ \nu_1 & 0 & -\nu_3\\ 0 & 2\nu_1 & 2\nu_2\end{pmatrix}.$$

This system can be decoupled, using the eigenvectors matrix of $\mathbf{M}$ ($\mathbf{E}$). By the change of variables $\mathbf{u}(t)=(u_1(t),u_2(t),u_3(t))^T=\mathbf{E}^{-1}\mathbf{s}(t)$, the decoupled system of equations obtained by this change of variables is

$$\dot{\mathbf{u}}=\mathbf{W}\mathbf{u},\quad \mathbf{W}=\mathrm{diag}\left(0,-4\sqrt{\nu_2^2-\nu_1\nu_3},4\sqrt{\nu_2^2-\nu_1\nu_3}\right),$$

which has the solution for $\nu_1\nu_3-\nu_2^2\neq 0$, namely,

$$u_1(t)=C=\frac{\nu_1(\nu_1\sigma_{pp}(t)+2\nu_2\sigma_{pq}(t)+\nu_3\sigma_{qq}(t))}{2(\nu_1\nu_3-\nu_2^2)},\quad u_2(t)=u_2(0)\exp(-4\omega t),\ u_3(t)=u_3(0)\exp(4\omega t),\tag{11}$$

with $\omega=\sqrt{\nu_2^2-\nu_1\nu_3}$ which, in the case $\nu_1\nu_3>\nu_2^2$, allows periodic solutions. It is important to notice the invariance of $u_1(t)=C$, which is a constant of motion of our system. These expressions then lead to the following solutions for the covariance matrix elements:

$$\begin{aligned}\sigma_{pp}(t)&=\frac{\omega\cosh(4\omega t)(\nu_1\sigma_{pp}(0)-C\nu_3)+C\nu_3\omega-\nu_1\sinh(4\omega t)(\nu_2\sigma_{pp}(0)+\nu_3\sigma_{pq}(0))}{\omega\nu_1},\\ \sigma_{pq}(t)&=\cosh(4\omega t)\left(\frac{C\nu_2}{\nu_1}+\sigma_{pq}(0)\right)-\frac{C\nu_2}{\nu_1}+\frac{\sinh(4\omega t)(\nu_1\sigma_{pp}(0)-\nu_3\sigma_{qq}(0))}{2\omega},\\ \sigma_{qq}(t)&=(\sigma_{qq}(0)-C)\cosh(4\omega t)+C+\frac{\sinh(4\omega t)(\nu_1\sigma_{pq}(0)+\nu_2\sigma_{qq}(0))}{\omega}.\end{aligned}\tag{12}$$

These solutions for the covariances lead to the following solutions to the nonlinear equations (8), which can be written as

$$a(t)=\frac{1+4(d-i\sigma_{pq}(t))}{8\sigma_{qq}(t)},\qquad a^*(t)=\frac{1+4(d+i\sigma_{pq}(t))}{8\sigma_{qq}(t)},\qquad a_{12}(t)=\frac{4d-1}{4\sigma_{qq}(t)},\tag{13}$$

where $d=\sigma_{pp}(t)\sigma_{qq}(t)-\sigma_{pq}^2(t)$ is the invariant determinant of the covariance matrix.

Example

As a further example, we can take the evolution of a Gaussian state for a free particle Hamiltonian. This implies the following values for the Hamiltonian parameters $\nu_1=1/2$ and $\nu_2=\nu_3=0$. In this example, the equations for the covariances are $\dot{\sigma}_{pp}(t)=0$, $\dot{\sigma}_{pq}(t)=\sigma_{pp}(t)$, and $\dot{\sigma}_{qq}(t)=2\sigma_{pq}(t)$; with nonlinear Eqs. (8)

$$\begin{aligned}\dot{a}_1(t)&=i(a_{12}^2(t)-4a_1^2(t))/2,\\ \dot{a}_1^*(t)&=-i(a_{12}^2(t)-4a_1^{*2}(t))/2,\\ \dot{a}_{12}(t)&=2ia_{12}(t)(a_1^*(t)-a_1(t)).\end{aligned}\tag{14}$$

By taking into account the initial thermal light state with frequency ϕ, given by the following Gaussian function:

$$\langle x|\hat{\rho}|x'\rangle=\frac{\exp\left[-\frac{1}{2}(x^2+x'^2)\coth(\beta\phi)+xx'\mathrm{csch}(\beta\phi)\right]}{\pi(1-e^{-\beta\phi})\sqrt{1-e^{-2\beta\phi}}},\quad \beta=1/T,$$

then the initial parameters of the density matrix and the initial covariances are

$$a(0)=a^*(0)=\frac{1}{2}\coth(\beta\phi),\quad a_{12}(0)=\mathrm{csch}(\beta\phi),\quad \sigma_{pp}(0)=\sigma_{qq}(0)=\frac{1}{2}\coth\left(\frac{\beta\phi}{2}\right),\quad \sigma_{pq}(0)=0.$$

The evolution of the covariance matrix components can be obtained by direct integration or by taking the limit of Eq. (12) when $\omega \to 0$, in other words,

$$a(t) = \frac{\coth(\beta\phi) - it}{2(1+t^2)}, \quad a^*(t) = \frac{\coth(\beta\phi) + it}{2(1+t^2)}, \quad a_{12}(t) = \frac{\operatorname{csch}(\beta\phi)}{1+t^2},$$
$$\sigma_{pp}(t) = \frac{1}{2}\coth\left(\frac{\beta\phi}{2}\right), \quad \sigma_{qq}(t) = \frac{1}{2}\left(1+t^2\right)\coth\left(\frac{\beta\phi}{2}\right), \quad \sigma_{pq}(t) = \frac{1}{2}t\coth\left(\frac{\beta\phi}{2}\right). \tag{15}$$

From these results, one can see the time invariance of the determinant of the covariance matrix $\sigma_{pp}\sigma_{qq} - \sigma_{pq}^2 = \frac{1}{4}\coth^2(\beta\phi/2)$ and the constant of motion $u_1 = \sigma_{pp}$.

TWO-MODE CASE

In the two-mode case, the most general Gaussian state can be written as

$$\langle x_1, x_2|\hat{\rho}(0)|x_1', x_2'\rangle = N\exp\Big[-a_{11}x_1^2 - a_{11}^*x_1'^2 - a_{22}x_2^2 - a_{22}^*x_2'^2 + a_{12}x_1x_2 + a_{12}^*x_1'x_2' + a_{13}x_1x_1'$$
$$+ a_{14}x_1x_2' + a_{14}^*x_1'x_2 + a_{24}x_2x_2'\Big], \tag{16}$$

which satisfies the hermiticity condition $\langle x_1, x_2|\hat{\rho}(0)|x_1', x_2'\rangle = (\langle x_1', x_2'|\hat{\rho}(0)|x_1, x_2\rangle)^*$. The Hamiltonian of the system can be expressed as

$$\hat{H} = \omega_{11}\hat{p}_1^2 + \omega_{22}\hat{x}_1^2 + \omega_{33}\hat{p}_2^2 + \omega_{44}\hat{q}_2^2 + \omega_{12}(\hat{p}_1\hat{x}_1 + \hat{x}_1\hat{p}_1) + 2\omega_{13}\hat{p}_1\hat{p}_2 + 2\omega_{14}\hat{p}_1\hat{x}_2$$
$$+ 2\omega_{23}\hat{p}_2\hat{x}_1 + 2\omega_{24}\hat{x}_1\hat{x}_2 + \omega_{34}(\hat{p}_2\hat{x}_2 + \hat{x}_2\hat{p}_2) + c_1\hat{p}_1 + c_2\hat{x}_1 + c_3\hat{p}_2 + c_4\hat{x}_2. \tag{17}$$

For this quadratic Hamiltonian, the nonlinear equations for the density matrix parameters resulting from the study of the von Neumann equation (1) are

$$\dot{a}_{11} = i\omega_{22} - 4\omega_{12}a_{11} + 2\omega_{23}a_{12} + i\omega_{11}(-4a_{11}^2 + a_{13}^2) + 2i\omega_{13}(2a_{11}a_{12} + a_{13}a_{14}) - i\omega_{33}(a_{12}^2 - a_{14}^2),$$
$$\dot{a}_{22} = i\omega_{44} + 2\omega_{14}a_{12} - i\omega_{11}(a_{12}^2 - a_{14}^{*2}) - 4\omega_{34}a_{22} + 2i\omega_{13}(2a_{12}a_{22} + a_{14}^*a_{24}) + i\omega_{33}(-4a_{22}^2 + a_{24}^2),$$
$$\dot{a}_{12} = -2i\omega_{24} + 4\omega_{14}a_{11} - 2\omega_{12}a_{12} - 2\omega_{34}a_{12} - 2i\omega_{11}(2a_{11}a_{12} + a_{13}a_{14}^*) + 4\omega_{23}a_{22}$$
$$+ 2i\omega_{13}(a_{12}^2 - a_{14}a_{14}^* + 4a_{11}a_{22} - a_{13}a_{24}) - 2i\omega_{33}(2a_{12}a_{22} + a_{14}a_{24}), \tag{18}$$
$$\dot{a}_{13} = -4\omega_{12}a_{13} - 4i\omega_{11}(a_{11} - a_{11}^*)a_{13} - 2\omega_{23}(a_{14} + a_{14}^*) + 2i\omega_{13}((a_{12} - a_{12}^*)a_{13}$$
$$+ 2a_{11}^*a_{14} - 2a_{11}a_{14}^*) + 2i\omega_{33}(-a_{12}^*a_{14} + a_{12}a_{14}^*),$$
$$\dot{a}_{14} = -2\omega_{14}a_{13} - 2\omega_{12}a_{14} - 2\omega_{34}a_{14} - 2i\omega_{11}(a_{12}^*a_{13} + 2a_{11}a_{14}) - 2\omega_{23}a_{24}$$
$$+ 2i\omega_{13}((a_{12} - a_{12}^*)a_{14} + 2a_{13}a_{22}^* - 2a_{11}a_{24}) + 2i\omega_{33}(2a_{14}a_{22}^* + a_{12}a_{24}),$$
$$\dot{a}_{24} = -2\omega_{14}(a_{14} + a_{14}^*) + 2i\omega_{11}(a_{12}a_{14} - a_{12}^*a_{14}^*) - 4\omega_{34}a_{24} + 4i\omega_{33}(-a_{22} + a_{22}^*)a_{24}$$
$$+ 2i\omega_{13}(-2a_{14}a_{22} + 2a_{14}^*a_{22}^* + (a_{12} - a_{12}^*)a_{24}), \tag{19}$$

which satisfy the initial conditions defined by the parameters of the initial Gaussian state. As in the one-dimensional case, the solutions for this set of equalities can be obtained, using the time evolution of the covariance matrix. As the covariance matrix satisfies a system of linear, classical-like differential equations, this method is a way to linearize the previous set of nonlinear equations.

In such a case, the covariances read

$$\sigma_{p_1p_1} = 2a_{11} - (-2a_{11}+a_{13})^2\sigma_{q_1,q_1} - (a_{12}+a_{14})^2\sigma_{q_2,q_2} + 2(2a_{11}-a_{13})(a_{12}+a_{14})\sigma_{q_1,q_2}\,,$$
$$\sigma_{p_1q_1} = \frac{-i\left(-4(a_{11}-a_{11}^*)(a_{22}+a_{22}^*-a_{24}) + (a_{12}+a_{14})^2 - (a_{12}^*+a_{14}^*)^2\right)}{4(a_{11}+a_{11}^*-a_{13})(a_{22}+a_{22}^*-a_{24}) - 2\left((a_{12}+a_{12}^*+a_{14}+a_{14}^*)^2\right)}\,,$$
$$\sigma_{p_1p_2} = -a_{12} + (2a_{22}-a_{24})(a_{12}+a_{14})\sigma_{q_2,q_2} + (2a_{11}-a_{13})(a_{12}+a_{14}^*)\sigma_{q_1,q_1}\,,$$
$$\sigma_{p_1q_2} = \frac{i(2a_{11}(a_{12}^*+a_{14}^*) - 2a_{11}^*(a_{12}+a_{14}) + a_{13}(a_{12}-a_{12}^*+a_{14}-a_{14}^*))}{4(a_{11}+a_{11}^*-a_{13})(a_{22}+a_{22}^*-a_{24}) - (a_{12}+a_{12}^*+a_{14}+a_{14}^*)^2}\,,$$
$$\sigma_{q_1q_1} = \frac{2(a_{22}+a_{22}^*-a_{24})}{4(a_{11}+a_{11}^*-a_{13})(a_{22}+a_{22}^*-a_{24}) - (a_{12}+a_{12}^*+a_{14}+a_{14}^*)^2}\,,$$
$$\sigma_{p_2q_1} = \frac{-i(a_{24}(-a_{12}+a_{12}^*+a_{14}-a_{14}^*) + 2a_{22}^*(a_{12}+a_{14}^*) - 2a_{22}(a_{12}^*+a_{14}))}{4(a_{11}+a_{11}^*-a_{13})(a_{22}+a_{22}^*-a_{24}) - (a_{12}+a_{12}^*+a_{14}+a_{14}^*)^2}\,,$$
$$\sigma_{q_1q_2} = \frac{a_{12}+a_{12}^*+a_{14}+a_{14}^*}{4(a_{11}+a_{11}^*-a_{13})(a_{22}+a_{22}^*-a_{24}) - (a_{12}+a_{12}^*+a_{14}+a_{14}^*)^2}\,,$$
$$\sigma_{p_2p_2} = 2a_{22} - (-2a_{22}+a_{24})^2\sigma_{q_2,q_2} - (a_{12}+a_{14}^*)^2\sigma_{q_1,q_1} - 2(-2a_{22}+a_{24})(a_{12}+a_{14}^*)\sigma_{q_1,q_2}\,,$$
$$\sigma_{p_2q_2} = \frac{i\left(-4(a_{22}-a_{22}^*)(a_{11}+a_{11}^*-a_{13}) + (a_{12}+a_{14}^*)^2 - (a_{12}^*+a_{14})^2\right)}{2\left((a_{12}+a_{12}^*+a_{14}+a_{14}^*)^2 - 4(a_{11}+a_{11}^*-a_{13})(a_{22}+a_{22}^*-a_{24})\right)}\,,$$
$$\sigma_{q_2q_2} = \frac{2(a_{11}+a_{11}^*-a_{13})}{4(a_{11}+a_{11}^*-a_{13})(a_{22}+a_{22}^*-a_{24}) - (a_{12}+a_{12}^*+a_{14}+a_{14}^*)^2}\,. \tag{20}$$

In view of these results, an analogous method, as in the one-dimensional case, can be applied to obtain the solutions to the nonlinear equations (19) following the steps:

1. Obtain the differential equations for the covariance matrix $\dot{\boldsymbol{\sigma}}(t) = 2i(\boldsymbol{\sigma}(t)\boldsymbol{\Omega}\mathbf{D} - \mathbf{D}\boldsymbol{\Omega}\boldsymbol{\sigma}(t))$ and rewrite them as $\dot{\mathbf{v}} = \mathbf{M}\mathbf{v}$, with $\mathbf{v}^T = (\sigma_{p_1p_1}, \sigma_{p_1q_1}, \sigma_{p_1p_2}, \sigma_{p_1q_2}, \sigma_{q_1q_1}, \sigma_{p_2q_1}, \sigma_{q_1q_2}, \sigma_{p_2p_2}, \sigma_{p_2q_2}, \sigma_{q_2q_2})$.

2. Using the eigenvalues of $\mathbf{M}$ ($\mathbf{E}$), one can decouple the linear system in the case of a time-independent Hamiltonian ($\boldsymbol{\Omega} \neq \boldsymbol{\Omega}(t)$). This is done by the change of variables $\mathbf{u} = \mathbf{E}^{-1}\mathbf{v}$. The system in the new coordinates is $\dot{\mathbf{u}} = \mathbf{W}\mathbf{u}$, with $\mathbf{W}$ being the matrix made of the eigenvalues of $\mathbf{M}$.

3. The solutions given $\mathbf{u}(t) = \mathbf{u}(0)e^{\mathbf{W}t}$ are then used by applying $\mathbf{E}$, i.e., $\mathbf{v}(t) = \mathbf{E}\mathbf{u}(t)$.

4. Finally, the solutions for the nonlinear differential equations for the density matrix parameters are obtained by the inversion of Eqs. (20).

In the cases, where it is difficult to obtain the solutions following this procedure, one can solve numerically the linear differential equations for the covariance matrix elements without the decoupling of the variables and then, by the inversion of Eq. (20), arrive at the solution for the density matrix parameters a_{jk}.

Example

As an example of the procedure elaborated, we present a pair of coupled harmonic oscillators with degenerated frequency ω and coupling constant ω_{12}, which can be characterized by the Hamiltonian

$$\hat{H}(t) = \frac{1}{2}(\hat{p}_1^2 + \hat{p}_2^2 + \omega^2(\hat{q}_1^2 + \hat{q}_2^2)) + \omega_{12}\hat{q}_1\hat{q}_2. \tag{21}$$

We introduce the matrix

$$\Omega = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega^2 & 0 & 2\omega_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 2\omega_{12} & 0 & \omega^2 \end{pmatrix}.$$

Following the procedure of decoupling the differential equations described above for the covariance matrix components, we arrive at the conclusion that there exist two different linear constants of motion in this system, which can be written as follows:

$$u_1 = \frac{1}{2}(\sigma_{p_1p_1}(t) + \sigma_{p_2p_2}(t) + \omega^2(\sigma_{q_1q_1}(t) + \sigma_{q_2q_2}(t))) + 2\omega_{12}\sigma_{q_1q_2}(t),$$
$$u_2 = (\sigma_{p_1p_1}(t) + \sigma_{p_2p_2}(t))\omega^2 + (\sigma_{q_1q_1}(t) + \sigma_{q_2q_2}(t))\omega^4 + 4\omega_{12}(\sigma_{p_1p_2}(t) + (\sigma_{q_1q_1}(t) + \sigma_{q_2q_2}(t))\omega^2). \quad (22)$$

If we consider the initial coherent state $|\alpha_1, \alpha_2\rangle$ given by the following Gaussian:

$$\langle x_1, x_2|\hat{\rho}|x_1', x_2'\rangle = N \exp\left\{-\frac{1}{2}(x_1^2 + x_2^2 + x_1'^2 + x_2'^2 - 2\sqrt{2}(\alpha_1^* x_1 + \alpha_1 x_1' + \alpha_2^* x_2 + \alpha_2 x_2'))\right\},$$

with the normalization constant

$$N = \frac{1}{\pi}\exp\left(-\frac{1}{2}(\alpha_1^2 + \alpha_1^{*2} + \alpha_2^2 + \alpha_2^{*2} + 2|\alpha_1|^2 + 2|\alpha_2|^2)\right),$$

then the solutions for the covariances differential equations $\dot{\boldsymbol{\sigma}}(t) = 2i(\boldsymbol{\sigma}(t)\boldsymbol{\Omega}\mathbf{D} - \mathbf{D}\boldsymbol{\Omega}\boldsymbol{\sigma}(t))$ can be listed as

$$\sigma_{p_1p_1}(t) = \frac{1}{8}\left(-(f_1-1)\cos\left(2\sqrt{f_1}t\right) - (f_2-1)\cos\left(2\sqrt{f_2}t\right) + 2\left(\omega^2+1\right)\right),$$
$$\sigma_{p_1q_1}(t) = -\frac{1}{8}\left(\frac{(f_1-1)\sin\left(2\sqrt{f_1}t\right)}{\sqrt{f_1}} + \frac{(f_2-1)\sin\left(2\sqrt{f_2}t\right)}{\sqrt{f_2}}\right),$$
$$\sigma_{p_1p_2}(t) = \frac{1}{8}\left(-(f_1-1)\cos\left(2\sqrt{f_1}t\right) + (f_2-1)\cos\left(2\sqrt{f_2}t\right) + 4\omega_{12}\right),$$
$$\sigma_{p_1q_2}(t) = \frac{1}{8}\left(\frac{(f_2-1)\sin\left(2\sqrt{f_2}t\right)}{\sqrt{f_2}} - \frac{(f_1-1)\sin\left(2\sqrt{f_1}t\right)}{\sqrt{f_1}}\right),$$
$$\sigma_{q_1q_1}(t) = \frac{(f_1-1)f_2\cos\left(2\sqrt{f_1}t\right) + f_1(f_2-1)\cos\left(2\sqrt{f_2}t\right) + 2\left(f_1f_2+\omega^2\right)}{8f_1f_2},$$
$$\sigma_{p_2q_1}(t) = \frac{1}{8}\left(\frac{(f_2-1)\sin\left(2\sqrt{f_2}t\right)}{\sqrt{f_2}} - \frac{(f_1-1)\sin\left(2\sqrt{f_1}t\right)}{\sqrt{f_1}}\right),$$
$$\sigma_{q_1q_2}(t) = \frac{(f_1-1)f_2\cos\left(2\sqrt{f_1}t\right) - f_1(f_2-1)\cos\left(2\sqrt{f_2}t\right) - 4\omega_{12}}{8f_1f_2},$$
$$\sigma_{p_2p_2}(t) = \frac{1}{8}\left(-(f_1-1)\cos\left(2\sqrt{f_1}t\right) - (f_2-1)\cos\left(2\sqrt{f_2}t\right) + 2\left(\omega^2+1\right)\right),$$
$$\sigma_{p_2q_2}(t) = -\frac{1}{8}\left(\frac{(f_1-1)\sin\left(2\sqrt{f_1}t\right)}{\sqrt{f_1}} + \frac{(f_2-1)\sin\left(2\sqrt{f_2}t\right)}{\sqrt{f_2}}\right),$$
$$\sigma_{q_2q_2}(t) = \frac{(f_1-1)f_2\cos\left(2\sqrt{f_1}t\right) + f_1(f_2-1)\cos\left(2\sqrt{f_2}t\right) + 2\left(f_1f_2+\omega^2\right)}{8f_1f_2}, \quad (23)$$

where the constants $f_{1,2} = \omega^2 \pm 2\omega_{12}$.

From this covariance matrix entries, we can see that the density matrix parameters are

$$a_{11}(t) = a_{22}(t) = \frac{1}{4}\left(\frac{i(f_1-1)\sin\left(\sqrt{f_1}t\right)}{\sqrt{f_1}\cos\left(\sqrt{f_1}t\right) + i\sin\left(\sqrt{f_1}t\right)} + \frac{i(f_2-1)\sin\left(\sqrt{f_2}t\right)}{\sqrt{f_2}\cos\left(\sqrt{f_2}t\right) + i\sin\left(\sqrt{f_2}t\right)} + 2\right),$$
$$a_{12}(t) = \frac{i}{2}\left(\frac{(f_2-1)\sin\left(\sqrt{f_2}t\right)}{\sqrt{f_2}\cos\left(\sqrt{f_2}t\right) + i\sin\left(\sqrt{f_2}t\right)} - \frac{(f_1-1)\sin\left(\sqrt{f_1}t\right)}{\sqrt{f_1}\cos\left(\sqrt{f_1}t\right) + i\sin\left(\sqrt{f_1}t\right)}\right),$$
$$a_{13}(t) = a_{14}(t) = a_{24}(t) = 0, \quad (24)$$

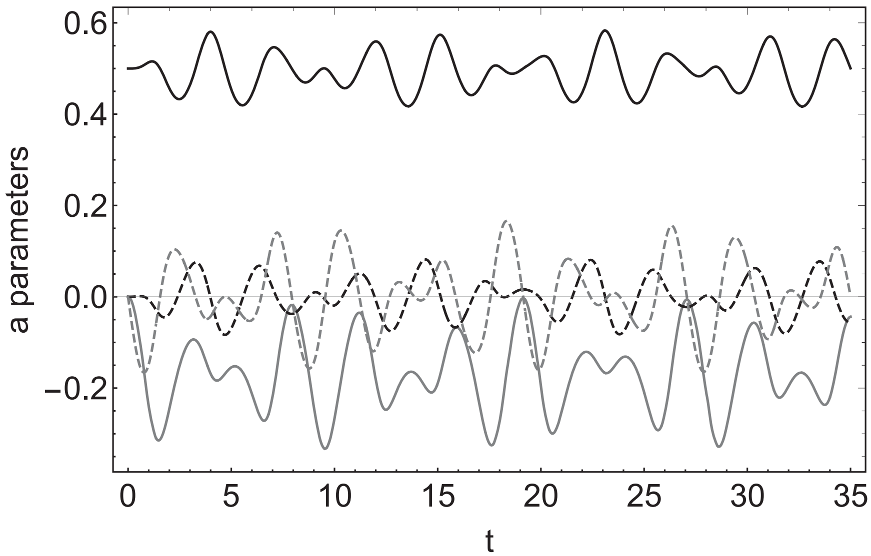

FIGURE 1. Evolution of the real and imaginary parts of the density matrix parameters $a_{11}(t) = a_{22}(t)$ and $a_{12}(t)$ in a pair of interacting harmonic oscillators of Eq. (21), with $\omega = 1$ and $\omega_{12} = 1/6$. Here, $\mathrm{Re}(a_{11})$ is shown by the black solid curve and $\mathrm{Re}(a_{12})$ is shown by the gray solid curve, while $\mathrm{Im}(a_{11})$ and $\mathrm{Im}(a_{12})$ are shown by the black and gray dashed curves, respectively.

which are the solutions to the following set of nonlinear differential equations:

$$\begin{gathered}\dot{a}_{11}(t) = \frac{1}{2} i \left(-4a_{11}^2 - a_{12}^2 + a_{13}^2 + a_{14}^2 + \omega^2\right), \ \dot{a}_{22}(t) = \frac{1}{2} i \left(-a_{12}^2 + a_{14}^{*2} - 4a_{22}^2 + a_{24}^2 + \omega^2\right), \\ \dot{a}_{12}(t) = -i(2a_{12}(a_{11} + a_{22}) + a_{13}a_{14}^* + a_{14}a_{24} + \omega 12), \ \dot{a}_{13}(t) = i(2a_{13}(a_{11}^* - a_{11}) + a_{12}a_{14}^* - a_{12}^* a_{14}), \\ \dot{a}_{14}(t) = i(2a_{14}(a_{22}^* - a_{11}) + a_{12}a_{24} - a_{12}^* a_{13}), \ \dot{a}_{24}(t) = i(a_{12}(t)a_{14} - a_{12}^* a_{14}^* + 2a_{24}(a_{22}^* - a_{22})),\end{gathered} \tag{25}$$

with the initial conditions $a_{11}(0) = a_{22}(0) = 1/2$ and the rest parameters equal to zero at zero time.

In fig. 1, the time evolution of the real and imaginary parts of the density matrix parameters are shown. One can observe here the periodicity of the system even though these parameters are the solutions of a highly nonlinear set of equations. This property is due to the periodicity of the solutions of the classical problem for two coupled harmonic oscillators, which are encoded in the covariance matrix.

CONCLUSIONS

The evolution of the density matrix is described by the von Neumann equation. The solutions to this equation give rise to a set of nonlinear equations for the parameters of the density matrix in the position representation. This set of nonlinear equations can be solved, and the solutions to these nonlinear equations correspond to the solutions to the linear von Neumann equation; this correspondence provides a possibility to develop a new tool for explicitly obtaining the evolution of quantum systems.

We showed that the nonlinear differential equations for the density matrix parameters of the quantum-system Gaussian states can be solved, using the solutions to the covariance matrix evolution equations. From this property, one can conclude that the mentioned nonlinear equations can be linearized by the definition of different equations (the covariance matrix entries defined by the density matrix parameters) and then solved. This also implies that there exists a whole family of nonlinear differential equations that can be solved by this method.

As examples, we gave explicit solutions for one-dimensional and two-dimensional quantum systems. In the two-dimensional case, the Hamiltonian for two coupled harmonic oscillators was studied. We demonstrated that this system has two constants of motion related to the covariance matrix parameters and derived the expressions for these constants of motion.

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