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NONLINEAR DYNAMIC BUCKLING OF A COMPRESSED ELASTIC COLUMN*

By

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1. Introduction. In a recent paper [1] the nonlinear dynamic buckling of an elastic column subjected to a constant, compressive, axial displacement was considered. The model was derived from a modified Euler–Bernoulli beam theory [2, 3, 4, 5], including damping and neglecting axial inertia. Assuming the axial displacement to be slightly above that value at which linear theory first predicts instability (the first bifurcation point), the authors in [1] predict the leading term in a formal asymptotic expansion of the response of the column using a multi-time technique [6] along the lines presented in [7]. A restriction on the initial data had to be imposed in order for the solution generated to be bounded. As is noted in [1], if this restriction is violated a different asymptotic expansion is presumably required to solve the problem.

We present here a solution of this problem using the technique of averaging [8, 9, 10]. In the formal development of this solution we note that it is not necessary to impose any restrictions on the initial data. In addition, this approach yields more information about the solution than the multi-time procedure even in the case when the above mentioned restriction on the initial data is made. This problem is therefore an example of a case where multi-timing is inapplicable while the averaging procedure does work.

In Sec. 2 we present the equations to be solved and first consider the undamped case with monochromatic initial data. The purpose of this is to motivate the scaling which follows. Then in Sec. 3, using appropriate scalings and a set of new variables, we put the problem into the form of a coupled system of ordinary differential equations. We then formally apply the method of averaging to derive the equations governing the leading terms in an asymptotic expansion of the solution. Finally, in Sec. 5 we discuss these equations and point out why the attempt in [1] led to the above-mentioned restriction on the initial data.

The application of the averaging procedure to dynamic bifurcation problems appears to be new and is presently being applied to other problems in elastic vibrations and fluid dynamics by the author and his co-workers at Rensselaer Polytechnic Institute.

2. Undamped equations with monochromatic initial data. The non-dimensional equations governing the transverse displacement of a bar are [1]

$$w_{xxxx} + \lambda w_{xx} + w_{tt} + 2\epsilon\gamma w_t = 0, \qquad 0 < x < 1, \quad t > 0, \tag{1a}$$

$$\lambda = 2ck - \frac{k}{2} \int_0^1 w_x^2 \, dx, \qquad t > 0, \tag{1b}$$

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$$w = w_{xx} = 0 \quad \text{for} \quad x = 0, 1, \qquad t > 0, \tag{1c}$$

$$w(x, 0) = \epsilon f(x), \qquad 0 \le x \le 1, \tag{1d}$$

$$w_t(x, 0) = \epsilon g(x), \qquad 0 \le x \le 1, \tag{1e}$$

where $w(x, t)$ is the non-dimensional transverse displacement, $\lambda(t)$ the non-dimensional axial stress, c the non-dimensional compressive ($c > 0$) axial displacement applied at the ends, k a dimensionless parameter related to the physical constants and dimensions, $2\epsilon\gamma$ is a small damping coefficient and $\epsilon f(x)$ and $\epsilon g(x)$ are small initial conditions. The object is to determine the asymptotic nature of the displacement w as $\epsilon \to 0$.

An analysis of the nonlinear, static equations in [11] reveals the existence of a number of different buckled (non-zero w) states of small norm. A new pair appears each time the end displacement c passes through the values $c_n = n^2\pi^2/2k$ ($n = 1, 2, \cdots$). When c is less than c_1, the only steady state is the unbuckled one ($w = 0$).

We now assume that the end displacement c exceeds c_1, but is close to it. That is, following [1], we will assume

$$c = (\pi^2/2k) + \epsilon^2, \tag{2}$$

and attempt to find an asymptotic approximation for the solution of Eqs. (1) when w is subject to the small-amplitude initial data given by (1d, e).

We first consider the special case of monochromatic initial data with no damping. That is, in (1) we put

$$\gamma = 0, f(x) = f_n \sin (n\pi x), g(x) = g_n \sin (n\pi x). \tag{3.1–3.3}$$

We do this because, as is shown below, the equations for $w(x, t)$ can be reduced to a single ordinary differential equation whose solution can be easily discussed. This will provide motivation for the proper scales to be used to solve the problem with arbitrary initial data (this is done in the next section).

With initial conditions given by (3.2) and (3.3), the solution to (1) can be written as

$$w(x, t) = w_n(t) \sin (n\pi x), \tag{4}$$

where $w_n(t)$ satisfies

$$\ddot{w}_n + n^2\pi^2\left[(n^2 - 1)\pi^2 - 2k\epsilon^2 + \frac{n^2\pi^2k}{4} w_n^2\right]w_n = 0, \tag{5a}$$

$$w_n(0) = \epsilon f_n, \qquad \dot{w}_n(0) = \epsilon g_n. \tag{5b, c}$$

In (5a), "$\cdot$" denotes d/dt and we have made use of (2).

A first integral of (5) is easily determined to be

$$\dot{w}_n^2 + n^2\pi^2[(n^2 - 1)\pi^2 - 2k\epsilon^2]w_n^2 + \frac{n^4\pi^4k}{8} w_n^4$$
$$= \epsilon^2 g_n^2 + n^2\pi^2\epsilon^2 f_n^2[(n^2 - 1)\pi^2 - 2k\epsilon^2] + \frac{n^4\pi^4k\epsilon^4 f_n^4}{8}. \tag{6}$$

Solving for w_n^2 from (6), we find that

$$w_n^2 = \frac{-4}{n^2\pi^2k}[(n^2 - 1)\pi^2 - 2k\epsilon^2] \pm \frac{1}{2}\left[\frac{64}{n^4k^2}(n^2 - 1)^2 - \frac{32\dot{w}_n^2}{n^4\pi^4k}\right.$$

$$+ \frac{\epsilon^2}{n^2\pi^2 k}\left[\frac{8g_n^2}{n^2\pi^2} + 32(n^2-1)\pi^2 f_n^2 - \frac{256(n^2-1)}{n^2}\right] + \epsilon^4\left[\frac{256}{n^4\pi^4} + 4f_n^4 - \frac{16f_n^2}{n^2\pi^2}\right]\Bigg]^{1/2}. \quad (7)$$

From (7), we can easily show that, for $\epsilon \ll 1$, the requirement that w_n be real-valued leads to

$$\begin{aligned} w_n &= 0(\epsilon), \qquad n \neq 1, \\ &= 0(g_1\epsilon^{1/2}) + O(\epsilon), \qquad n = 1, \end{aligned} \quad (8)$$

$$\dot{w}_n = 0(\epsilon), \quad \text{for all} \quad n, \quad (9)$$

where "O" denotes the usual order symbol.

It might be pointed out here that w_1 changes from $O(\epsilon)$ when $g_1 = 0$ to $O(\epsilon^{1/2})$ when $g_1 \neq 0$.

3. The nonlinear dynamic theory: arbitrary initial data. If, in Eqs. (1), we expand $f(x)$, $g(x)$, and $w(x, t)$ in the eigenfunction expansions

$$f(x) = \sum_{n=1}^{\infty} f_n \sin(n\pi x), \quad (10.1)$$

$$g(x) = \sum_{n=1}^{\infty} g_n \sin(n\pi x), \quad (10.2)$$

$$w(x, t) = \sum_{n=1}^{\infty} w_n(t) \sin(n\pi x), \quad (10.3)$$

then it is an easy matter to show that the Fourier coefficients $w_n(t)$ satisfy the following infinite coupled system of ordinary differential equations:

$$\ddot{w}_n + 2\epsilon\gamma\dot{w}_n + n^2\pi^2 w_n\left[\pi^2(n^2-1) - 2k\epsilon^2 + \frac{k\pi^2}{4}\sum_{j=1}^{\infty} j^2 w_j^2\right] = 0, \quad (11.1)$$

$$w_n(0) = \epsilon f_n, \ \dot{w}_n(0) = \epsilon g_n, \quad (11.2, 11.3)$$

where "$\cdot$" means d/dt.

The object now is to transform the above equations into a first-order system to which we can apply the formalities of averaging. To do this we define the real functions u_1 and v_1, and the complex function u_n by

$$w_1 = \epsilon^{1/2} u_1, \ \dot{w}_1 = \epsilon v_1, \quad (12.1, 12.2)$$

$$w_n = \frac{\epsilon}{n}(u_n \exp(i\omega_n t) + \bar{u}_n \exp(-i\omega_n t)), \qquad n \geq 2, \quad (12.3)$$

$$\dot{w}_n = \frac{i\epsilon\omega_n}{n}(u_n \exp(i\omega_n t) - \bar{u}_n \exp(-i\omega_n t)), \qquad n \geq 2, \quad (12.4)$$

where

$$\omega_n = n\pi^2(n^2-1)^{1/2}, \quad n \geq 2, \quad (13)$$

and the bar denotes complex conjugate.

The definitions of u_1 and v_1 above are motivated by the discussion in Sec. 2. To motivate the definition of u_n, we observe from (11.1) that, for $n \geq 2$, the equation for w_n is a perturbation from an harmonic oscillator equation with natural frequency ω_n.

Eqs. (12.3) and (12.4) are then just standard transformations used in averaging [9].

The equations governing u_1, v_1 and u_n (the equation governing $\bar{u}_n$ is just the complex conjugate of that for u_n) can easily be shown to be

$$\dot{u}_1 = \epsilon^{1/2} v_1, \tag{14.1}$$

$$\dot{v}_1 = -\frac{k\pi^4}{4}\epsilon^{1/2}u_1^3 - 2\epsilon\gamma v_1 - k\pi^2\epsilon^{3/2}u_1 \cdot\left[-2 + \frac{\pi^2}{4}\sum_{j=2}^{\infty}(u_j \exp(i\omega_j t) + \bar{u}_j \exp(-i\omega_j t))^2\right], \tag{14.2}$$

$$\dot{u}_n = \epsilon\left[-\gamma(u_n - \bar{u}_n \exp(-2i\omega_n t)) + \frac{in^2\pi^4 k}{8\omega_n}u_1^2(u_n + \bar{u}_n \exp(-2i\omega_n t))\right] + \frac{in^2\pi^2 k\epsilon^2}{2\omega_n}(u_n + \bar{u}_n \exp(-2i\omega_n t)) \cdot\left[-2 + \frac{\pi^2}{4}\sum_{j=2}^{\infty}(u_j \exp(i\omega_j t) + \bar{u}_j \exp(-i\omega_j t))^2\right], \quad n \geq 2 \tag{14.3}$$

$$u_1(0) = \epsilon^{1/2} f_1, \tag{14.4}$$

$$v_1(0) = g_1, \tag{14.5}$$

$$u_n(0) = \frac{n}{2}\left(f_n - \frac{ig_n}{\omega_n}\right), \quad n \geq 2. \tag{14.6}$$

These equations are now in a form which is amenable to a formal application of the averaging technique, where the small parameter is $\epsilon^{1/2}$. Thus, we formally write

$$u_1 = \xi_1 + \sum_{j=1}^{\infty}\epsilon^{j/2}P_1^{(j)}(\xi_1, \eta_1, \xi, \bar{\xi}; t), \tag{15.1}$$

$$v_1 = \eta_1 + \sum_{j=1}^{\infty}\epsilon^{j/2}q_1^{(j)}(\xi_1, \eta_1, \xi, \bar{\xi}; t), \tag{15.2}$$

$$u_n = \xi_n + \sum_{j=1}^{\infty}\epsilon^{j/2}P_n^{(j)}(\xi_1, \eta_1, \xi, \bar{\xi}; t), \quad n \geq 2, \tag{15.3}$$

$$\bar{u}_n = \bar{\xi}_n + \sum_{j=1}^{\infty}\epsilon^{j/2}\bar{P}_n^{(j)}(\xi_1, \eta_1, \xi, \bar{\xi}; t), \quad n \geq 2, \tag{15.4}$$

where ξ_1, η_1 and ξ_n are to satisfy

$$\dot{\xi}_1 = \sum_{j=1}^{\infty}\epsilon^{j/2}A_1^{(j)}(\xi_1, \eta_1, \xi, \bar{\xi}), \tag{16.1}$$

$$\dot{\eta}_1 = \sum_{j=1}^{\infty}\epsilon^{j/2}B_1^{(j)}(\xi_1, \eta_1, \xi, \bar{\xi}), \tag{16.2}$$

$$\dot{\xi}_n = \sum_{j=1}^{\infty}\epsilon^{j/2}A_n^{(j)}(\xi_1, \eta_1, \xi, \bar{\xi}), \quad n \geq 2, \tag{16.3}$$

and where ξ and $\bar{\xi}$ symbolically represent the infinite vectors whose components are ξ_j and $\bar{\xi}_j$, $j \geq 2$.

Very briefly, the point is to substitute (15) and (16) into (14), equate coefficients of like powers of ϵ, and to choose the unknowns $P_1^{(i)}$, $q_1^{(i)}$, $P_n^{(i)}$, $A_1^{(i)}$, $B_1^{(i)}$, and $A_n^{(i)}$ in such a way that no secular terms are introduced. Then, once the $A_1^{(i)}$, $B_1^{(i)}$, and $A_n^{(i)}$ are determined, Eqs. (16) can be solved for ξ_1, η_1, and ξ_n. We will not dwell on the details here since they are amply presented in [10]. Let us note, however, that we must at least go up to the $\epsilon^{3/2}$ terms in order to bring into focus the interesting features of equations (14). The calculations were done up to $O(\epsilon^2)$ and the results are presented in Appendix A.

First, taking into account only terms up to $O(\epsilon^{3/2})$, the equations for ξ_1, η_1 and ξ_n are (see Eqs. (16) and Appendix A):

$$\dot{\xi}_1 = \epsilon^{1/2}\eta_1 , \tag{17.1}$$

$$\dot{\eta}_1 = \frac{-k\pi^4}{4}\epsilon^{1/2}\xi_1^3 - 2\epsilon\gamma\eta_1 - k\pi^2\epsilon^{3/2}\xi_1\left[-2 + \frac{\pi^2}{4}\sum_{j=2}^{\infty}|\xi_j|^2\right], \tag{17.2}$$

$$\dot{\xi}_n = \epsilon\xi_n\left[-\gamma + \frac{in^2\pi^4k}{8\omega_n}\xi_1^2\right], \qquad n \geq 2. \tag{17.3}$$

$$\dot{\bar{\xi}}_n = \epsilon\bar{\xi}_n\left[-\gamma - \frac{in^2\pi^4k}{8\omega_n}\xi_1^2\right]. \tag{17.4}$$

The initial conditions for the above quantities, up to $O(\epsilon^{2/2})$, are easily computed from Eqs. (14) and (15), using Appendix A. They are

$$\xi_1(0) = \epsilon^{1/2}f_1 , \tag{18.1}$$

$$\eta_1(0) = g_1 , \tag{18.2}$$

$$\xi_n(0) = \frac{n}{2}\left(f_n - \frac{ig_n}{\omega_n}\right) - \frac{i\epsilon\gamma n}{4\omega_n}\left(f_n + \frac{ig_n}{\omega_n}\right), \qquad n \geq 2. \tag{18.3}$$

We note that the coupling in Eqs. (17.1) and (17.2) with the ξ_n is only through $|\xi_n|^2$. This quantity can be computed from Eqs. (17.3) and (17.4), even though these two equations are coupled to the solution of (17.1) and (17.2). To see this, multiply (17.3) by $\bar{\xi}_n$, (17.4) by ξ_n and add to get the following single equation for $|\xi_n|^2$:

$$\frac{d}{dt}|\xi_n|^2 = -2\gamma\epsilon\,|\xi_n|^2, \qquad n \geq 2. \tag{19}$$

Thus we have that

$$|\xi_n|^2 = |\xi_n(0)|^2 \exp(-2\epsilon\gamma t), \quad n \geq 2, \tag{20}$$

where $|\xi_n(0)|^2$ is easily computed from Eq. (18) to be, up to $O(\epsilon^{3/2})$,

$$|\xi_n(0)|^2 = \frac{n^2}{4}\left(f_n^2 + \frac{g_n^2}{\omega_n^2}\right) + \frac{\epsilon\gamma n^2 f_n g_n}{2\omega_n^2}. \tag{21}$$

Inserting the right-hand side of Eq. (20) into Eq. (17.2), and eliminating η_1 between (17.1) and (17.2), we get the following equations for ξ_1 and ξ_n:

$$\ddot{\xi}_1 + 2\epsilon\gamma\dot{\xi}_1 + \frac{\epsilon k\pi^4}{4}\xi_1\left[\xi_1^2 - \frac{8\epsilon}{\pi^2} + \epsilon\sum_{j=2}^{\infty}|\xi_n(0)|^2\exp(-2\epsilon\gamma t)\right], \tag{22.1}$$

$$\dot{\xi}_n = \epsilon\xi_n\left[-\gamma + \frac{in^2\pi^4k}{8\omega_n}\xi_1^2\right], \qquad n \geq 2. \tag{22.2}$$

These equations are now decoupled in that Eq. (22.1) can be solved for ξ_1 (e.g. numerically) and in terms of the now-known ξ_1, Eq. (22.2) easily yields

$$\xi_n = \xi_n(0)\exp(-\gamma\epsilon t)\exp\left[\frac{in^2\pi^4\epsilon k}{8\omega_n}\int_0^t \xi_1^2(\tau)\,d\tau\right]. \tag{23}$$

If we now take into account terms up to $O(\epsilon^2)$, Eqs. (17) become

$$\dot{\xi}_1 = \epsilon^{1/2}\eta_1, \tag{24.1}$$

$$\dot{\eta}_1 = \frac{-k\pi^4}{4}\epsilon^{1/2}\xi_1^3 - 2\epsilon\gamma\eta_1 - k\pi^2\epsilon^{3/2}\xi_1\left[-2 + \frac{\pi^2}{4}\sum_{j=2}^{\infty}|\xi_j|^2\right], \tag{24.2}$$

$$\dot{\xi}_n = \epsilon\xi_n\left[-\gamma + \frac{in^2\pi^4k}{8\omega_n}\xi_1^2\right] - \frac{i\epsilon^2\xi_n}{2\omega_n} \cdot\left[\left|\gamma - \frac{in^2\pi^4k\xi_1^2}{8\omega_n}\right|^2 - n^2\pi^2k\left(\frac{\pi^2}{4}|\xi_n|^2 - 2 + \frac{\pi^2}{2}\sum_{j=2}^{\infty}|\xi_j|^2\right)\right], \qquad n \geq 2, \tag{24.3}$$

with the equation for $\bar{\xi}_n$ just being the complex conjugate of (24.3).

Let us make the following interesting observations about Eqs. (24). First of all, the equations for ξ_1 and η_1 are precisely as before (see (17.1) and (17.2)). Secondly, the equation for ξ_n has a non-trivial additional term coupling this equation even further to those for ξ_1 and all of the $|\xi_j|^2$ ($j \geq 2$). However, this coupling occurs as a purely imaginary term in (24.3) and we can, as before, derive

$$|\xi_n|^2 = |\xi_n(0)|^2 e^{-2\epsilon\gamma t}, \quad n \geq 2, \tag{25}$$

where $|\xi_n(0)|^2$ is again computed, up to $O(\epsilon^2)$, from (14), (15) and Appendix A.

With Eq. (25), Eqs. (24) are again decoupled, enabling us to derive a single equation (the same one as (22.1)) for ξ_1, and to integrate the remaining equation for ξ_n in terms of the "known" function ξ_1. In fact, we get

$$\ddot{\xi}_1 + 2\epsilon\gamma\dot{\xi}_1 + \frac{\epsilon k\pi^4}{4}\xi_1\left[\xi_1^2 - \frac{8\epsilon}{\pi^2} + \epsilon\sum_{j=2}^{\infty}|\xi_n(0)|^2\exp(-2\epsilon\gamma t)\right], \tag{26.1}$$

$$\xi_n = \xi_n(0)\exp(-\epsilon\gamma t)\exp i\left[\frac{\epsilon n^2\pi^4k}{8\omega_n}\int_0^t \xi_1^2(\tau)\,d\tau - \frac{\epsilon^2n^4\pi^8k^2}{64\omega_n^2}\int_0^t \xi_1^4(\tau)\,d\tau - \left(\frac{n^2\pi^2k}{\omega_n} + \frac{\gamma^2}{2\omega_n}\right)\epsilon^2 t - \frac{\epsilon n^2\pi^4k}{16\gamma\omega_n}\left(|\xi_n(0)|^2 + 2\sum_{j=2}^{\infty}|\xi_j(0)|^2\right)(\exp(-2\gamma\epsilon t) - 1)\right] \qquad n \geq 2. \tag{26.2}$$

4. Discussion of results. From Eqs. (10.3) and (12) we have that

$$w(x, t) = \sum_{n=1}^{\infty} w_n(t)\sin(n\pi x), \tag{27}$$

where

$$w_1(t) = \epsilon^{1/2}u_1, \tag{28.1}$$

$$w_n(t) = \frac{\epsilon}{n}(u_n\exp(i\omega_n t) + \bar{u}_n\exp(-i\omega_n t)), \qquad n \geq 2. \tag{28.2}$$

Also, from Eqs. (15), the leading term in the asymptotic expansion of u_n is given by ξ_n, which are to satisfy equations (22) to $O(\epsilon^{3/2})$.

We first note that nowhere in the formal derivations given in the previous section was it necessary to demand that $g_1 = 0$, where g_1 is the Fourier coefficient of the first mode in the initial velocity (see Eq. (10.2)). As was noted in the introduction, the authors in [1] had to make this assumption when using the formal procedure of multi-timing. To try and understand why this condition had to be met in [1], let us put our Eq. (22.1) for ξ_1 into one involving those variables used in [1]. Thus define

$$\theta = \epsilon t, \qquad b(\theta) = \frac{1}{\epsilon^{1/2}} u_1(t). \tag{29.1, 29.2}$$

Then (22.1), together with (17.1) and (18.1) and (18.2), yields

$$b_{\theta\theta} + 2\gamma b_\theta + \frac{k\pi^4 b}{4}\left[b^2 - \frac{8}{\pi^2} + \sum_{j=2}^{\infty} |\xi_n(0)|^2 \exp(-2\epsilon\gamma t)\right] = 0, \tag{30.1}$$

$$b(0) = f_1, \qquad b_\theta(0) = g_1/\epsilon. \tag{30.2, 30.3}$$

The variable θ is the slow time scale introduced in [1] and $b(\theta)$ is the amplitude. We note that (30.1) is the same as (7.1) in [1], with the same initial conditions only if $g_1 = 0$. If this is not the case, then the initial slow time-scale velocity for b is asymptotically large ($O(1/\epsilon)$) and cannot be treated by the methods of [1]. The actual breakdown in the multi-time procedure when $g_1 \neq 0$ occurred, we believe, because of the assumption that all the modes, including w_1, had amplitude $O(\epsilon)$ for all time (see Eq. (6.5) in [1]). If the assumption $g_1 = 0$ was not made, the solution generated there appeared to grow with t. What really happens is that, due to the "large" ($g_1 \neq 0$) initial velocity of the first mode, the amplitude of this mode will change from $O(\epsilon)$ initially to $O(\epsilon^{1/2})$ at some later time (see the discussion in Sec. 2). This is manifested mathematically by an apparent growth on the "t" scale because this mode was assumed to have an amplitude which was not large enough for all time.

We also note that the phase plane diagrams given in [1] can still be used here to describe the solutions of Eqs. (30).

The averaging procedure yields another additional improvement. Namely, from Eq. (22.2), we not only get information about the amplitude of the higher-order modes (the same as that derived in [1]), but also have some idea of how these modes oscillate as they decay.

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Appendix

$$P_1^{(1)} = q_1^{(1)} = P_n^{(1)} = 0,$$

$$A_1^{(1)} = \eta_1,$$

$$B_1^{(1)} = \frac{-k\pi^4}{4}\xi_1^3,$$

$$A_n^{(1)} = B_n^{(1)} = 0, \qquad n \geq 2,$$

$$P_1^{(2)} = q_1^{(2)} = 0,$$

$$P_n^{(2)} = \frac{i\bar{\xi}_n \exp(-2i\omega_n t)}{2\omega_n}\left(\gamma + \frac{in^2\pi^4 k\xi_1^2}{8\omega_n}\right), \qquad n \geq 2,$$

$$A_1^{(2)} = 0,$$

$$B_1^{(2)} = -2\gamma\eta_1$$

$$A_n^{(2)} = \xi_n\left(-\gamma + \frac{in^2\pi^4 k\xi_1^2}{8\omega_n}\right), \qquad n \geq 2,$$

$$P_1^{(3)} = 0,$$

$$q_1^{(3)} = \frac{ik\pi^4\xi_1}{8}\sum_{j=2}^{\infty}\frac{1}{\omega_j}(\xi_j^2 \exp(2i\omega_j t) - \bar{\xi}_j^2 \exp(-2i\omega_j t)),$$

$$P_n^{(3)} = \frac{ik\pi^4 n^2}{16\omega_n^3}\xi_1\eta_1\bar{\xi}_n \exp(-2i\omega_n t), \qquad n \geq 2,$$

$$A_1^{(3)} = 0,$$

$$B_1^{(3)} = k\pi^2\xi_1\left[2 - \frac{\pi^2}{2}\sum_{j=2}^{\infty}|\xi_j|^2\right],$$

$$A_n^{(3)} = 0, \qquad n \geq 2,$$

$$P_1^{(4)} = \frac{k\pi^4\xi_1}{16}\sum_{j=2}^{\infty}\frac{1}{\omega_j^2}(\xi_j^2 \exp(2i\omega_j t) + \bar{\xi}_j^2 \exp(-2i\omega_j t)),$$

$$q_1^{(4)} = \frac{-k\pi^4\eta_1}{16}\sum_{j=2}^{\infty}\frac{1}{\omega_j^2}(\xi_j^2 \exp(2i\omega_j t) + \bar{\xi}_j^2 \exp(-2i\omega_j t)),$$

$$p_n^{(4)} = \frac{n^2\pi^4 k\xi_n}{16\omega_n}\sum_{j=2}^{\infty}\frac{1}{\omega_j}(\xi_j^2 \exp(2i\omega_j t) - \bar{\xi}_j^2 \exp(-2i\omega_{j\,t}))$$

$$+\frac{n^2\pi^4k\bar{\xi}_n}{16\omega_n}\sum_{\substack{j=2\\j\neq n}}^{\infty}\left[\frac{\xi_j^{\,2}\exp\left(2i(\omega_j-\omega_n)t\right)}{\omega_j-\omega_n}-\frac{\bar{\xi}_j^{\,2}\exp\left(-2i(\omega_j+\omega_n)t\right)}{\omega_j+\omega_n}\right]$$

$$+\frac{n^2\pi^2k\bar{\xi}_n\exp\left(-2i\omega_n t\right)}{128\omega_n^{\,4}}\left[\begin{array}{c}4\pi^2\eta_1^{\,2}-8i\gamma\pi^2\xi_1^{\,2}\omega_n+k\pi^6\xi_1^{\,4}(n^2-1)\\ +64\omega_n^{\,2}-16\pi^2\omega_n^{\,2}\sum_{j=2}^{\infty}|\xi_j|^2\end{array}\right],$$

$$A_1^{(4)}=B_1^{(4)}=0$$

$$A_n^{(4)}=\frac{-i\xi_n}{2\omega_n}\left[\left|\gamma-\frac{in^2\pi^4k\xi_1^{\,2}}{8\omega_n}\right|^2-n^2\pi^2k\left(-2+\frac{\pi^2}{4}|\xi_n|^2+\frac{\pi^2}{2}\sum_{j=2}^{\infty}|\xi_j|^2\right)\right],\qquad n\geq 2.$$

In the above, "||" denotes absolute value.