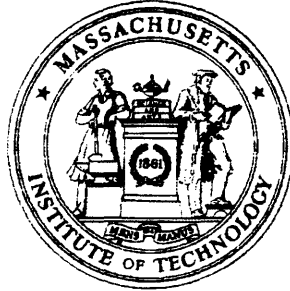


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Nonlinear Dynamic Maximum Power Theorem

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John L. Wyatt, Jr.

Research Laboratory of Electronics
Massachusetts Institute of Technology
Cambridge, MA 02139 USA

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Earlier versions of this work appeared in References (1), (2).

Nonlinear Dynamic Maximum Power Theorem

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ABSTRACT

This paper considers the problem of maximizing the energy or average power transfer from a nonlinear dynamic source. The main theorem includes as special cases the standard linear result $\tilde{Y}_{\text{load}} = \tilde{Y}_{\text{source}}^*$ and a recent finding for nonlinear resistive networks. An operator equation for the optimal output voltage $\hat{y}(\cdot)$ is derived, and a numerical method for solving it is given.

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Earlier versions of this work appeared in [1], [2].

The author is with the Research Laboratory of Electronics and the Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, MA 02139.

I. Introduction

This paper addresses the problem of extracting the maximum energy or average power from a nonlinear dynamic source with the topology shown in Fig. 1. The main results are i) a simple iterative scheme for finding the optimal output voltage waveform $\hat{v}(\cdot)$ for any given current source waveform $i_s(\cdot)$; and ii) an expression for the (noncausal) optimal load admittance operator in terms of the source admittance. The first result can be useful in engineering practice because it specifies the optimal performance that is possible in principle and because $\hat{v}(\cdot)$ itself is a concrete design goal: any load for which the output voltage closely approximates $\hat{v}(\cdot)$ will absorb nearly the maximum possible energy or average power. The second result has no immediate impact on applications because the optimal load is noncausal. But it has some theoretical significance because it generalizes and unifies previous work: the 1-port versions of the standard result $Y_{\text{load}} = Y_{\text{source}}^*$ for dynamic linear¹ time-invariant (LTI) systems [3] and a recent theorem for resistive nonlinear systems [4] fall out as special cases.

The body of this paper addresses the topic rigorously by giving sufficient conditions for existence, uniqueness and global optimality of the network solution, along with a convergence proof for the iterative algorithm. But the remainder of this introduction is utterly nonrigorous and enables the reader to sample the results before (or instead of) delving into the mathematics used to establish them.

1.1) Informal Description of Results

Restricting attention to sources with the special "Norton form" topology

1. Reference [3] actually deals with the dual network, where the source appears in Thévenin form.

shown in Fig. 1 makes it possible to derive a load admittance operator G_{opt} that is optimal for any current source waveform $i_s(\cdot)$. The description of G_{opt} involves the linearized behavior of the source admittance operator F about any nominal voltage waveform $v_{nom}(\cdot)$, i.e.,

$$F(v_{nom}(\cdot) + \delta v(\cdot)) \approx F(v_{nom}(\cdot)) + \int_{-\infty}^{\infty} h_{v_{nom}}(t, \tau) \delta v(\tau) d\tau, \quad (1)$$

to first order in $\delta v(\cdot)$, where the second term on the right is linear in $\delta v(\cdot)$ and, in general, time-varying. Under certain assumptions on F described later, the optimal load turns out to be

$$G_{opt}(v(\cdot)) = \int_{-\infty}^{\infty} h_v(\tau, t) v(\tau) d\tau. \quad (2)$$

Note that:

- i) in the causal LTI case where F is characterized by an impulse response $h_v(t, \tau) = h(t - \tau)$ with Fourier transform $Y_{source}(j\omega)$, Eq. (2) describes an anti-causal LTI load with impulse response $h(\tau - t)$ and thus an admittance $Y_{load} = Y_{source}^*(j\omega)$, in agreement with the classical result [3];
- ii) if the source is a nonlinear resistor, $i = f(v)$, then the optimal load is a nonlinear resistor $i = v \cdot f'(v)$, in agreement with [4];
- iii) $h_v(\cdot, \cdot)$ represents the behavior of F linearized about the independent variable $v(\cdot)$ as a nominal input, so G_{opt} is nonlinear in general;
- iv) for causal F , G_{opt} is unrealizable in all but the purely resistive case since the roles of t and τ in Eq. (2) are reversed from Eq. (1);
- v) Eq. (2) is exact for large-signal behavior, despite the fact that Eq. (1) is only a valid approximation for small $\delta v(\cdot)$;
- vi) first appearances notwithstanding, G_{opt} is time-invariant if F is; and

vii) the optimal load admittance operator is a linear function of the source admittance. Thus the optimal load for a parallel connection of source admittances is the parallel connection of the optimal loads for each source separately.

Given $i_s(\cdot)$, the optimal voltage $\hat{v}(\cdot)$ is the solution to the network in Fig. 1, i.e.,

$$i_s(\cdot) = F(\hat{v}(\cdot)) + G_{\text{opt}}(\hat{v}(\cdot)) \triangleq H(\hat{v}(\cdot)) , \quad (3)$$

where H is the combined admittance loading the current source. And Eq. (3) can be solved in practice by the iterative procedure,

$$v_{j+1}(\cdot) = v_j(\cdot) + r(i_s(\cdot) - H(v_j(\cdot))) , \quad (4)$$

which is guaranteed to converge given any initial guess $v_0(\cdot)$ if $r > 0$ is sufficiently small and certain technical conditions are satisfied. The solution $\hat{v}(\cdot)$ gives both the circuit behavior when optimally loaded and the performance bound for average power transfer, which can be calculated as

$$\bar{P}_{\text{max}} = \frac{1}{T} \int_0^T \hat{v}(t) [G_{\text{opt}}(\hat{v}(\cdot))] (t) dt , \quad (5)$$

for a drive with period T .

In general, no causal load will produce the optimal output voltage for every $i_s(\cdot)$, but in practice one frequently encounters the more restricted problem of maximizing power transfer for some single $i_s(\cdot)$. In this case there may be a variety of causal loads that will do the job, i.e., load admittances G_{causal} such that Eq. (3) and

$$i_s(\cdot) = F(v(\cdot)) + G_{\text{causal}}(v(\cdot)) \quad (6)$$

have the same solution $\hat{v}(\cdot)$ for the particular $i_s(\cdot)$ of interest. In

designing a circuit to realize G_{causal} , $\hat{v}(\cdot)$ can serve as a design goal and \bar{P}_{max} as a performance standard.

1.2) Example

Suppose the source takes the specific form shown in Fig. 2, with the resistor curves given by $i = g_k(v) = v|v|^{k-1}$, $k = 1, 2, 3$, as shown in Fig. 3. Then:

$$[F_k(v(\cdot))](t) = g_k(v(t)) + \int_{-\infty}^{\infty} 1(t-\tau) e^{-(t-\tau)} v(\tau) d\tau, \quad (7)$$

where $1(x)$ is the unit step function vanishing for $x < 0$. The optimal load admittance is

$$[G_{\text{opt}_k}(v(\cdot))](t) = v(t) \cdot g'_k(v(t)) + \int_{-\infty}^{\infty} 1(\tau-t) e^{-(\tau-t)} v(\tau) d\tau. \quad (8)$$

One can check that the resistor curves are, in fact, continuously differentiable, with derivatives given by

$$g'_k(v) = k|v|^{k-1}, \quad k = 1, 2, 3. \quad (9)$$

The optimal output voltage $\hat{v}(\cdot)$ was determined by numerically carrying out the iterative procedure (Eq. 4), which in this case takes the form

$$v_{j+1}(t) = r \left[6 \sin(t) - (k+1)v_j(t)|v_j(t)|^{k-1} - \int_{-\infty}^{\infty} e^{-|t-\tau|} v_j(\tau) d\tau \right] + v_j(t), \quad k = 1, 2, 3. \quad (10)$$

Since g_1 represents a linear resistor, it follows from the traditional linear theorem that $\hat{v}(t) = 2 \sin(t)$ for $k=1$, in agreement with the numerical solution. Note that the instantaneous current drawn by

the nonlinear source resistor increases in magnitude with k for $|v| > 1$ but decreases for $|v| < 1$. Thus it is intuitively reasonable that with increasing k the optimal output spends a progressively greater percentage of time in the region $|v| < 1$, as seen in Fig. 4.

1.3) Generality and Limitations

The results in this paper extend in a straightforward way to the dual case, i.e., "Thévenin form" topology, consisting of an independent voltage source in series with a circuit element characterized by an impedance operator. The extension to multiport sources is also straightforward.

For source networks with the topology shown, the key restriction is that H be monotone increasing, i.e., incrementally passive. But the assumed topology is perhaps a greater restriction, since the results have been shown to depend on this topology in a fundamental way [5] and since there is no general nonlinear analog of the transformation used to put any linear circuit in Thévenin or Norton form.

The difficult nonlinear version of the linear "broadband matching problem" [6-9], in which the goal is to choose a causal load admittance that optimizes power transfer for a set of inputs, is not addressed in this paper and remains entirely open, to the best of the author's knowledge.

II. Results

2.1) Notation and Definitions

Let L be any real inner product space and \hat{L} any linear subspace of L . An operator $F: \hat{L} \rightarrow L$ is said to be

a) strictly increasing if

$$\langle F(y) - F(x), y-x \rangle > 0, \forall x \neq y \in \hat{L}, \quad (11)$$

b) uniformly increasing if for some $\delta > 0$,

$$\langle F(y) - F(x), y-x \rangle \geq \delta \|y-x\|^2, \forall x, y \in \hat{L}, \quad (12)$$

c) Lipschitz continuous if for some $K \geq 0$,

$$\|F(y) - F(x)\| \leq K \|y-x\|, \forall x, y \in \hat{L}. \quad (13)$$

Let L, L' be any real inner product spaces, and $L(L, L')$ denote the space of continuous linear maps from L to L' , with the operator norm [10, p.316]. For $A \in L(L, L')$, let A^{adj} denote the adjoint of A .

The Hilbert space L^2 is the set of all measurable, square-integrable functions $x: \mathbb{R} \rightarrow \mathbb{R}$, equipped with the usual inner product $\langle x, y \rangle$ and norm $\|x\|$.

For each $T > 0$, L^2_T is the set of all periodic measurable functions $x: \mathbb{R} \rightarrow \mathbb{R}$ with period T such that the integral of x^2 over one period is finite. It is a Hilbert space with the "average power" inner product

$$\langle x, y \rangle_T \triangleq \frac{1}{T} \int_0^T x(t) \cdot y(t) dt \quad . \quad (14)$$

The norm on L^2_T is denoted

$$\|x\|_T \triangleq (\langle x, x \rangle_T)^{1/2} \quad .$$

Linearization of the source admittance operator about a nominal input waveform is conventionally accomplished using the Fréchet derivative [10, sect. 3.1], [11, sect. 2.1c]. Unfortunately, one cannot encompass nonlinear resistors in a framework that requires differentiation on L^2 or L_T^2 . It is a little known fact that if $f: \mathbb{R} \rightarrow \mathbb{R}$ and the memoryless operator $F: x(t) \mapsto f(x(t))$ maps L^2 to L^2 , then F is not Fréchet differentiable unless $f = ax + b$ [12, Appendix]. For this reason the theory in this paper is based on the weaker Gâteaux derivative [10, sec. 4.1], [11, sect. 2.1c].

Given L, L' as above, an operator $F: L \rightarrow L'$, and $x, h \in L$, suppose there exists an element denoted $\delta F(x, h)$ of L' such that

$$\lim_{t \rightarrow 0^+} \left\| \frac{F(x+th) - F(x)}{t} - \delta F(x, h) \right\|_{L'} = 0 .$$

Then $\delta F(x, h)$ is called the Gâteaux variation of F at x for the increment h [10, p.251]. If $\delta F(x, h)$ exists for all $x, h \in L$, and if for each $x \in L$ the map $h \rightarrow \delta F(x, h)$ is an element of $L(L, L')$, then F is said to be Gâteaux differentiable on L . In this case the map $x \rightarrow \delta F(x, \cdot)$ is called the Gâteaux derivative of F and denoted $DF: L \rightarrow L(L, L')$ [10, pp.255-256]. Similarly, $\delta F(x, \cdot)$ is denoted $DF(x) \in L(L, L')$, and $\delta F(x, h)$ is denoted $(DF(x))h \in L'$.

2.2) Main Theorem

Theorem 1 (Maximum Average Power in Periodic Steady State)

Fix $T > 0$ and let N_s in Fig. 1 be characterized by an admittance operator $F: L_T \rightarrow L_T$, where L_T is any linear subspace of L_T^2 . Suppose F is Gâteaux differentiable on L_T and the associated operator $H: L_T \rightarrow L_T$,

-
2. Thus, if $v(\cdot)$ has period T and lies in L_T^2 , the steady-state response $i(\cdot)$ of N_s cannot have subharmonics.

given by

$$H: v \mapsto F(v) + (DF_{(v)})^{\text{adj}} v, \quad (15)$$

is strictly increasing.

Then for each $i_s \in H(L_T)$ there is a unique solution $\hat{v}(i_s) \in L_T$ to the network equation,

$$i_s = H(v), \quad (16)$$

and the average power³ absorbed by the load,

$$\bar{P}(v) \triangleq \langle i_o, v \rangle_T = \langle i_s - F(v), v \rangle_T \quad (17)$$

has a unique global maximum over L_T , which is attained at $v = \hat{v}(i_s)$.

Corollary (Maximum Total Energy for Transients)

Let L be a linear subspace of L^2 and substitute L for L_T in the assumptions of Theorem 1. Then the same conclusions⁴ hold, but with $\hat{v}(i_s) \in L$ maximizing the total energy $E(v) \triangleq \langle i_s - F(v), v \rangle$ over L .

Note that H is the sum of the source admittance F and the optimal load admittance operator

$$G_{\text{opt}}(v) \triangleq (DF_{(v)})^{\text{adj}} v, \quad (18)$$

as stated less formally in Eq. (2).

In applications one might wish to restrict attention to currents and voltages in L_T^2 with additional properties such as continuity or boundedness. This is the reason for introducing $L_T \subset L_T^2$ in the formulation of Theorem 1.

-
3. A more explicit, but cumbersome, notation would be $\bar{P}(v, i_s)$. Using it, Theorem 1 states that $\forall v, i_s \in L_T, \bar{P}(v, i_s) < \bar{P}(\hat{v}(i_s), i_s)$ if $v \neq \hat{v}(i_s)$.
 4. For the Corollary, the adjoint is of course taken with respect to the inner product on L^2 rather than $\langle \cdot, \cdot \rangle_T$.

The essential idea behind the theorem is that a solution $\hat{v}(\cdot)$ of Eq. (16) is a stationary point of $\bar{P}: L_T \rightarrow \mathbb{R}$, and the monotonicity assumption on H guarantees that \bar{P} is strictly concave. Details follow.

Proof of Theorem 1

Uniqueness of the solution to Eq. (16) follows from the fact that H is strictly increasing. By the chain rule for the composition of Fréchet- and Gâteaux-differentiable functions [10,p.253], \bar{P} is Gâteaux differentiable and for all $x, h \in L_T$,

$$\begin{aligned} (D\bar{P}(x))h &= \langle i_s - F(x), h \rangle_T - \langle (DF(x))h, x \rangle_T = \\ &\langle i_s - F(x) - DF(x)^{\text{adj}} x, h \rangle_T = \\ &\langle i_s - H(x), h \rangle_T . \end{aligned} \tag{19}$$

Thus, if $i_s \in H(L_T)$,

- a) $D\bar{P}(\hat{v}(i_s)) = 0 \in L(L_T, L_T)$,
- b) given any $x, y \in L_T$, the map $\lambda \rightarrow \bar{P}[x + \lambda(y-x)]$ is differentiable at each $\lambda \in \mathbb{R}$, and
- c) $\frac{d}{d\lambda} \bar{P}[x + \lambda(y-x)] = \langle i_s - H[x + \lambda(y-x)], (y-x) \rangle_T$.

To show that $\hat{v}(i_s)$ globally optimizes \bar{P} , fix $i_s \in H(L_T)$, let $\hat{v} = \hat{v}(i_s)$, and choose any $v \in L_T$, $v \neq \hat{v}$. Then

$$\begin{aligned} \bar{P}(v) - \bar{P}(\hat{v}) &= \\ \bar{P}[\hat{v} + \lambda(v - \hat{v})] \Big|_{\lambda=1} - \bar{P}[\hat{v} + \lambda(v - \hat{v})] \Big|_{\lambda=0} &= \\ \int_0^1 \left\{ \frac{d}{d\lambda} \bar{P}[\hat{v} + \lambda(v - \hat{v})] \right\} d\lambda . \end{aligned} \tag{20}$$

Using c), the integrand above is

$$\langle i_s - H[\hat{v} + \lambda(v - \hat{v})], v - \hat{v} \rangle_T =$$

(since $i_s = H(\hat{v})$)

$$\frac{-1}{\lambda} \langle H[\hat{v} + \lambda(v - \hat{v})] - H(\hat{v}), [\hat{v} + \lambda(v - \hat{v})] - [\hat{v}] \rangle_T, \forall \lambda > 0,$$

and the integrand vanishes at $\lambda=0$. The inner product above is strictly positive for $\lambda \neq 0$ since H is strictly increasing by assumption. Thus, the integrand in Eq. (20) is negative for $\lambda > 0$ and zero for $\lambda=0$, so $\bar{P}(v) < \bar{P}(\hat{v})$ as claimed. ■

The proof of the Corollary is essentially identical and hence is omitted.

2.3) Iterative Algorithm for Determining $\hat{v}(\cdot)$

Equation (19) shows that $i_s - H(v)$ is the gradient [10, p.196], [11, sect. 2.5], [13, p.54ff] of \bar{P} at v . This suggests that we attempt to maximize \bar{P} by a simple "hill-climbing" algorithm of the form

$$x_{j+1} = r(i_s - H(x_j)) + x_j \triangleq M(x_j) \tag{21}$$

for some $r > 0$. Note that under the assumptions of Theorem 1, if $\{v_j\}$ converges to some $\tilde{v} \in L_T$ and H is continuous, then $i_s = H(\tilde{v})$ and \tilde{v} globally maximizes \bar{P} . By tightening the assumptions a little further, we can guarantee convergence for all sufficiently small positive r .

Theorem 2

Strengthen the assumptions of Theorem 1 by supposing further that L_T is closed and H is uniformly increasing and Lipschitz continuous on L_T . Then for any $i_s \in L_T$, any initial guess $v_0 \in L_T$, and any $r \in (0, 2\delta/K^2)$, the

sequence generated by Eq. (21) converges to $\hat{v}(i_s)$.

Note that Theorem 2 also guarantees existence of a solution to Eq. (16) for all $i_s \in L_T$, i.e., $H(L_T) = L_T$. The proof is a straightforward application of [14] and is similar to that in [15]. It is given in detail in [2] and will be omitted here.

Note that the example in Sect. 1.2) satisfies all the assumptions of Theorem 2 except that of global Lipschitz continuity, since the derivatives $g_2'(\cdot)$ and $g_3'(\cdot)$ are unbounded. (Because they are bounded on every bounded subset of \mathbb{R} , a more detailed argument, omitted here, shows that the solutions obtained maximize \bar{P} over $L_T^\infty \cap L_T^2$, which is certainly sufficient in practice.)

IV. Concluding Remarks

The interested reader may wish to compare these results with those obtained by a describing function method in [16].

Acknowledgment

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Figure Captions

- Fig. 1 The optimal nonlinear source admittance, given in Eqs. (2) and (18), is independent of the current source waveform $i_s(\cdot)$ and extracts the maximum power from the source for every $i_s(\cdot)$. Although G_{opt} is noncausal in general, it can be useful in designing a realizable load that maximizes power for some particular $i_s(\cdot)$.
- Fig. 2 Theorem 2 enables one to numerically determine the optimal output voltage $\hat{v}(\cdot)$ for this circuit when the resistor curves are as shown in Fig. 3.
- Fig. 3 The three resistor curves for the circuit in Fig. 2 are $g_k(v) \triangleq v|v|^{k-1}$, $k=1,2,3$, with $g_1(0) \triangleq 0$.
- Fig. 4 One period of the optimal output voltages for the circuit in Fig. 2.

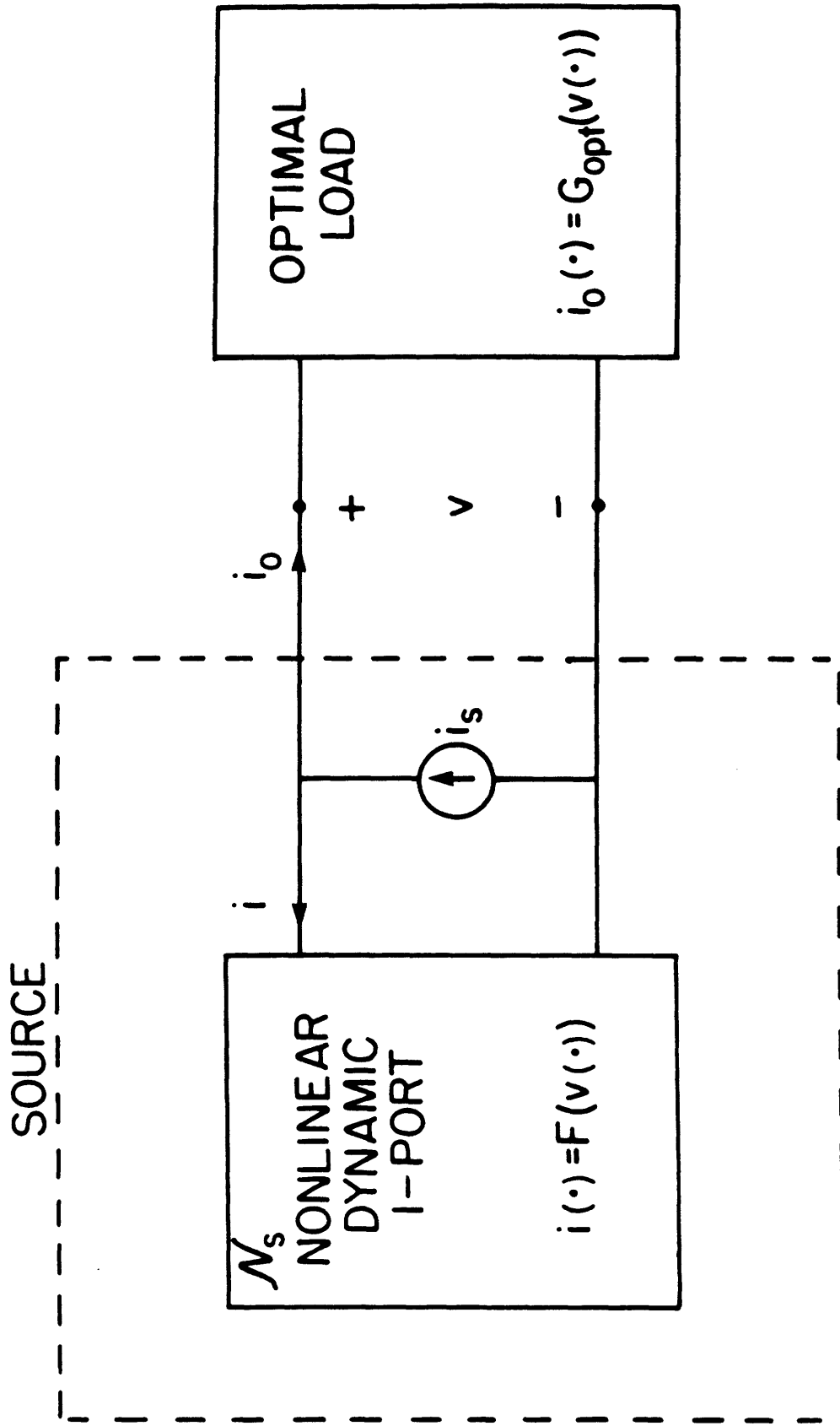


FIG. 1

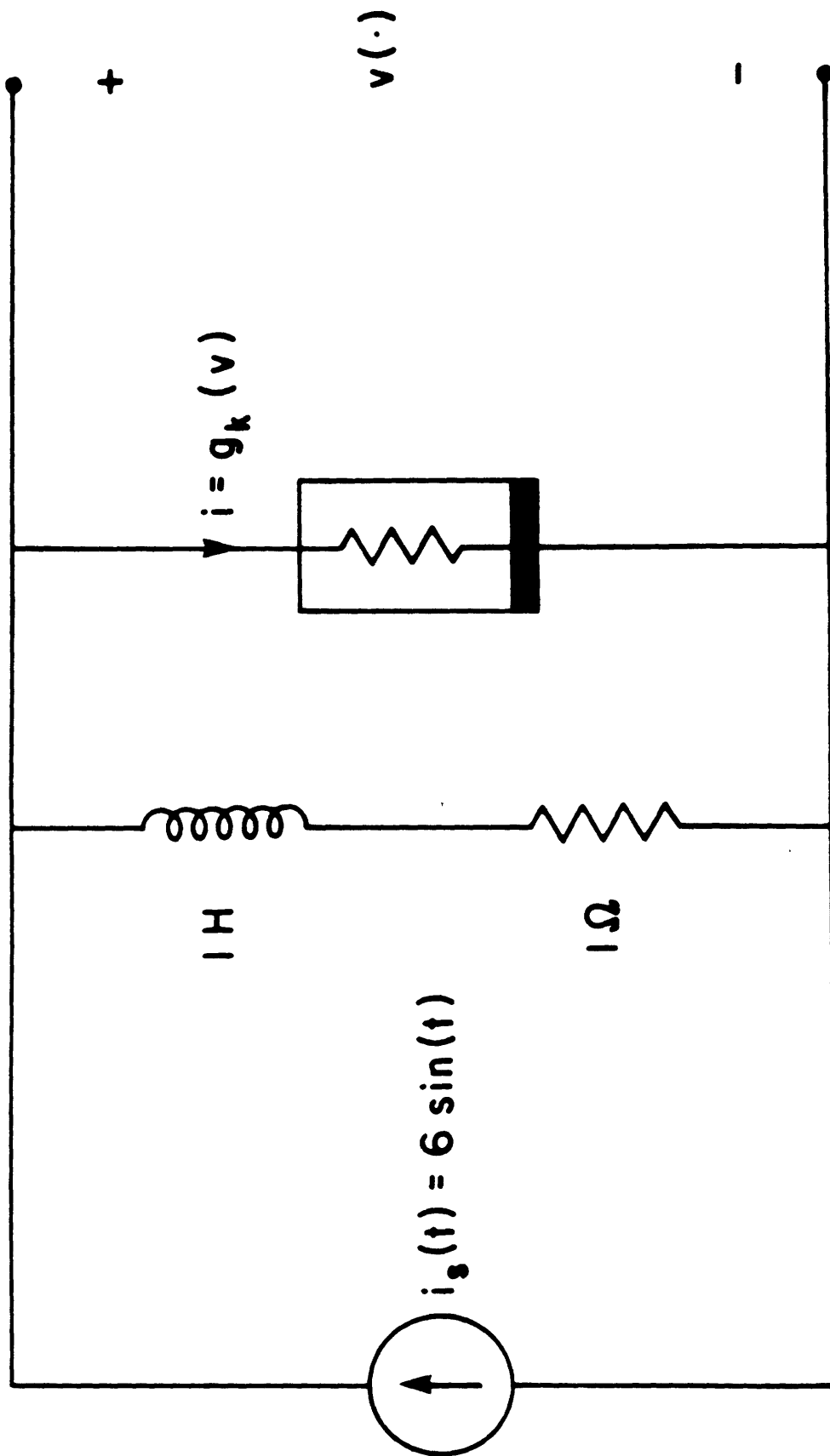


FIG. 2

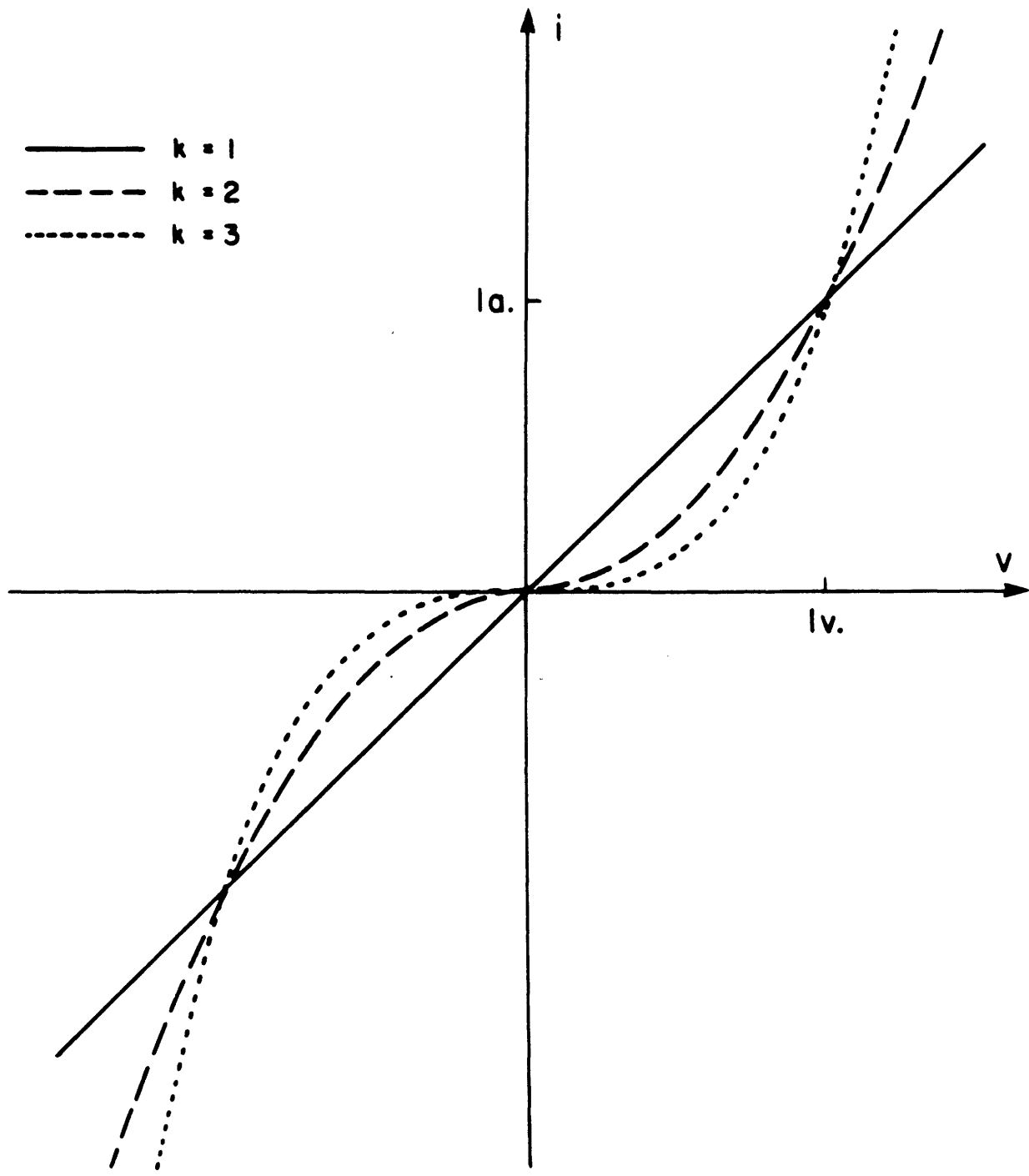


FIG. 3

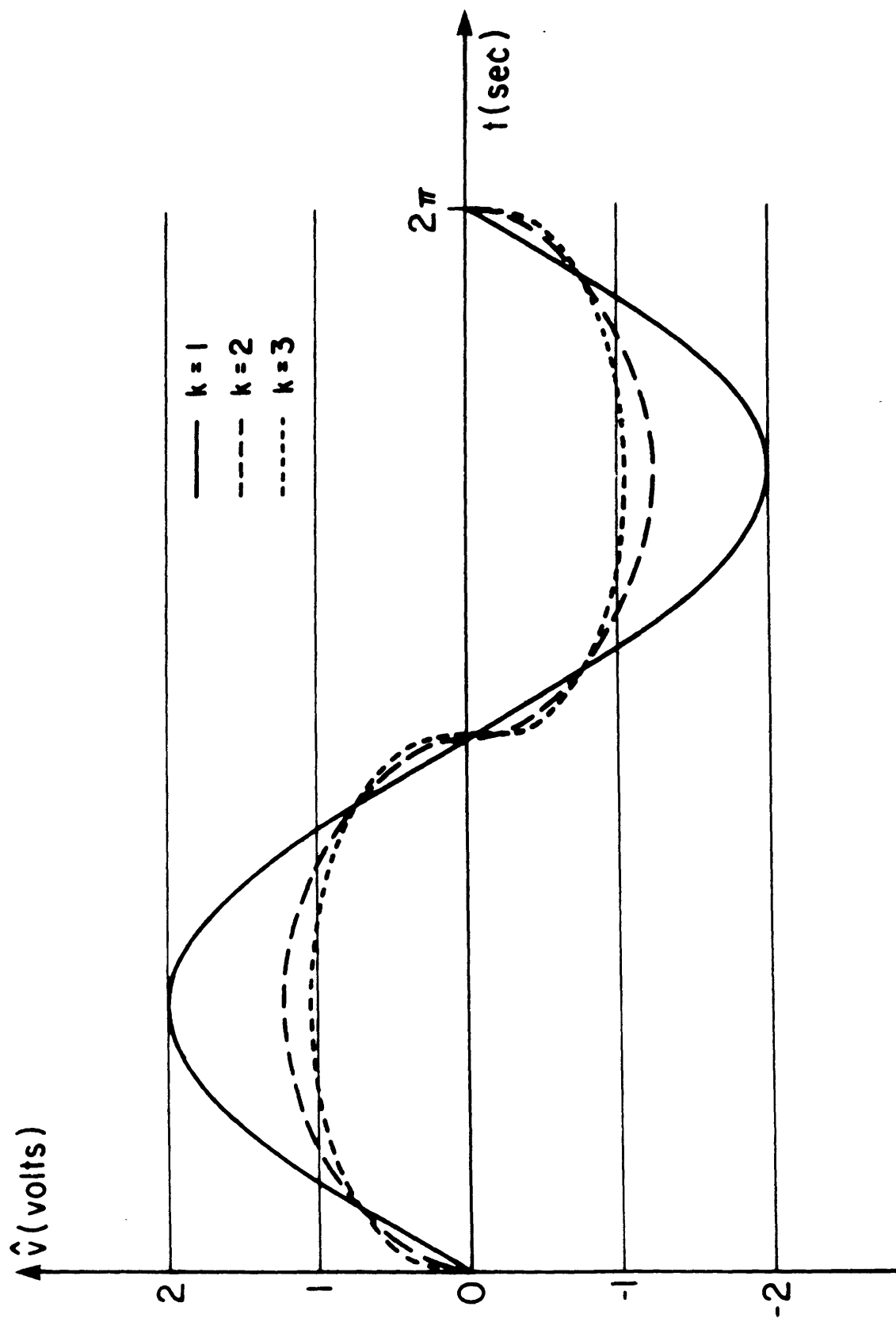


FIG. 4