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ORIGINAL PAPER

# Nonlinear dynamics and control of a variable order oscillator with application to the van der Pol equation

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**Abstract** We investigate the dynamics and control of a nonlinear oscillator that is described mathematically by a Variable Order Differential Equation (VODE). The dynamic problem in question arises from the dynamical analysis of a variable viscoelasticity oscillator. The dynamics of the model and the behavior of the variable order differintegrals are shown in variable phase space for different parameters. Two different controllers are developed for the VODEs under study in order to track an arbitrary reference function. A generalization of the van der Pol equation using the VODE formulation is analyzed under the light of the methods introduced in this work.

**Keywords** Fractional derivatives and integrals · Variable order differential equations · Control of nonlinear oscillators · van der Pol equation

## 1 Introduction

The subject of fractional order calculus or the mathematical analysis of differentiation and integration to an arbitrary noninteger order has attracted much interest during the past 3 decades. Lately, fractional calculus has developed significantly and important applications have been found in a variety of different fields (e.g., Oldham and Spanier [1], Miller and Ross [2], Podlubni [3], Hu [5], Hilfer [4], and Kilbas et al. [6]). Physical evidence of fractional order behavior has been found in areas such as fluid mechanics, mechanical systems, rheology, electromagnetism, electrochemistry, and biology. Beyond the fact that it provides superior modeling capability of memory-intense and delay systems, fractional modeling has been associated with the exact description of complex transport phenomena. For example, Coimbra et al. [7] and L'Esperance et al. [8] provide definitive experimental evidence of fractional history effects in the unsteady viscous motion of small particles in suspension.

Several authors have utilized fractional order calculus to analyze the dynamics and control of physical systems [9–11]. Charef et al. [12] developed a method of singularity function to represent fractional slopes on the log-log Bode plot. The concept of fractional order $PI^{\lambda}D^{\mu}$ controller is discussed in Podlubny [13]. Petras [14] developed a controller for fractional-order Chua's circuit, which exhibits chaotic behavior with total order less than three. Hwang et al. [15] described two numerical methods for inverting fractional-order Laplace transforms that generate accurate solutions for Fractional Differential Equations (FDEs). The dynamics and control of initialized fractional-order systems was analyzed by Hartley and Lorenzo [16]. In

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their paper, the initialization process is described in order to account for the history effects of fractional order systems. Also, the stability properties of these systems were presented using the complex $w$-plane, which is a transformation of the $s$-plane. Ahmad et al. [17] described the stabilization of three different types of chaotic fractional order systems, Cao et al. [18] presented an optimization method based on genetic algorithms for the design of a $PI^{\lambda}D^{\delta}$ controller, and fractional adaptive control is described by Ladaci and Charef [19]. Fractional order derivatives have also been applied to the analysis of the van der Pol equation [20, 21]. Finally, several schemes for the solution of multi-order fractional differential equations have been proposed [22, 23].

Compared to fractional order systems, the study of systems for which the order of the derivative changes with respect to either the dependent or the independent variables has not received much attention. From the mathematical point of view, interest in this area started with the studies by Samko, Ross, and collaborators [24, 25]. These pioneering publications focused on mathematical properties of possible candidates for a Variable-Order (VO) differintegral operator. According to Ref. [24], "To the best knowledge of the(se) authors, no one has published any work on this topic, possibly because there are as yet no applications. This work is stimulated by intellectual curiosity". Until recently, this first set of publications by Samko and collaborators seemed to have escaped the attention of other authors that independently developed similar concepts from a more physical standpoint (e.g., [26, 28–31]).

Ingman et al. [26] expanded an existing fractional constitutive model by means of a differintegral operator of time-dependent order (variable order on the independent variable) to describe an elastoplastic indentation problem. Ingman and Suzdalnitsky [27] studied the vibrations of a one-degree-of-freedom oscillator with a differential operator of variable order. Lorenzo and Hartley [28] analyzed mathematically some important properties of candidate VO operators (invariance, memory retentiveness, adherence to the exponent rule, etc.), essentially within the same spirit that motivated the work of Samko and Ross. Coimbra [30] described the mechanics of an oscillating mass subjected to a variable viscoelasticity damper and a linear spring, a physical problem that yields a VODE on the dependent variable (position). In [30], a consistent VO differential operator for mechanical systems was proposed, and a comparison of the VODE behavior with an interpolative solution of a nonlinear fixed-order differential equation is provided. Soon et al. [31] extended the work developed in [30] by proposing a second-order accurate method for the solution of a number of initial value problems arising from modeling a variable viscoelasticity oscillator. As far as the present authors know, these more physically-oriented works were developed independently of the mathematically-oriented publications referred earlier. Ingman and Suzdalnitsky [29] later developed the technique described in [26] to describe the behavior of a polymeric material. Finally, Ramirez and Coimbra [32] recently developed a statistical mechanics model that yields a macroscopic constitutive relation for a viscoelastic composite material undergoing compression at varying strain rates. The physical model presented in [32] adds a new layer of motivation for the study of VODEs since the statistical mechanics formulation results in a much simpler constitutive equation that is intrinsically of variable order.

In the present paper, we extend the analysis of the nonlinear dynamics of a VO system [30, 31] in order to design two different controllers for tracking a reference function for a nonlinear VODE oscillator. A generalization of the van der Pol equation using a master VODE is also presented.

## 2 Choice of variable order operator

It is well known that there are different ways of defining a fractional differential operator [22, 35], and this means that there are even more options for defining a variable order differential operator (for example, placing the Gamma function term inside or outside of the integral sign in the equation below is immaterial for fractional operators, as opposed to the case of variable order operators). In the present work, we use the variable order operator defined by Coimbra [30]:

$$\mathcal{D}^{q(x(t))}x(t) = \frac{1}{\Gamma(1-q(x(t)))}\int_{0+}^{t}(t-\sigma)^{-q(x(t))}\mathcal{D}^{1}x(\sigma)\,d\sigma + \frac{(x(0^{+})-x(0^{-}))t^{-q(x(t))}}{\Gamma(1-q(x(t)))}, \quad (1)$$

which is valid for $q(x(t)) < 1$.

The operator defined above returns the correct value of the zeroth order derivative ($x(0)$) when $q(x(t)) = 0$ and approaches the correct values of the first derivative when $q(x(t)) \to 1$. Also, the operator returns the $p$-th derivative of $x(t)$ when $q(x(t)) = p$, a necessary property for dynamical modeling that is often overlooked when the focus is placed on mathematical properties that resemble fixed order differential operators (for example, whether the operators abide by the exponent rule, etc.) Also of great importance to dynamic modeling is the fact that Coimbra's operator is dynamically consistent with causal behavior in the initial conditions. In other words, when $x(t)$ is a true constant from $-\infty$ to the initial time ($t = 0^+$), the operator in Eq. 1 returns zero for all values of $q(x(t))$. However, if $x(t)$ is not continuous between $t = 0^-$ and $t = 0^+$, the operator returns the appropriate Heaviside contribution to the integral value of $\mathcal{D}^{q(x(t))}x(t)$. In accordance with this causal definition, we take the value of the physical variable $x(t)$ to be identically null from $-\infty$ to $0^-$ as a representation of dynamic equilibrium. A nonzero initial condition is treated as a Heaviside function at $t = 0$, and, therefore, included in the second term of the definition of the operator (Eq. 1).

## 3 Model description

The physical model we analyze corresponds to an oscillating mass on a guide that is covered with a nonuniform viscoelastic film such that there is a continuous variation of the order of the frictional force as the mass translates from the viscous (order 1) to the purely half-order viscoelastic (order 1/2) portion of the guide [30]. A nondimensional version of the model oscillator is given in [30, 31]:

$$\mathcal{D}^2 x(t) + \mathcal{D}^{(1+x^2)/2} x(t) + \mathcal{D}^0 x(t) = u(t), \tag{2}$$

where $|x| < 1$ is the nondimensional position of the mass given by $x = X/L$. $L$ corresponds to half of the displacement of the guide and $X$ is the dimensional position of the mass. The variable $t$ is the dimensionless time and $u(t)$ corresponds to the dimensionless forcing acceleration (the forcing function divided by the mass). The initial value problem is made consistent by application of Newtonian initial conditions to the equation, so that $x(0) = 0$ and $\mathcal{D}^1 x(0) = 0$.

## 4 Dynamics of the oscillator

In order to solve the VO differential equation given by Eq. 2, we discretize the variable order derivative with a second-to-first $(2 - q)$ order approximation calculated by the algorithm proposed by [31] (which follows the quadrature method proposed in [33, 34]):

$$\mathcal{D}^q x_n = \frac{\Delta t^{1-q}}{\Gamma(3-q)} \sum_{i=0}^{n} a_{i,n} \mathcal{D}^1 x_i + \frac{(x_{0^+} - x_{0^-})(t_n)^{-q}}{\Gamma(1-q)}, \tag{3}$$

with quadrature weights given by

$$a_{i,n} = \begin{cases} (n-1)^{2-q} - n^{1-q}(n+q-2), & \text{if } i = 0, \\ (n-i-1)^{2-q} - 2(n-i)^{2-q} \\ \quad + (n-i+1)^{2-q}, & \text{if } 0 < i < n, \\ 1, & \text{if } i = n, \end{cases} \tag{4}$$

and the integer order derivatives with a second-order Runge–Kutta method [31].

The discretized equation for the oscillator is then simply constructed as [30]:

$$x_{n+2} = (\Delta t)^2 \big[u(t) - \mathcal{D}^q x_n - x_n\big] + 2x_{n+1} - x_n, \tag{5}$$

where $\Delta t$ is the increment in dimensionless time and $n$ is the time step.

Figure 1 shows the dynamic behavior of the viscous-viscoelastic oscillator due to different values of the forcing function $u(t)$. The dashed line corresponds to $u(t) = 0.5\sin(t)$, the dotted line corresponds to $u(t) = \sin(2t)$, and the solid line corresponds to $u(t) = 5\sin(10t)$. The order of differentiation varies with the function $q(x(t)) = (1 + x^2)/2$.

Figure 2(a) shows the variation of the value of the variable order derivative with respect to the position, $x(t)$, for $u(t) = \sin(2t)$ and $q(x(t)) = (1 + x^2)/2$. For this forcing function, the system does not reach the ends of the guide ($|x(t)| < 1$), and the solution shows a bounded oscillatory behavior. Figure 2(b) shows the variation of the value of the VO derivative with respect to the order of the derivative, $q(x(t))$. Since the order of the derivative is a function of the position and the position does not reach the ends of the guide, the value of $q(x(t))$ is kept under unity at all times, with values restricted to the viscoelastic-viscous range of $0.5 \le q \le 1$. Figure 3 shows the magnitude of the VO derivative together with the value of the position, $x(t)$,

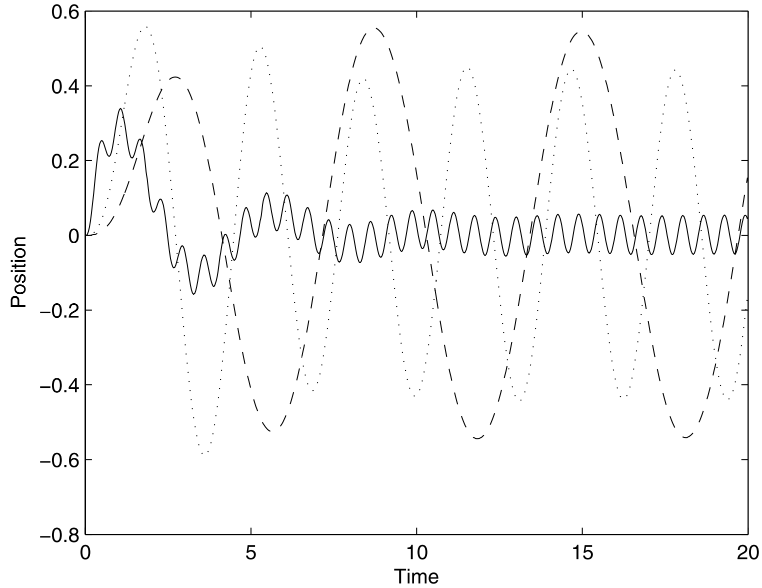



**Fig. 1** Dynamic behavior of the viscous-viscoelastic oscillator given by Eq. 2. - - $u(t) = 0.5\sin(t)$, ... $u(t) = \sin(2t)$, — $u(t) = 5\sin(10t)$. $q(x(t)) = (1 + x^2)/2$

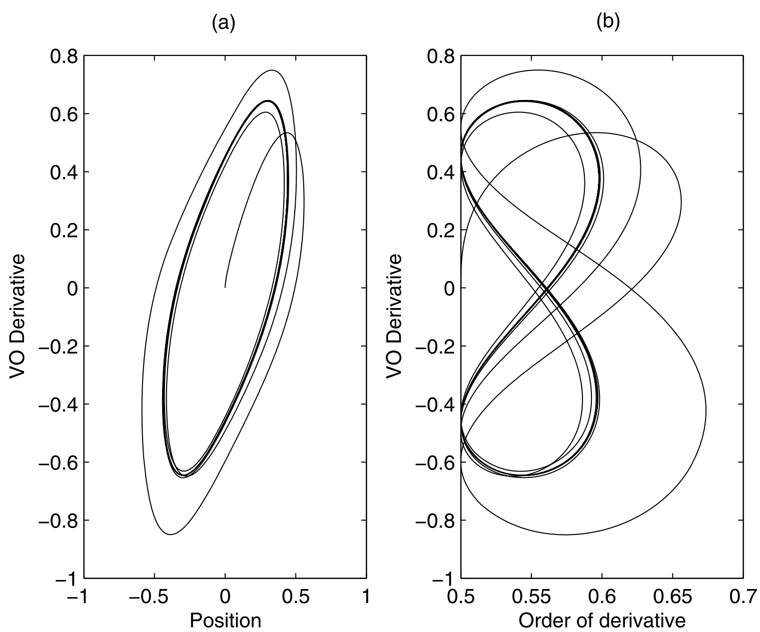



**Fig. 2** Dynamics of VOD system. (**a**) Variation of VO derivative, $\mathcal{D}^{q(x(t))}x(t)$, with respect to position, $x(t)$, (**b**) order of derivative, $q(x(t))$, for $u(t) = \sin(2t)$ and $q(x(t)) = (1 + x^2)/2$

as a function of time. More information about the dynamics of variable order oscillators can be found in [30, 31].

## 5 Control of the oscillator

In this section, we make use of the dimensionless forcing function $u(t)$ as a control variable. Equation 2 is rewritten as follows:

$$\frac{d^2x}{dt^2} + x = u - \frac{1}{\Gamma(1-q)} \int_{0+}^{t} (t-\sigma)^{-q} \frac{dx(\sigma)}{d\sigma} d\sigma - \frac{(x(0^+) - x(0^-))t^{-q}}{\Gamma(1-q}. \tag{6}$$

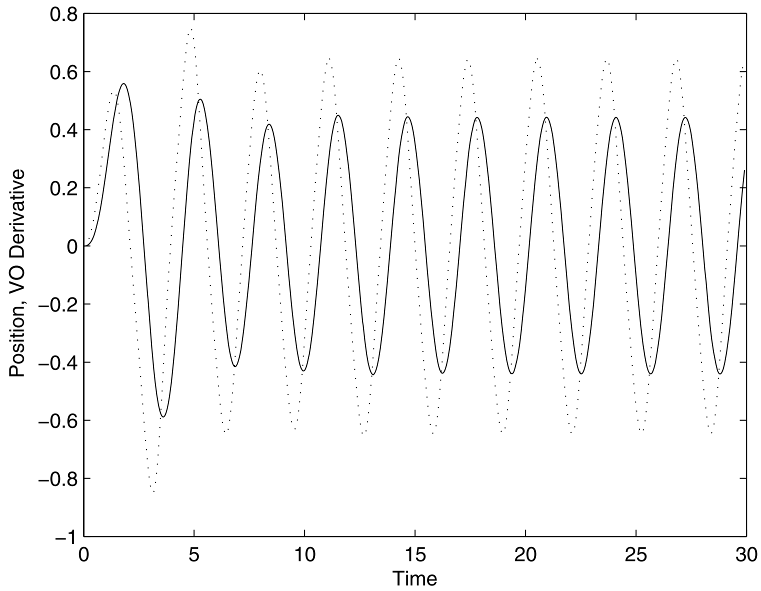



**Fig. 3** Position ($x(t) =$ —) and VO derivative ($\mathcal{D}^{q(x(t))}x(t) =$ ...) versus time for $u(t) = \sin(2t)$ and $q(x(t)) = (1 + x^2)/2$

After imposing Newtonian initial conditions, Eq. 6 is simplified to:

$$\dot{x}_0 = x_1, \tag{7}$$

$$\dot{x}_1 = -x_0 + u - \mathcal{D}^{(1+x_0^2)/2}x_0, \tag{8}$$

which can be written in matrix form as:

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathcal{D}^q x_0$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathcal{D}^{(1+x_0^2)/2}x_0 \tag{9}$$

with output values

$$\mathbf{y} = C\mathbf{x} = [1 \quad 0]\mathbf{x}. \tag{10}$$

Equation 9 represents a quasi-linear integrodifferential equation for which sufficient conditions for local controllability in Banach spaces have been established in [36]. Linear controllers for quasi-linear equations with bounded state-dependent nonlinear disturbances have been studied recently [37–39]. Recognizing that the last (nonlinear) term in Eq. 9 is bounded, we develop a linear controller by choosing the location of the close-loop system eigenvalues considering only the linear part of the problem. The effectiveness of this linearization is illustrated in the following subsections.

We close the loop by choosing a controller of the form

$$u = -K\mathbf{x} + Gr = -k_0x_0 - k_1x_1 + Gr, \tag{11}$$

where $K = [k_0 \ k_1]$ and $k_0$, $k_1$ are constants that are used for the location of the desired values of the closed-loop eigenvalues and $G$ is the feedforward gain for the reference, $r$. Thus, the closed-loop system is represented as follows:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1-k_0 & -k_1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} Gr - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathcal{D}^{(1+x_0^2)/2}x_0, \tag{12}$$

$$\mathbf{y} = C\mathbf{x}. \tag{13}$$

Since the linear system $(A, B)$ is controllable [37–39] we locate the eigenvalues of the closed-loop system by equating the characteristic polynomial with an arbitrary polynomial that has eigenvalues located at a desired location. We choose a polynomial with eigenvalues located at $\lambda = -5$.

$$|\lambda I - A + BK| = \begin{vmatrix} \lambda & -1 \\ 1+k_0 & \lambda + k_1 \end{vmatrix} = (\lambda + 5)(\lambda + 5). \tag{14}$$

Thus, $k_0 = 24$ and $k_1 = 10$.

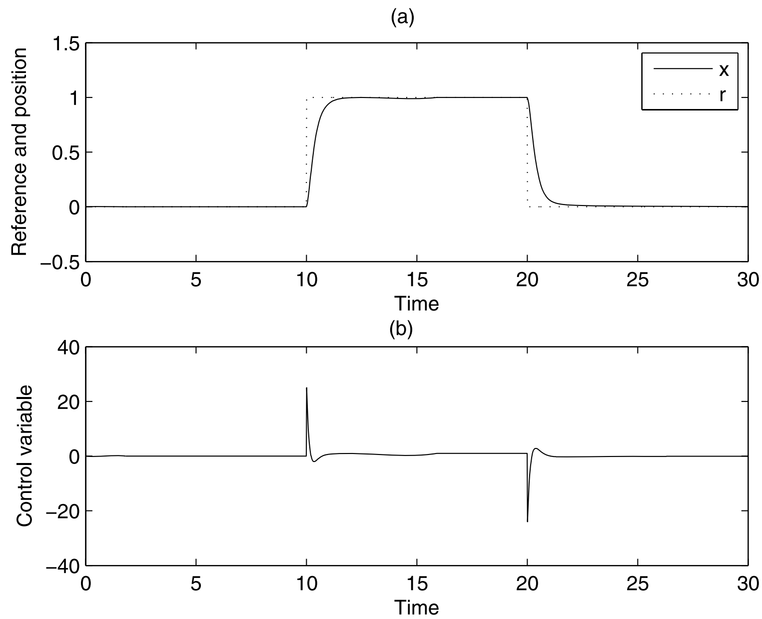



**Fig. 4** Steady state tracking. (**a**) Tracking of reference $r$ with $q(x(t)) = (1+x^2)/2$ for $\lambda_{1,2} = -5$, (**b**) control variable $u$

To obtain the value of the feedforward gain, we match closed-loop dc gain $H_{CL}(0) = -C(A - BK)^{-1}BG$ with the open-loop dc gain $H_{OL}(0) = -CA^{-1}B$ [40]. For the open-loop case, $H_{OL}(0) = 1$. For the closed-loop, $H_{CL}(0) = \frac{1}{25}G$. Thus, to match the dc gains, we choose $G = 25$ as the value of the gain associated with the reference.

### 5.1 Steady-state tracking

Here, we describe the steady-state tracking of an arbitrary reference, $r$, for the VO system described by Eqs. 12 and 13 . The reference is varied according to Eq. 15.

$$r = \begin{cases} 0, & \text{if } t < 10, \\ 1, & \text{if } 10 \leq t < 20. \\ 0, & \text{if } t \geq 20. \end{cases} \quad (15)$$

Figure 4(a) shows the controlled variable $x$ tracking reference $r$. Figure 4(b) shows the value of the control variable $u$ for the process shown in Fig. 4(a).

The closed loop system becomes unstable if the closed-loop eigenvalues are chosen to be located on the positive real axis of the complex plane. To show this, we find new values of $k_0$ and $k_1$ so that the closed-loop eigenvalues are located at arbitrary location chosen as $\lambda_{1,2} = 0.5$. Figure 5(a) shows the behavior of the VO system with unstable closed-loop eigenvalues for the same sequence of values of the reference $r$ shown in Fig. 4. Figure 5(b) shows the control variable $u$ for this case.

### 5.2 Oscillatory reference

We test the controller developed in Sect. 5 with an oscillatory reference similar in shape to the forcing functions used in Fig. 1. Figure 6(a) shows the performance of the controller for reference $r(t) = 0.5\sin(t)$. The controlled variable, $x(t)$, lags reference $r(t)$ and has a slightly lower amplitude of oscillation. Figure 6(b) depicts the oscillatory character of the control action, $u(t)$. Simulations using reference functions with higher frequency values, such as $r(t) = \sin(2t)$, show that the performance of this controller deteriorates as the frequency of reference $r(t)$ is increased. To improve the performance of the controller, we increase the magnitude of the closed-loop eigenvalues. Figure 7(a) shows the improved performance of the controller for $r(t) = 0.5\sin(t)$. The control action $u(t)$, for this case, is shown in Fig. 7(c). Phase and amplitude are matched by the controlled variable. Figures 7(b) and (d) depict the performance of the controller for $r(t) = \sin(2t)$ and its control action, respectively. An error of 1% between the maximum value of the reference with respect to the controlled variable, $x(t)$, has been chosen for these simulations.

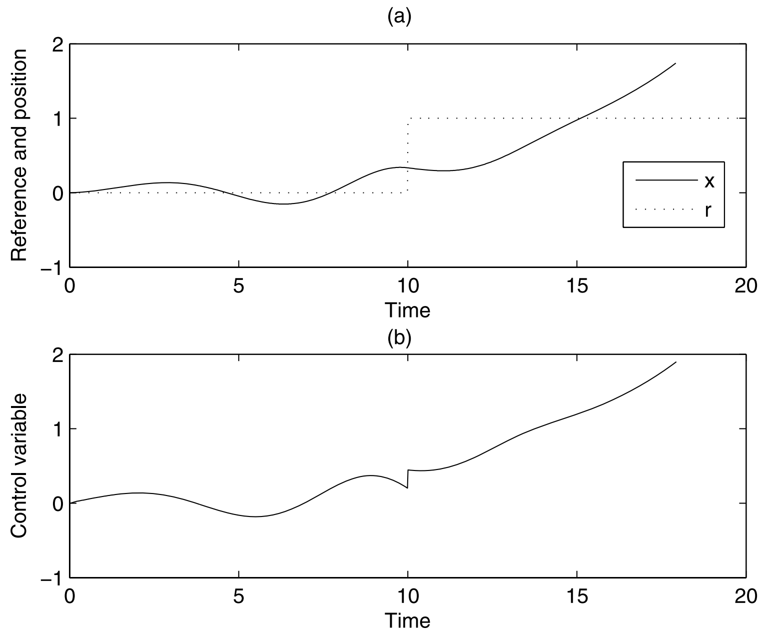



**Fig. 5** Unstable closed-loop. (**a**) Tracking of reference $r$ with $q(x(t)) = (1 + x^2)/2$ for $\lambda_{1,2} = 0.5$, (**b**) control variable $u$

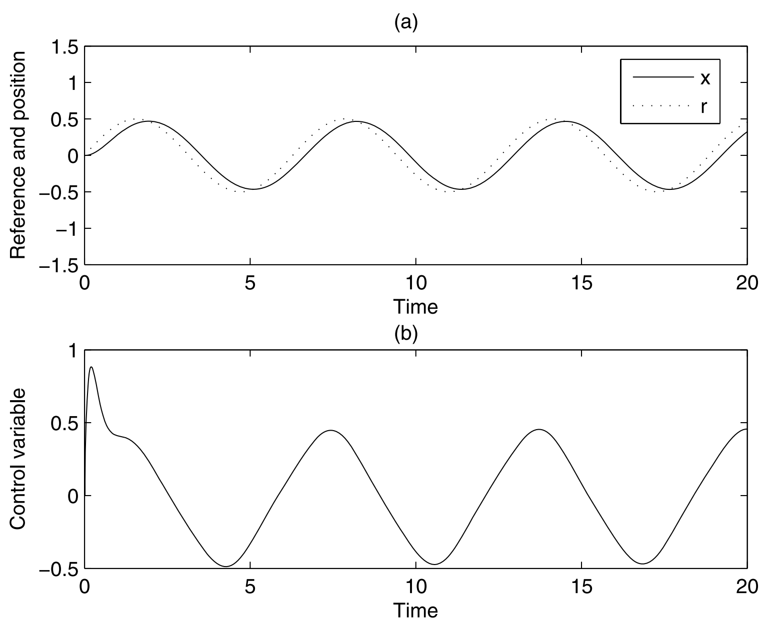



**Fig. 6** Oscillatory reference. (**a**) Tracking of reference $r$ with $q(x(t)) = (1 + x^2)/2$, $r = 0.5\sin(t)$, (**b**) control variable $u$. Note that the controller with $\lambda_{1,2} = -5$ does not track $r(t)$ accurately. Compare this performance with the controller in Fig. 7(a)

This performance was obtained with $\lambda_{1,2} = -20$ for $r(t) = \sin(2t)$.

## 5.3 Optimal control

In this section, we apply optimal tracking control to the same dynamical system analyzed in the previous section.

Consider a linear system of the form:

$$\begin{aligned} \dot{x} + \mathcal{D}^q x &= Ax + Bu, \\ y &= Cx. \end{aligned} \tag{16}$$

We transform this system by using $\tilde{x} = x - \bar{x}$ and $\tilde{u} = u - \bar{u}$, where $\bar{x}$ and $\bar{u}$ are the values of $x$ and $u$ at steady state, respectively. For a reference value, $r$, we assume

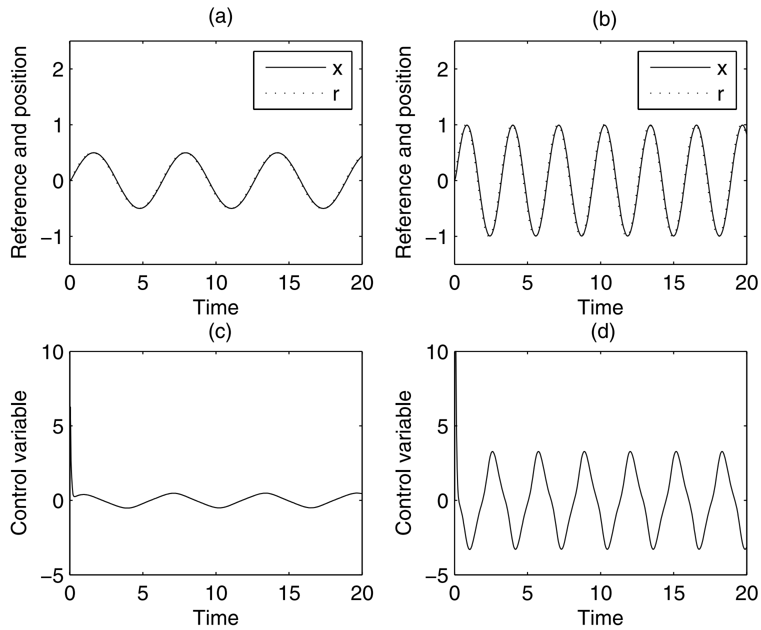



**Fig. 7** Tracking of oscillatory reference $r$ with $q(x(t)) = (1+x^2)/2$ and closed-loop eigenvalues at $\lambda_{1,2} = -20$. (**a**) $r = 0.5\sin(t)$, (**b**) $r = \sin(2t)$, (**c**) $u(t)$ for $r = 0.5\sin(t)$, and (**d**) $u(t)$ for $r = \sin(2t)$

that in steady state $\bar{y} = r$ so that

$$\bar{x} = -A^{-1}B\bar{u}, \tag{17}$$

$$\bar{u} = -\left[CA^{-1}B\right]^{-1}r. \tag{18}$$

We find the linear optimal tracker that minimizes the performance index

$$J = \frac{1}{2}\int_0^\infty (\tilde{x}'Q\tilde{x} + \tilde{u}'R\tilde{u})\,dt, \tag{19}$$

where $Q \geq 0$ and $R > 0$. The control law becomes $\tilde{u} = -K\tilde{x}$, where $K$ is given by

$$K = R^{-1}B'S, \tag{20}$$

and $S$ is the solution to the algebraic Riccati equation

$$SA + A'S - SBR^{-1}B'S + Q = 0. \tag{21}$$

Using the definitions of $\tilde{x}$ and $\tilde{u}$, we obtain $u = -K(x - \bar{x}) + \bar{u}$. We then substitute the values of $\bar{x}$ and $\bar{u}$ from Eqs. 17 and 18 to arrive at $u = -Kx + K_r r$, where $K_r = [KA^{-1}B - I][CA^{-1}B]^{-1}$.

### 5.4 Performance of the tracking optimal controller

In this section, we describe the steady-state tracking of an arbitrary reference, $r$, for the VO system described by Eq. 16 using the optimal controller. The reference is varied in the same way as in Sect. 5.1, i.e., $r$ follows the behavior described by Eq. 15. Figure 8(a) shows the controlled variable $x$ tracking reference $r$ for the VO system with $q(x(t)) = (1 + x^2)/2$. Matrices $Q$ and $R$ have been chosen to obtain a one percent overshoot for the VO system with $q(x(t)) = (1 + x^2)/2$. The same controller can be used for a similar system for which the variable order derivative, $q(x(t))$, behaves as a different function of the position. Figure 8(b) shows the performance of the controller in the viscous-viscoelastic range with $q(x(t)) = (2 - x^2)/2$.

## 6 VODE formulation of the van der Pol equation

The methodologies employed in the previous sections can be utilized to generate a variable order version of the van der Pol equation [41]. Equation 2 resembles the forced van der Pol oscillator equation, which is given by:

$$\mathcal{D}^2 y(t) + \alpha\left(y^2 - 1\right)\mathcal{D}^1 y(t) + \mathcal{D}^0 y(t) = u(t). \tag{22}$$

Equation 2 is then rewritten as follows,

$$\mathcal{D}^2 x(t) + \alpha\left(\beta\mathcal{D}^{q(x(t))}x(t)\right) + \mathcal{D}^0 x(t) = u(t), \tag{23}$$

where $\beta$ is a binary coefficient that can only take the values $-1$ or $+1$. It is possible to satisfy Eq. 23 by determining the value of the variable order $q(x(t))$ that

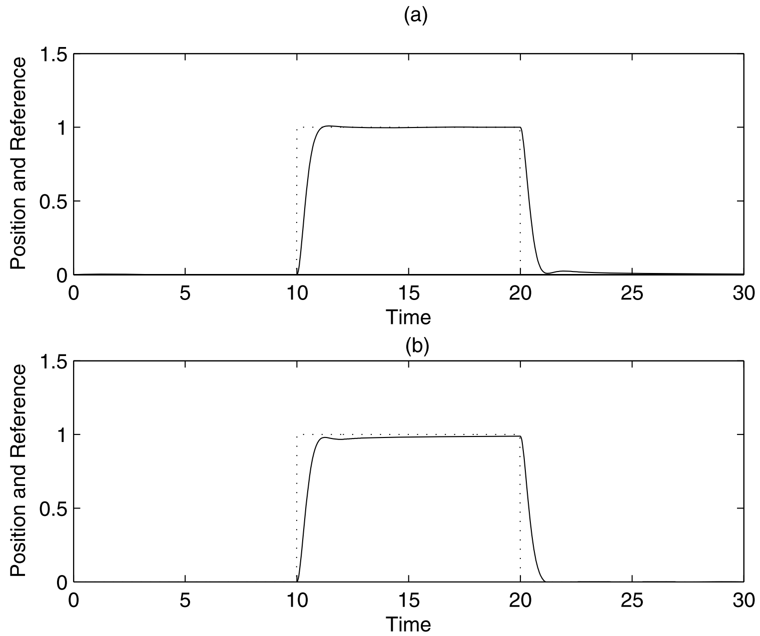



**Fig. 8** Tracking optimal control. (**a**) $q(x(t)) = (1 + x^2)/2$, (**b**) $q(x(t)) = (2 - x^2)/2$

matches the solution of Eq. 22 at the same instant in time, $t$. We employ a minimization algorithm for a single-variable nonlinear-function that finds the value of $q(x(t))$ while solving the identity equation

$$\left(y^2 - 1\right)\mathcal{D}^1 y(t) = \beta \mathcal{D}^{q(x(t))} x(t), \tag{24}$$

where the variable $y$ is used to denote the numerical solution of the standard van der Pol equation, and the variable $x$ denotes the numerical solution of the VODE.

Because the highest order of the van der Pol equation is higher than one, we modify our numerical algorithm for evaluation of Coimbra's operator to allow for covering a wider range of values of $q(t)$, namely $q(t) < 2$. Thus, we modify Eq. (3) as:

$$\mathcal{D}^q x_n = \frac{\Delta t^{2-q}}{\Gamma(4-q)} \sum_{i=0}^{n} a_{i,n} \mathcal{D}^2 x_i + \frac{x_{0^+}(1-q)(t_n)^{-q} + \mathcal{D}^1 x(0^+)(t_n)^{1-q}}{\Gamma(2-q)}, \tag{25}$$

with quadrature weights given by

$$a_{i,n} = \begin{cases} (3-q)n^{2-q} - n^{3-q} + (n-1)^{3-q}, \\ \quad \text{if } i = 0, \\ (n-i-1)^{3-q} - 2(n-i)^{3-q} \\ \quad + (n-i+1)^{3-q}, \quad \text{if } 0 < i < n. \\ 1, \quad \text{if } i = n, \end{cases} \tag{26}$$

where it is implied that both $x(t)$ and $\mathcal{D}^1 x(t)$ are identically null for the time interval between $-\infty$ and $0^-$ (for the reasons given in the discussion following Eq. 1), and where $q = q(t)$.

We test the algorithm to verify that it calculates the correct values of $\mathcal{D}^0 x(t)$, $\mathcal{D}^1 x(t)$ , and $\mathcal{D}^2 x(t)$. Figure 9 shows the results of the simulations for the case $u(t) = \sin(t)$ and $\alpha = 5$ with a step size $\Delta t = 0.002$. Values of $\mathcal{D}^q x(t)$ for fractional orders $q = 1.5$ and 1.75 are also shown in the figure. It is observed that $\mathcal{D}^0 x(t)$ matches $y(t)$, $\mathcal{D}^1 x(t)$ matches $\frac{dy}{dt}$, and $\mathcal{D}^2 x(t)$ approximates $\frac{d^2y}{dt^2}$ accurately. We used $q = 1.999$ to approximate the second derivative. It is also seen that the fractional-order derivatives approach $\frac{d^2y}{dt^2}$ as $q(t)$ increases from 1.5 to 1.75 and then to 2.

We run simulations for a base case where $u(t) = \sin(t)$, $\alpha = 5$, and $\Delta t = 0.002$ and for $t$ in the range $[0, 20]$. The value of $q(x(t))$ is allowed to vary between $-2$ to 2. The algorithm developed for obtaining the solution of Eq. 24 obtains the value of $\mathcal{D}^q x(t)$ at a number of points within the range of $q$. Single solutions, shown in Fig. 10(a) for $t = 4.5$, or multiple solutions, shown in Fig. 10(b) for $t = 11$, can be obtained. The solid line represents the value of $(y^2 - 1)\frac{dy}{dt}$ that needs to be matched and the dashdot line represents the values of $\beta \mathcal{D}^q x(t)$ for -2< $q$ <2. The circle indicates the chosen solution. For multiple solutions, we consider the value of $q(t)$ that is closer to $q(t - \Delta t)$ to

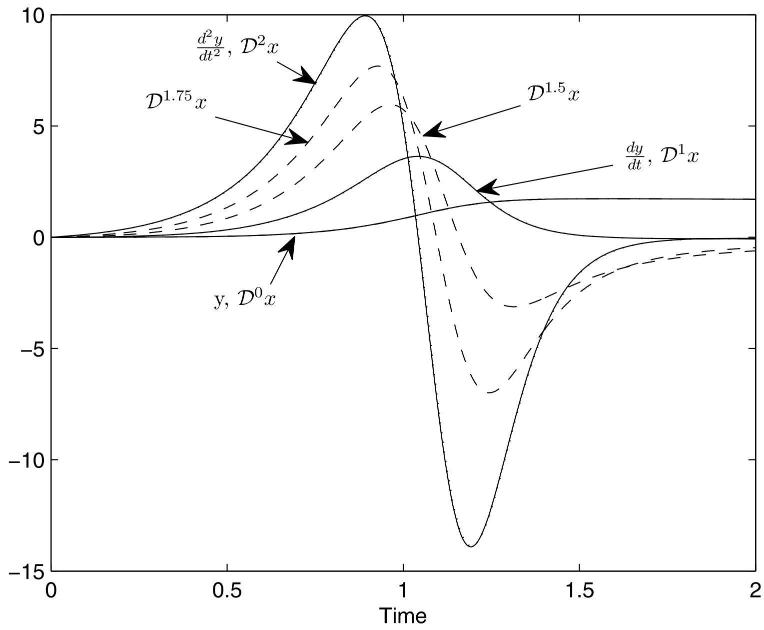



**Fig. 9** Fractional order derivatives for $q(t) = 0$, 1, 1.5, 1.75, and 2, compared to $y$, $\frac{dy}{dt}$, and $\frac{d^2y}{dt^2}$

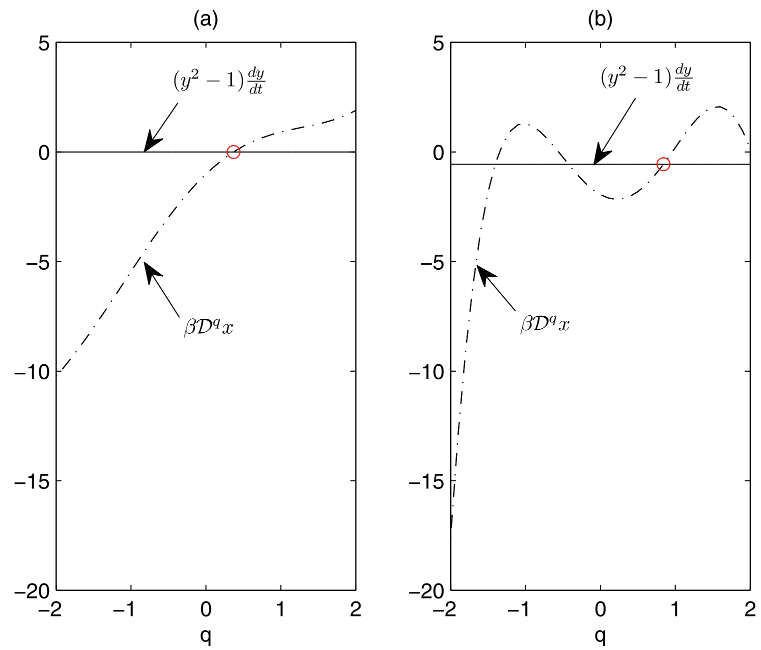



**Fig. 10** Single or multiple solutions obtained for Eq. 24. (**a**) Single solution at $t = 4.5$, (**b**) Multiple solutions at $t = 11$. The *circle* denotes the chosen solution

minimize sharp changes in the order of the derivative as a function of time.

Figure 11 shows the results obtained using the matching algorithm developed to solve Eq. 24, where the horizontal axis corresponds to the nondimensional time. The solid line denotes $(y^2-1)\frac{dy}{dt}$ and the dash-dot line corresponds to $\beta\mathcal{D}^{q(x(t))}x(t)$. The results show that appropriate values of $q$ that solve Eq. 24 exactly are always found for the time intervals under consideration. Figure 12 shows the comparison between the results for the numerical solution of the standard van der Pol equation and the VODE. Figure 12(a) shows that there is excellent agreement between $x$ and $y$. This is also true in Fig. 12(b) for $\frac{dy}{dt}$ and $\frac{dx}{dt}$,

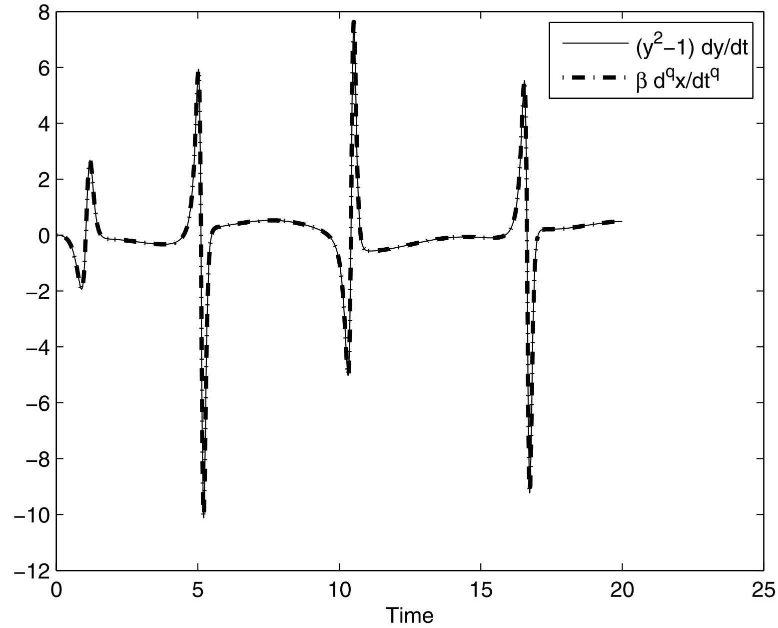



**Fig. 11** Solution of the matching algorithm for Eq. 24. — $= (y^2-1)\frac{dy}{dt}$, and -.- $= \beta \frac{d^q x}{dt^q}$

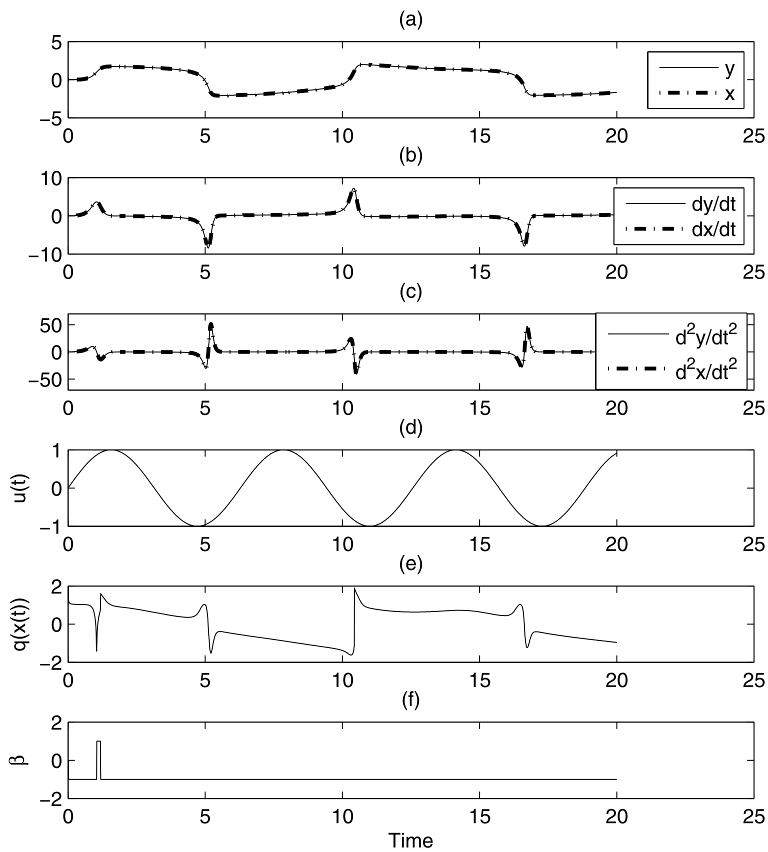
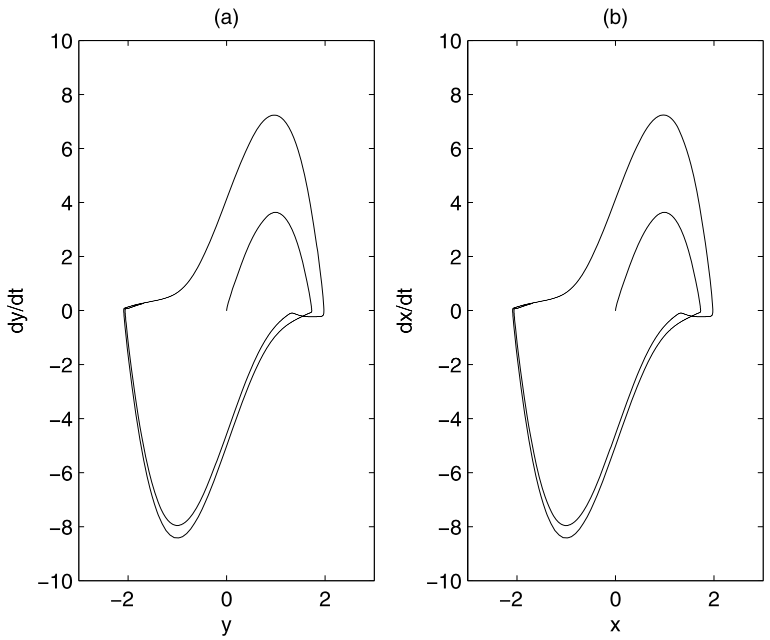



**Fig. 12** Comparison between the numerical solution of the VODE and the standard van der Pol equation. (**a**) $y, x$, (**b**) $\frac{dy}{dt}$, $\frac{dx}{dt}$, (c) $\frac{d^2y}{dt^2}$, $\frac{d^2x}{dt^2}$, (**d**) $u(t)$, (**e**) $q(x(t))$, and (**f**) $\beta$

**Fig. 13** Comparison of phase diagrams. (**a**) Standard van der Pol equation, (**b**) VODE van der Pol equation




and for $\frac{d^2y}{dt^2}$ and $\frac{d^2x}{dt^2}$ shown in Fig. 12(c). As a reference, Fig. 12(d) shows the forcing function $u(t)$, and Fig. 12(e) shows the corresponding values of $q(x(t))$ that solve Eq. 24. Large variations in the value of $q(x(t))$ are observed during fluctuations of $\frac{d^2x}{dt^2}$ and $\frac{dx}{dt}$. The values of the coefficient $\beta$ are shown in Fig. 12(f) where it is seen that only a portion of the interval studied finds solutions for which $\beta = +1$. This explains the opposite behavior of $q(x(t))$ compared to $\frac{d^2x}{dt^2}$ during sharp changes of the latter.

The comparison of the results between the VODE and standard van der Pol equation is also shown with phase diagrams in Fig. 13. Figure 13(a) shows $\frac{dy}{dt}$ versus $y$ and Fig. 13(b) shows $\frac{dx}{dt}$ versus $x$. It is seen that the results are identical.

## 7 Conclusions

Traditional applications of control theory involve differential equations of constant integer or fractional order. Over the past few decades, there has been a growing interest in fractional order controllers, which allow for the representation of memory-laden systems (e.g., viscoelastic behavior). The present work expands on previous contributions in control theory by considering a controller for a dynamical system represented by a Variable-Order Differential Equation (VODE). The dynamical behavior of a viscous-viscoelastic oscillator [30, 31] is discussed and two different control schemes are applied for tracking a step and oscillatory reference function. A variable order formulation of the van der Pol oscillator is also discussed. The value of the variable order derivative, $q(x(t))$, that matches the numerical solution of the standard forced van der Pol equation exactly is obtained.

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