Nonlinear Dynamics in Signals derived from Bessel Functions

Sai Venkatesh Balasubramanian

Sree Sai Vidhya Mandhir, Mallasandra, Bengaluru-560109, Karnataka, India.
saivenkateshbalasubramanian@gmail.com

Abstract

The present work purports to the formulation and characterization of signal based chaos based on Bessel Functions. Specifically, the variable in these functions is viewed as an additively coupled sum of sinusoidal signals, with competing frequencies. By adapting the regular and modified Bessel Functions of the first and second kinds into signals, the derivatives are computed and used to form the corresponding iterative maps, which are studied using phase portraits. It is seen that the phase portraits of the regular and modified Bessel functions of the first kind exhibit rich, ornamental patterns, characteristic of quasiperiodicity and chaos. Using these, the bifurcation diagrams are plotted, and the chaotic behavior is quantitatively characterized using largest Lyapunov Exponents. It is seen that the nature of chaos in the generated signals indeed depend on the frequency ratio of the driving signals, thus pertaining to a case of signal based chaos, which has the key advantage of easy tunability, which forms the novelty of the present work.

Keywords: Nonlinear Dynamics, Bessel Functions, Frequency Controlled Chaos, Bifurcation Analysis

1. Introduction

Chaos Theory, the flagship of Nonlinear Dynamics, with its characteristic properties of determinism and an extreme sensitivity to initial conditions, has emerged as one of the defining highlights of twentieth century science, with a plethora of applications covering biology, astrophysics, engineering and mechanics [1]-[16]. The development of tools and techniques to characterize and study nonlinear dynamics such as Bifurcation Plots, Iterative Maps and Lyapunov Exponents, stemming from the development of computation and visualization technologies have enabled visualizations of complex and intricate patterns of long term evolution [1, 2].

In the engineering domain, the ability to generate chaotic signals using op-amp based physical realizations of nonlinear differential equations, such as the Chua Circuits, has subsequently been translated into real-time applications of chaos theory such as secure communications and cryptography [17]-[21]. However, such implementations typically use physical system-based parameters such as resistors and capacitors as the initial conditions or 'control parameters', with the looming disadvantage of difficulty in tunability when implemented at high frequencies as Integrated Circuits (IC) [22, 23].

A radical and innovative way to address this issue would be to use signal based chaos, where the initial conditions are not physical parameters, but rather the signal properties (amplitude, frequency and phase) of the input signals in a driven chaotic system. This forms the key motivation for the present work, where we explore various forms of the Bessel Function, one of the most popular special mathematical functions, with applications as diverse as electromagnetic waves in a cylindrical waveguide, heat conduction in cylindrical objects, lattice diffusion and acoustic radiation patterns and dynamics of floating bodies [24]-[31]. Specifically, the regular and modified Bessel functions of the first and second kinds are considered, and in each case for the general form $B(x)$, the variable x is set to an additively coupled sinusoidal signal with competing frequencies, which then becomes the 'driving signal' of the system, with the frequency ratio between the input signals acting as the control parameter r. Various orders from 1 to 4 are considered for each form, and for each case, the phase portrait is plotted for an r value of π. The presence of ornamental patterns in the phase portrait indicate the presence of quasi-periodicity and chaos in the system, and this is observed for the Bessel and modified Bessel functions of the first kind. For these cases, the iterative map is then formed by computing the derivative of $B(x)$ and expressing it as a difference equation. Using the iterative map, the Bifurcation plots are then plotted, which indicate the regions of order and chaos in these signal forms. Finally, the computation of Largest Lyapunov Exponents assertively establishes the trends of order and chaos in these signal forms as a function of r.

The results discussed in the present work indicate that for specific values of r in some of the Bessel function derived signal forms, chaotic behavior is observed, thus pertaining to the case of signal based chaos, which is physically realized using FPGA's, with much simpler circuitry and easier tunability than conventional chaos generator circuits, and this forms the novelty of the present work.

2. The Bessel Functions and Phase Portraits

We start with the most general Bessel's Differential Equation first defined by Bernoulli and later generalized by Bessel and given as follows, of which the Bessel Functions are the canonical solutions $y(x)$ [24]-[31].

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \tag{1}$$

where n is typically an integer, defined as the 'Order' of the function and for such cases, the Bessel Functions are termed the 'Cylinder Functions'. Based on the finiteness at the origin ($x = 0$) and divergence of x towards zero or negative non-integer n, the solutions of the above equation are obtained as regular and modified Bessel Functions of the first and second kinds [24]-[31]. By assuming a generalized form $B(x)$ the following procedure is used for the investigation of nonlinear dynamics in the various Bessel Functions:

1. The variable x is denoted as an additively coupled signal of two sinusoids of frequencies f_1 and $f_2 = rf_1$, as
$$x = sin(2\pi f_1 t) + sin(2\pi r f_1 t) \tag{2}$$
with r denoting the ratio between the frequencies, and acting as the key control parameter.
2. Using this substitution, $B(x)$ is rewritten as a time-varying signal $B(t)$, and its time derivative $B'(t)$ is computed.
3. The dynamics of $B(x)$ are studied using the Phase Portrait, which is a plot of $B'(t)$ in terms of $B(t)$ for a given r, illustrating the phase space dynamics and qualitatively serving as a tool to assess various chaotic parameters such as sensitivity and ergodicity. Since r denotes the frequency ratio of the driving signals, and since the fundamental property of chaos is mismatches in growing evolutions, an irrational number such as π is set as the value of r, in order to maximize the frequency and phase mismatches between the driving signals.
4. The detection of ornamental and rich patterns in a phase portrait is a clear indicator of the presence of chaos in the corresponding iterative map.
5. In order to form the iterative map, $B(t)$ and $B'(t)$ are discretized into $B(i)$ and $B'(i)$ respectively and a difference equation of the form $B'(i) = B(i+1) - B(i)$ is formed. This difference equation is rearranged to give the expression of 'next' sample $B(i + 1)$ in terms of 'current' samples $B(i)$, as $B(i + 1) = B(i) + B'(i)$, this equation termed the 'Iterative Map' due to its recurrence nature.
6. For systems depicting phase portraits indicative of chaos, the bifurcation diagram, plotting the output B as a function of r is obtained. This diagram clearly illustrates for what values of r, the system exhibits chaotic and non-chaotic behavior.

We start the investigation with the regular Bessel function of the First Kind $J_n(x)$, given by the Frobenius method to Bessel's Equation defining the function by its series expansion around $x = 0$ as follows [24]-[31]:

$$J_n(x) = \sum_{m=0}^{\infty} \frac{-1^m}{m!\Gamma(m + n + 1)}(x/2)^{2m+n} \tag{3}$$

where the Gamma Function $\Gamma(t)$ is given as $\Gamma(t) = (t - 1)!$ for positive integers and as follows for all complex numbers except non-positive integers:

$$\Gamma(t) = \frac{1}{t}\prod_{n=1}^{\infty} \frac{(1 + \frac{1}{n})^t}{1 + \frac{t}{n}} \tag{4}$$

By substituting x with the additive sum of sinusoids as mentioned above, we obtain the 'Regular Bessel Signal of the First Kind' $J_n(t)$. Using this, the time derivative $J_n'(t)$ is computed, and is plotted for an r value of π, for four orders $n = 1, 2, 3, 4$ in Fig. (1).

As seen from the phase portraits, it can be observed that all the four orders show ornamental patterns characteristic of either quasi-periodic or chaotic behavior, where to distinguish between these two, further bifurcation analysis is required.

Next, the Bessel function of the second kind $Y_n(x)$, multivalued and with a singularity at the origin ($x = 0$) given as follows, as the limit of a non-integral a tending to integer n for the n-th order is considered [24]-[31].

$$Y_a(x) = \frac{J_a(x)cos(a\pi) - J_{-a}(x)}{sin(a\pi)} \tag{5}$$

$$Y_n(x) = \lim_{a\to n} Y_a(x) \tag{6}$$

In a similar fashion as above, the phase portraits of the 'Regular Bessel Signal of the Second Kind' $Y_n(t)$ are plotted for four orders $n = 1, 2, 3, 4$ in Fig. (2).

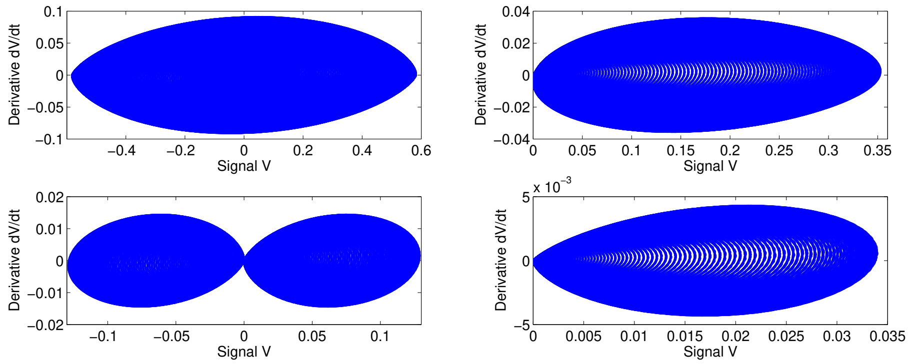

Figure 1: Phase Portrait of $J_n(t)$ with $n = 1$ (top left), $n = 2$ (top right), $n = 3$ (bottom left) and $n = 4$ (bottom right)

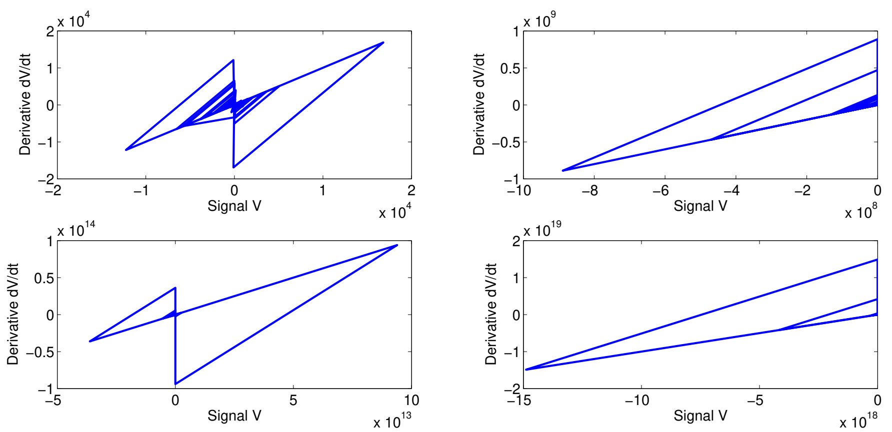

Figure 2: Phase Portrait of $Y_n(t)$ with $n = 1$ (top left), $n = 2$ (top right), $n = 3$ (bottom left) and $n = 4$ (bottom right)

Unlike the first kind Bessel signal forms, the phase portraits of the second kind Bessel forms do not display ornamental patterns sufficient to be characterized as quasi-periodic or chaotic.

We now turn our attention towards the modified Bessel Function of the first kind $I_n(x)$, alternatively called the Hyperbolic Bessel Function of the First Kind, given as follows for non-integral a and integral n orders [24]-[31].

$$I_a(x) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+a+1)} (x/2)^{2m+a} \tag{7}$$

$$I_n(x) = \lim_{a \to n} I_a(x) \tag{8}$$

Deriving from this form, the phase portraits of the signal $I_n(t)$ are plotted for four orders $n = 1, 2, 3, 4$ in Fig. (3).

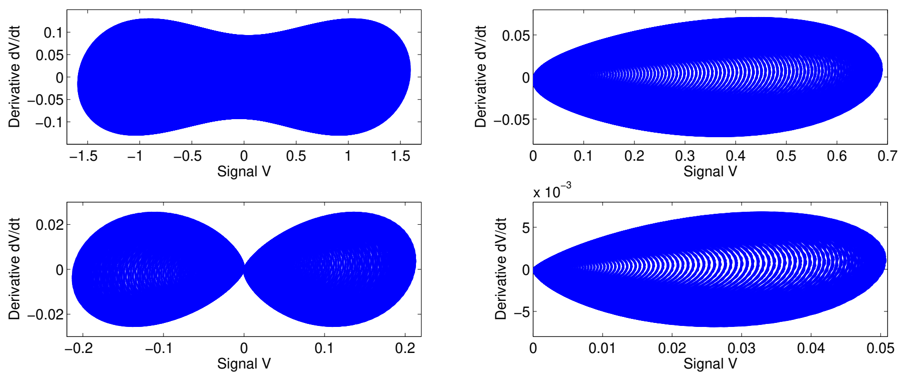

Figure 3: Phase Portrait of $I_n(t)$ with $n = 1$ (top left), $n = 2$ (top right), $n = 3$ (bottom left) and $n = 4$ (bottom right)

Similar to the $J_n(t)$ signals, the phase portraits of $I_n(t)$ also display ornamental patterns characteristic of quasiperiodic and chaotic behavior.

Finally, the modified Bessel Function of the Second Kind, given by $K_n(x)$ is considered [24]-[31].

$$K_a(x) = \frac{\pi}{2} \frac{I_{-a}(x) - I_a(x)}{sin(a\pi)} \tag{9}$$

$$K_n(x) = \lim_{a \to n} K_a(x) \tag{10}$$

Using this form, the phase portraits of the corresponding signals $K_n(t)$ are plotted for four orders $n = 1, 2, 3, 4$ in Fig. (4).

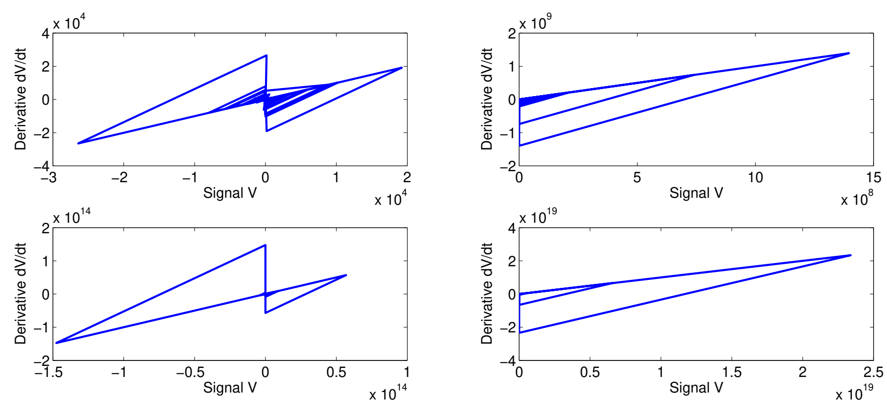

Figure 4: Phase Portrait of $K_n(t)$ with $n = 1$ (top left), $n = 2$ (top right), $n = 3$ (bottom left) and $n = 4$ (bottom right)

Similar to the $Y_n(t)$ signals, the phase portraits of $K_n(t)$ too lack ornamental richness of pattern.

Thus, in summary, it is seen that signals derived from the Regular and Modified Bessel Functions of the First Kind namely, $J_n(t)$ and $I_n(t)$ exhibit ornamental phase portraits for the four orders $n = 1, 2, 3, 4$, characteristic of quasiperiodic or chaotic behavior.

3. Bifurcation Analysis

In this section, the bifurcation analyses for $J_n(t)$ and $I_n(t)$ are explored. It must be noted that, on account of the coupled frequency formation of x as given earlier, these forms do not give rise to bifurcations in the traditional sense; rather they exhibit a quasi-periodic route to chaos, as typically seen in other coupled phase based chaotic systems such as the standard circle map [32, 33, 34, 35, 36, 37].

In order to compute the time derivative of $J_n(t)$, the following identity is used

$$J'_n(x) = \frac{1}{2}(J_{n-1}(x) - J_{n+1}(x)) \tag{11}$$

Thus, we obtain the following difference equation by discretizing the above identity and setting $J'_n(i) = J_n(i+1) - J_n(i)$.

$$J_n(i+1) = J_n(i) + \frac{1}{2}(J_{n-1}(i) - J_{n+1}(i)) \tag{12}$$

The above equation is termed the 'iterative map' of the First Kind Bessel Function, and the corresponding bifurcation diagram is plotted for J_n as a function of r in Fig. (5).

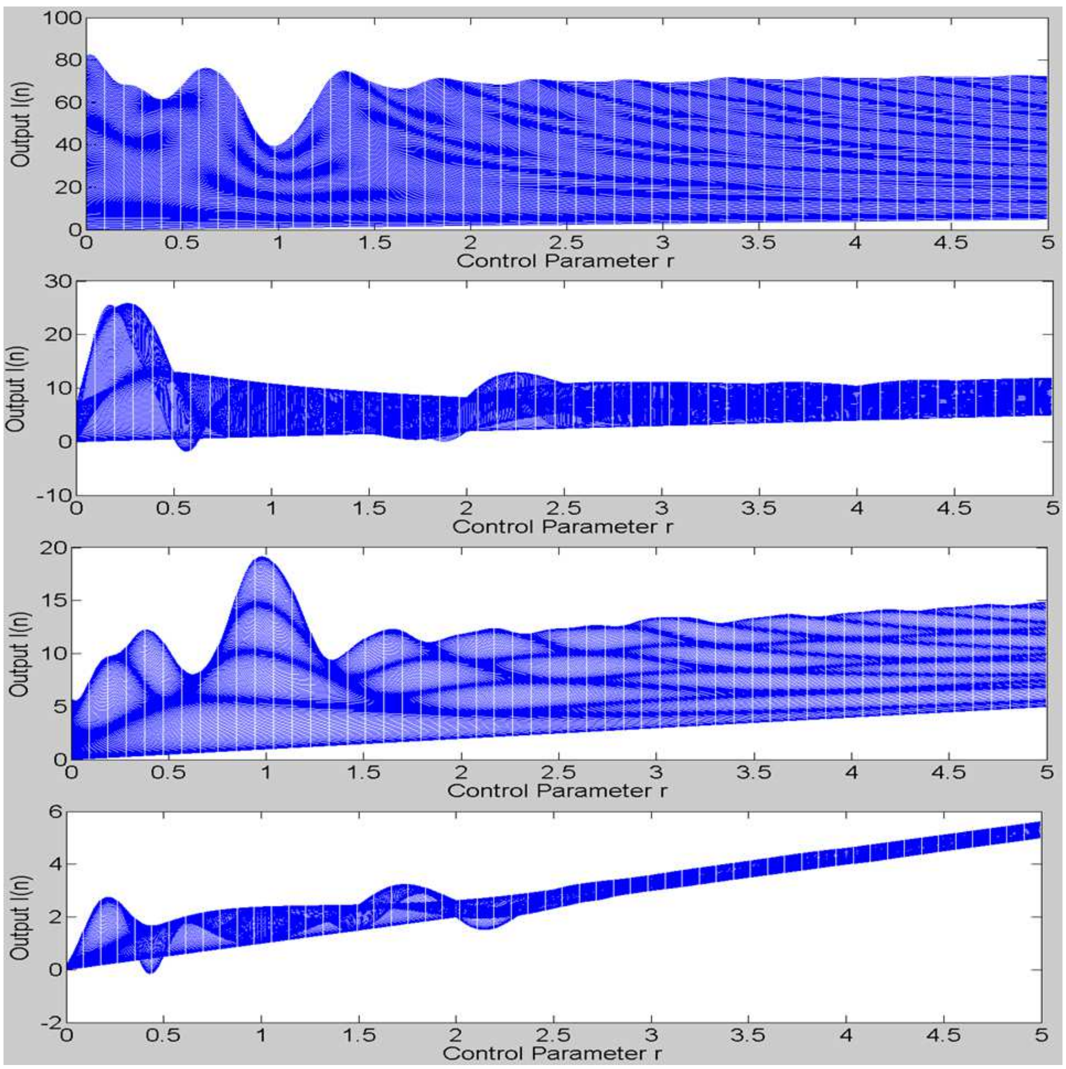

Figure 5: Bifurcation Plots of J_n forms for (from top to bottom) $n = 1, 2, 3, 4$

From the bifurcation plots, it is seen that while the odd orders $n = 1, 3$ exhibit largely quasiperiodic, 'wrinkled landscape' formations, the even order bifurcation plots $n = 2, 4$ exhibit well-bounded bifurcation trends, characteristic of chaos specifically at r values such as 0.6 and 1.4.

In a similar fashion, the iterative map for signals based on the modified Bessel Function of the first kind $I_n(t)$ is given as follows, and the corresponding bifurcation plots for the four orders are given in Fig. (6).

$$I'_n(x) = \frac{1}{2}(I_{n-1}(x) + I_{n+1}(x)) \tag{13}$$

$$I_n(i+1) = I_n(i) + \frac{1}{2}(I_{n-1}(i) + I_{n+1}(i)) \tag{14}$$

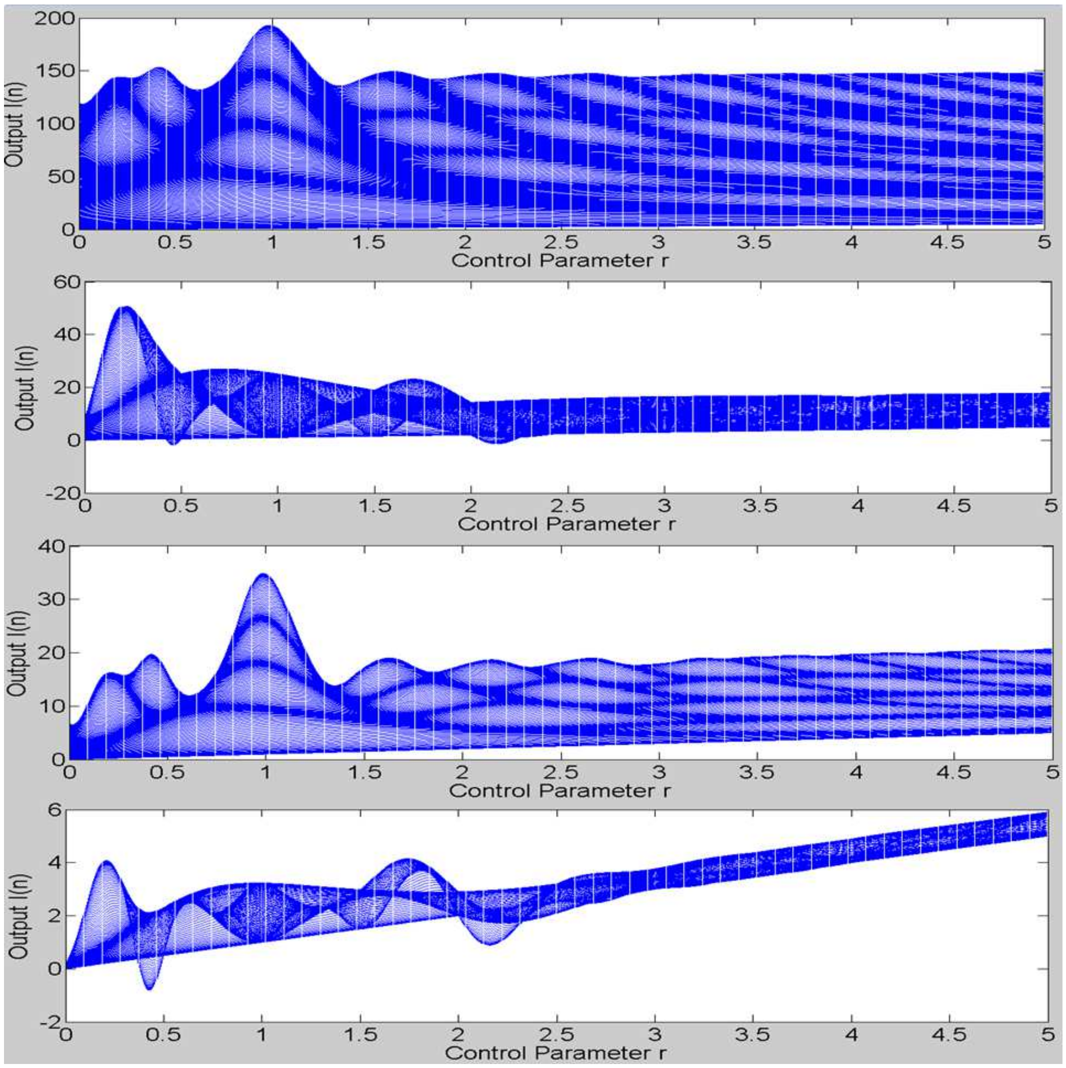

Figure 6: Bifurcation Plots of I_n forms for (from top to bottom) $n = 1, 2, 3, 4$

It is seen from the plots that the trends of quasiperiodicity and chaos are very similar to that of the $J_(n)$ forms, albeit with slightly more dense and 'grassy' patterns in all orders.

4. Chaotic Characterization using Lyapunov Exponents

In order to understand the dependence of the chaotic nature in the generated signals on r, we use the Largest Lyapunov Exponent (LLE), quantifying a systems sensitive dependence on initial conditions [38, 39]. The Rosensteins algorithm is used to compute the Lyapunov Exponents λ_i from the signal, where the sensitive dependence is characterized by the divergence samples $d_j(i)$ between nearest trajectories represented by j given as $d_j(i) = C_j e^{\lambda_i(i\delta t)}$, C_j being a normalization constant [38, 39]. With a positive value of LLE assertively establishing the presence of chaos, the LLE's of the $J_n(t)$ and $I_n(t)$ forms with $n = 1, 2, 3, 4$ for various r values from 0.1 to 1.5 are computed and plotted as a graph in Fig. (7).

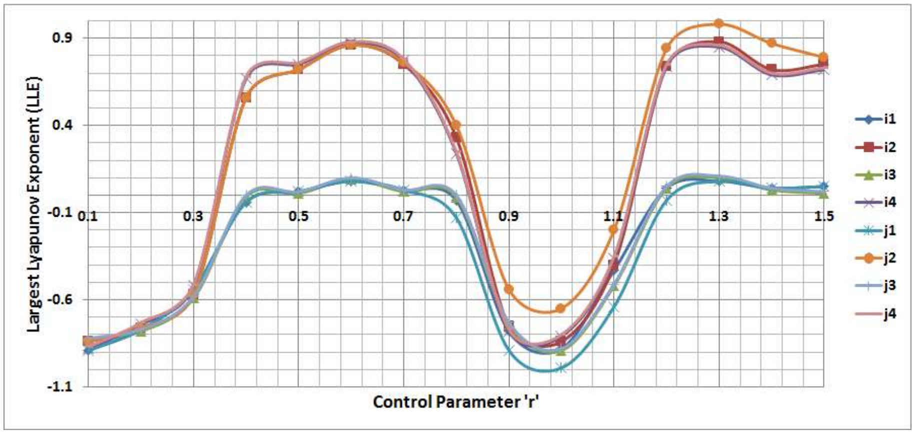

Figure 7: Largest Lyapunov Exponent Values of $J_n(t)$ and $I_n(t)$ for $n = 1, 2, 3, 4$ as function of r

As seen from the plot, the trends of LLE for all the orders of $J_n(t)$ and $I_n(t)$ are largely similar with the highest and lowest LLE values of the lot seen for $J_2(t)$ and $J_1(t)$ respectively. Of the selected range of r, the most chaotic values are invariably seen for r values of 0.6 and 1.4.

5. Conclusion

Considering concerns of tunability in system based chaos generation circuits such as the Chua Diode, the present work purports to a radical and innovative solution using signal based chaos, and to achieve this, the regular and modified Bessel's Functions of the first and second kinds are considered. Specifically, the Bessel function forms are adapted into signals by substituting the variable x as an additively coupled sinusoidal signal,viewing the output as a representative of a driven nonlinear coupled system. Following this, the derivative of the output is computed and used to form a difference equation, which yields the iterative map of the proposed system. This iterative map is studied using phase portraits, where it is seen that of the four Bessel forms chosen, signals derived from the regular and modified Bessel functions of the first kind, namely $J_n(t)$ and $I_n(t)$ exhibit rich ornamental phase portrait patterns characteristic of quasiperiodic or chaotic behavior. Hence, the bifurcation analysis of these three forms are presented following which the dependence of chaotic nature on the frequency ratio r is studied using largest Lyapunov Exponents.

Finally, it is noteworthy that since the behavior of the output signal depends on frequency ratio r, this ratio serves as a secure 'key', enabling the use of the Bessel Function based 'Frequency Controlled Chaos' in secure communication and encryption systems. The signal oriented approach to generating chaos from mathematical functions, coupled with the easy tunability hence obtained forms the novelty of the present work.

References

[1] M. Ausloos, M. Dirickx, *The Logistic Map and the Route to Chaos: From the Beginnings to Modern Applications*, (Springer, US, 2006).

[2] S. H. Strogatz, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering* ,(Westview Press, Cambridge, 2008).

[3] F. Cramer, *Chaos and Order the Complex Structure of Living Systems.* (Springer, 1993).

[4] D. S. Coffey, *Self-organization, complexity and chaos: the new biology for medicine.* Nature medicine **4**, 882-885 (1998).

[5] G. Contopoulos, *Order and chaos in dynamical astronomy*, (Springer Science and Business Media, 2002).

[6] J. Laskar, *Large-scale chaos in the solar system*, Astronomy and Astrophysics **287**, L9-L12 (1994).

[7] A. B. Cambel, *Applied chaos theory-A paradigm for complexity.* (Academic Press, Inc., 1993).

[8] K. Aihara, *Chaos engineering and its application to parallel distributed processing with chaotic neural networks.* Proceedings of the IEEE, **90** 919-930 (2002).

[9] G. Chen, *Controlling chaos and bifurcations in engineering systems.* (CRC press, 1999).

[10] R. B. Stull, *An introduction to boundary layer meteorology.* (Springer Science and Business Media, 1988).

[11] M. F. Barnsley, A. D. Sloan, *Chaotic Compression*, Computer Graphics World, **3** (1987).

[12] K. E. Barner G. R. Arce, *Nonlinear Signal and Image Processing: Theory, Methods, and Applications*, (CRC Press, U.S, 2003).

[13] S. Saini, J. S. Saini. *Secure communication using memristor based chaotic circuit.* Parallel, Distributed and Grid Computing (PDGC), 2014 International Conference on. IEEE, (2014).

[14] S. Shaerbaf, S. A. Seyedin. *Nonlinear Multiuser Receiver for Optimized Chaos-Based DS-CDMA Systems.*, Iranian Journal of Electrical andElectronic Engineering **7**, 149 (2011): 149.

[15] L. Kocarev, *From chaotic maps to encryption schemes.* Circuits and Systems, (1998).

[16] G. Jakimoski, L. Kocarev. *Chaos and cryptography: block encryption ciphers based on chaotic maps.* IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications **48** 163-169 (2001).

[17] E.Bilotta and P.Pantano, *A gallery of Chua attractors*, (World Scientific, Singapore, 2008).
[18] L. Chua, *A universal circuit for studying and generating chaos. I. Routes to chaos.* Circuits and Systems I: Fundamental Theory and Applications, IEEE Transactions on **40**732-744 (1993).
[19] G. Kolumban, M. P. Kennedy, L. O. Chua. *The role of synchronization in digital communications using chaos. II. Chaotic modulation and chaotic synchronization*, Circuits and Systems I: Fundamental Theory and Applications, IEEE Transactions on **45**, 1129-1140 (1998).
[20] L. Chua, *Chaos synchronization in Chua's circuit*, Journal of Circuits, Systems, and Computers **3**, 93-108 (1993).
[21] L. Chua, *Experimental chaos synchronization in Chua's circuit.* International Journal of Bifurcation and Chaos **2**, 705-708 (1992).
[22] B. Razavi, *RF Microelectronics*, (Prentice Hall, US, 2011).
[23] M.Chan, K.Hui, C.Hu, P.K.Ko, *A robust and physical BSIM3 non quasi static transient and AC small signal model for circuit simulation*, IEEE Transactions on Electron Devices. **45**, 834 (1998).
[24] M. Abramowitz, I. A. Stegun, *Bessel Functions J and Y (Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables)*, New York: Dover, (1972).
[25] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, (1966).
[26] P. M. Morse, H. Feshbach, *Methods of Theoretical Physics, Part I* McGraw-Hill, (1953).
[27] J. E. Goell, *A circularharmonic computer analysis of rectangular dielectric waveguides*, Bell System Technical Journal **48**, 2133-2160 (1969).
[28] M. N. Ozisik, *Heat conduction*, John Wiley and Sons, (1993).
[29] I. Webman, Itzhak, *Effective-medium approximation for diffusion on a random lattice*, Physical Review Letters **47**, 1496 (1981).
[30] K. Yosioka, Y. Kawasima, *Acoustic radiation pressure on a compressible sphere*, Acta Acustica **5**, 167-173 (1955).
[31] F. John, *On the motion of floating bodies II. Simple harmonic motions*, Communications on pure and applied mathematics **3**, 45-101 (1950).
[32] A. Kudrolli, A, J. P. Gollub. *Patterns and spatiotemporal chaos in parametrically forced surface waves: a systematic survey at large aspect ratio.* Physica D: Nonlinear Phenomena **97**, 133-154 (1996).
[33] J. Briggs, *Fractals: The patterns of chaos: A new aesthetic of art, science, and nature.* (Simon and Schuster, 1992).
[34] M. Lakshmanan, S. Rajaseekar, *Nonlinear dynamics: integrability, chaos and patterns.* (Springer Science and Business Media, 2012).
[35] I R. Epstein, K. Showalter. *Nonlinear chemical dynamics: oscillations, patterns, and chaos.* The Journal of Physical Chemistry *100* 13132-13147 (1996).
[36] R. Gilmore, M. Lefranc, *The Topology of Chaos*, (Wiley,US, [2002]).
[37] J. M. T. Thompson, H. B. Stewart, *Nonlinear Dynamics and Chaos* (Wiley,UK, [2002]).
[38] R. G. James, K. Burke, J. P. Crutchfield, *Chaos forgets and remembers: Measuring information creation, destruction, and storage*, Int. J Bifurcation Chaos. **378**, 2124 (2014).
[39] M. T. Rosenstein, J. J. Collins, C. J. De Luca, *A practical method for calculating largest Lyapunov exponents from small data sets*, Physica D, **65**, 117, (1993).