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Nonlinear dynamics of a rotating shaft with a breathing crack

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Abstract In this paper, the effects of a breathing crack on the vibratory characteristics of a rotating shaft are investigated. A new, simple and robust model composed of two rigid bars connected with a nonlinear flexural spring is proposed. The nonlinear spring, located at the cracked transverse section position, concentrates the global stiffness of the cracked shaft. The breathing mechanism of the crack is described by a more realistic periodic variation of the global stiffness depending not only but substantially on the system vibratory response. It is based on an energy formulation of the problem of 3D elasticity with unilateral contact conditions on the crack lips. A possible partial opening and closing of the crack is considered which makes the approach more appropriate for deep cracks modeling. The harmonic balance method, direct time-integration schemes and nonlinear dynamics tools are used to characterize the global dynamics of the system. The effects of the crack depth and rotating frequency have been meticulously examined and it was found that the cracked shaft never exhibits chaotic or quasi-periodic vibratory response.

Keywords Rotating shaft · breathing crack · nonlinear dynamics · stability · Floquet theory · period doubling · bifurcation diagram · Poincaré section

1 Introduction

The development and propagation of a crack represents the most common and trivial beginning of integrity losses in engineering structures. For rotating shafts, a propagating fatigue crack can have detrimental effects on the reliability of a process or utility plant where theses vital parts are subjected to very arduous working conditions in harsh environment. It is one of the most serious causes of accidents and, an early warning is essential to extend the durability and increase the reliability of these machines. According to Bently and Muszynska [1986], in the 70s and till the beginning of the 80s, at least 28 shaft failures due to cracks were registered in the USA energy industry. It is

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today well understood that a crack or a local defect in a structural member introduces a local flexibility that affects its vibration response. Thus, the dynamic characteristics has to be analyzed in order to detect the existence and progression of cracks and develop a monitoring methodology.
Vibration-based monitoring for crack detection is gaining an increasing interest since it allows to inspect a machine health without dismantling its parts. Also, by being carried out online, it permits to avoid coastdown and running up through the shaft bending critical speeds which generally leads to a rubbing at the internal seals (efficiency loss) and a propagation of existing cracks.

Because of the increasing need of energy, the plants installed by electricity supply utilities throughout the world are becoming larger and more highly stressed. Thus, the risk of turbogenerator shaft cracking is increasing also. Since the early 1970s when investigations on the vibrational behavior of cracked rotors began, numerous papers on this subject have been published, as a literature survey by Dimarogonas [1996] shows. The excellent papers by Dimarogonas [1996], Wauer [1990] and Bachschmid et al. [2000], and books by Dimarogonas and Paipetis [1983] and Bachschmid et al. [2009] cover many aspects of this area and summarize the most relevant analytical, experimental and numerical works conducted in the last three decades and related to the cracked structures modeling.
The vibration analysis and modeling of the shaft and cracks are necessary for a reliable identification of the crack location and depth to avoid catastrophic failures. In fact, cracks can develop and propagate to relevant depths without affecting consistently the normal operating conditions of the shaft (Bachschmid and Pennacchi [2008]).
The local flexibility induced by the presence of a crack has been most of the time determined by application of the linear fracture mechanics theory [Gross and Srawley, 1965, Anifantis and Dimarogonas, 1983, Dimarogonas and Paipetis, 1983, Dimarogonas, 1996, Papadopoulos and Dimarogonas, 1987a,b,c, Papadopoulos, 2004]. Obviously, the first work was done in the early 1970s by Dimarogonas [1970, 1971] and Pafelias [1974] at the General Electric Company. A good review on this method called Energy Release Rate Approach (ERRA) is presented by Papadopoulos [2008]. However, this approach has some limitation (Abraham et al. [1994]) when the crack depth exceeds the shaft radius.
Another feature related to the problem of modeling cracked rotating shafts is the consideration of the opening−closing phenomenon of the crack during the shaft rotation. Many researchers have been concerned with this mechanism and discussed different procedures used to accurately compute the time history of local flexibilities associated with a breathing crack (Dimarogonas [1996], Bachschmid et al. [2008a,b], Georgantzinos and Anifantis [2008], Papadopoulos [2008]). When this phenomena is neglected, the overall behaviour of the system can be considered linear. Also, when the breathing mechanism is governed by static loads which are independent from the vibration response of the shaft, a linear model is sufficient for the system dynamics analysis. However, the consideration of a breathing mechanism that depends on the shaft vibration makes the problem nonlinear: the behaviour depends also on the system response. For an accurate prediction of the time-variant flexibility, Andrieux and Varé [2002] from Electricité De France (EDF), proposed an original method for deriving a lumped model for a cracked beam section. Based on three-dimensional computations, the procedure incorporates more realistic behavior on the cracks than the previous models, namely the unilateral contact conditions on the cracks lips and the breathing mechanism of

the cracks under variable loading. The method was derived from three-dimensional formulation of the general problem of elasticity with unilateral contact conditions on the cracks lips. The authors identified the characteristics of a one-dimensional (beam) model from accurate three-dimensional FE model. The transverse cracked section is replaced by a nonlinear flexural spring for which they established the constitutive equations. Great attention is paid to the capability of such a model to take into account the real 3D geometry of a crack and to represent, as general as possible, the effects of different load components on its nonlinear behavior.
The experimental validation of the approach is presented by Stoisser and Audebert [2008]. The authors reported that the model reproduces with good accuracy the overall behavior of the shaft line in presence of cracks. The experimental validation allows the use of the model, with confidence and reliability, for the determination of the dynamics of supposed cracked rotors.

Recently, many researchers have been dealing with the bifurcation and chaotic behavior of a cracked rotating shaft. Chen and Dai [2007] investigated the nonlinear dynamics of a cracked rotor system with viscoelastic supports. A linear spring is used to model the cracked transverse section, and truncated time-varying cosine series are utilized to account for the breathing mechanism of the crack. With various combinations of the rotor system parameters, the authors observed bifurcation, periodic of NT period, quasi-periodic and chaotic dynamical response.

With the aim of developing an online crack detection method, Zuo [1992] and Zuo and Curnier [1994] have used a bilinear model to characterize the vibrational response of a cracked rotating shaft. The authors, first, examined the 1 degree of freedom (dof) system then studied the behavior of a system with 2 dof. By extending the Rosenberg normal mode notion for smooth and symmetric nonlinear systems to conwise linear systems, they defined the nonlinear modes of the bilinear system which were calculated numerically, and for simple cases, analytically. The application of this original method to systems with high number of dof is complicated and would lead to high computation costs.
Muller et al. [1994] observed chaotic motions and strange attractors by applying the theory of Lyapunov exponents to a system with the switching crack model. Ishida [2008] presented an overview of the most observed phenomena when examining the response of a cracked rotor. Harmonic, superharmonic, subharmonic and supersubharmonic resonances are well commented and explained using a simple rotor model similar to Jeffcott rotor. Chaotic and quasi-periodic response has been observed by Patel and Darpe [2008] using a switching crack model. The authors examined the effect of the unbalance eccentricity level, the crack depth and damping on the bifurcation characteristics of a cracked rotor. It was reported that, unlike the switching crack model, with a more realistic breathing mechanism of the crack the chaotic, quasi-periodic and subharmonic vibrations are not observed on the response of the cracked rotor for the similar set of parameters.

The work presented in this paper is based on the original approach recently developed by Andrieux and Varé [2002] which is more comprehensive and easier to implement numerically. This new approach avoids the difficulty of accurately computing the Stress Intensity Factors required by the ERRA.

El Arem and Maitournam [2008, 2007] have presented a method of construction of a cracked beam finite element which they used afterwards for the stability analysis of cracked shafts. Unlike the approach of Andrieux and Varé [2002], the authors distributed the additional energy due to the cracked section on the entire length of the cracked beam finite element. Considerable gain in computing efforts was reached compared to the nodal representation of the cracked section when dealing with the numerical integration of differential equations in structural dynamics. The authors considered a system of five beam FE (16 dof) to explore de vibrational response of a cracked rotating shaft. In this paper, and by considering a simpler dynamical system of two dof, the computing efforts are remarkably reduced and deeper investigations are carried out to foresee bifurcation, instability and propose parameters for online crack detection. Moreover, the breathing mechanism considered here is depending not only on the crack angular position but also on the dynamical response of the cracked shaft.

The main purposes of this work are :

1. To explore the steady-state vibratory response of a cracked rotating shaft assuming:
   (a) small motions near the equilibrium position under the shaft's own weight (assumption H_0),
   (b) a specific function to describe the breathing mechanism of the crack (assumption H_1). For better description of the origins of this function, its identification and all the developments behind it, the works of Andrieux and Varé [2002], El Arem [2006] and El Arem and Maitournam [2007, 2008] could be referred.
2. second, to give a qualitative description of the effects of a crack propagation on the behavior of the shaft. The motivation of this paper is to present general tendencies of a cracked rotating shaft dynamics. It is not our intention to present a quantitative results for special cases like observed in experimental investigations.

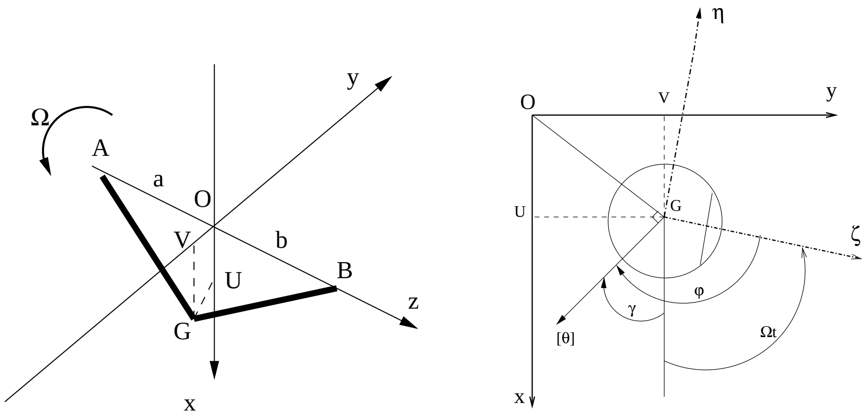
2 Mechanical system: description and equilibrium equations

As described by Figure 5, the uncracked parts of the shaft are modeled by two rigid bars AG and GB of circular section S, of respective lengths a and b, and distributed mass m. The transverse cracked section of the shaft is modeled by the nonlinear elastic flexural spring placed in G and concentrating the global stiffness of the cracked shaft. The system is simply supported at A and B, rotating at the frequency Ω about the bars axis and subjected to the effects of its own weight. In the inertial frame, the small movements of G are described by $(U(t), V(t), 0)$.
Under the H_0 assumption, the rotations gaps in G could be written as:

$$[\boldsymbol{\theta}] = \begin{cases} [\theta]_x \\ [\theta]_y \end{cases} = \begin{cases} \theta_x(0^+) - \theta_x(0^-) = (\frac{1}{a} + \frac{1}{b})V \\ \theta_y(0^+) - \theta_y(0^-) = -(\frac{1}{a} + \frac{1}{b})U \end{cases} \tag{1}$$

We can notice in particular that: $[\boldsymbol{\theta}] \perp \boldsymbol{OG}$.
The crack orientation is defined by the $\boldsymbol{G\zeta}$ direction given by: $(\boldsymbol{Ox}, \boldsymbol{G\zeta}) = \Omega t$. El Arem [2009] showed that the shearing effects on the breathing mechanism of the crack are negligible and, accordingly, will not be considered in what follows.
The elastic energy of the system could be written as quadratic form of the rotations gaps :

$$W([\theta]_x, [\theta]_y) = \frac{1}{2} k(\varphi)([\theta]_x^2 + [\theta]_y^2) \tag{2}$$



(a) Cracked rotating shaft (b) Cracked transverse section

Fig. 1 *Mechanical system. (Oxy) inertial frame and (Gζη) rotating frame*

where $k(\varphi)$ represents a directional stiffness, 2π-periodic function of the angle

$$\begin{aligned}\varphi = \arctan(\frac{[\theta]_\eta}{[\theta]_\zeta}) &= (\boldsymbol{G\zeta}, [\boldsymbol{\theta}]) = (\boldsymbol{G\zeta}, \boldsymbol{Ox}) + (\boldsymbol{Ox}, [\boldsymbol{\theta}]) \\ &= -\Omega t + \gamma \in [0, 2\pi[\textit{ modulo } 2\pi \qquad (3)\end{aligned}$$

where $[\theta]_\zeta$ and $[\theta]_\eta$ are the rotations gaps in G expressed in the rotating shaft fixed frame $(G\zeta\eta)$. γ is given by $\gamma = -\arctan(\frac{U}{V}) + n\pi$, with $n \in \{0, 1, 2\}$, cf. figure 5.
The flexural moments associated with the rotations at time t are:

$$M_x = \frac{\partial W}{\partial [\theta]_x} \text{ and } M_y = \frac{\partial W}{\partial [\theta]_y}$$

In the rotating, shaft fixed frame $(\zeta\eta)$, the following constitutive equations is then obtained :

$$\begin{pmatrix} M_\zeta \\ M_\eta \end{pmatrix} = \begin{pmatrix} k(\varphi) & -\frac{1}{2}k'(\varphi) \\ \frac{1}{2}k'(\varphi) & k(\varphi) \end{pmatrix} \begin{pmatrix} [\theta]_\zeta \\ [\theta]_\eta \end{pmatrix} \qquad (4)$$

This nonlinear relation leads to

$$\begin{pmatrix} M_x \\ M_y \end{pmatrix} = \begin{pmatrix} k(\varphi) & -\frac{1}{2}k'(\varphi) \\ \frac{1}{2}k'(\varphi) & k(\varphi) \end{pmatrix} \begin{pmatrix} [\theta]_x \\ [\theta]_y \end{pmatrix} \qquad (5)$$

in the inertial non-rotating frame (xy).
With the switching crack model, only stiffness reduction in the weak direction of the crack $(\boldsymbol{G\zeta})$ is considered, but it is well known that the rotor is weakened not only in the crack direction but also in the perpendicular direction as the crack propagates.
By writing the constitutive equations of the cracked section in this form:

- the additional flexibilities in both directions ($\boldsymbol{G\zeta}$ and $\boldsymbol{G\eta}$) are taken into consideration,
- the breathing mechanism is governed not only but substantially by the system response (nonlinear behavior),

- the cracked section flexibility is represented by a single parameter, k,
- based on an energy approach (Andrieux and Varé [2002], El Arem [2006]), this representation is independent of the crack form or cracks number affecting the same transverse section of the shaft,
- with an appropriate choice of the periodic variation of function k, the partial opening-closing of the crack is considered.

The dynamical equilibrium equations are given by the Virtual Powers Principle as:

$$\begin{cases} M_0\ddot{U}(t) + D\dot{U}(t) + k(\varphi)U(t) - \frac{1}{2}k'(\varphi)V(t) = m_2 g \\ M_0\ddot{V}(t) + D\dot{V}(t) + k(\varphi)V(t) + \frac{1}{2}k'(\varphi)U(t) = 0 \end{cases} \tag{6}$$

where $M_0 = \frac{\rho S}{3}(a+b)\mathfrak{L}^2 + \rho I \mathfrak{L}, m_2 = m\frac{a+b}{2}\mathfrak{L}^2$ and $\frac{1}{\mathfrak{L}} = \frac{1}{a} + \frac{1}{b}$, $D = 2dM_0 w_0$ is the viscous damping coefficient, d the reduced (dimensionless) damping coefficient and $w_0 = \sqrt{\frac{K_0}{M_0}}$ the natural frequency of the system with the global stiffness K_0.
Under the H_0 assumption, $|U| \gg V$. Consequently, γ is close to $\frac{3\pi}{2}$ so that the approximation

$$\varphi = \frac{3\pi}{2} - \Omega t \tag{7}$$

is justified and (6) becomes:

$$\begin{cases} M_0\ddot{U}(t) + D\dot{U}(t) + k(\varphi)U(t) - \frac{1}{2}k'(\varphi)V(t) = m_2 g \\ M_0\ddot{V}(t) + D\dot{V}(t) + k(\varphi)V(t) + \frac{1}{2}k'(\varphi)U(t) = 0 \\ \varphi = \frac{3\pi}{2} - \Omega t \end{cases} \tag{8}$$

By considering approximation (7), it becomes possible to solve system (8) using the harmonic balance method.

3 Periodic solutions with H_0 and H_1 assumptions

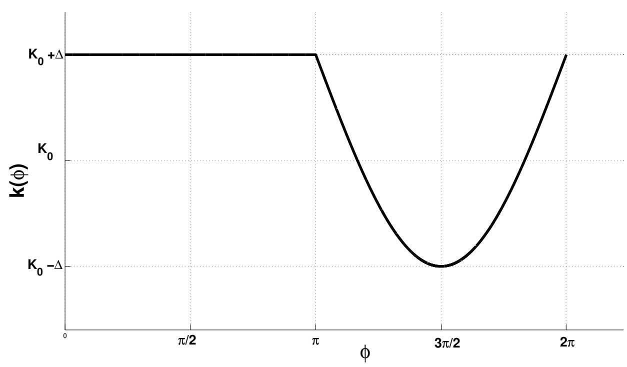


Fig. 2 *Periodic global stiffness function k: breathing mechanism*

A crack presence excites the vibrations of the rotating shaft and changes their characteristics. The way the crack opens and closes, has an important effect on these characteristics. When captured and measured, the implied modification of the vibratory response is a significant symptom of cracking. Accurate modeling of the breathing mechanism is, therefore, essential for a reliable prediction and correct simulations of shafts dynamics.

Based on previous development and observation carried out by the authors (El Arem and Nguyen [2006], El Arem [2006], El Arem and Maitournam [2008]) and colleagues from EDF (Andrieux and Varé [2002]), an original description of the breathing mechanism of a crack in a rotating shaft is given in this paper.
The global stiffness of the system is depending on the shaft vibratory response through $\varphi = \arctan(\frac{[\theta]_\eta}{[\theta]_\varsigma})$. It is concentrated at the nonlinear flexural spring and described by the periodic function (Figure 3):

$$k(\varphi) = \begin{cases} K_0 + \Delta & \forall \ \varphi \in [0, \pi[\\ K_0 + \Delta + 2\Delta \sin\varphi & \forall\varphi \ \ \in [\pi, 2\pi[\end{cases} \tag{9}$$

For $\varphi \in [0, \pi]$, the crack is totally closed and $k(\varphi) = K_s = K_0 + \Delta$. For $\varphi \in [\pi, 2\pi[$, the crack can open or be closed partially. It opens totally at $\varphi = \frac{3\pi}{2}$. K_s is the global stiffness of the non-cracked shaft.
By adopting this way of modeling the crack, the resort to the penalization technique when the crack closes totally is avoided. This technique could indeed lead to exorbitant computational costs (El Arem [2006]). It is also important to notice that only one measurement is needed to identify $k(\varphi)$. In fact, knowing the mass M_0 and the first critical frequency $w_s = \sqrt{\frac{K_s}{M_0}}$ of the new shaft, we need only to measure approximately the current critical frequency $w_0 \approx \sqrt{\frac{K_0}{M_0}}$ of the rotating shaft to identify the loss of global stiffness Δ by $\Delta = K_s - K_0$.
Under the H_1 assumption, The vibratory response of the shaft can also be split,

$$U(t) = U_0 + u(t), \qquad\qquad V(t) = V_0 + v(t) \tag{10}$$

where ($U_0 = \frac{m_2 g}{K_0}, V_0 = 0$) is the static response of the shaft with the global stiffness K_0 subjected to its own weight. When k is written as a Fourier series using the approximation 7, system (8) becomes :

$$\begin{cases} M_0\ddot{u}(t) + D\dot{u}(t) + \\ \left(k_0 + \Delta(1-\cos\Omega t) - \dfrac{2\Delta}{\pi} + \dfrac{4\Delta}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{(4n^2-1)}\cos(2n\Omega t)\right)u(t) - \\ \left(-\dfrac{\Delta}{2}\sin\Omega t + \dfrac{4\Delta}{\pi}\sum_{n=1}^{\infty}(-1)^n\dfrac{n}{(4n^2-1)}\sin(2n\Omega t)\right)v(t) \\ = U_0(\dfrac{2\Delta}{\pi} - \Delta(1-\cos\Omega t) - \dfrac{4\Delta}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{(4n^2-1)}\cos(2n\Omega t)) \\ M_0\ddot{v}(t) + D\dot{v}(t) + \\ \left(k_0 + \Delta(1-\cos\Omega t) - \dfrac{2\Delta}{\pi} + \dfrac{4\Delta}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{(4n^2-1)}\cos(2n\Omega t)\right)v(t) \\ \quad + \left(-\dfrac{\Delta}{2}\sin\Omega t + \dfrac{4\Delta}{\pi}\sum_{n=1}^{\infty}(-1)^n\dfrac{n}{(4n^2-1)}\sin(2n\Omega t)\right)u(t) \\ = -\dfrac{4\Delta U_0}{\pi}\sum_{n=1}^{\infty}(-1)^n\dfrac{n}{(4n^2-1)}\sin(2n\Omega t) + U_0\dfrac{\Delta}{2}\sin\Omega t \end{cases} \tag{11}$$

When using the harmonic expansion method and considering only the N first harmonics, the dynamical response of the system

$$z(t) = u(t) + iv(t) = \frac{U_0 \Delta}{\pi k_0} \sum_{j \in Z} z_j e^{ij\Omega t} \text{ (with } i^2 = -1)$$

satisfies:

$$\sum_{j=-N}^{j=N} (1 + \frac{\Delta}{k_0} - (j\xi)^2 + 2ijd\xi) z_j e^{ij\Omega t} - \frac{\Delta}{k_0} \sum_{j-N}^{j=N} z_j (\frac{3}{4} e^{i(1+j)\Omega t}$$
$$+ \frac{1}{4} e^{-i(1-j)\Omega t}) + \frac{2\Delta}{\pi k_0} \sum_{j-N}^{j=N} z_j \sum_{k=-N}^{k=N} (-1)^k \frac{1+k}{4k^2-1} e^{i(2k+j)\Omega t}$$
$$= -\pi(1 - \frac{3}{4} e^{i\Omega t} - \frac{1}{4} e^{-i\Omega t}) - 2 \sum_{n=-N}^{n=N} (-1)^n \frac{1+n}{4n^2-1} e^{i2n\Omega t} \quad (12)$$

The problem leads to the determination of a solution of an algebraic system of $(2N+1)$

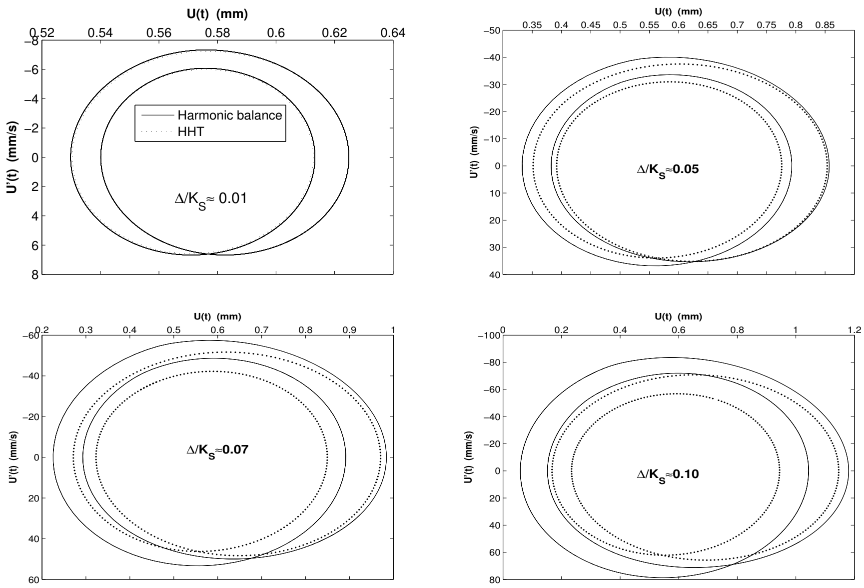
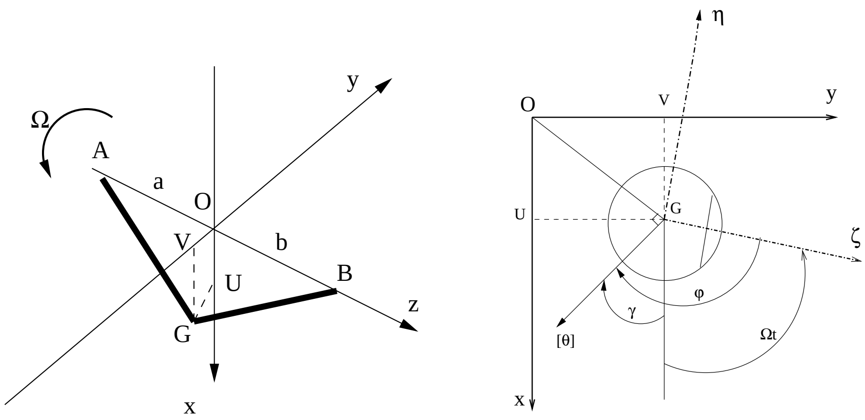


Fig. 3 *Validity of approximation (13), d=0.03,* $\xi = \frac{\Omega}{w_0} = 0.50$

unknowns, the z_j with $-N \leq j \leq N$.
The next step is to check the validity of the approximation we have made so far to make the previous developments possible:

$$\varphi = \frac{3\pi}{2} - \Omega t \quad (13)$$

With approximation (13), the breathing mechanism of the crack is only depending on the shaft rotation angle (Ωt). Consequently, the effects of the shaft vibrations on the opening-closing of the crack are neglected. This hypothesis has been always adopted by research scientists dealing with rotating cracked shafts. It is believed that this hypothesis is suitable for the case of weight-dominant rotors. Figure 4 shows a good agreement between the solution of system (6) using the the Hilber−Hughes−Taylor (HHT) $\alpha-$method direct time-integration scheme and that of system (8) using the balance harmonic method: approximation (13) is consequently justified for small crack depths ($\frac{\Delta}{K_S} < 0.07$). As shown in Figures 6 & 7, the super-harmonic resonance phenomenon



(a) Cracked rotating shaft (b) Cracked transverse section

Fig. 4 *111 111* 121 *Mechanical system. (Oxy) inertial frame and (Gζη) rotating frame*

is observed when the rotating frequency is a submultiple of the critical rotating speed of the shaft. In this case, the superharmonics reach higher vibration amplitudes (Figure 7): when $\xi = \frac{\Omega}{w_0} \approx \frac{1}{N}, N \in \mathbb{N}$, a resonance peak lies at the frequency $w = N\Omega$ and the shaft orbit is composed of N loops, cf. Figure 6. Gasch [1993] and Patel and Darpe [2008] have observed the same behavior by using a switching crack model. By considering the crack model of Mayes and Davies [1984], Sinou and Lees [2005] noticed changes of the harmonic components of the vibratory response of a cracked shaft and an evolution of its orbits with the rotating frequency. Also, El Arem [2006], El Arem and Maitournam [2008], Chen and Dai [2007], Chen et al. [2007] made the same observation by using a more realistic breathing mechanism of the crack. In previous works, the authors (El Arem and Nguyen [2006]) have considered a simple model with a step function (Figure 8) to describe the breathing mechanism of the crack (assumption H_2). The partial opening and closing of the crack were not taken into account: the transition from fully open to fully close configuration was considered instantaneously. The main difference with the switching model is that the authors considered the effects of the system vibrations on the opening-closing of the crack. Figure 9 shows the phase diagrams with different breathing mechanisms of the crack. It is obvious that with

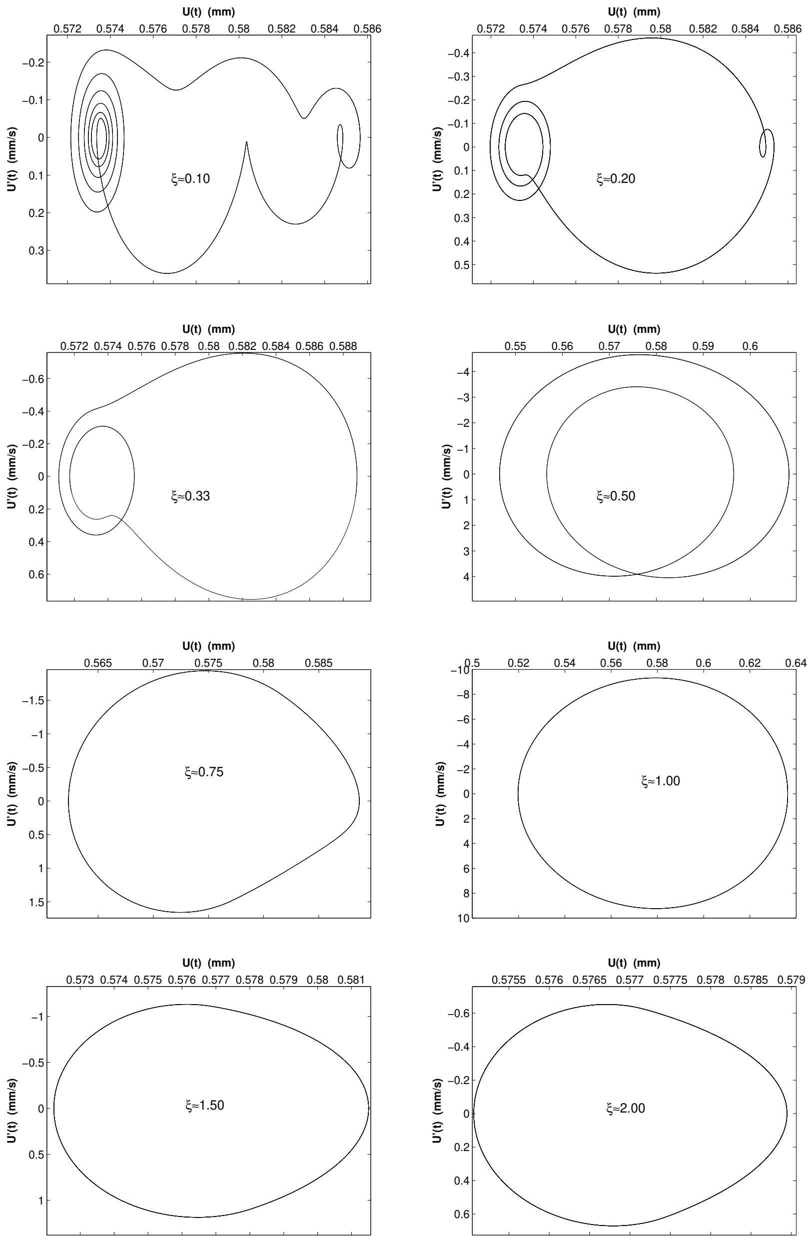


Fig. 5 *Examples of the shaft phase portrait, d=0.05,* $\frac{\Delta}{K_s} = 0.01$

the possible partial opening-closing of the crack, the current model leads to a globally stiffer structure.

4 Stability analysis of the periodic solutions

The dynamic equilibrium equations (6) governing the behavior of the considered system are linear Ordinary Differential Equations (ODE) with periodic coefficients. The

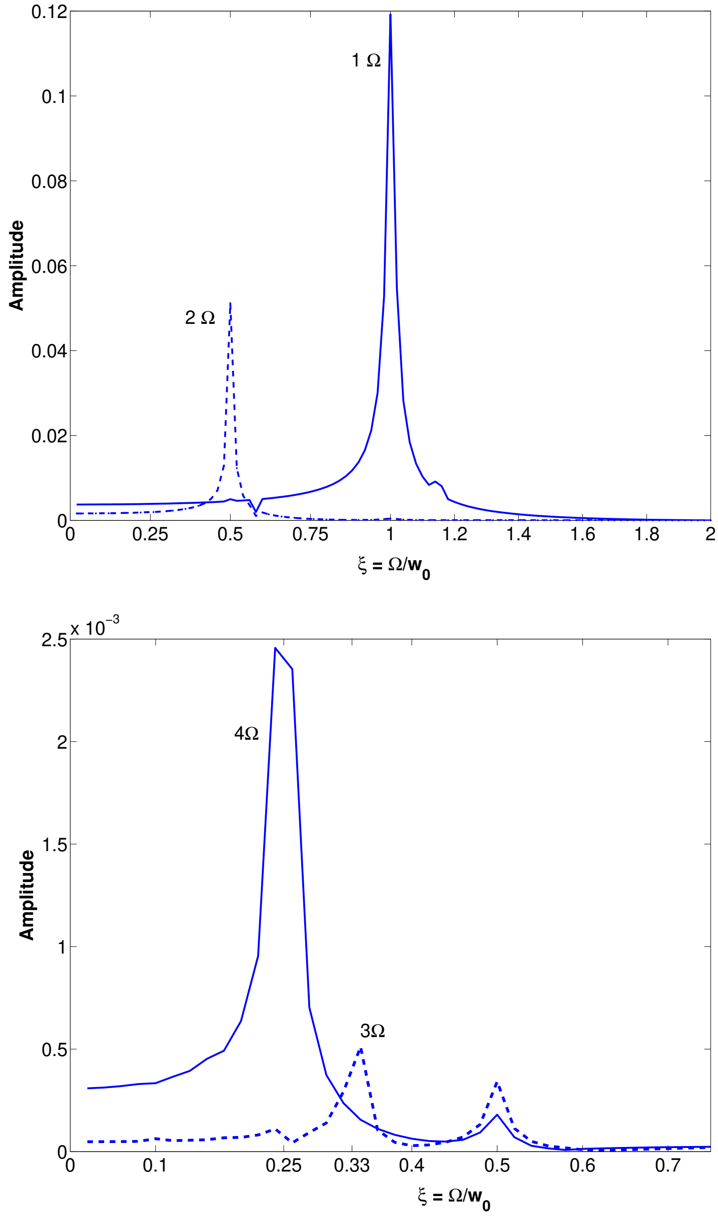


Fig. 6 *Harmonics amplitudes, d=0.01,* $\frac{\Delta}{K_s} = 0.01$

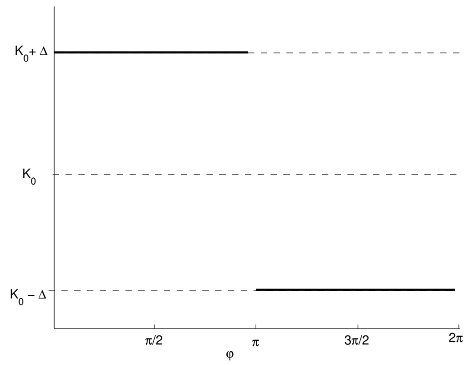


Fig. 7 *Periodic global stiffness function used by El Arem [2006]: Assumption* H_2

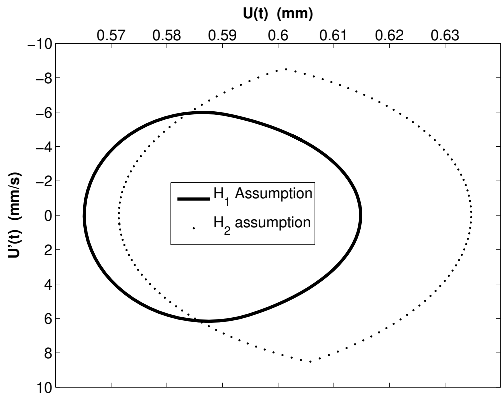
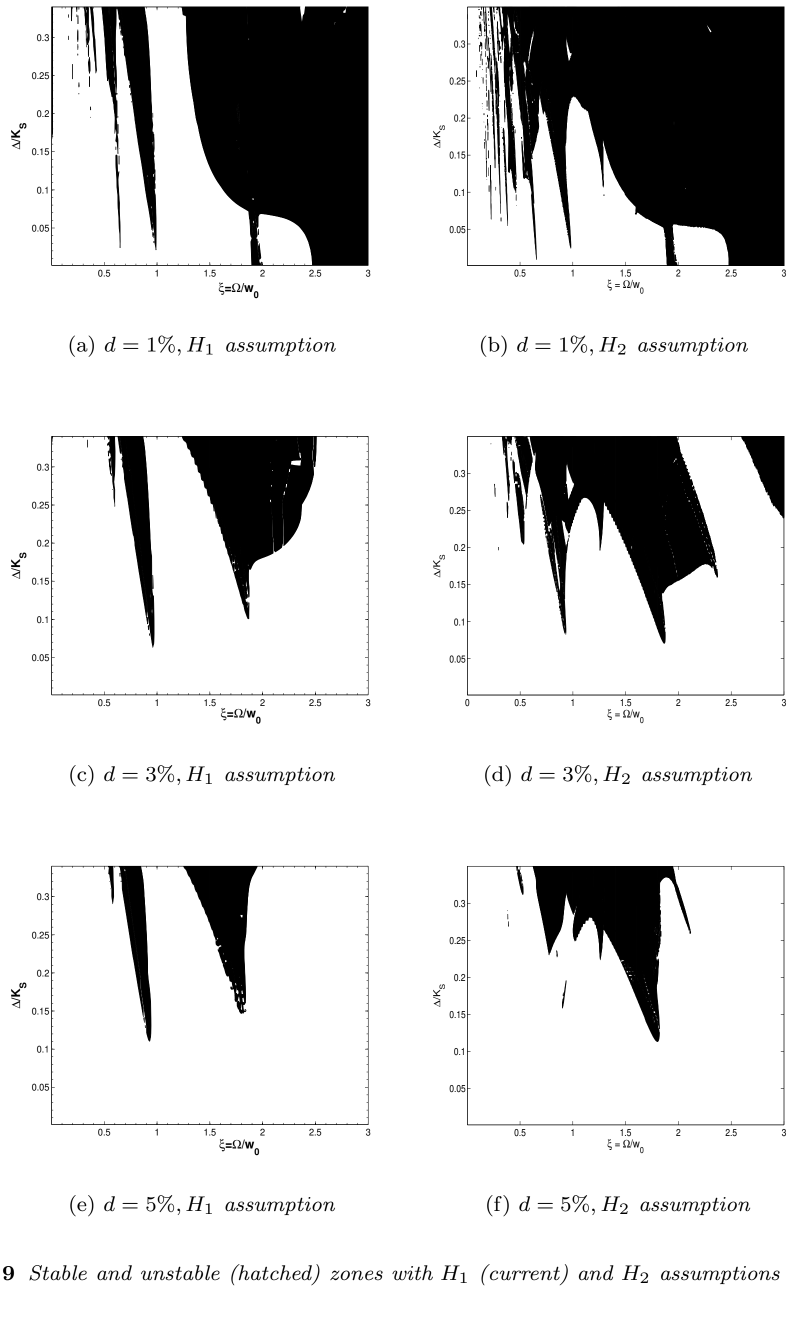


Fig. 8 *Phase portraits comparison,* $\xi = \dfrac{\Omega}{w_0} = 1.50,\ \dfrac{\Delta}{k_0} = 0.05,\ d = 0.03.$

stability analysis of the periodic solutions of this class of ODE have been generally carried out based on Floquet's theory (Nayfeh and Mook [1979]). The HHT scheme is used for the time integration of the system. This direct time-integration method is an elegant way to introduce damping in the Newmark method without degrading the order of accuracy. Moreover, very high sampling frequency is considered by dividing the forcing period $\frac{1}{\Omega}$ into 20 000 integration steps.

Figure 10 shows the stable and unstable (hatched) zones for three different values of the dissipation coefficient d. From Figure 10, we could easily see that a slight damping will raise effectively the instability regions. Mainly, we distinguish three zones of instabilities: the first one corresponds to subcritical excitation frequencies (around $\xi < \frac{2}{3}$). The second one is located around the exact resonance ($\xi \approx 1$). The third zone is located at super-critical rotating speeds close to $\xi \approx 2$.
For lower rotating speeds ($\xi < 0.5$), the instability occurs only for deep cracks and near an integer fraction of the shaft bending frequency. This instability region disap-



(a) $d = 1\%, H_1$ *assumption* (b) $d = 1\%, H_2$ *assumption*

(c) $d = 3\%, H_1$ *assumption* (d) $d = 3\%, H_2$ *assumption*

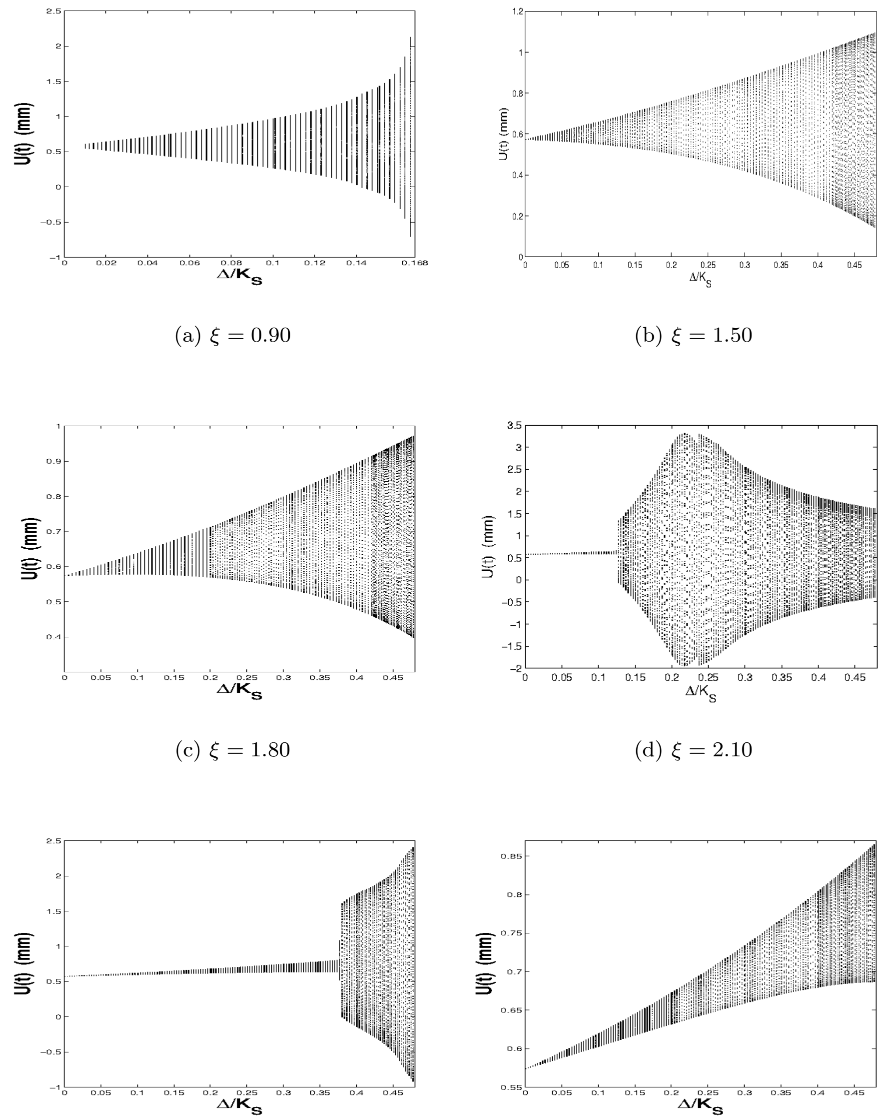
(e) $d = 5\%, H_1$ *assumption* (f) $d = 5\%, H_2$ *assumption*

Fig. 9 *Stable and unstable (hatched) zones with* H_1 *(current) and* H_2 *assumptions*

pear when considering a higher damping coefficients ($d = 3\%$ or 5%): at lower speeds, the shaft could endure larger crack depths, the instability occurs at higher rotating frequencies. Similar instability zones and behavior have been observed by Huang et al.

[1993] who analyzed the stability of a cracked shaft.
Unlike the model of Gasch [1993] which does not foresee instabilities for $\xi > 2$, the current model shows a large zone of instability for this speed range, cf. Figure 10(a). This zone which disappears for high dissipation values, cf. Figure 10(e), has been also noticed by El Arem and Maitournam [2008] in the study of the response of a cracked shaft using a new finite element with a stiffness variation deduced from three-dimensional finite element calculation taking into consideration the unilateral contact conditions between the crack lips. However, real shafts are often exploited at $\xi < 3$, thus we don't focus on the instability zone for $\xi > 3$.
To show the influence of the breathing mechanism of the crack, stability diagrams of the periodic solutions of the same system when considering the stiffness variation of Figure 3 (assumption H_2) are also given (Figures 10(b), 10(d) & 10(f)). Assumption H_2 leads to larger zones of instability.

For the bifurcation analysis, it is important that the response data corresponds to a fully stabilized shaft motion. Hence, for the amplitude spectra plots, data of more than 5 000 rotations is considered.
Figure 11 shows the bifurcation diagrams for six different rotating frequencies. In all cases, the first 3 000 rotations have been discarded to consider fully stabilized shaft motion only. For a given rotating frequency, the vertical displacement $U(t)$ is presented versus the crack depth. It appears that the system response remains bounded for sub- and super-critical excitations even after the loss of stability of the periodic solution.
According to Figure 10(c), at $\xi \approx 0.90$, the periodic solution becomes unstable at $\frac{\Delta}{K_s} \approx 0.10$ where one of the Floquet matrix eigenvalues leaves the unit circle through 1. The system response remains bounded (Figure 11(a)) and periodic of period one (Figure 12) till $\frac{\Delta}{K_s} \approx 0.16$. Thereafter, the response is unbounded and two eigenvalues leave the unit circle through 1.
At $\xi \approx 1.50$, 1.80, and 3.0 the vibration amplitude increases with the crack depth but the system response remains bounded (cf. Figures 11(b), 11(c) & 11(f)) and periodic of period $T = \frac{1}{\Omega}$ even for very deep cracks (cf. Figure 13).
At $\xi \approx 2.10$, the vibration amplitude remains bounded all the time (Figure 11(d)). The dynamical response is T-periodic till $\frac{\Delta}{K_s} \approx 0.1274$ where it becomes $2T$-periodic as it could be deduced from the two points Poincaré section of Figure 14. The cracked shaft is running through a subharmonic resonance: the period doubling phenomena that is observed here and which is a known route to chaotic behavior. This phenomena appears only when $\frac{\Delta}{K_s} \approx 0.38$ when the shaft is rotating at $\xi \approx 2.50$ (Figure 11(e)). Changes in the response periodicity when varying the cracked shaft speed have also been noticed by Patel and Darpe [2008], Chen and Dai [2007], Foong et al. [2003].
In their exploration of the influence of the crack model on the nonlinear dynamics of a cracked shaft, Patel and Darpe [2008] have found that in the subcritical speed range, chaotic, quasi-periodic and subharmonic responses are completely absent when considering a breathing crack model. These phenomena have been noticed only with a switching crack model and might be due to the instantaneous stiffness change when the crack opens and closes. Figures 10(c) & 10(e) show that for subcritical rotating frequencies (with $d = 3\%$ or 5%), the response of the shaft is T-periodic and stable even for very deep cracks which confirms the observations of Patel and Darpe [2008].
An unobvious phenomena could noticed in Figure 11(d): the vibration amplitude increases with the crack depth till $\frac{\Delta}{K_s} \approx 0.22$, then decreases gradually for deeper cracks. In fact, around this crack depth, the rotating frequency (Ω) is very close to twice the natural frequency of the cracked shaft (w): ($\Omega \approx 2w$). The subharmonic resonance



(a) $\xi = 0.90$

(b) $\xi = 1.50$

(c) $\xi = 1.80$

(d) $\xi = 2.10$

(e) $\xi = 2.50$

(f) $\xi = 3.00$

Fig. 10 *Bifurcation diagrams, $d = 3\%$*

taking place leads to higher vibration amplitude values.
It is obvious to notice, given this results, that a stable shaft rotating at $\xi > 1.50$ could collapse when slowing down to stop the machine for example since the speed will go through critical values ($\xi \approx 0.90$). The same shaft stable at subcritical excitations will

also collapse while speeding up if its lost stiffness is that $\frac{\Delta}{K_S} > 0.16$. Consequently, for cracked shafts, the passage near the exact resonance ($\xi \approx 1$) should be made with a very selective attention to avoid the machine collapse.

We have also examined the stability and routes to chaos of the current cracked shaft system with the stiffness function of Figure 8. We have observed chaotic features that might be due to the rough variations of the stiffness matrix terms that take place for very deep cracks. The same observation were made by Patel and Darpe [2008]. We think that theses features could not be observed with real shafts and that a more realistic breathing mechanism needs to be considered for deep cracks which motivated the present work.

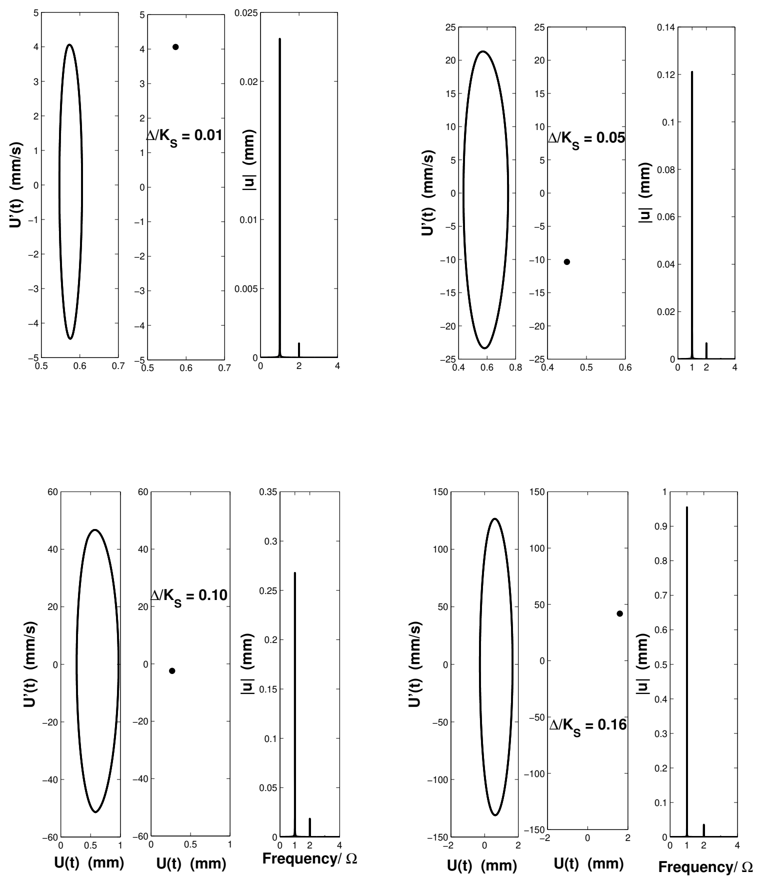


Fig. 11 *Shaft response: phase portraits, Poincaré sections, amplitude spectra,* $\xi = 0.90$, $d = 3\%$

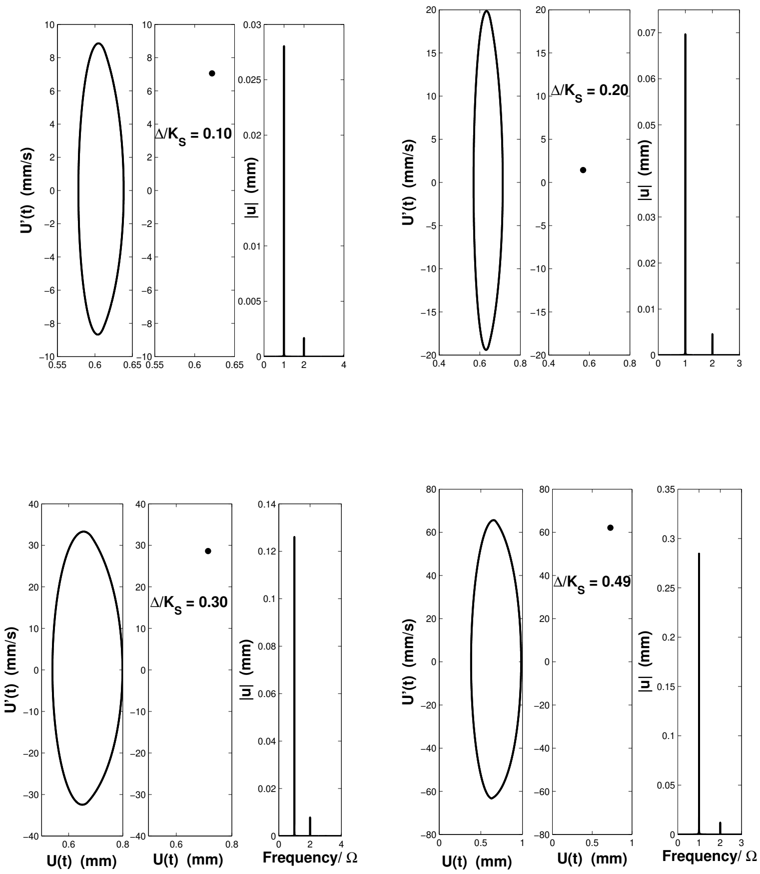


Fig. 12 *Shaft response: phase portraits, Poincaré sections, amplitude spectra,* $\xi = 1.80$, $d = 3\%$

5 Conclusions

In this paper, an elegant, simple and comprehensive model is suggested for the nonlinear dynamics and stability analysis of a cracked rotating shaft. It is known that the gap between the lips of a fatigue-induced crack are very small and that closure of the crack occurs when the shaft rotates. Accordingly, the breathing mechanism of the crack is taken into account in this article by considering a more realistic function describing the periodic variation of the global stiffness of the shaft. This new approach which was validated experimentally at the Rotating Machinery Department of Electritié De France (EDF), is based on an energy formulation of the problem of a breathing crack with unilateral contact conditions between the lips. In particular, a partial opening−closing of the crack in both directions is allowed since a switching crack model is not adequate to examine the stability of shafts with deep cracks. Moreover, the breathing mechanism

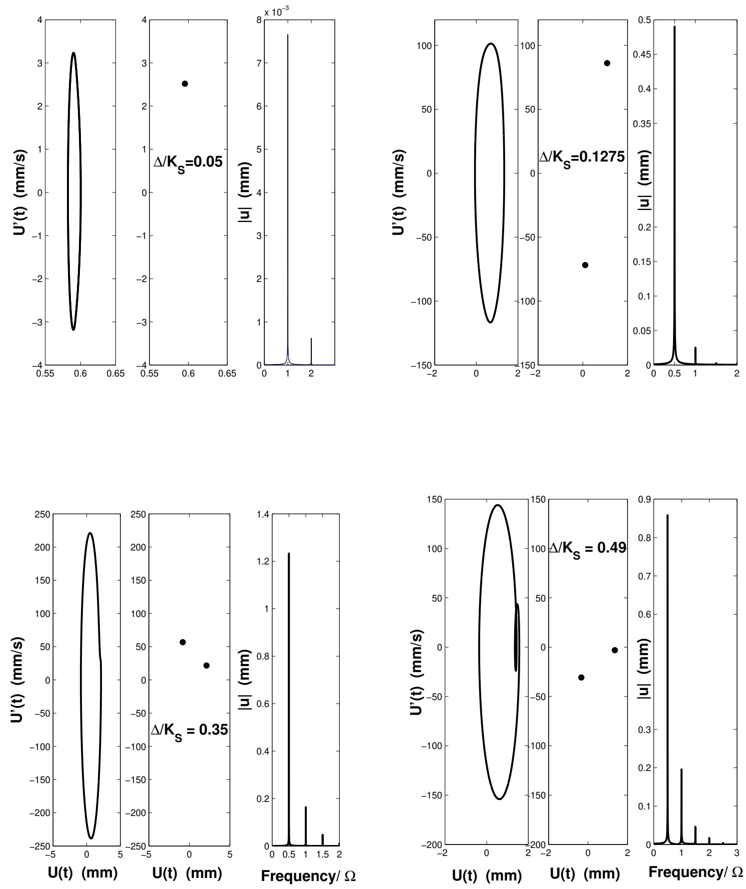


Fig. 13 *Shaft response: phase portraits, Poincaré sections, amplitude spectra,* $\xi = 2.10$, $d = 3\%$

is depending not only but substantially on the shaft dynamics response and, thus, the obtained equilibrium equations are nonlinear. The representation of the cracked section is original, quite simple and its parameters could be easily identified for application of this approach to real machines. It was found, as well-established in the literature, that the breathing crack induces higher harmonics in the vibratory response of the cracked shaft. For super-critical excitations, the increase of the static deflection and the vibratory amplitude could be used as cracks presence indicators as proposed by El Arem [2006], El Arem and Nguyen [2006]. The stability analysis was carried out using the Floquet theory. Mainly, three zones of instabilities were found: the first one corresponds to subcritical excitation frequencies (around $\xi < \frac{2}{3}$). The second one is located around the exact resonance ($\xi \approx 1$). The third zone is located at super-critical rotating speeds close to $\xi \approx 2$. It is important to notice that the vibratory response of the system remains bounded for sub- and super-critical excitations even for very high loss of global stiffness (very deep cracks). However, near the principal resonance speed, the response becomes unbounded as soon as the system periodic solutions leave the stable zones. We have also shown the effects of the breathing mechanism on the overall vibratory behavior of the shaft and its stability. As it could be inferred from the bifurcation diagrams and Poincaré sections, the system considered in this paper never exhibits chaotic or quasi-periodic response.

Additional work is required for deeper exploration of this simple mechanical system and to establish quantitative results and diagrams that could be useful for engineers in power stations industry.

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