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NONLINEAR EFFECTS IN ELECTROHYDRODYNAMIC SURFACE WAVE PROPAGATION*

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The author has recently made a number of mathematical studies of the effects of electrostatic surface stresses on the equilibrium of menisci and on wave motions at the surface of highly conducting fluids. One aspect of such problems which is of interest is that electrostatic boundary conditions can be applied in different ways, and one may contrast the effects on wave motion when electric fields are produced by insulated charged conductors or by conductors with maintained electrical potentials. A discussion of the problem for linearized theory has recently been given by the author [1], and so far as small-amplitude waves are concerned, there is no difference between these two cases. However, it seems inconceivable that such a similarity could extend into wave motions which are not small, and the following work takes the analysis of plane wave motion to a higher order, with a view to showing how the differing electrostatic conditions do produce different effects.

In Sec. 2 a discussion is given for waves of permanent form, which shows that the behavior of the electric field begins to differ in these two cases in the second-order effects, and that the difference is reflected in different phase velocities. The other features of the wave motion, such as the surface elevation, are indistinguishable at this stage of approximation, and it seems that differences in these features only begin to appear in third- or higher-order approximations.

In Sec. 3 a similar analysis is given for unstable standing waves. Similar second-order differences appear in the electric field, but the effect is now simply to change the mean level of the fluid pressure. The growth rate and the surface elevation are indistinguishable in the second-order analysis.

2. Propagation of progressive waves without change of form. We consider the propagation of waves on a conducting fluid of height a with a conducting plate at a distance b above the surface. The undisturbed free surface of the fluid is at $z = 0$, the conducting plate at $z = b$, and the fluid is supported on a conducting plate at $z = -a$. In the first place let the upper plate be maintained at a fixed potential $V = V_0$ above the fluid for which $V = 0$.

We study a plane progressive wave of wave length $2\pi/k$ travelling without change of form in the horizontal x direction with phase velocity U. We shall refer the analysis to the Newtonian frame moving with speed U relative to the boundaries, in which the wave disturbance is steady, and steady-state equations apply.

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Let the electrostatic potential in the air space be ϕ and the velocity potential in the fluid Ω. The surface elevation $z = \eta(x)$ due to waves will be a periodic function of wavelength $2\pi/k$, having components of the form

$$\left.\begin{matrix}\sin\\ \cos\end{matrix}\right| mkx, \quad \text{where } m \text{ is an integer,}$$

but no constant terms since the mean level of the surface cannot change when the fluid is incompressible.

In terms of a small parameter ϵ which we shall identify later, we write

$$\phi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \cdots, \tag{1}$$

$$\eta = \epsilon\eta_1 + \epsilon^2\eta_2 + \cdots, \tag{2}$$

$$\Omega = \Omega_0 + \epsilon\Omega_1 + \epsilon^2\Omega_2 + \cdots, \tag{3}$$

where $\phi_0 = V_0 z/b$, the undisturbed electrostatic potential.

In the chosen frame of reference the fluid has a mean velocity U in the x direction, where

$$U = U_0 + \epsilon U_1 + \epsilon^2 U_2 + \cdots \tag{4}$$

with the notation that the fluid velocity $\mathbf{v} = \text{grad}\ \Omega$, $\Omega_0 = -U_0 x$.

The solution of the problem to all orders in ϵ will satisfy the following conditions:

(i) $$\int_0^{2\pi/k} \eta\, dx = 0,$$

(ii) $$\nabla^2\phi = 0,$$

(iii) $$\nabla^2\Omega = 0,$$

(iv) $\partial\Omega/\partial z = 0$ at $z = -a$, implying that $\partial\Omega_1/\partial z = \partial\Omega_2/\partial z = \partial\Omega_3/\partial z = \cdots = 0$ at $z = -a$.

(v) For steadiness $\partial\Omega/\partial n$, representing the normal velocity of the fluid, is zero at $z = \eta$.

(vi) The normal stress is continuous at the free surface.

The electrostatic boundary conditions may be of different forms. In (a) and (b) below we consider two cases in turn, the first in which the potentials of the conducting surfaces are kept fixed, and the second when the charge is kept fixed.

(a) *Oscillations at fixed potentials.* Here the electrostatic condition will be

(vii) $\phi = \phi_0$ at $z = b$, which implies that $\phi_1 = \phi_2 = \phi_3 = \cdots = 0$ at $z = b$.

(viii) $\phi = 0$ at $z = \eta$.

Starting with condition (v), if ψ is the inclination of the surface to the horizontal, $d\eta/dx = \tan\psi$, and the condition is

$$\frac{\partial\Omega}{\partial x}\frac{d\eta}{dx} - \frac{\partial\Omega}{\partial z} = 0. \tag{5}$$

The condition is exact provided we take $\partial\Omega/\partial x$ and $\partial\Omega/\partial z$ at $z = \eta$. To express it correctly in powers of ϵ we write

$$\left\{\left(\frac{\partial \Omega}{\partial x}\right)_0 + \eta \frac{\partial}{\partial z}\left(\frac{\partial \Omega}{\partial x}\right)_0 + \frac{\eta^2}{2!}\frac{\partial^2}{\partial z^2}\left(\frac{\partial \Omega}{\partial x}\right)_0 + \cdots\right\} \frac{\partial \eta}{\partial x} - \left\{\left(\frac{\partial \Omega}{\partial z}\right)_0 + \eta\left(\frac{\partial^2 \Omega}{\partial z^2}\right)_0 + \frac{\eta^2}{2!}\left(\frac{\partial^3 \Omega}{\partial z^3}\right)_0 + \cdots\right\} = 0,$$

in which the suffix 0 denotes that the derivatives of Ω refer to $z = 0$. It is necessary to substitute for η from (2) and equate to zero the coefficients of successive powers of ϵ. We find these conditions, up to the terms in ϵ^2, to be

$$\epsilon_1{}^0 : \quad \frac{\partial \Omega_0}{\partial z} = 0, \tag{6}$$

$$\epsilon_1{}^1 : \quad \left(\frac{\partial \Omega_0}{\partial x}\right)\frac{d\eta_1}{dx} - \frac{\partial \Omega_1}{\partial z} - \eta_1 \frac{\partial^2 \Omega_0}{\partial z^2} = 0, \tag{7}$$

$$\epsilon_1{}^2 : \quad \left(\frac{\partial \Omega_1}{\partial x}\right)\frac{d\eta_1}{dx} + \left(\frac{\partial \Omega_0}{\partial x}\right)\frac{d\eta_2}{dx} - \frac{\partial \Omega_2}{\partial z} - \eta_1 \frac{\partial^2 \Omega_1}{\partial z^2} = 0, \tag{8}$$

where it is now implied that the derivatives of Ω, and later of ϕ, refer to $z = 0$ without writing in the suffix 0 throughout.

For condition (viii) we have

$$\phi_{z=\eta} = \phi_0 + \eta\left(\frac{\partial \phi}{\partial z}\right)_0 + \frac{\eta^2}{2!}\left(\frac{\partial^2 \phi}{\partial z^2}\right)_0 + \cdots,$$

and using (1) and (2) we find the following conditions up to the order ϵ^2:

$$\epsilon^0 : \quad \phi_0 = 0, \tag{9}$$

$$\epsilon_1{}^1 : \quad \phi_1 + \eta_1 \frac{\partial \phi_0}{\partial z} = 0, \tag{10}$$

$$\epsilon_1{}^2 : \quad \phi_2 + \eta_2 \frac{\partial \phi_0}{\partial z} + \eta_1 \frac{\partial \phi_1}{\partial z} = 0. \tag{11}$$

In condition (vi) the dynamic pressure p in the fluid is given by Bernoulli's equation

$$\rho + \frac{1}{2} p(\text{grad } \phi)^2 = \text{constant},$$

where ρ is the density. Thus, allowing for surface tension, electrical, and gravitational contributions to the stress we have the condition

$$\frac{1}{8\pi}\left(\frac{\partial \phi}{\partial n}\right)^2 + T\frac{\dfrac{d^2\eta}{dx^2}}{\left\{1 + \left(\dfrac{d\eta}{dx}\right)^2\right\}^{3/2}} - \rho g \eta - \frac{\rho}{2}(\text{grad } \Omega)^2 = \text{constant},$$

at $z = \eta$. This condition is to be treated as shown previously. The algebra is lengthy and will be omitted here. The results are as follows, up to terms of order ϵ^2:

$$\epsilon_1{}^0 : \frac{1}{8\pi}\left(\frac{\partial \phi_0}{\partial z}\right)^2 - \frac{\rho}{2}\left(\frac{\partial \Omega_0}{\partial x}\right)^2 = \text{constant}, \tag{12}$$

$$\epsilon_1{}^1 : \frac{1}{4\pi}\frac{\partial \phi_0}{\partial z}\frac{\partial \phi_1}{\partial z} + T\frac{d^2\eta_1}{dx^2} - \rho g \eta_1 - \rho\left(\frac{\partial \Omega_0}{\partial x}\right)\left(\frac{\partial \Omega_1}{\partial x}\right) = 0, \tag{13}$$

$$\epsilon^2: \frac{1}{8\pi}\left\{\left(\frac{\partial\phi_1}{\partial z}\right)^2 + 2\frac{\partial\phi_0}{\partial z}\left(\frac{\partial\phi_2}{\partial z} + \eta_1\frac{\partial^2\phi_1}{\partial z^2} - \frac{d\eta_1}{dx}\frac{\partial\phi_1}{\partial x}\right) - \left(\frac{\partial\phi_0}{\partial z}\right)^2\left(\frac{d\eta_1}{dx}\right)^2\right\}$$

$$+ T\frac{d^2\eta_2}{dx^2} - \rho g\eta_2 - \frac{\rho}{2}\left\{\left(\frac{\partial\Omega_1}{\partial x}\right)^2 + \left(\frac{\partial\Omega_1}{\partial z}\right)^2 + 2\frac{\partial\Omega_0}{\partial x}\left(\frac{\partial\Omega_2}{\partial x} + \eta_1\frac{\partial^2\Omega_1}{\partial z\,\partial x}\right)\right\} = 0. \quad (14)$$

The solution to order ϵ^0 is the undisturbed state given by $\Omega_0 = -U_0x$, $\phi_0 = V_0z/b$. For ϵ^1 terms we designate

$$\eta_1 = \cos kx. \quad (15)$$

This implies that ϵ has the dimension of length, being the amplitude of this component of the surface elevation. Condition (11) is then $\phi_1 = (-V_0/b)\cos kx$ at $z = 0$. Hence

$$\phi_1 = \frac{V_0}{b}\frac{\sinh k(z-b)\cos kx}{\sinh kb}. \quad (16)$$

Also (7) gives $\partial\Omega_1/\partial z = kU_0\sin kx$ at $z = 0$, and the solution for Ω_1 becomes

$$\Omega_1 = U_0\frac{\sin kx\cosh k(z+a)}{\sinh ka}. \quad (17)$$

The remaining condition (13) then shows that

$$\rho k\, U_0^2\coth ka - Tk^2 - \rho g + \frac{V_0^2k}{4\pi b^2}\coth kb = 0, \quad (18)$$

which gives us the value of U_0. A term of the form U_1x could have been added to Ω_1 in (17), but then (13) requires $U_1 = 0$.

We turn now to the conditions of order ϵ^2. Using (15), (16), (17), we find that conditions (8) and (11) are, respectively,

$$\frac{\partial\Omega_2}{\partial z} + U_0\frac{d\eta_2}{dx} + k^2U_0\coth ka\sin 2kx = 0, \quad (19)$$

$$\phi_2 + \frac{V_0\eta_2}{b} + \frac{V_0k}{2b}\coth kb\,(1 + \cos zkx) = 0. \quad (20)$$

After some reduction (14) becomes

$$\rho U_0\frac{\partial\Omega_2}{\partial x^1} + T\frac{d^2\eta_2}{dx^2} - \rho g\eta_2 + \frac{V_0}{4\pi b}\frac{\partial\phi_2}{\partial z} = l + m\cos 2kx, \quad (21)$$

where

$$l = \left(\frac{\rho k^2U_0^2}{4} - \frac{V_0^2k^2}{16\pi b^2}\right)(\coth^2 ka - 1),$$

and

$$m = \left(\frac{\rho k^2U_0^2}{4} - \frac{V_0^2k^2}{16\pi b^2}\right)(\coth^2 ka - 3).$$

Since η_2 can have no mean part we write

$$\eta_2 = r\cos 2kx + s\sin 2kx. \quad (22)$$

Also let

$$\left.\begin{aligned}\phi_2 &= \alpha + \beta \cos 2kx + \gamma \sin 2\,kx \\ \Omega_2 &= \lambda x + \mu \cos 2kn + \nu \sin 2kx\end{aligned}\right\} \text{ at } z = 0.$$

Solutions for ϕ_2 and Ω_2 are then

$$\phi_2 = \alpha\left(1 - \frac{z}{b}\right) - \frac{\sinh 2k(z - b)}{\sinh 2\,kb}\{\beta \cos 2kx + \gamma \sin 2kx\}, \tag{23}$$

$$\Omega_2 = \lambda x + \frac{\cosh 2k(z + a)}{\cosh 2ka}(\mu \cos 2kx + \gamma \sin 2kx). \tag{24}$$

Eqs. (22), (23), (24) can then be used to satisfy (19), (20) and (21) at $z = 0$, by ensuring that the coefficients of sin $2kx$, cos $2kx$, and the constant terms are zero in each. Eq. (19) gives

$$2U_0 r - 2\gamma \tanh 2ka = U_0\, k \coth ka, \tag{25}$$

$$2U_0 s + 2\mu \tanh 2ka = 0. \tag{26}$$

From (20), we find

$$\alpha = -\frac{V_0 k}{2b}\coth kb, \tag{27}$$

$$\beta + \frac{V_0 r}{b} + \frac{V_0 k}{2b}\coth kb = 0, \tag{28}$$

$$\gamma + \frac{V_0 s}{b} = 0. \tag{29}$$

Eq. (21) is satisfied when

$$\rho U_0 \lambda - \frac{V_0 \alpha}{4\pi b^2} = l, \tag{30}$$

$$2k\rho U_0 \mu + 4k^2 Ts + \rho gs - \frac{2kV_0\gamma}{4\pi b}\coth 2kb = 0, \tag{31}$$

$$2k\rho\, U_0 \nu + 4k^2 Tr - \rho gr - 2kV_0\beta\,\frac{\coth 2kb}{4\pi b} = m. \tag{32}$$

Eqs. (25)–(32) may now be solved for the unknown coefficients α, β, γ, λ, μ, ν, r and s. The coefficients of most interest are those which give rise to the mean changes to this order, namely α and λ. Eq. (23) shows that there is a second-order change in the mean electrostatic field given by

$$\phi_2 = -\frac{V_0 k \coth kb}{2b}\left(1 - \frac{z}{b}\right). \tag{33}$$

Thus to the order ϵ^2 the mean electric field is raised to

$$\frac{V_0}{b}\left\{1 + \frac{\epsilon^2 k}{2b}\coth kb\right\},$$

and this is accompanied by additional charge on the upper plate of amount

$$\frac{V_0 k \epsilon^2}{8\pi b^2} \coth kb.$$

The coefficient λ measures the second-order change in the phase velocity of the wave. We have

$$\rho U_0 \lambda = l + \frac{V_0}{4\pi b^2}\alpha = \left(\frac{\rho k^2 U_0^2}{4} - \frac{V_0^2 k^2}{16\pi b^2}\right)\operatorname{cosech}^2 ka - \frac{V_0^2 k \coth kb}{8\pi b^3}.$$

This gives rise to an additional phase velocity $-\partial\Omega_2/\partial x = -\lambda$, so that to order ϵ^2 the phase velocity of the wave is

$$U = U_0 - \frac{\epsilon^2}{\rho U_0}\left\{\left(\frac{\rho k^2 U_0^2}{4} - \frac{V_0^2 k^2}{16\pi b^2}\right)\operatorname{cosech}^2 ka - \frac{V_0^2 k}{8\pi b^3}\coth kb\right\}. \tag{34}$$

When $V_0 = 0$ this reduces to the Stokes [2] result, and it is noticeable that the electrostatic effect is to increase this speed in the ϵ^2 terms.

(b) *Charge-maintained oscillations.* The above analysis shows that when the potentials of the conducting surfaces are fixed a change in the mean charge density of order ϵ^2 occurs. If the conductors are insulated this change cannot occur, and we might then expect that there will occur a change of order ϵ^2 occurring the potential difference between the fluid and the plate instead.

When the conducting plate at $z = b$ is insulated, conditions (vii) and (viii) must now be replaced so that

(ix) $\phi_1 = c_1, \quad \phi_2 = c_2, \quad \phi_3 = c_3, \cdots \quad$ at $\quad z = b,$

where $c_1, c_2, c_3 \cdots$ are constants. This represents the condition that the potential at the plate is constant, but not necessarily unchanged in the wave motion. Further, $c_1, c_2, c_3 \cdots$ must be such that the total charge per wavelength is unchanged to all orders in ϵ.

(x) Similarly, we have ϕ = constant at $z = \eta$, with the same condition of invariance on the total charge per wavelength.

We shall trace through the changes in the previous analysis when (ix) and (x) apply instead of (vii) and (viii), without repeating all the steps in full.

The boundary conditions on ϕ_1 now become $\phi_1 = c_1$ at $z = b$, and, instead of (10) $\phi_1 + \eta_1(\partial\phi_0/\partial z) = d_1$, say, at $z = 0$, where d_1 is a constant. Evidently if $d_1 \neq c_1$, a uniform electric field of order ϵ would be present, which would give a change in the mean surface charge density of order ϵ, which is inadmissible. Hence $d_1 = c_1$ and we may without loss of generality let $c_1 = d_1 = 0$ on the basis that the fluid potential remains unchanged. Assuming η_1 again as in (15), this leaves the solution (16) for ϕ_1 unchanged. Eq. (17) for Ω_1 is also unchanged.

As boundary conditions for ϕ_2 we now have $\phi_2 = c_2$ at $z = b$, and $\phi_2 + \eta_2(\partial\phi_0/\partial z) + \eta_1(\partial\phi_1/\partial z) = d_2$ at $z = 0$ instead of (11), where d_2 is another constant. Condition (19) remains unchanged, but for (20) we now have

$$\phi_2 + \frac{V_0\eta_2}{b} + \frac{V_0 k}{2b}\coth kb(1 + \cos 2kx) = d_2 \quad \text{at} \quad z = 0.$$

Eqs. (21) and (22) are unaltered, so now

$$\phi_2 = d_2 - \frac{V_0}{b}(r\cos 2kx + s\sin 2kx) - \frac{V_0 k}{2b}\coth kb(1+\cos 2kx) \quad \text{at} \quad z = 0.$$

No change in the mean value of ϕ_2 can occur between $z = 0$ and $z = b$ in this case, since there can be no change in mean charge density. Hence

$$d_2 - \frac{V_0 k}{2b}\coth kb = c_2 .$$

With the fluid at earthed potential, we take $d_2 = 0$ and $c_2 = (-V_0 k/2b)\coth kb$. Thus

$$\phi_2 = -\frac{V_0 k}{2b}\coth kb - \frac{\sinh 2k(z-b)}{\sinh 2kb}\{\beta\cos 2kx + \gamma\sin 2kx\}, \tag{35}$$

which shows a mean loss in potential difference $(\epsilon^2 V_0 k/2b)\coth kb$, to the order ϵ^2. The solution for Ω_2 given by (24) is unaltered, and (22), (24) and (35) must now be used to satisfy the conditions (19), (20) and (21) at $z = 0$. We find that Eqs. (25), (26), (28), (29), (31) and (32) are unaltered. But (30) is changed because $\partial\phi_2/\partial z$ now has no constant component, so that now

$$\rho U_0 \lambda = l = \frac{1}{\sinh^2 ka}\left(\frac{\rho k^2 U_0^2}{4} - \frac{V_0^2 k^2}{16\pi b^2}\right). \tag{36}$$

The second-order contribution to the phase velocity is therefore lower, and we have

$$U = U_0 - \frac{\epsilon^2}{\rho U_0}\left(\frac{\rho k^2 U_0^2}{4} - \frac{V_0^2 k^2}{16\pi b^2}\right)\operatorname{cosech}^2 ka, \quad \text{to order } \epsilon^2.$$

We note that, otherwise, the solution for β, γ, μ, ν, r, s remains the same as in case (a), so that, for example, there is no distinction at this order between the surface elevation or the motion of the fluid in the wave, as seen relative to the rest frame in each case.

3. Unstable standing waves. When the phase velocity U_0 is imaginary, wave propagation without change of form becomes impossible, and progressive waves with phase of the form $\exp i(kx \pm \omega t)$ change to the form $\exp(ikx \pm \sigma t)$ where σ is real. The most natural form of wave function to study in this case is of the form $f(x)\exp(ct)$, where again $f(x)$ is a periodic function with wavelength $2\pi/k$. The approach to this problem must change because there will not exist a frame of reference in which the motion is steady. There is then no advantage in changing the frame of reference from the natural one in which the fluid is at rest in the absence of the wave. However, there is the complication in this case that the wave motion is now unsteady. As in Sec. 2, we shall consider the growth of the wave in the different electrostatic conditions, but without repeating steps which are the same as in Sec. 2.

We may again write expansions for ϕ, η as (1) and (2), with $\phi_0 = V_0 z/b$, but (3) we now write as

$$\Omega = \epsilon\Omega_1 + \epsilon^2\Omega_2 + \epsilon^3\Omega_3 + \cdots . \tag{37}$$

The terms in Ω are now unsteady.

Conditions (i), (ii), (iii), (iv) and (vi) remain, but (v) now becomes

(xi) $\left(\dfrac{\partial\Omega}{\partial n}\right)_{z=y}$ = normal velocity of the surface.

As in Sec. 2 we make a distinction between the electrostatic conditions under subsections (a) and (b).

(a) *Waves at fixed potentials.* Conditions (vii) and (viii) will apply in this case. Condition (xi) may be written

$$\left(\frac{\partial \Omega}{\partial z} \cos \psi - \frac{\partial \Omega}{\partial x} \sin \psi\right) = \frac{\partial \eta}{\partial t} \cos \psi,$$

or

$$\left(\frac{\partial \Omega}{\partial z} - \frac{\partial \zeta}{\partial x}\frac{\partial \Omega}{\partial x}\right) = \frac{\partial \eta}{\partial t} .$$

Now we suppose that the wave growth occurs on a time scale c where

$$c = c_0 + \epsilon c_1 + \epsilon^2 c_2 + \cdots, \tag{38}$$

and write $t = c\tau$ where $\partial/\partial\tau \sim 1$. Then

$$(c_0 + \epsilon c_1 + \epsilon^2 c_2 + \cdots)\left(\frac{\partial \Omega}{\partial z} - \frac{\partial \eta}{\partial x}\frac{\partial \Omega}{\partial x}\right) = \frac{\partial \eta}{\partial t} \quad \text{at} \quad z = \eta .$$

To order ϵ^2 this becomes

$$\epsilon^1: \qquad c_0\left(\frac{\partial \Omega_1}{\partial z}\right) = \frac{\partial \eta_1}{\partial \tau} \tag{39}$$

$$\epsilon^2: \qquad c_1\frac{\partial \Omega_1}{\partial z} + c_0\left(\frac{\partial \Omega_2}{\partial z} + \eta_1\frac{\partial^2 \Omega_1}{\partial z_2} - \frac{\partial \Omega_1}{\partial x}\frac{\partial \eta_1}{\partial x}\right) = \frac{\partial \eta_2}{\partial \tau} . \tag{40}$$

Conditions (39), (40) replace (7) and (8). The equations (9), (10) and (11) for condition (viii) remain unchanged. But the stress condition (vi) now takes a different form because there is a contribution to the pressure from the unsteadiness of the fluid motion. We have

$$p = f(t) - \tfrac{1}{2}\rho(\text{grad } \Omega)^2 - \rho(\partial\Omega/\partial t).$$

When this is put in the surface stress condition, and when we allow for the expansion of c in powers of ϵ_1, we find to order ϵ^2:

$$\epsilon^0: \qquad \frac{1}{8\pi}\left(\frac{\partial \phi_0}{\partial z}\right)^2 = \text{constant} \tag{41}$$

$$\epsilon^1: \qquad c_0\left(T\frac{\partial^2 \eta_1}{\partial x^2} - \rho g \eta_1 + \frac{1}{4\pi}\frac{\partial \phi_0}{\partial z}\frac{\partial \phi_1}{\partial z}\right) - \rho\frac{\partial \Omega_1}{\partial \tau} = 0, \tag{42}$$

$$\begin{aligned} \epsilon^2: \quad & c_0\left\{T\frac{\partial^2 \eta_2}{\partial x^2} - \rho g \eta_2 - \frac{\rho}{2}\left\{\left(\frac{\partial \Omega_1}{\partial x}\right)^2 + \left(\frac{\partial \Omega_1}{\partial z}\right)^2\right\}\right. \\ & \left. + \frac{1}{8\pi}\left\{\left(\frac{\partial \phi_1}{\partial z}\right)^2 + 2\frac{\partial \phi_0}{\partial z}\left(\frac{\partial \phi_2}{\partial z} + \eta_1\frac{\partial^2 \phi_1}{\partial z^2} - \frac{\partial \eta_1}{\partial x}\frac{\partial \phi_1}{\partial x}\right) - \left(\frac{\partial \phi_0}{\partial z}\right)^2\left(\frac{\partial \eta_1}{\partial x}\right)^2\right\}\right\} \\ & + c_1\left\{T\frac{\partial \eta_1}{\partial x^2} - \rho g \eta_1 + \frac{1}{4\pi}\frac{\partial \phi_0}{\partial z}\frac{\partial \phi_1}{\partial z}\right\} - \rho\frac{\partial \Omega_2}{\partial \tau} - \rho\eta_1\frac{\partial^2 \Omega_1}{\partial z\, \partial \tau} = 0. \end{aligned} \tag{43}$$

Eqs. (41), (42) and (43) replace (12), (13) and (14).

For the first-order solution let

$$\eta_1 = g(\tau) \cos kx. \tag{44}$$

It follows from (10) that $\phi_1 = (-V_0/b)\, g(\tau) \cos kx$ at $z = 0$. Thus

$$\phi_1 = \frac{V_0}{b} g(\tau) \frac{\sinh k(z-b)}{\sinh kb} \cos kx. \tag{45}$$

Also following from (39) and (44) we find

$$\Omega_1 = \frac{g'(\tau) \cosh k(z+a) \cos kx}{kc_0 \sinh ka}, \tag{46}$$

where $g'(\tau) = dg/d\tau$, $g''(\tau) = d^2g/d\tau^2$, etc. Eq. (42) gives

$$g''(\tau) = kc_0^2 \tanh ka \left[\frac{V_0^2 k}{4\pi b^2} \coth kb - Tk^2 - \rho g\right] g(\tau).$$

The appropriate form of solution is to let c_0 be determined by the equation

$$kc_0^2 \tanh ka \left[\frac{V_0^2 k}{4\pi b^2} \coth kb - Tk^2 - \rho g\right] = 1, \tag{47}$$

so that $g''(\tau) = g(\tau)$. The condition for instability in linear form, is that $c_0^2 > 0$; that is,

$$\frac{V_0^2 k}{4\pi b^2} \coth kb > Tk^2 + \rho g.$$

For the ϵ^2 terms we have from (11)

$$\phi_2 + \frac{V_0}{b}\eta_2 + \frac{kV_0}{2b} \coth kb g^2(\tau)(1 + \cos 2kx) = 0, \quad (z = 0). \tag{48}$$

We note that η_2 can be of the form

$$\left.\begin{matrix}\sin\\ \cos\end{matrix}\right| 2kx,$$

with no mean part. Hence the mean part of the last term of (48) must be matched by a mean part of ϕ_2 at $z = 0$. Now (40) becomes, after substitution,

$$\frac{\partial \eta_2}{\partial \tau} - c_0 \frac{\partial \Omega_2}{\partial z} - kgg' \coth ka \cos 2kx - \frac{c_1}{c_0} \cos kx g'(\tau) = 0. \tag{49}$$

Since Ω_2, like η_2, can contain only terms in x of the form

$$\left.\begin{matrix}\sin\\ \cos\end{matrix}\right| 2kx,$$

the $\cos kx$ term in (49) is unbalanced, and therefore $c_1 = 0$, so that

$$\frac{\partial \eta_2}{\partial \tau} - c_0 \frac{\partial \Omega_2}{\partial z} = kgg' \coth ka \cos 2kx. \tag{50}$$

The last condition of order ϵ^2 is (43) which becomes, after substitution,

$$\frac{\partial^2 \eta_2}{\partial x^2} - \rho g \eta_2 - \frac{\rho}{c_0} \frac{\partial \Omega_2}{\partial \tau} + \frac{V_0}{4\pi b}\left(\frac{\partial \phi_2}{\partial z}\right) = \frac{\rho (g')^2}{4c_0^2} \{(\coth^2 ka + 1) + (1 - \coth^2 ka) \cos 2kx\}$$

$$- \frac{V_0^2 k^2 g^2}{16\pi b^2} [(\coth^2 ka - 1) + (\coth^2 kb - 3) \cos 2kx] + \frac{\rho g g''}{2c_0^2} (1 + \cos 2kx), \quad \text{at} \quad z = 0. \tag{51}$$

Eq. (51) shows that Ω_2 has a component which is a function of τ only which must be used to match the terms independent of x on the right-hand side of (51). This does not affect the velocity distribution, but does represent a change in the mean pressure to the second order.

We shall confine our discussion at this stage to the mean components, which are of most interest. Clearly from (48) a second-order mean electric field arises from the requirements that

$$\phi_2 = -\frac{kV_0}{2b} g^2(\tau) \coth kb \text{ at } z = 0,$$

$$\phi_2 = 0 \qquad \text{at } z = b.$$

This solution is therefore

$$\phi_2 = \frac{kV_0}{2b} g^2(\tau) \coth kb \left(\frac{z}{b} - 1\right),$$

and represents an increase in the mean field to

$$\frac{V_0}{b}\left\{1 + \frac{\epsilon^2 k}{2b} g^2(\tau) \coth kb\right\}.$$

From (51) we find that the mean part of $\partial\Omega_2/\partial\tau$ is given by

$$-\rho \frac{\partial \Omega_2}{\partial \tau} = \frac{\rho (g')^2}{4c_0} (\coth^2 ka + 1) + \frac{\rho g g''}{2c_0} - \frac{V_0^2 k^2 g^2(\tau) c_0}{16\pi b^2}\left\{\coth^2 kb + \frac{2 \coth kb}{kb} - 1\right\}. \tag{52}$$

Thus the right-hand side of (52) represents the increase in mean pressure to the second order.

For the parts of the second-order solution periodic in x we again write η_2, ϕ_2, Ω_2 in terms of

$$\left.\begin{matrix}\sin\\ \cos\end{matrix}\right| 2kx \text{ with the coefficients } r, s, \beta, \gamma, \mu, \nu$$

as used in Sec. 2. But here these coefficients are functions of τ. The periodic parts of ϕ_2 and Ω_2 follow as in (23) and (24). The equations for the coefficients are now

$$\beta + \frac{V_0}{b} r + \frac{kV_0}{2b} \coth kb[g(\tau)]^2 = 0, \tag{53}$$

$$\gamma + \frac{V_0}{b} s = 0, \tag{54}$$

$$\frac{dr}{d\tau} - 2kc_0\mu \tanh 2ka - khgg' \coth ka = 0, \tag{55}$$

$$\frac{ds}{d\tau} - 2\, kc_0\nu \tanh 2ka = 0, \tag{56}$$

$$- 4k^2Tr - \rho gr - \frac{\rho}{c_0}\frac{d\mu}{d\tau} - \frac{V_0}{4\pi b}\cdot 2k\beta \coth 2kb$$
$$= \frac{\rho(g')^2}{4c_0^2}(1 - \coth^2 ka) + \frac{\rho gg''}{2c_0^2} - \frac{V_0^2k^2g^2}{16\pi b^2}\;(\coth^2 kb - 3), \tag{57}$$

$$-4k^2Ts - \rho gs - \frac{\rho}{c_0}\frac{d\nu}{d\tau} - \frac{V_0}{4\pi b}\, 2k\gamma \coth 2kb = 0. \tag{58}$$

(*b*) *Charge-maintained oscillations.* The changes needed to apply the alternative boundary conditions on ϕ follow much as in Sec. 2. It is easily seen that (45) for ϕ still holds, with the proviso that the fluid is at earthed potential. Also, Eq. (46) for Ω_1 and the equation for $g(\tau)$ are unchanged. With the same significance for c_2 and d_2 as in Sec. 2, we now have

$$d_2 - \frac{kV_0}{2b}\, g^2(\tau) \coth kb = c_2,$$

or, with $d_2 = 0$,

$$c_2 = -\frac{kV_0}{2b} \coth kb\, g^2(\tau).$$

Thus the mean loss of potential is $(\epsilon^2 V_0/2b)[g(\tau)]^2 \coth kb$ in this case. Eqs. (48) and (49) remain. So also does (51), though now, since $\partial\phi_2/\partial z$ has no mean part, the equation for the mean pressure change is now

$$-\frac{\rho\,\partial\Omega_2}{\partial t} = \frac{\rho(g')^2}{4c_0}(\coth^2 ka + 1) + \frac{\rho gg''}{2c_0} - \frac{V_0^2k^2g^2c_0}{16\pi b^2}\{\coth^2 kb - 1\}. \tag{59}$$

The increase in mean pressure in (59) as compared with (52) is due directly to the absence of electrostatic stress at the free surface arising from the mean part of ϕ_2. Eqs. (53)–(58) for the periodic coefficients remain unaltered, and we see that the form of the free-surface is again the same to order ϵ^2 in both cases (*a*) and (*b*). As a final remark we note that $g(\tau)$ has been left unspecified. Its particular form is determined by the initial conditions.

References

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