NONLINEAR EIGENVALUE PROBLEMS

BY

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Contents

Introduction

Let $A, B_0, ..., B_h$ be compact linear operators on a Hilbert space H and let p and q be integers, $p \geqslant 0$ and $q > 0$. Consider the operator

$$C(\lambda) = \lambda^q I - A - \sum_{k=0}^{h} \lambda^{p+k} B_k, \tag{0.1}$$

where I is the identity operator and λ is any complex number. If for some λ there is a non-zero element u such that $C(\lambda)u = 0$ then we say that λ is an *eigenvalue* and u an *eigenvector* of $C(\lambda)$. In case $C(\lambda) = \lambda I - A$ and A is self-adjoint, a classical result asserts that the eigenvectors of $C(\lambda)$ (or of A) are complete. There is also a completeness theorem in case A is not

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self-adjoint but is in some class C_r (cf. Lemma 7.3 below). But in this case, one can only assert that certain generalized eigenvectors of A are complete (see Sec. 2 for definitions).

In this paper we shall generalize these two results to the case where $\lambda I - A$ is replaced by the $C(\lambda)$ of (0.1). In Chapter 2 we consider the case where A is self-adjoint and we obtain various completeness theorems for the generalized eigenvectors of (0.1). In Chapter 3 we consider the case where A is not self-adjoint. We then obtain a generalization of the result mentioned above for $\lambda I - A$, with A in C_r.

In Chapter 1 we prove some preliminary theorems for operators of the form (0.1), theorems analogous to well known results for $\lambda I - A$. Thus, for instance, it is shown that each complex number $\lambda \neq 0$ is either an eigenvalue or else $C^{-1}(\lambda)$ is a bounded operator.

Chapter 2 deals with the case where A is self-adjoint. In Sec. 2 we define the concept of generalized eigenvector and, under certain hypotheses, prove the following fact. Let v be any element of H orthogonal to all the generalized eigenvectors with eigenvalues in a disc $|\lambda| < \Lambda$. Let C_* denote the adjoint of C. Then $C_*^{-1}(\bar{\lambda})v$ is an analytic function of λ, regular in the disc $|\lambda| < \Lambda$. The idea of studying the analyticity properties of $C_*^{-1}(\bar{\lambda})v$ is due to Agmon [0] and to Dunford and Schwartz [4], who used it to study completeness of the generalized eigenvectors of operators having the simple form: $\lambda I - A$.

The result of Sec. 2 is used in Sec. 3 to derive various completeness theorems, the main ones being Theorems 3.1 and 3.2. Theorem 3.2 is subject to a number of far-reaching generalizations, and the next two sections are devoted to a number of these, culminating in Theorem 5.1, which we consider to be the most profound result of this paper.

It is obtained in the following way. The results of Sec. 3 are limited to the case $p > q$. In Sec. 4, we introduce certain transformations which, while preserving generalized eigenvectors, have the effect of increasing p. Completeness theorems valid even when $p \leq q$ can therefore be obtained by applying the results of Sec. 3 to the transformed versions of the original operators. This method of transformation also gives new results when it is applied directly to the situation of Sec. 3 in which $p > q$.

In Sec. 5 we apply a process we call "linearization" that transforms any equation for generalized eigenvectors to a system of equations. In the computationally rather involved Lemma 5.1 we show that the generalized eigenvectors of the resulting system are related in a simple way to the generalized eigenvectors of the original operator. Therefore, the results of Secs. 3 and 4 can be applied to the system of equations, and this gives results for the generalized eigenvectors of (0.1). Finally, Theorem 5.1, mentioned above, is obtained as a limiting case of the linearization process.

In Chapter 3 we drop the hypothesis that A (in (0.1)) is self-adjoint. We restrict ourselves to the case where $p + h < q$. The main result is stated in Theorem 6.2. The proof of this theorem is given in Sec. 7. The method we use is that of "linearization" of operators (0.1) with respect to λ, which means, in this context, the reduction of (0.1) to a matrix operator of the form $\lambda I - A$. This technique has been used before by several authors, notably by Agmon and Nirenberg [1]. In that paper they prove a completeness theorem for a reduced weighted elliptic system. Their method depends upon a lemma of Agmon [0] concerning the growth properties of the resolvent of a compact operator of a certain class in Hilbert spaces $\Pi_{i=1}^{k} H^{m_i}(\Omega)$. The proof of that lemma makes use of the C_r theory of Dunford and Schwartz [4]. Our method also employs this theory, but in a different way.(1)

In Chapter 4 we give a few applications of the results of Chapters 2 and 3. We believe that the methods and results of this paper should lend themselves to many other applications.

We conclude this introduction with a brief survey of the literature. Completeness theorems were derived for operators of the form

$$I - \lambda A - \lambda^2 B + \sum_{k=1}^{m} \frac{\lambda^2}{\lambda - a_k} H_k \tag{0.2}$$

by a number of authors, the first of whom was Miranda [11], who considered the case of integral and integro-differential operators. Harazov [8] considered the more general case where A, B, H_k are compact operators in a Hilbert space. He assumed that B, $(1/a_k)H_k$ are positive Hilbert-Schmidt operators, that A is a Hilbert-Schmidt operator, and that the range of each H_k is finite-dimensional. More recently, Müller [12] relaxed the last condition on the H_k. His paper also contains a thorough bibliography on the subject.

The results of Secs. 3 and 4 are related to the work of Shinbrot [14], [15]. In [14] he considered the eigenvalue problem

$$\lambda u = Au + \lambda^\alpha B(\lambda) u \quad (\alpha > 1) \tag{0.3}$$

where A is a compact, self-adjoint operator with simple eigenvalues, and $B(\lambda)$ is a bounded operator satisfying a uniform Lipschitz condition in λ. He proved that the closed subspace spanned by the eigenvectors of (0.3) has finite codimension if

$$\sum \left| \frac{\mu_n^\alpha}{\delta_n} \right|^2 < \infty \quad (\delta_n = \min_{j \neq n} |\mu_n - \mu_j|), \tag{0.4}$$

where μ_n are the eigenvalues of A. In addition, Shinbrot showed that if one adjoins a finite number of eigenvectors of A to those of (0.3), the resulting system of vectors is complete.

(1) See also Addendum at the end of this paper.

His method is based on a perturbation of the eigenvectors v_n of A. The results were generalized, in [15], to include the case $\alpha \leqslant 1$.

A more recent result is that of Turner [16], who considered (0.1) with $q=1$, $p \geqslant 2$. He assumes that A and the B_k's are compact, self-adjoint and positive. In addition, he assumes either that A is in C_r for some $r<1/2$ or that A is in C_r for some $r<2/3$ and that the B_k's are in C_2. He then proves that the eigenvectors corresponding to real eigenvalues form an unconditional basis. He also shows that the non-negative eigenvalues can be characterized by variational principles provided the eigenvectors form a basis. Related results for the case $q=1$, $p=2$, $h=0$ were obtained in [17], [18].

Chapter 1. Preliminary results on the spectrum

Let H be a Hilbert space and let A, $B_0, \ldots, B_h$ be compact operators on H. We shall establish completeness of certain sets of elements u of H related to solutions of the equation

$$\lambda^q u = Au + \sum_{k=0}^{h} \lambda^{p+k} B_k. \tag{1.1}$$

Here p and h are non-negative integers, while q is a positive integer. We introduce the operator

$$C(\lambda) = \lambda^q I - A - \sum_{k=0}^{h} \lambda^{p+k} B_k, \tag{1.2}$$

and begin with preliminary results which show that the structure of the "spectrum" of $C(\lambda)$, like that of the more familiar operator $\lambda I - A$, consists only of "eigenvalues" of finite multiplicity, forming at most a countable sequence without finite points of accumulation, except possibly 0.

We need a few definitions. Let $\mathbf{C}$ denote the complex plane. We define two subsets of $\mathbf{C}$:

$$\sigma(A; B_0, \ldots, B_h) = \{\lambda \in \mathbf{C};\ C(\lambda) \text{ is not one-to-one}\},$$

$$\varrho(A; B_0, \ldots, B_h) = \{\lambda \in \mathbf{C};\ C(\lambda) \text{ is one-to-one and onto}\}.$$

Note that if A, $B_0, \ldots, B_h$ are merely closed and $\lambda \in \varrho(A; B_0, \ldots, B_h)$, then $C^{-1}(\lambda)$ (the inverse of $C(\lambda)$) is a bounded operator in H (by the closed graph theorem).

Clearly $\sigma(A; B_0, \ldots, B_h)$ and $\varrho(A; B_0, \ldots, B_h)$ are generalizations of the ordinary notions of point spectrum and resolvent set of an operator.

THEOREM 1.1. *Suppose $p+h<q$. Let A, $B_0, \ldots, B_h$ be bounded operators. Then $\sigma(A; B_0, \ldots, B_h)$ is a bounded set.*

Proof. Let $\lambda \in \sigma(A; B_0, \ldots, B_h)$. Then there exists an element u of H with $\|u\| = 1$ satisfying (1.1). Hence

$$|\lambda|^q \leqslant \|A\| + \sum_{k=0}^{h} |\lambda|^{p+k} \|B_k\|.$$

The right-hand side is a polynomial in $|\lambda|$ of degree $< q$. Hence $|\lambda|$ is bounded.

THEOREM 1.2. *Let $A, B_0, \ldots, B_h$ be compact operators. Then*

$$\mathbf{C} = \{0\} \cup \varrho(A; B_0, \ldots, B_h) \cup \sigma(A; B_0, \ldots, B_h).$$

Proof. Let λ_0 be a non-zero complex number which does not belong to $\varrho(A; B_0, \ldots, B_h)$. Introduce the operator

$$D = A + \sum_{k=0}^{h} \lambda_0^{p+k} B_k.$$

Then $\lambda_0^q I - D$ is either not one-to-one or is not onto. In either case λ_0^q is in the spectrum of D. Since $\lambda_0^q \neq 0$ and D is compact, this means that λ_0^q is an eigenvalue of D. Therefore, $\lambda_0^q I - D$ is not one-to-one. But this means that $\lambda_0 \in \sigma(A; B_0, \ldots, B_h)$.

THEOREM 1.3. *Let $A, B_0, \ldots, B_h$ be compact operators and assume that $\varrho(A, B_0, \ldots, B_h)$ is not empty. Then the set $\sigma(A; B_0, \ldots, B_h)$ is either finite or countable. In the latter case it has no finite limit-points except possibly $\lambda = 0$. If $\lambda \in \sigma(A; B_0, \ldots, B_h)$ then the space of solutions of* (1.1) *is finite dimensional.*

Note that, by Theorem 1.1, if $p+h<q$ then the set $\varrho(A; B_0, \ldots, B_h)$ is not empty.

Proof. Our proof follows one of the classical proofs for the case where $C(\lambda) = \lambda I - A$. It suffices to show that, for any $\Lambda > 1$, the set

$$S_\Lambda = \sigma(A; B_0, \ldots, B_k) \cap \left\{\lambda \in \mathbf{C};\ \frac{1}{\Lambda} \leqslant |\lambda| \leqslant \Lambda\right\}$$

is finite. Since $A, B_0, \ldots, B_h$ are compact, they can be decomposed as follows:

$$A = A^{(1)} + A^{(2)},$$
$$B_k = B_k^{(1)} + B_k^{(2)},$$

where the ranges $R(A^{(1)})$ and $R(B_k^{(1)})$ are finite-dimensional, and the norms of $A^{(2)}$ and $B_k^{(2)}$ can be made smaller than any prescribed positive number. We can thus assume that

$$\Lambda^q [\|A^{(2)}\| + \sum_{k=0}^{h} \Lambda^{p+k} \|B_k^{(2)}\|] \leqslant \tfrac{1}{2}. \tag{1.3}$$

Set
$$B(\lambda) = A + \sum_{k=1}^{h} \lambda^{p+k} B_k,$$

$$B^{(i)}(\lambda) = A^{(i)} + \sum_{k=0}^{h} \lambda^{p+k} B_k^{(i)} \quad (i = 1, 2).$$

An equation of the form $\lambda^q u = B(\lambda) u + f$ is equivalent to

$$\left[I - \frac{1}{\lambda^q} B^{(2)}(\lambda)\right] u = \frac{1}{\lambda^q} [B^{(1)}(\lambda) u + f]. \tag{1.4}$$

Because of (1.3) we have, for any λ in the region $\Lambda^{-1} \leqslant |\lambda| \leqslant \Lambda$,

$$\left\| \frac{1}{\lambda^q} B^{(2)}(\lambda) \right\| \leqslant \Lambda^q \left(\| A^{(2)} \| + \sum_{k=0}^{h} \Lambda^{p+k} \| B_k^{(2)} \| \right) \leqslant \tfrac{1}{2}.$$

Therefore, $[I - \lambda^{-q} B^{(2)}(\lambda)]^{-1}$ exists and is a bounded operator. We can then write (1.4) in the form

$$u = [\lambda^q I - B^{(2)}(\lambda)]^{-1} [B^{(1)}(\lambda) u + f]. \tag{1.5}$$

Let $\{\varphi_m\}$ be an orthonormal basis for the finite-dimensional subspace

$$R \equiv \left\{ v \in H;\ v = v_0 + \sum_{k=0}^{h} u_k,\ v_0 \in R(A^{(1)}),\ u_k \in R(B_k^{(1)}) \right\}.$$

Denote the dimension of R by N and the scalar product in H by (,). Since $B^{(1)}(\lambda) u$ is in R, (1.5) is equivalent to

$$u = [\lambda^q I - B^{(2)}(\lambda)]^{-1} f + \sum_{n=1}^{N} (B^{(1)}(\lambda) u, \varphi_n) [\lambda^q I - B^{(2)}(\lambda)]^{-1} \varphi_n. \tag{1.6}$$

Set $x_k(\lambda) = (B^{(1)}(\lambda) u, \varphi_k)$. Then, (1.6) has a unique solution if and only if one can solve uniquely the equations

$$x_k - \sum_{n=1}^{N} (B^{(1)}(\lambda) [\lambda^q I - B^{(2)}(\lambda)]^{-1} \varphi_n, \varphi_k) x_n = (B^{(1)}(\lambda) [\lambda^q I - B^{(2)}(\lambda)]^{-1} f, \varphi_k) \quad (k = 1, \ldots, N), \tag{1.7}$$

i.e., if the condition

$$\det \{ I - (B^{(1)}(\lambda) [\lambda^q I - B^{(2)}(\lambda)]^{-1} \varphi_n, \varphi_k) \} = 0 \tag{1.8}$$

is *not* satisfied. In fact, applying $B^{(1)}(\lambda)$ to both sides of (1.6) and taking the scalar product with φ_k we obtain the system (1.7). On the other hand, if $(x_1(\lambda), \ldots, x_N(\lambda))$ forms a unique solution of (1.7) then the element

$$u = [\lambda^q I - B^{(2)}(\lambda)]^{-1} [f + \sum_{k=1}^{N} x_k(\lambda) \varphi_k] \tag{1.9}$$

is easily seen to satisfy (1.6). The correspondence $(x_1(\lambda), \ldots, x_N(\lambda)) \to u$ is one-to-one.

In view of Theorem 1.2 it follows that $\lambda \in \sigma(A; B_0, \ldots, B_h)$ if and only if it satisfies (1.8).

Note now that $B^{(1)}(\lambda)$ is a polynomial in λ. Also, $[\lambda^q I - B^{(2)}(\lambda)]^{-1}$ is analytic in λ for $\Lambda^{-1} \leqslant |\lambda| \leqslant \Lambda$, since the Neumann series for this operator converges there. Thus the relation

(1.8) has the form $P(\lambda)=0$ where $P(\lambda)$ is an analytic function for $\Lambda^{-1}\leqslant|\lambda|\leqslant\Lambda$. Since $\varrho(A; B_0, \ldots, B_h)$ is not empty, $P(\lambda)\not\equiv 0$. Hence the equation (1.9) has at most a finite number of solutions for $\Lambda^{-1}\leqslant|\lambda|\leqslant\Lambda$.

Finally, the last assertion of the theorem is a consequence of the fact that any solution u of (1.1) (with $\Lambda^{-1}\leqslant|\lambda|\leqslant\Lambda$) must have the form (1.6) with $f=0$.

Remark. Note that the $x_k(\lambda)$ occurring in (1.9) are analytic functions for $\lambda\in\varrho(A; B_0, \ldots, B_h)$. This implies that $C^{-1}(\lambda)$ is analytic for $\lambda\in\varrho(A; B_0, \ldots, B_h)$.

Chapter 2. The case of A self-adjoint

2. Fundamental lemmas

In this chapter we shall continue to study equations of the form

$$\lambda^q u = Au + \sum_{k=0}^{h} \lambda^{p+k} B_k u, \tag{2.1}$$

but the basic hypothesis that A is self-adjoint will be made. As before, introduce the operator

$$C(\lambda) = \lambda^q I - A - \sum_{k=0}^{h} \lambda^{p+k} B_k. \tag{2.2}$$

Assume that $\varrho(A; B_0, \ldots, B_h)$ is non-empty. Then, by Theorem 1.3, it is an open set. As we remarked at the end of the last section, $C^{-1}(\lambda)$ is an analytic function in C, regular on the open set $\varrho(A; B_0, \ldots, B_h)$. As (1.7) shows, the functions $x_k(\lambda)$ occurring in (1.9) are ratios of determinants of regular analytic functions. Therefore, by (1.9), $C^{-1}(\lambda)$ has isolated singularities at the points of $\sigma(A; B_0, \ldots, B_h)$. Furthermore, if $\lambda_0\in\sigma(A; B_0, \ldots, B_h)$, then, in a punctured neighborhood of λ_0, $C^{-1}(\lambda)$ is a finite sum of terms of the form $(p_i(\lambda)/q(\lambda))\Gamma_i$ where Γ_i is a bounded operator independent of λ and $p_i(\lambda)$, $q(\lambda)$ are regular analytic functions, $q(\lambda)$ having an isolated zero at λ_0. Consequently, $C^{-1}(\lambda)$ has a pole at λ_0.

Definition. Let $\lambda_0\in\sigma(A; B_0, \ldots, B_h)$, $\lambda_0\neq 0$, and let n be the order of the pole of $C^{-1}(\lambda)$ at $\lambda=\lambda_0$. Let $'=(d/d\lambda)$. By a *packet of generalized eigenvectors* of $C(\lambda)$ (or of (2.1)) at $\lambda=\lambda_0$, we shall mean a vector $(u_0, u_1, \ldots, u_{n-1})$ whose components are elements of H satisfying the following system of equations:

$$\begin{cases} C(\lambda_0)u_0 = 0, \\ C(\lambda_0)u_1 = -C'(\lambda_0)u_0, \\ \cdots \\ C(\lambda_0)u_{n-1} = -C'(\lambda_0)u_{n-2} - \dfrac{1}{2!}C''(\lambda_0)u_{n-3} - \cdots - \dfrac{1}{(n-1)!}C^{(n-1)}(\lambda_0)u_0. \end{cases} \tag{2.3}$$

The non-zero components u_i of each such vector $(u_0, u_1, \ldots, u_{n-1})$ will be called *generalized eigenvectors* of $C(\lambda)$ at $\lambda=\lambda_0$.

It should be noted that if a set $\{u_0, \ldots, u_{k-1}\}$ satisfies the first k equations in (2.3), where $k<n$, and if there do not exist elements $u_k, \ldots, u_{n-1}$ such that the last $n-k$ equations of (2.3) have a solution, then the elements $u_0, \ldots, u_{k-1}$ are not necessarily generalized eigenvectors.

Consider the special case where $C(\lambda)=\lambda I-A$. In this case Dunford and Schwartz [3] have also defined the notion of a generalized eigenvector. They call an element $v\in H$ a generalized eigenvector if v satisfies an equation

$$(A-\lambda_0 I)^{k+1}v=0 \tag{2.4}$$

for some non-negative integer k and $\lambda_0\in\mathbf{C}$. We wish to point out that when $C(\lambda)=\lambda I-A$, our notion coincides with the notion of Dunford and Schwartz.

If $C(\lambda)=\lambda I-A$, equations (2.3) reduce to

$$\begin{cases}(A-\lambda_0 I)\,u_0=0,\\ (A-\lambda_0 I)\,u_1=u_0,\\ \cdots\\ (A-\lambda_0 I)\,u_{n-1}=u_{n-2}.\end{cases} \tag{2.5}$$

It follows immediately that each u_k $(0\leqslant k\leqslant n-1)$ satisfies $(A-\lambda_0 I)^{k+1}u_k=0$ and is, therefore, a generalized eigenvector in the sense of Dunford and Schwartz.

Suppose conversely that v is an element of H satisfying (2.4) for some $k\geqslant 0$. By [3] it follows that v satisfies (2.4) with (possibly another) k such that $0\leqslant k\leqslant n-1$. Set $u_i=(A-\lambda_0 I)^{n-1-i}v$ $(0\leqslant i\leqslant n-1)$. Then $(u_0, \ldots, u_{n-1})$ is a solution of (2.5) and is, therefore, a packet of generalized eigenvectors of $C(\lambda)=\lambda I-A$ at λ_0. Since $u_{n-1}=v$, it follows that v is a generalized eigenvector in our sense. Thus, the Dunford–Schwartz definition coincides with our definition.

Definition. Denote by $\mathrm{sp}_R(C)$ the closed subspace spanned by the generalized eigenvectors of the operator $C(\lambda)$ (given by (2.2)) when λ varies in the disc $|\lambda|<R$, and write $\mathrm{sp}(C)=\mathrm{sp}_\infty(C)$. Similarly denote by $\mathrm{sp}_R(C)$ the closed subspace spanned by the eigenvectors of $C(\lambda)$, when λ varies in the disc $|\lambda|<R$, and write $\mathrm{sp}^0(C)=\mathrm{sp}^0_\infty(C)$.

Let $\lambda_0\in\sigma(A; B_0, \ldots, B_h)$, $\lambda_0\neq 0$ and let $u(\lambda)$ be any function, with values in H, analytic in a neighborhood of $\lambda=\lambda_0$. If we denote by n the order of the pole of $C^{-1}(\lambda)$ at $\lambda=\lambda_0$, then we have

$$C^{-1}(\lambda)\,u(\lambda)=\frac{u_0}{(\lambda-\lambda_0)^n}+\frac{u_1}{(\lambda-\lambda_0)^{n-1}}+\cdots+\frac{u_{n-1}}{\lambda-\lambda_0}+w(\lambda), \tag{2.6}$$

where $w(\lambda)$ is analytic at $\lambda=\lambda_0$.

Notation. We shall denote by $G(C(\lambda_0))$ the closed subspace spanned by all the generalized eigenvectors of $C(\lambda)$ at $\lambda=\lambda_0$. We shall denote by $\tilde{G}(C(\lambda_0))$ the closed subspace spanned by all the elements $u_0, \ldots, u_{n-1}$ occurring in the expansion (2.6), when $u(\lambda)$ varies over the set of all functions analytic in a neighborhood of $\lambda=\lambda_0$.

LEMMA 2.1. $G(C(\lambda_0))=\tilde{G}(C(\lambda_0))$.

Proof. If we apply $C(\lambda)$ to both sides of (2.6), we conclude that the function

$$g(\lambda)\equiv C(\lambda)\left(\frac{u_0}{(\lambda-\lambda_0)^n}+\frac{u_1}{(\lambda-\lambda_0)^{n-1}}+\ldots+\frac{u_{n-1}}{\lambda-\lambda_0}\right) \tag{2.7}$$

is analytic at $\lambda=\lambda_0$. Developing $C(\lambda)$ into Taylor's series about $\lambda=\lambda_0$ we find that (2.3) holds. This shows that $\tilde{G}(C(\lambda_0))\subset G(C(\lambda_0))$.

Suppose conversely that $(u_0, \ldots, u_{n-1})$ is a packet of generalized eigenvectors of $C(\lambda)$ at $\lambda=\lambda_0$. Then the function $g(\lambda)$ defined in (2.7) is analytic at $\lambda=\lambda_0$. Since (2.6) clearly holds with $u(\lambda)=g(\lambda)$, $w(\lambda)\equiv 0$, it follows that the u_i belong to $\tilde{G}(C(\lambda_0))$. Thus we have proved that $G(C(\lambda_0))\subset\tilde{G}(C(\lambda_0))$.

Lemma 2.1 gives a useful characterization of generalized eigenvectors. We shall use it to prove the following lemma.

LEMMA 2.2. *Let k be any positive integer and set $\Gamma(\lambda)=C(\lambda^k)$. Then $\lambda_0(\neq 0)$ is a pole of $C^{-1}(\lambda)$ of order n if and only if each k-th root $\lambda_0^{1/k}$ is a pole of $\Gamma^{-1}(\lambda)$ of order n. Moreover, in that case,*

$$G(\Gamma(\lambda_0^{1/k}))=G(C(\lambda_0)). \tag{2.8}$$

COROLLARY. $\mathrm{sp}_R(\Gamma)=\mathrm{sp}_{R^k}(C)$.

Proof. Let λ_0 be a pole of order n of $C^{-1}(\lambda)$, $\lambda_0\neq 0$, and let $u(\lambda)$ be an analytic function in a neighborhood of $\lambda=\lambda_0$. Then (2.6) holds. For any k-th root $\mu_0=\lambda_0^{1/k}$, let $\mu=\mu(\lambda)=\lambda^{1/k}$ be the uniquely defined analytic function in a neighborhood of μ_0 with $\mu(\lambda_0)=\mu_0$. Setting $v(\mu)=u(\lambda)$, we obtain from (2.6),

$$\Gamma^{-1}(\mu)\,v(\mu)=C^{-1}(\lambda)\,u(\lambda)=\frac{u_0}{(\mu^k-\mu_0^k)^n}+\frac{u_1}{(\mu^k-\mu_0^k)^{n-1}}+\ldots+\frac{u_{n-1}}{\mu^k-\mu_0^k}+w_0(\mu), \tag{2.9}$$

where $w_0(\mu)$ is analytic at $\mu=\mu_0$. Note that $v(\mu)$ is analytic at $\mu=\mu_0$. From (2.9) we get

$$\Gamma^{-1}(\mu)\,v(\mu)=\frac{v_0}{(\mu-\mu_0)^n}+\frac{v_1}{(\mu-\mu_0)^{n-1}}+\ldots+\frac{v_{n-1}}{\mu-\mu_0}+w_1(\mu), \tag{2.10}$$

where $w_1(\mu)$ is analytic at $\mu=\mu_0$, and the v_i are linear combinations of the u_j $(0\leqslant j\leqslant i)$ and vice versa. Clearly, when $u(\lambda)$ varies over the set of all functions analytic at $\lambda = \lambda_0$, $v(\mu)$ varies over the set of all functions analytic at $\mu = \mu_0$. It follows that

$$\tilde{G}(\Gamma(\lambda_0^{1/k})) = \tilde{G}(C(\lambda_0)). \tag{2.11}$$

Suppose now that $\Gamma^{-1}(\mu)$ has a pole at some point $\mu_0 \neq 0$. We claim that there is a pole of $C^{-1}(\lambda)$ at the point $\lambda_0 = \mu_0^k$. Indeed, the map $\lambda \to \mu(\lambda) = \lambda^{1/k}$ (with $\mu(\lambda_0^{1/k}) = \mu_0$) is a one-to-one analytic transformation from a neighborhood of λ_0 into a neighborhood of μ_0. It takes $C^{-1}(\lambda)$ into $\Gamma^{-1}(\mu)$. Since the latter is not analytic at $\mu = \mu_0$, $C^{-1}(\lambda)$ must have a pole at λ_0.

We have so far proved all the assertions of Lemma 2.2 with (2.11) instead of (2.8). Now use Lemma 2.1 to deduce (2.8).

LEMMA 2.3. *Assume that $\varrho(A; B_0, ..., B_h)$ is non-empty. Then there is only a finite number of linearly independent generalized eigenvectors of $C(\lambda)$ at λ_0.*

Proof. By Theorem 1.3, there is a finite number, say N, of solutions u_0 for the first equation in (2.3). For each such u_0, consider the second equation in (2.3). If there is at least one solution, then the number of linearly independent solutions is at most N. Repeating this process a finite number of times, we cover all the equations (2.3) and obtain at most a finite number of generalized eigenvectors at each step. The lemma follows from this.

Definition. Let A be a compact self-adjoint operator. Denote by $\{\lambda_n\}$ the sequence of eigenvalues of A. If

$$\sum_{n=1}^{\infty} |\lambda_n|^r < \infty \tag{2.12}$$

for some $r > 0$, then we say that A is in class C_r.

We can now state our main lemma.

LEMMA 2.4. *Let $p > q$. Suppose that A, B_0, ..., B_h are compact, that A is self-adjoint and one-to-one, and in class C_r for some $r < (p/q) - 1$. Then the set $\varrho(A; B_0, ..., B_h)$ is non-empty. Furthermore, if an element v in H is orthogonal to all the generalized eigenvectors of $C(\lambda)$, $\lambda \in \mathbf{C}$, then the function*

$$v(\lambda) \equiv C_*^{-1}(\bar{\lambda})v \quad (C_* \text{ denotes the adjoint of } C) \tag{2.13}$$

is an entire analytic function of λ.

Note that the right-hand side of (2.13) is well defined and regular on $\varrho(A; B_0, ..., B_h)$. The last assertion of the lemma is the statement that the right-hand side of (2.13) can be extended into the whole finite plane so as to become an entire function.

In proving Lemma 2.4 we shall need the following lemma.

LEMMA 2.5. *Let A be a compact self-adjoint operator with zero null space, and suppose that $A \in C_r$ for some $r > 0$. Then there exists a sequence $\{\mu_n\}$, $\mu_n \searrow 0$, such that*

$$\|(\lambda I - A)^{-1}\| \leq \frac{C}{|\lambda|^{1+r}} \quad \textit{whenever } |\lambda| = \mu_n,\ 1 \leq n < \infty. \tag{2.14}$$

Proof. Denote the eigenvalues of A by λ_n, and let u_n be an eigenvector corresponding to λ_n such that $\{u_n\}$ is a complete orthonormal sequence. Then we have

$$(\lambda I - A)^{-1} u = \sum_{n=1}^{\infty} \frac{(u, u_n)}{\lambda - \lambda_n} u_n. \tag{2.15}$$

Therefore, if $\delta(\lambda) = \min_n |\lambda_n - \lambda|$,

$$\|(\lambda I - A)^{-1} u\|^2 \leq \frac{1}{\delta^2(\lambda)} \sum_{n=1}^{\infty} |(u, u_n)|^2.$$

It follows that

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{\delta(\lambda)}. \tag{2.16}$$

Let $\{\nu_n\}$ be the sequence of distinct values of $|\lambda_n|$, $\nu_{n+1} \leq \nu_n$, and define

$$\gamma_n = \tfrac{1}{2}(\nu_n + \nu_{n+1}).$$

Then, if $|\lambda| = \gamma_n$,

$$\|(\lambda I - A)^{-1}\| \leq \frac{2}{\nu_n - \nu_{n+1}}.$$

Therefore, it suffices to show that there are infinitely many values of n such that $\nu_n - \nu_{n+1} \geq c \cdot \nu_n^{1+r}$, where c is a positive constant.

If this were false then, for some $n_0 > 0$,

$$\nu_{n+1} > \nu_n(1 - c\nu_n^r) \quad \text{for all } n \geq n_0. \tag{2.17}$$

Since $A \in C_r$, $\sum \nu_n^r < \infty$. It follows that $n\nu_n^r \to 0$. Therefore, for any $\varepsilon > 0$ there exists a number $n_1(\geq n_0)$ such that

$$\nu_n < \frac{\varepsilon}{n^{1/r}} \quad \text{if } n \geq n_1. \tag{2.18}$$

From (2.17), (2.18) we get

$$\nu_{n+1} > \nu_n\left(1 - \frac{c\varepsilon^r}{n}\right).$$

Inductively we obtain

$$\nu_n > \nu_{n_1}\left(1 - \frac{c\varepsilon^r}{n_1}\right)\left(1 - \frac{c\varepsilon^r}{n_1 + 1}\right)\cdots\left(1 - \frac{c\varepsilon^r}{n - 1}\right) = \nu_{n_1} \cdot \frac{\Gamma(n_1)}{\Gamma(n_1 - c\varepsilon^r)} \cdot \frac{\Gamma(n - c\varepsilon_r)}{\Gamma(n)},$$

where Γ denotes the Γ-function.

Using the asymptotic formula (see, for instance, [5; p. 47])

$$\Gamma(z)=e^{-z}\,e^{(z-1/2)\log z}\sqrt{2\pi}\left(1+O\left(\frac{1}{z}\right)\right)\quad(1<z<\infty),$$

we find that

$$\frac{\Gamma(z+a)}{\Gamma(z)}=(z+a)^a\left(1+O\left(\frac{1}{z}\right)\right).$$

This relation with $z=n-c\varepsilon^r$, $a=c\varepsilon^r$ gives

$$\nu_n>\nu_{n_1}\cdot\frac{\Gamma(n_1)}{\Gamma(n_1-c\varepsilon^r)}\frac{1}{2n^{c\varepsilon^r}}$$

if n is sufficiently large. Choosing ε such that $\varepsilon^r<1/rc$, we find that, for some positive constant c_1,

$$n\nu_n^r\geqslant c_1 n^{1-rc\varepsilon^r}\to\infty\quad\text{if } n\to\infty.$$

This contradicts the fact that $n\nu_n^r\to 0$ as $n\to\infty$.

We now proceed with the proof of Lemma 2.4. Since A satisfies the conditions of Lemma 2.5, we have

$$\|(\lambda^q I-A)^{-1}\|\leqslant\frac{c}{|\lambda|^{q(1+r)}}\quad\text{for } |\lambda|=\mu_n^{1/q}, \tag{2.19}$$

where $\{\mu_n\}$ is a sequence which decreases to 0 and c is a constant. Hence,

$$\left\|\sum_{k=0}^{h}(\lambda^q I-A)^{-1}\lambda^{p+k}B_k\right\|\leqslant c|\lambda|^{p-q(1+r)}\quad\text{for } |\lambda|=\mu_n^{1/q}. \tag{2.20}$$

Since $r<\dfrac{p}{q}-1$, we have $p-q(1+r)>0$. Hence the right-hand side of (2.20) goes to 0 as $\mu_n\to 0$. It follows that

$$(I-\sum_{k=0}^{h}(\lambda^q I-A)^{-1}\lambda^{p+k}B_k)^{-1} \tag{2.21}$$

exists if $|\lambda|=\mu_n^{1/q}$ and n is sufficiently large. It can now be immediately verified that, whenever $|\lambda|=\mu_n^{1/q}$, $C^{-1}(\lambda)$ also exists, and is given by

$$C^{-1}(\lambda)=(I-\sum_{k=0}^{h}(\lambda^q I-A)^{-1}\lambda^{p+k}B_k)^{-1}(\lambda^q I-A)^{-1}. \tag{2.22}$$

We have thus proved that the set $\varrho(A;B_0,\ldots,B_h)$ is non-empty.

From (2.20) it follows that the norm of the operator in (2.21) is bounded by 2 if $|\lambda|=\mu_n^{1/q}$ and, say, $n\geqslant n_1$. Hence, from (2.19), (2.22) we get

$$\|C^{-1}(\lambda)\|\leqslant\frac{c}{|\lambda|^{q(1+r)}}\quad\text{for } |\lambda|=\mu_n^{1/q},\ n\geqslant n_1. \tag{2.23}$$

Now, let v be an element in H orthogonal to all the generalized eigenvectors of $C(\lambda)$, $\lambda \in \mathbf{C}$. Consider the function

$$\varphi(\lambda) = (C^{-1}(\lambda)u, v), \tag{2.24}$$

where u is a fixed element of H.

$\varphi(\lambda)$ is an analytic function, regular in the open set $\varrho(A; B_0, ..., B_h)$, and $\varphi(\lambda)$ can have singularities only at the points 0, ∞ and the points of $\sigma(A; B_0, ..., B_h)$. If $\lambda_0 \in \sigma(A; B_0, ..., B_h)$, then

$$C^{-1}(\lambda)u = \frac{u_0}{(\lambda-\lambda_0)^n} + \frac{u_1}{(\lambda-\lambda_0)^{n-1}} + \ldots + \frac{u_{n-1}}{\lambda-\lambda_0} + w(\lambda), \tag{2.25}$$

where n is the order of the pole of $C^{-1}(\lambda)$ at λ_0, and $w(\lambda)$ is regular at λ_0. Lemma 2.1 shows that $u_0, ..., u_{n-1}$ are generalized eigenvectors of $C(\lambda)$ at λ_0. v is therefore orthogonal to them. It follows that $\varphi(\lambda) = (w(\lambda), v)$, so that $\varphi(\lambda)$ is regular at $\lambda = \lambda_0$.

We next consider the singularity of $\varphi(\lambda)$ at $\lambda = 0$. Introduce the function $\chi(\lambda) = \lambda^p \varphi(\lambda)$. From (2.23) we see that

$$|\chi(\lambda)| \leq |\lambda|^p \|C^{-1}(\lambda)\| \, \|u\| \, \|v\| \leq c$$

on the boundary of each ring $\mu_{n+1}^{1/q} < |\lambda| < \mu_n^{1/q}$, $n \geq n_1$. Since $\chi(\lambda)$ is regular in each such ring, the maximum principle implies that $\chi(\lambda)$ is uniformly bounded in a punctured neighborhood of $\lambda = 0$. It follows that $\chi(\lambda)$ has a removable singularity at $\lambda = 0$. Consequently, $\varphi(\lambda)$ has a pole at $\lambda = 0$ of order $\leq p$. Since $|\varphi(\lambda)| \leq c|\lambda|^{-q(r+1)}$ and $q(r+1) < p$, the order of the pole is actually $\leq p-1$.

We thus have
$$\varphi(\lambda) = \frac{a_{p-1}}{\lambda^{p-1}} + \frac{a_{p-2}}{\lambda^{p-2}} + \ldots + \frac{a_1}{\lambda} + \psi(\lambda), \tag{2.26}$$

where $\psi(\lambda)$ is an entire function and

$$a_k = \frac{1}{2\pi i} \int_{|\lambda|=R} \lambda^{k-1} (C^{-1}(\lambda)u, v) \, d\lambda, \quad R > 0.$$

If we choose $R = \mu_n^{1/q}$ for some sufficiently large *fixed* n and make use of (2.23), we find that

$$|a_k| \leq \int_{|\lambda| = \mu_n^{1/q}} \frac{c}{\mu_n^{1+r}} \|u\| \, \|v\| \, |d\lambda| \leq \text{const.} \, \|u\|.$$

Thus, $a_k = a_k(u)$, which is obviously a linear functional of u, is also a bounded functional. It follows that there exists a unique element w_k in H such that $a_k(u) = (u, w_k)$.

Next, if $\lambda \in \varrho(A; B_0, \ldots, B_h)$ then

$$|\varphi(\lambda)| \leq \|C^{-1}(\lambda)\| \, \|u\| \, \|v\|,$$

i.e., $\varphi(\lambda) \equiv \varphi(\lambda; u)$ is a bounded linear functional in u. From (2.26) we then conclude that there exists an element $v(\lambda)$ in H such that $\psi(\lambda) \equiv \psi(\lambda; u)$ has the form $(u, v(\lambda))$ for all $\lambda \in \varrho(A; B_0, \ldots, B_h)$. Hence,

$$C_*^{-1}(\bar{\lambda})\, v = \frac{w_{p-1}}{\lambda^{p-1}} + \frac{w_{p-2}}{\lambda^{p-2}} + \ldots + \frac{w_1}{\lambda} + v(\lambda) \tag{2.27}$$

for all $\lambda \in \varrho(A;\, B_0, \ldots, B_h)$.

The function $v(\lambda)$ is regular in $\varrho(A;\, B_0, \ldots, B_k)$. We claim that at each point λ_0 of $\sigma(A;\, B_0, \ldots, B_k)$, $v(\lambda)$ has a removable singularity. Indeed, for any $u \in H$, (2.26) gives

$$|(u, v(\lambda))| \leqslant c \quad (c = c(u)) \tag{2.28}$$

for all λ in some punctured neighborhood V of λ_0. The principle of uniform boundedness implies that $\|v(\lambda)\| \leqslant c$ in V. Therefore $v(\lambda)$ has a removable singularity at λ_0. Defining $v(\lambda_0)$ by continuity, we thus have that $v(\lambda)$ is regular at $\lambda = \lambda_0$. We now define $C_*^{-1}(\bar{\lambda})v$ at $\lambda = \lambda_0$ such that (2.27) holds also at $\lambda = \lambda_0$.

Since $(u, v(\lambda)) = \psi(\lambda)$ is regular at $\lambda = 0$, (2.28) holds in a punctured neighborhood of $\lambda = 0$. We conclude, as before, that $v(\lambda)$ is regular also at $\lambda = 0$. Thus, $v(\lambda)$ *is an entire analytic function.*

We now apply $C_*(\bar{\lambda})$ to both sides of (2.27) and obtain:

$$v = \left(\lambda^q I - A - \sum_{k=0}^{h} \lambda^{p+k} B_k^*\right)\left(\frac{w_{p-1}}{\lambda^{p-1}} + \frac{w_{p-2}}{\lambda^{p-2}} + \ldots + \frac{w_1}{\lambda}\right) + C_*(\lambda)\, v(\lambda).$$

The only coefficient of λ^{-p+1} is $-Aw_{p-1}$. Hence $Aw_{p-1} = 0$. Since A is one-to-one, we get $w_{p-1} = 0$. In the same way it follows that $w_{p-2} = 0$, and, in general, that all the w_j are zero. The last assertion of the lemma then follows from (2.27).

From the proof of Lemma 2.4 we also obtain the following result:

LEMMA 2.6. *Let A, $B_0, \ldots, B_h$ be as in Lemma 2.4. If an element v of H is orthogonal to all the generalized eigenvectors of $C(\lambda)$ for which $|\lambda| < \Lambda$, then the function $v(\lambda) \equiv C_*^{-1}(\bar{\lambda})v$ is a regular analytic function in the disc $|\lambda| < \Lambda$.*

3. Completeness theorems in case $p > q$

Our first completeness theorem is for equations of the form

$$\lambda^q u = Au + \lambda^p Bu \quad (1 \leqslant q < p). \tag{3.1}$$

THEOREM 3.1. *Let A, B be compact self-adjoint operators with zero null space. Assume that A is in C_r and B is in C_s where $r < (p-q)/q$, $s < q/(p-q)$. Then the generalized eigenvectors of* (3.1) *are complete.*

Proof. Let v be an element of H orthogonal to all the generalized eigenvectors of (3.1). We have to show that $v=0$. For any $u\in H$, consider the function $\varphi(\lambda)=(C^{-1}(\lambda)u, v)$. By Lemma 2.4, $v(\lambda)\equiv C_*^{-1}(\bar\lambda)v$ is an entire function. Setting $\lambda=1/\mu$ we find that $\varphi(\lambda)=\mu^p\varphi_0(\mu)$ where

$$\varphi_0(\mu)=((\mu^{p-q}I-B-\mu^pA)^{-1}u, v).$$

In view of our assumptions on B, we can now apply the considerations of Lemma 2.4 to the function $\varphi_0(\mu)$. We conclude (cf. (2.26)) that $\varphi_0(\mu)$ has at most a pole of order $p-1$ at $\mu=0$. It follows that $\varphi(\lambda)\to 0$ as $\lambda\to\infty$. This implies that $v(\lambda)$ is bounded in a neighborhood of $\lambda=\infty$. Hence, by Liouville's theorem, $v(\lambda)\equiv$ const. Since, however, $(u, v(\lambda))=\varphi(\lambda)\to 0$ as $\lambda\to\infty$, $v(\lambda)\equiv 0$. From (2.13), with fixed λ in $\varrho(A; B)$, we then conclude that $v=0$.

Remark. Theorem 3.1 can also be stated in the following way. Consider the equation

$$\lambda u=Au+\lambda^\alpha B\quad (\alpha>1) \tag{3.2}$$

where α is a rational number p/q. If A, B are compact, self-adjoint and one-to-one, and if $A\in C_r$, $B\in C_s$ where $r<\alpha-1$, $s<1/(\alpha-1)$, then the generalized eigenvectors of $\lambda^qI-A-\lambda^pB$ are complete. Lemma 2.2 shows that the generalized eigenvectors of $\lambda^qI-A-\lambda^pB$ and of $\lambda^{kq}I-A-\lambda^{kp}B$ span the same subspace. Thus the above result does not depend on the representation $\alpha=p/q$ of α as a ratio of integers.

We shall next consider the general case (2.1), and prove a completeness theorem in case $B_0, ..., B_h$ have sufficiently small norms. We first derive some auxiliary results.

Let v be an element of H, and let $\{v_n\}$ be a sequence of elements of H $(0\leqslant n<\infty)$. Let $\{D_k\}$ be a sequence of bounded operators in H $(0\leqslant k<\infty)$ and assume that D_0 is one-to-one. Let v, v_n, D_k satisfy the formal relation

$$v=\left(D_0-\lambda^qI+\sum_{k=1}^{\infty}\lambda^{q+k}D_k\right)\left(\sum_{n=0}^{\infty}\lambda^n v_n\right), \tag{3.3}$$

i.e.,
$$v=\sum_{n=0}^{\infty}\lambda^nD_0v_n-\sum_{n=q}^{\infty}\lambda^n v_{n-q}+\sum_{n=q+1}^{\infty}\lambda^nD_1v_{n-q-1}+\sum_{n=q+2}^{\infty}\lambda^nD_2v_{n-q-2}+\dots. \tag{3.4}$$

Introduce the following column vectors with q components:

$$\mathbf{v}=\begin{pmatrix} v\\ 0\\ \vdots\\ 0\end{pmatrix},\qquad \mathbf{v}_m=\begin{pmatrix} v_{mq}\\ v_{mq+1}\\ \vdots\\ v_{(m+1)q-1}\end{pmatrix}. \tag{3.5}$$

These vectors are elements of the space $H^q=H\times\dots\times H$ (q factors). Denote by $\mathbf{I}$ the unit matrix in H^q and set $\mathbf{D}_0 = D_0 \mathbf{I}$. Then, the first q equations obtained from (3.4) can be written in the form

$$\mathbf{D}_0 \mathbf{v}_0 = \mathbf{v}. \tag{3.6}$$

The next set of q equations can be written in the form

$$\mathbf{D}_0 \mathbf{v}_1 = \mathbf{v}_0 - \mathbf{D}_1 \mathbf{v}_0, \tag{3.7}$$

where

$$\mathbf{D}_1 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ D_1 & 0 & 0 & \dots & 0 & 0 \\ D_2 & D_1 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ D_{q-1} & D_{q-2} & D_{q-3} & \dots & D_1 & 0 \end{pmatrix}. \tag{3.8}$$

Note that if $q = 1$ then $\mathbf{D}_1 = 0$.

By equating the coefficients of λ^{nq+j} on both sides of (3.4), we get generally

$$D_0 v_{nq+j} = v_{(n-1)q+j} - D_1 v_{(n-1)q+j-1} - \dots - D_{(n-1)q+j} v_0. \tag{3.9}$$

Taking $j = 0, 1, \dots, q-1$ we can write these q equations in the matrix form

$$\mathbf{D}_0 \mathbf{v}_n = \mathbf{v}_{n-1} - \sum_{j=0}^{n-1} \mathbf{D}_{n-j} \mathbf{v}_j, \tag{3.10}$$

where, as is easily verified by induction,

$$\mathbf{D}_m = \begin{pmatrix} D_{(m-1)q} & D_{(m-1)q-1} & \dots & D_{(m-1)q-(q-1)} \\ D_{(m-1)q+1} & D_{(m-1)q} & \dots & D_{(m-1)q-(q-2)} \\ \dots & \dots & \dots & \dots \\ D_{(m-1)q+(q-1)} & D_{(m-1)q+(q-2)} & \dots & D_{(m-1)q} \end{pmatrix} \tag{3.11}$$

for all $m \geqslant 2$.

The relations (3.6), (3.7), (3.10) enable us to solve for $\mathbf{v}_n$ uniquely in terms of $\mathbf{v}$. In fact, $\mathbf{v}_n$ is clearly a linear function of $\mathbf{v}$, say $\mathbf{v}_n = \mathbf{W}_n \mathbf{v}$, where the operators $\mathbf{W}_n$ are uniquely determined. We introduce the linear operator $\mathbf{V}_n$ by setting $\mathbf{V}_n = \mathbf{D}_0^{n+1} \mathbf{W}_n$, so that

$$\mathbf{v}_n = \mathbf{D}_0^{-(n+1)} \mathbf{V}_n \mathbf{v}. \tag{3.12}$$

If we can construct operators $\mathbf{V}_n$ such that

$$\mathbf{D}_0^{-n} \mathbf{V}_n = \mathbf{D}_0^{-n} \mathbf{V}_{n-1} - \sum_{j=0}^{n-1} \mathbf{D}_{n-j} \mathbf{D}_0^{-(j+1)} \mathbf{V}_j, \tag{3.13}$$

then the $\mathbf{v}_n$ defined by (3.12) will satisfy (3.10). If we take

$$\mathbf{V}_0 = \mathbf{I} \tag{3.14}$$

then (3.7) will also be satisfied. Thus, the unique solution of (3.7), (3.10) will be given by (3.12) provided the $\mathbf{V}_n$ satisfy (3.13), (3.14).

We shall try to construct the $\mathbf{V}_n$ in the form

$$\mathbf{D}_0^{-n}\mathbf{V}_n = (\mathbf{I}-\mathbf{X}_n\mathbf{D}_0)\mathbf{D}_0^{-1}(\mathbf{I}-\mathbf{X}_{n-1}\mathbf{D}_0)\mathbf{D}_0^{-1}\dots\mathbf{D}_0^{-1}(\mathbf{I}-\mathbf{X}_1\mathbf{D}_0)\mathbf{D}_0^{-1}(\mathbf{I}-\mathbf{X}_0\mathbf{D}_0), \tag{3.15}$$

For $n=0$ this relation becomes $\mathbf{V}_0=\mathbf{I}-\mathbf{X}_0\mathbf{D}_0$. Recalling (3.14) we conclude that $\mathbf{X}_0=0$.

LEMMA 3.1. *Assume that, for some* $0<\theta<1$,

$$\sum_{k=1}^{\infty}\|\mathbf{D}_k\|\left(\frac{\|\mathbf{D}_0\|}{1-\theta}\right)^{k-1}\leqslant\theta. \tag{3.16}$$

Then the system of equations (3.13), (3.14) *has a solution of the form* (3.15). *The* $\mathbf{X}_n$ *are uniquely determined and satisfy the inequalities:*

$$\|\mathbf{X}_n\mathbf{D}_0\|\leqslant\theta \quad (0\leqslant n<\infty). \tag{3.17}$$

Proof. We proceed by induction on n. The case $n=0$ was considered before: (3.14) has a solution $\mathbf{V}_0=\mathbf{I}-\mathbf{X}_0\mathbf{D}_0$ with $\mathbf{X}_0=0$. We now proceed from $n-1$ to n. We try to determine $\mathbf{X}_n$ by substituting the expression (3.15) and the similar expressions (assumed by the inductive hypothesis) for the $\mathbf{D}_0^{-j}\mathbf{V}_j$ $(0\leqslant j\leqslant n-1)$ into (3.13). We find that

$$\mathbf{X}_n\mathbf{D}_0\mathbf{D}_0^{-1}(\mathbf{I}-\mathbf{X}_{n-1}\mathbf{D}_0)\mathbf{D}_0^{-1}\dots\mathbf{D}_0^{-1}(\mathbf{I}-\mathbf{X}_0\mathbf{D}_0)=\sum_{j=0}^{n-1}\mathbf{D}_{n-j}\mathbf{D}_0^{-1}(\mathbf{I}-\mathbf{X}_j\mathbf{D}_0)\mathbf{D}_0^{-1}\dots\mathbf{D}_0^{-1}(\mathbf{I}-\mathbf{X}_0\mathbf{D}_0).$$

Since, by the inductive assumption, $\|\mathbf{X}_j\mathbf{D}_0\|\leqslant\theta$ $(0\leqslant j\leqslant n-1)$, each operator $\mathbf{I}-\mathbf{X}_j\mathbf{D}_0$ in the last equation has an inverse. Thus $\mathbf{X}_n$ is uniquely determined and

$$\mathbf{X}_n\mathbf{D}_0=\mathbf{D}_1+\sum_{j=0}^{n-2}\mathbf{D}_{n-j}(\mathbf{I}-\mathbf{X}_{j+1}\mathbf{D}_0)^{-1}\mathbf{D}_0\dots\mathbf{D}_0(\mathbf{I}-\mathbf{X}_{n-1}\mathbf{D}_0)^{-1}\mathbf{D}_0.$$

Using the inequalities $$\|(\mathbf{I}-\mathbf{X}_j\mathbf{D}_0)^{-1}\|\leqslant\frac{1}{1-\theta}$$

and (3.16), we get $$\|\mathbf{X}_n\mathbf{D}_0\|\leqslant\sum_{j=0}^{n-1}\|\mathbf{D}_{n-j}\|\left(\frac{\|\mathbf{D}_0\|}{1-\theta}\right)^{n-j-1}\leqslant\theta.$$

This completes the proof of the lemma.

LEMMA 3.2. *Let* $\{D_k\}$ *be bounded operators in* H *with* D_0 *one-to-one, and assume that* (3.16) *holds for some* $0<\theta<1$. *Let* $\{v_n\}$ *be a sequence of elements in* H *satisfying:*

$$\limsup_{n\to\infty}\|v_n\|^{1/n}=\frac{1}{R}, \quad \textit{where } R>\frac{\|\mathbf{D}_0\|}{1-\theta}. \tag{3.18}$$

If v *is an element of* H *which satisfies* (3.3) *formally, then* $v=0$.

Proof. For any $\mathbf{z}$ in H^q,

$$\|\mathbf{D}_0^{-1}\mathbf{z}\| \geqslant \frac{\|\mathbf{z}\|}{\|\mathbf{D}_0\|}.$$

Also, by Lemma 3.1,

$$\|(\mathbf{I}-\mathbf{X}_j\mathbf{D}_0)\mathbf{z}\| \geqslant (1-\theta)\|\mathbf{z}\| \quad (j\geqslant 1),\ \mathbf{X}_0=0.$$

Therefore, we obtain from (3.15)

$$\|\mathbf{D}_0^{-n}\mathbf{V}_n\mathbf{v}\| \geqslant \left(\frac{1-\theta}{\|\mathbf{D}_0\|}\right)^n \|\mathbf{v}\|$$

for any $\mathbf{v}\in H^q$. Now let $\mathbf{v}$ be related to v by (3.5). Then, (3.12) shows that

$$\|\mathbf{D}_0\mathbf{v}_n\| = \|\mathbf{D}_0^{-n}\mathbf{V}_n\mathbf{v}\| \geqslant \left(\frac{1-\theta}{\|\mathbf{D}_0\|}\right)^n \|\mathbf{v}\| = \left(\frac{1-\theta}{\|\mathbf{D}_0\|}\right)^n \|v\|.$$

Hence

$$\|v\|^{1/n} \leqslant \frac{\|\mathbf{D}_0\|}{1-\theta}\|\mathbf{D}_0\|^{1/n}\|\mathbf{v}_n\|^{1/n}.$$

Using (3.18), we conclude that $\limsup_{n\to\infty}\|v\|^{1/n}<1$. But this is possible only if $\|v\|=0$.

For the rest of this chapter, a certain hypothesis on the operators A, B_0, ..., B_h will be repeatedly made. Therefore, it will be convenient to give this hypothesis a name.

Definition. We shall say that $C(\lambda)$, given by (2.2), satisfies the hypothesis $\mathcal{H}_r$ if the operators A, B_0, ..., B_h are all compact while A is one-to-one, self-adjoint and in class C_r.

We return to the equation (2.1) and introduce the following notation:

$$\begin{cases} \beta_n = \left\{ \sum_{j=1}^{2q-1} \|B_{(n-2)q+j-(p-q)}\|^2 \right\}^{1/2} & \text{if } n \geqslant 2, \\ \beta_1 = \left\{ \sum_{j=1}^{q-1} \|B_{j-(p-q)}\|^2 \right\}^{1/2}, \end{cases} \tag{3.19}$$

where, by definition, $B_i=0$ if either $i<0$ or if $i>h$. Note that $\beta_n=0$ if $n>1+(p+h)/q$.

We can now state the following completeness theorem.

THEOREM 3.2. *Let $C(\lambda)$ satisfy the hypothesis $\mathcal{H}_r$ for some $r<(p/q)-1$. If for some $\theta<1$*

$$\sum_{k=1}^{n_0} \beta_k \left(\frac{\|A\|}{1-\theta}\right)^{k-1} \leqslant \theta \quad \left(n_0 = \left[\frac{p+h}{q}\right]+1\right) \tag{3.20}$$

then $\mathrm{sp}_R(C)=H$ for any $R>\|A\|/(1-\theta)$. In particular, the conclusion holds if $\|B_0\|$, ..., $\|B_h\|$ are small enough.

Proof. Let v be orthogonal to all the generalized eigenvectors of (2.1) with $|\lambda| < R$. We have to show that $v=0$. By Lemma 2.6, there exists an analytic function $v(\lambda)$, regular in the disc $|\lambda| < R$, such that

$$v = C_*^{-1}(\bar{\lambda})\, v(\lambda).$$

Writing $v(\lambda)=\sum_{n=0}^{\infty} v_n \lambda^n$, we have

$$v = \left(\lambda^q I - A - \sum_{k=0}^{h} \lambda^{p+k} B_k^*\right)\left(\sum_{n=0}^{\infty} v_n \lambda^n\right) \quad \text{if } |\lambda| < R. \tag{3.21}$$

We now define the D_j in (3.3) by

$$D_0 = A, \quad D_k = 0 \text{ if } 1 \leqslant k \leqslant p-q-1 \text{ (provided } p \geqslant q+2),$$

$$D_{k+p-q} = B_k^* \text{ if } 0 \leqslant k \leqslant h, \quad D_{k+p-q} = 0 \text{ if } k > h.$$

Recalling (3.8), (3.11), and the fact that the norm of any matrix-operator $\mathbf{D} = (D_{ij})$ satisfies

$$\|\mathbf{D}\| \leqslant \sup_i \{\sum_j \|D_{ij}\|^2\}^{1/2},$$

we easily find that (3.16) is a consequence of (3.19), (3.20).

Since $v(\lambda)$ is regular for $|\lambda| < R$, $\limsup_{n\to\infty} \|v_n\|^{1/n} = 1/R$. We can therefore apply Lemma 3.2 and conclude that $v=0$.

From Theorems 1.3, 3.2 and Lemma 2.3 we obtain:

COROLLARY 1. *Let the assumptions of Theorem* 3.2 *hold. Then, for any* $\varepsilon > 0$, *the subspace* $\mathrm{sp}_\varepsilon(C)$ *has finite codimension.*

We shall now obtain some additional results in the interesting special case

$$\lambda u = Au + \lambda^2 Bu. \tag{3.22}$$

In this case $\beta_1 = 0$, $\beta_2 = \|B\|$, and the inequality (3.20) becomes

$$\frac{\|A\|\,\|B\|}{1-\theta} < \theta. \tag{3.23}$$

Since we have $q=1$, $p=2$, Theorem 3.2 yields:

COROLLARY 2. *Let* A, B *be compact operators and assume that* A *is one-to-one, self-adjoint and of class* C_r *for some* $r<1$. *If*

$$\|A\|\,\|B\| < \tfrac{1}{4} \tag{3.24}$$

then $\mathrm{sp}(C) = H$. *Even more,* $\mathrm{sp}_R(C) = H$ *if* $R > \|A\|/(1-\theta)$ *for any* θ *for which* (3.23) *holds.*

COROLLARY 3. *Let A, B be as in Corollary 2. Assume, in addition, that B is self-adjoint. Then* $\mathrm{sp}^0(C)=H$. *Even more,* $\mathrm{sp}^0_R(C)=H$ *if* $R>\|A\|/(1-\theta)$ *for any* θ *for which* (3.23) *holds.*

Proof. It suffices to show that if $\lambda_0 \in \sigma(A;B)$ and if u_0 is an eigenvector corresponding to λ_0, then the second equation of (2.3) cannot be satisfied. Indeed, this will show that $C^{-1}(\lambda)$ has a pole of order 1 at λ_0. Therefore every generalized eigenvector is an eigenvector. Using Corollary 2 it will then follow that $\mathrm{sp}^0_R(C)=\mathrm{sp}_R(C)=H$ if $R>\|A\|/(1-\theta)$.

We may suppose that $\|u_0\|=1$. From $C(\lambda_0)u_0=0$, we get

$$\lambda_0=(Au_0,u_0)+\lambda_0^2(Bu_0,u_0).$$

Hence
$$\lambda_0=\frac{1\pm\sqrt{1-4(Au_0,u_0)(Bu_0,u_0)}}{2(Bu_0,u_0)} \tag{3.25}$$

Since A and B are self-adjoint and $\|A\|\,\|B\|<1/4$, λ_0 is real.

Since λ_0 is real, $C(\lambda_0)$ is self-adjoint. Hence in order for the second equation in (2.3) to have a solution, it is necessary that $C'(\lambda_0)u_0$ be orthogonal to u_0, i.e. that

$$1=(u_0,u_0)=2\lambda_0(Bu_0,u_0).$$

This equation and (3.25) imply that $4(Au_0,u_0)\,(Bu_0,u_0)=1$. This is impossible if $\|A\|\cdot\|B\|<1/4$, since $\|u_0\|=1$.

Combining Corollary 3, Theorem 1.3 and Lemma 2.3, we obtain:

COROLLARY 4. *Let A, B be compact self-adjoint operators, and let A be one-to-one and in class C_r for some $r<1$. Assume further that $4\|A\|\,\|B\|<1$. Then, for any $\varepsilon>0$, $\mathrm{sp}^0_\varepsilon(C)$ has finite codimension.*

4. Completeness theorems for general p

In this section we shall prove some completeness theorems for the generalized eigenvectors of (2.2), without any restriction on the integer $p\geqslant 0$. We begin with the following lemma.

LEMMA 4.1. *Let $C(\lambda)$, $\tilde{C}(\lambda)$ be any two polynomials having the form (2.2). Let $D(\lambda)$ be an operator-valued function analytic in a neighborhood V of a point λ_0, such that $D(\lambda_0)$ maps H onto H and has a bounded inverse. Assume that*

$$\tilde{C}(\lambda)=D(\lambda)C(\lambda)\ \ in\ V, \tag{4.1}$$

that $\tilde{C}^{-1}(\lambda)$ exists and is analytic in a punctured neighborhood of $\lambda=\lambda_0$, and that $C^{-1}(\lambda)$ is an analytic function and has a pole of order n at $\lambda=\lambda_0$. Then $\tilde{C}^{-1}(\lambda)$ also has a pole of order n at

$\lambda = \lambda_0$ and every packet of generalized eigenvectors of $C(\lambda)$ at λ_0 is also a packet of generalized eigenvectors of $\tilde{C}(\lambda)$ at λ_0.

Proof. Clearly
$$C^{-1}(\lambda) = \tilde{C}^{-1}(\lambda)\, D(\lambda).$$
Since $C^{-1}(\lambda)$ has a pole at $\lambda = \lambda_0$, $\tilde{C}^{-1}(\lambda)$ cannot be regular there. For the same reason, the singularity of $\tilde{C}^{-1}(\lambda)$ at $\lambda = \lambda_0$ must be a pole. Let the order of the pole of $\tilde{C}^{-1}(\lambda)$ be m. Then, we can write
$$\tilde{C}^{-1}(\lambda) = \frac{\tilde{C}_{-m}}{(\lambda-\lambda_0)^m} + \frac{\tilde{C}_{-m+1}}{(\lambda-\lambda_0)^{m-1}} + \dots \quad (\tilde{C}_{-m} \neq 0)$$
for $|\lambda - \lambda_0|$ sufficiently small, $\lambda \neq \lambda_0$. We also have
$$D(\lambda) = D(\lambda_0) + (\lambda - \lambda_0)\, D'(\lambda_0) + \dots$$
in V. From the relation $C^{-1}(\lambda) = \tilde{C}^{-1}(\lambda)\, D(\lambda)$, we then get
$$C^{-1}(\lambda) = \frac{\tilde{C}_{-m}\, D(\lambda_0)}{(\lambda-\lambda_0)^m} + \dots .$$
Since the range of $D(\lambda_0)$ is H, $\tilde{C}_{-m}\, D(\lambda_0) \neq 0$. Hence $m = n$.

Now let $(u_0, \dots, u_{n-1})$ be a packet of generalized eigenvectors of $C(\lambda)$ at λ_0. Then
$$\sum_{j=0}^{k} \frac{1}{(k-j)!}\, C^{(k-j)}(\lambda_0)\, u_j = 0 \quad (0 \leqslant k \leqslant n-1). \tag{4.2}$$
In view of (4.1),
$$D(\lambda_0)\, C^{(k-j)}(\lambda_0) = \tilde{C}^{(k-j)}(\lambda_0) - \sum_{i=0}^{k-j-1} \binom{k-j}{i} D^{(k-j-i)}(\lambda_0)\, C^{(i)}(\lambda_0).$$
Therefore, multiplying (4.2) by $D(\lambda_0)$ we find that
$$\begin{aligned}\sum_{j=0}^{k} \frac{1}{(k-j)!}\, \tilde{C}^{(k-j)}(\lambda_0)\, u_j &= \sum_{j=0}^{k} \frac{1}{(k-j)!} \sum_{i=0}^{k-j-1} \binom{k-j}{i} D^{(k-j-i)}(\lambda_0)\, C^{(i)}(\lambda_0)\, u_j \\ &= \sum_{j=0}^{k} \sum_{i=1}^{k-j} \frac{1}{(k-j-i)!\, i!}\, D^{(i)}(\lambda_0)\, C^{(k-j-i)}(\lambda_0)\, u_j \\ &= \sum_{i=1}^{k} \frac{1}{i!}\, D^{(i)}(\lambda_0) \sum_{j=0}^{k-i} \frac{1}{(k-i-j)!}\, C^{(k-i-j)}(\lambda_0)\, u_j = 0.\end{aligned}$$
Therefore, $(u_0, \dots, u_{n-1})$ is a packet of generalized eigenvectors of $\tilde{C}(\lambda)$ at $\lambda = \lambda_0$. This proves Lemma 4.1.

In what follows we shall make frequent use of Theorem 3.2. Since the condition (3.20) occurring in the statement of that theorem is somewhat complicated, it will be convenient to introduce a parameter η in $C(\lambda)$ and to state the condition (3.20) as the condition that $|\eta|$ is small enough. Thus, in this section we shall consider, instead of (2.1), the equation

$$\lambda^q u = Au + \eta \sum_{k=0}^{h} \lambda^{p+k} B_k u \tag{4.3}$$

where η is a complex parameter. Setting

$$Q(\lambda) = \sum_{k=0}^{h} \lambda^k B_k, \tag{4.4}$$

we can write (4.3) in the simpler form

$$\lambda^q u = Au + \eta \lambda^p Q(\lambda) u. \tag{4.5}$$

As before, we introduce the operator

$$C(\lambda) = \lambda^q I - A - \eta \lambda^p Q(\lambda). \tag{4.6}$$

In this new notation, Theorem 3.2 becomes:

THEOREM 4.1. *Define $C(\lambda)$ by (4.6) and assume that it satisfies the hypothesis $\mathcal{H}_r$ for some $r < (p/q) - 1$. If η is small enough, then $\mathrm{sp}_R(C) = H$ for all R sufficiently large.*

In this section we shall proceed by transforming the equation (4.3) into another equation having the same form, but for which p is increased. Then we shall apply Theorem 4.1 to obtain completeness theorems for the original equation (4.3). Lemma 4.1 will be used to show that the transformation preserves the set of generalized eigenvectors.

Our first transformation of (4.3) is obtained by multiplying this equation by $I - \eta\lambda^p Q(\lambda) A^{-1}$. In that way we shall obtain a result for the case $p \geqslant q$, which shows that if the assumptions on $B_0, \ldots, B_h$ are increased then A can be allowed to belong to a larger C_r class.

THEOREM 4.2. *Let $p \geqslant q$ and let $C(\lambda)$ satisfy the hypothesis $\mathcal{H}_r$ for some $0 < r < (p/q) + k - 1$, where k is a non-negative integer. Suppose further that $Q(\lambda) A^{-k}$ is compact (for all λ). If η is sufficiently small then $\mathrm{sp}_R(C) = H$ for all R sufficiently large.*

COROLLARY. *Let the assumptions of Theorem 4.2 hold and let η be sufficiently small. Then, for any $\varepsilon > 0$, $\mathrm{sp}_\varepsilon(C)$ has finite codimension.*

The corollary follows immediately upon using Theorem 1.3 and Lemma 2.3.

Note that the assumption that $Q(\lambda) A^{-k}$ is compact is equivalent to the assumption that each of the operators $B_i A^{-k}$ $(0 \leqslant i \leqslant h)$ is compact.

Proof of Theorem 4.2. Define

$$(S_p Q)(\lambda) = Q(\lambda) A^{-1} - \eta \lambda^{p-q} Q(\lambda) A^{-1} Q(\lambda)$$

and set $$C_k(\lambda) = \lambda^q I - A - \eta\lambda^{p+kq}(S_{p+(k-1)q}\, S_{p+(k-2)q} \ldots S_p Q)(\lambda) \quad (k \geqslant 1), \tag{4.7}$$

$$C_0(\lambda) = C(\lambda). \tag{4.8}$$

One easily verifies that

$$C_k(\lambda) = [I - \eta\lambda^{p+(k-1)q}(S_{p+(k-2)q}\, S_{p+(k-3)q} \ldots S_p Q)\, A^{-1}]\, C_{k-1}(\lambda) \tag{4.9}$$

if $k \geqslant 2$, whereas
$$C_1(\lambda) = [I - \eta\lambda^p Q(\lambda)\, A^{-1}]\, C_0(\lambda). \tag{4.10}$$

Noting that $C_k(\lambda)$ has the form (4.7), we can apply Theorem 4.1. We thus conclude that $\mathrm{sp}_R(C_k) = H$ if $|\eta| \leqslant \eta_0$ where η_0 is sufficiently small and R is any sufficiently large positive number.

From (4.9), (4.10) we see that
$$C_k(\lambda) = D_k(\lambda)\, C(\lambda),$$

where $D_k(\lambda)$ is a bounded operator together with its inverse for all λ in the disc $|\lambda| \leqslant R$, provided $|\eta| \leqslant \eta_1$, where η_1 is sufficiently small. By Lemma 4.1 we then have $\mathrm{sp}_R(C) = \mathrm{sp}_R(C_k) = H$ provided $|\eta| \leqslant \min(\eta_0, \eta_1)$.

Our next theorem is based on the same transformations as before, but we now begin with an equation (4.3) for which $p \leqslant q$.

THEOREM 4.3. *Let $2^{-s} \leqslant p/q \leqslant 2^{-s+1}$, where s is a positive integer. Let $C(\lambda)$ satisfy the hypothesis $\mathcal{H}_r$ for some $0 < r < 2^s(p/q) + k - 1$, where k is a non-negative integer. Suppose further that $Q(\lambda)\, A^{-(s+k)}$ is compact. If η is sufficiently small then $\mathrm{sp}_R(C) = H$ for all sufficiently large R.*

COROLLARY 1. *Let the assumptions of Theorem 4.3 hold and let η be sufficiently small. Then, for any $\varepsilon > 0$, $\mathrm{sp}_\varepsilon(C)$ has finite codimension.*

Proof of Theorem 4.3. Define
$$(R_p Q)\,(\lambda) = Q(\lambda)\, A^{-1}\,(-\eta Q(\lambda) + \lambda^{q-p} I).$$
Then the operator
$$C_1(\lambda) = [I - \eta\lambda^p Q(\lambda)\, A^{-1}]\, C_0(\lambda)$$
can be written in the form
$$C_1(\lambda) = \lambda^q I - A - \eta\lambda^{2p}(R_p Q)\,(\lambda).$$

If $(q/2) \leqslant p \leqslant q$, we apply Theorem 4.2 to $C_1(\lambda)$ to conclude that $\mathrm{sp}_R(C_1) = H$ for sufficiently large R.

Next we introduce
$$C_2(\lambda) = [I - \eta\lambda^{2p}(R_p Q)\, A^{-1}]\, C_1(\lambda)$$

which can be written in the form

$$C_2(\lambda) = \lambda^q I - A - \eta\lambda^{2^s p}(R_{2p} R_p Q).$$

If $(q/2^2) \leqslant p \leqslant q/2$, we apply Theorem 4.2 to $C_2(\lambda)$ to conclude that $\mathrm{sp}_R(C_2) = H$ for sufficiently large R.

In this way we can proceed step by step, and thus show that, for any s, k as the statement of the theorem, $\mathrm{sp}_R(C_k) = H$ for all sufficiently large R provided $|\eta|$ is sufficiently small. We now use Lemma 4.1 to complete the proof of the theorem.

We shall derive from Theorem 4.3 a result in which s does not enter. In view of the inequality $2^s \geqslant q/p$, the condition $r < 2^s p/q + k - 1$ is surely satisfied if $r < k$. We also have

$$s \leqslant 1 + \left[\frac{\log q/p}{\log 2}\right],$$

where $[\alpha]$ denotes the largest integer $\leqslant \alpha$. Therefore, $Q(\lambda) A^{-(s+k)}$ is compact if $Q(\lambda) A^{-(k+1+\beta)}$ is compact, where

$$\beta = \left[\frac{\log q/p}{\log 2}\right].$$

We can now state:

COROLLARY 2. *Let $1 \leqslant p \leqslant q$. Let $C(\lambda)$ satisfy the hypothesis $\mathcal{H}_r$. Suppose that $Q(\lambda) A^{-(1+k+\beta)}$ is compact where k is an integer exceeding r and β is defined as before. If η is sufficiently small, then* $\mathrm{sp}_R(C) = H$ *for all sufficiently large R.*

In the previous two theorems we have assumed that all the operators $B_0 A^{-j}, \ldots, B_h A^{-j}$ are compact for an appropriate positive number j. In the next theorem we shall assume that only some of the operators $B_i A^{-j}$ are compact. However A will be assumed to belong to a class C_r with a smaller r than in the previous theorems. The transformation we shall use is that of multiplication of (4.3) by $I - \eta\lambda^p Q(0) A^{-1}$.

Definitions. We write $Q(\lambda) \in \pi_0(A)$ if $Q(0) A^{-1}$ is compact, i.e., if $B_0 A^{-1}$ is compact. For polynomials $Q(\lambda)$ in $\pi_0(A)$ we define a transformation $T_{p,1}$ by:

$$(T_{p,1} Q)(\lambda) = \frac{Q(\lambda) - Q(0)}{\lambda} + \lambda^{q-1} Q(0) A^{-1} - \eta\lambda^{p-1} Q(0) A^{-1} Q(\lambda). \tag{4.11}$$

Here p is any non-negative integer.

If $Q(\lambda) \in \pi_0(A)$ and $(T_{p,1} Q)(\lambda) \in \pi_0(A)$, we write $Q(\lambda) \in \pi_1(A)$ and define

$$(T_{p,2} Q)(\lambda) = (T_{p+1,1} T_{p,1} Q)(\lambda).$$

In general, we proceed inductively. If $Q(\lambda) \in \pi_k(A)$ and $(T_{p,k} Q)(\lambda) \in \pi_k(A)$, we write $Q(\lambda) \in \pi_{k+1}(A)$ and define

$$(T_{p,k+1}Q)(\lambda) = (T_{p+k,1}T_{p,k}Q)(\lambda). \tag{4.12}$$

If $Q(\lambda) \in \pi_k(A)$ $(k \geqslant 1)$ then we define an operator function $C_k(\lambda)$ by

$$C_k(\lambda) = \lambda^q I - A - \eta\lambda^{p+k}(T_{p,k}Q)(\lambda). \tag{4.13}$$

We also set $C_0(\lambda) = C(\lambda)$.

An easy calculation shows that

$$C_1(\lambda) = [I - \eta\lambda^p Q(0) A^{-1}]\, C_0(\lambda), \tag{4.14}$$

$$C_{k+1}(\lambda) = [I - \eta\lambda^{p+k}(T_{p,k}Q)(0) A^{-1}]\, C_k(\lambda) \quad (k \geqslant 1). \tag{4.15}$$

If we set $T_{p,0}Q = Q$ then (4.15) for $k=0$ reduces to (4.14).

Note the difference between the definitions of the $C_k(\lambda)$ in (4.13) and in (4.7). This difference is very clearly exhibited by comparing (4.14) with (4.10).

From (4.15) we get

$$C_k(\lambda) = D_k(\lambda)\, C(\lambda), \tag{4.16}$$

where

$$D_k(\lambda) = \prod_{j=0}^{k-1} [I - \eta\lambda^{p+j}(T_{p,j}Q)(0) A^{-1}]. \tag{4.17}$$

THEOREM 4.4. *Let $p+k \geqslant q$. Let $C(\lambda)$ satisfy the hypothesis $\mathcal{H}_r$ for some $0 < r < (p+k)/q - 1$. Suppose further that $Q(\lambda) \in \pi_k(A)$. If η is sufficiently small then $\mathrm{sp}_R(C) = H$ for all sufficiently large R.*

COROLLARY. *Let the assumptions of Theorem 4.4 hold and let η be sufficiently small. Then, for any $\varepsilon > 0$, $\mathrm{sp}_\varepsilon(C)$ has finite codimension.*

The proof of Theorem 4.4 is similar to the proofs of Theorems 4.2, 4.3 and is therefore omitted.

We conclude this section with two results that will be useful in the sequel.

THEOREM 4.5. *All the preceding results of Secs. 2, 3, 4, with the exception of Corollaries 2–4 of Theorem 3.2, remain correct if the hypothesis that A is self-adjoint is replaced, wherever it appears, by the hypothesis that A is normal.*

In fact all the proofs remain the same, except that now the eigenvalues of A are complex numbers.

Note that the definition of $A \in C_r$ for normal operators is the same as for self-adjoint operators.

It will be convenient to be able to refer easily to the condition $\mathcal{H}_r$ with the condition that A be self-adjoint replaced by the condition that A be normal. Therefore we make the following definition.

Definition. We shall say that $C(\lambda)$ satisfies the hypothesis $\mathcal{H}_r^*$ if the operators A, B_0, ..., B_h are compact, while A is one-to-one, normal and in C_r.

Note that for any compact self-adjoint (or normal) operator A and for any positive number α, there exists a fractional power A^α and this power is a normal operator. In fact, if

$$Au = \sum_{n=1}^{\infty} \lambda_n (u, \varphi_n)\, \varphi_n,$$

where λ_n are the eigenvalues of A, then we can take

$$A^\alpha u = \sum_{n=1}^{\infty} \lambda_n^\alpha (u, \varphi_n)\, \varphi_n. \tag{4.18}$$

A^α possesses the usual properties of fractional powers.

If $C(\lambda)$ has the form (4.6), so does $A^{-\alpha}C(\lambda)A^\alpha$. In fact,

$$A^{-\alpha}C(\lambda)A^\alpha = \lambda^q I - A - \eta\lambda^p A^{-\alpha}Q(\lambda)A^\alpha.$$

LEMMA 4.2. *Let $\alpha \geqslant 0$. Suppose that $C(\lambda)$ and $A^{-\alpha}C(\lambda)A^\alpha$ both satisfy $\mathcal{H}_r^*$. If* $\mathrm{sp}_R(A^{-\alpha}CA^\alpha) = H$, *then* $\mathrm{sp}_R(C) = H$.

Proof. If u is a generalized eigenvector of $A^{-\alpha}C(\lambda)A^\alpha$, then $A^\alpha u$ is a generalized eigenvector of $C(\lambda)$. Hence, if v is orthogonal to $\mathrm{sp}_R(C)$, then $A_*^\alpha v$ must be orthogonal to $\mathrm{sp}_R(A^{-\alpha} CA^\alpha)$. Therefore, $A_*^\alpha v = 0$, so that $v = 0$.

THEOREM 4.6. *Theorem 4.2 remains valid if the condition that $Q(\lambda)A^{-k}$ is compact is replaced by the condition that, for some $\alpha \geqslant 0$, $A^{-\alpha}Q(\lambda)A^{\alpha-k}$ is compact. Similarly, the conditions that $Q(\lambda)A^{-(s+k)}$ and $Q(\lambda)A^{-(1+k+\beta)}$ are compact, occurring in Theorem 4.3 and its Corollary 2, can be replaced by the conditions that $A^{-\alpha}Q(\lambda)A^{\alpha-(s+k)}$ and $A^{-\alpha}Q(\lambda)A^{\alpha-(1+k+\beta)}$ are compact, respectively.*

Proof. We shall carry out the proof as it applies to Theorem 4.2; the other parts of the proof are similar. Suppose, then, that $C(\lambda)$ satisfies $\mathcal{H}_r$ and that $A^{-\alpha}Q(\lambda)A^{\alpha-k}$ is compact. Then $A^{-\alpha}Q(\lambda)A^\alpha$ is clearly compact and $A^{-\alpha}C(\lambda)A^\alpha$ also satisfies $\mathcal{H}_r$. Theorem 4.2 then shows that $\mathrm{sp}_R(A^{-\alpha}CA^\alpha) = H$ for all R sufficiently large, provided η is sufficiently small. Lemma 4.2 then shows that $\mathrm{sp}_R(C) = H$.

Remark. Theorem 4.6 obviously extends to the case where the condition $\mathcal{H}_r$ is replaced everywhere by the condition $\mathcal{H}_r^*$.

5. Linearization and its consequences

We shall maintain the notation (4.3), and set

$$Q(\lambda) = \sum_{k=0}^{h} \lambda^k B_k, \tag{5.1}$$

as before. Then we have

$$C(\lambda) = \lambda^q I - A - \eta\lambda^p Q(\lambda). \tag{5.2}$$

We wish to analyse the generalized eigenvectors of $C(\lambda^k)$ for any positive integer k. We consider then the equation

$$\lambda^{kq} u_1 = Au_1 + \eta\lambda^{kp} Q(\lambda^k) u_1. \tag{5.3}$$

We shall transform (5.3) into a system of equations by considering the equations

$$\begin{cases} \lambda^q u_1 = A^{1/k} u_2, \\ \lambda^q u_2 = A^{1/k} u_3, \\ \dots \\ \lambda^q u_{k-1} = A^{1/k} u_k, \end{cases} \tag{5.4}$$

where $A^{1/k}$ is the normal operator introduced in Sec. 4. As before, A is assumed to be compact, one-to-one and either self-adjoint or normal. Then, (5.3) gives

$$\lambda^q u_k = A^{1/k} u_1 + \eta\lambda^{kp-(i-1)q} A^{-(k-1)/k} Q(\lambda^k) A^{(i-1)/k} u_i \tag{5.5}$$

for any $i, 1 \leqslant i \leqslant k$. Here, we have used the relation $\lambda^{(i-1)q} u_1 = A^{(i-1)/k} u_i$, which follows from (5.4).

Define

$$\mathbf{A} = \begin{pmatrix} 0 & A^{1/k} & 0 & \dots & 0 \\ 0 & 0 & A^{1/k} & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & A^{1/k} \\ A^{1/k} & 0 & 0 & \dots & 0 \end{pmatrix}$$

and let $\mathbf{Q}(\lambda)$ be the matrix having zero entries everywhere except in the kth row and the ith column, where it has the entry $A^{-(k-1)/k} Q(\lambda^k) A^{(i-1)/k}$. If we write $\mathbf{u}$ for the column vector $(u_1, \dots, u_k)$, then the system (5.3), (5.4) becomes

$$\lambda^q \mathbf{u} = \mathbf{A}\mathbf{u} + \eta\lambda^{kp-(i-1)q} \mathbf{Q}(\lambda)\mathbf{u}. \tag{5.6}$$

Set

$$\mathbf{C}(\lambda) = \lambda^q \mathbf{I} - \mathbf{A} - \eta\lambda^{kp-(i-1)q} \mathbf{Q}(\lambda), \tag{5.7}$$

where $\mathbf{I}$ is the identity operator in $H^q = H \times \dots \times H$ (q factors).

We shall say that the system (5.6) is a *linearization* of the equation (4.3). This process of linearization has important consequences for the generalization of the results of Sec. 4. Before going on to some of these, we shall state a result that shows that the generalized eigenvectors of (5.2) can be obtained in a simple way from the generalized eigenvectors of (5.7).

LEMMA 5.1. *Let $C(\lambda)$ satisfy the hypothesis $\mathcal{H}_r^*$ for some $r>0$. Assume further that for some fixed integers i and k with $1\leqslant i\leqslant k$, the operators*

$$A^{-(k-1)/k}B_j A^{(i-1)/k} \quad (j=0,1,\ldots,h)$$

are compact. Then $\mathbf{C}^{-1}(\lambda)$ has a pole of order n at a point $\lambda_0\neq 0$ if and only if $\Gamma(\lambda)\equiv C(\lambda^k)$ has a pole of order n at λ_0. If $(\mathbf{u}^0,\ldots,\mathbf{u}^{n-1})$ is a packet of generalized eigenvectors of $\mathbf{C}(\lambda)$ at $\lambda=\lambda_0$, denote the first components of these vectors by $u_1^0,\ldots,u_1^{n-1}$, respectively. Then $(u_1^0,\ldots,u_1^{n-1})$ is a packet of generalized eigenvectors of $C(\lambda^k)$ at $\lambda=\lambda_0$.

Note that $\mathbf{C}(\lambda)$ has the form

$$\lambda^q\mathbf{I}-\mathbf{A}-\eta\sum_{j=0}^{h}\lambda^{kp-(i-1)q}\lambda^j\mathbf{B}_j,$$

where the $\mathbf{B}_j$ are compact operators. Thus the results of Chapter 1 show that, if the resolvent set of $\mathbf{C}(\lambda)$ is not empty, $\mathbf{C}^{-1}(\lambda)$ is an analytic function in $\mathbf{C}$ except for a countable set of poles with no finite points of accumulation except, possibly, zero.

Before proving Lemma 5.1, we derive some consequences of it.

One easily verifies that $\mathbf{A}$ is a normal operator. Suppose A is in class C_r. Then $A^{1/k}$ is in class C_{kr}. As is easily seen, the sequence of eigenvalues of $\mathbf{A}$ is obtained from the sequence of eigenvalues of $A^{1/k}$ by repeating k times each eigenvalue of the latter. Hence $\mathbf{A}$ is also in class C_{kr}.

If i, k are chosen so that $i/k\leqslant p/q$ then Theorems 4.2 and 4.5 can be applied to $\mathbf{C}(\lambda)$. The result is that $\mathrm{sp}_R(\mathbf{C})=H^q$ for η sufficiently small and R sufficiently large, provided there is a non-negative integer l such that

$$kr<k\frac{p}{q}-i+l$$

while $\mathbf{QA}^{-l}$ is compact. From the definitions of $\mathbf{Q}$ and $\mathbf{A}$ it can be seen that $\mathbf{QA}^{-l}$ is compact if

$$A^{-(k-1)/k}Q(\lambda)A^{-(l-i+1)/k}$$

is compact. Here l is an arbitrary non-negative integer, k is an arbitrary positive integer, and i is an integer restricted by

$$\frac{1}{k} \leqslant \frac{i}{k} \leqslant \min\left(1, \frac{p}{q}\right).$$

Therefore the number $t = (l - i)/k$ can be chosen to be an arbitrary rational number subject only to the restriction

$$t \geqslant -\min\left(1, \frac{p}{q}\right).$$

Thus, if $A \in C_r$, $r < (p/q) + t$, while $A^{-(k-1)/k} Q A^{-t-1/k}$ is compact, then $\mathrm{sp}_R(\mathbf{C}) = H^q$ for η sufficiently small and R sufficiently large. From Lemma 5.1 and the fact that $\mathrm{sp}_R(\mathbf{C}) = H^q$, it follows that $\mathrm{sp}_R(\Gamma) = H$ where $\Gamma(\lambda) = C(\lambda^k)$. But Lemma 2.2 then shows that $\mathrm{sp}_r(C) = H$ for $r = R^h$. We sum up:

LEMMA 5.2. *Let $C(\lambda)$ satisfy $\mathcal{H}_r^*$. Suppose there is a rational number $t \geqslant -\min(1, p/q)$ such that $r < (p/q) + t$, while for some integer k, with kt an integer, the operators*

$$A^{-(k-1)/k} B_j A^{-(1/k)-t} \quad (0 \leqslant j \leqslant h)$$

are compact. If η is sufficiently small, then $\mathrm{sp}_R(C) = H$ for all R sufficiently large.

If instead of applying Theorem 4.2 we apply Corollary 2 of Theorem 4.3, then we obtain by the same method the following result:

LEMMA 5.3. *Let $C(\lambda)$ satisfy $\mathcal{H}_r^*$. Let i, k and l be integers such that*

$$\frac{p}{q} \leqslant \frac{i}{k} \leqslant \frac{p}{q} + \frac{q-1}{kq}, \quad r < \frac{l}{k}.$$

Assume that the operators

$$A^{-(k-1)/k} B_j A^{(i-l-2-\beta)/k} \quad (0 \leqslant j \leqslant h)$$

are compact, where

$$\beta = \left[\frac{\log q/(kp - (i-1)q)}{\log 2}\right].$$

If η is sufficiently small, then $\mathrm{sp}_R(C) = H$ for all R sufficiently large.

We shall now prove the following theorem.

THEOREM 5.1. *Let $C(\lambda)$ satisfy the hypothesis $\mathcal{H}_r^*$. Let s be a real number with $r < (p/q) + s$ and suppose that the operators*

$$A^{-1} B_j A^{-s} \quad (0 \leqslant j \leqslant h)$$

are bounded. If η is sufficiently small then $\mathrm{sp}_R(C) = H$ for all R sufficiently large.

Proof. We shall apply Lemma 5.3 with $i/k = p/q$. Choose a rational number t such that

$$t < s \quad \text{and} \quad r < \frac{p}{q} + t,$$

and set $\sigma = t + p/q$. We can represent σ as a fraction l/k with k so large that

$$t + \frac{2+\beta}{k} < s.$$

(Note that $\beta = 0$.) Since $i/k = p/q$, it follows that the operators

$$A^{-1} B_j A^{-t-(2+\beta)/k} = A^{-1} B_j A^{(i-l-2-\beta)/k} \quad (0 \leqslant j \leqslant h)$$

are bounded. Therefore, the operators

$$A^{(1/k)-1} B_j A^{(i-l-2-\beta)/k} \quad (0 \leqslant j \leqslant h)$$

are compact. Since $r < \sigma = l/k$, we can apply Lemma 5.3 to complete the proof of the theorem.

It is interesting to note that if we employ Lemma 5.2, instead of Lemma 5.3, then the above argument yields the assertion of Theorem 5.1 provided we make the additional assumption that $s > -\min(1, p/q)$.

We also note that variants of Lemmas 5.2, 5.3 and Theorem 5.1 can be derived by employing Lemma 4.2.

Finally, we remark that the assertions of either Lemma 5.2, or Lemma 5.3, or Theorem 5.1 imply that $\mathrm{sp}_\varepsilon(C)$ has finite codimension, for any $\varepsilon > 0$.

Proof of Lemma 5.1. Set

$$\mathbf{C}(\lambda)\,\mathbf{u} = \mathbf{v}; \tag{5.8}$$

where $\mathbf{v}$ is the column vector $(v_1, \ldots, v_k)$. A straightforward computation shows that

$$\lambda^{(j-1)q} u_1 = A^{(j-1)/k} u_j + \sum_{l=1}^{j-1} \lambda^{(j-1-l)q} A^{(l-1)/k} v_l \quad (1 \leqslant j \leqslant k). \tag{5.9}$$

Therefore,

$$\begin{aligned}
\lambda^{kq} u_1 &= A^{(k-1)/k} \lambda^q u_k + \sum_{l=1}^{k-1} \lambda^{(k-l)q} A^{(l-1)/k} v_l \\
&= A u_1 + \lambda^{kp-(i-1)q} Q(\lambda^k) A^{(i-1)/k} u_i + \sum_{l=1}^{k} \lambda^{(k-l)q} A^{(l-1)/k} v_l \\
&= A u_1 + \lambda^{kp} Q(\lambda^k) \left[u_1 - \sum_{l=1}^{i-1} \lambda^{-lq} A^{(l-1)/k} v_l \right] + \sum_{l=1}^{k} \lambda^{(k-l)q} A^{(l-1)/k} v_l.
\end{aligned}$$

Hence,

$$C(\lambda^q)\, u_1 = \sum_{l=1}^{k} \lambda^{(k-l)q} A^{(l-1)/k} v_l - \lambda^{kp} Q(\lambda^k) \sum_{l=1}^{i-1} \lambda^{-lq} A^{(l-1)/k} v_l.$$

From this and (5.9) we obtain

$$C(\lambda^k)\,A^{(j-1)/k}u_j=\sum_{l=1}^{k}\lambda^{(k+j-1-l)q}A^{(l-1)/k}v_l-\lambda^{kp}Q(\lambda^k)\sum_{l=1}^{j-1}\lambda^{(j-1-l)q}A^{(l-1)/k}v_l$$

$$-C(\lambda^k)\sum_{l=1}^{j-1}\lambda^{(j-1-l)q}A^{(l-1)/k}v_l. \tag{5.10}$$

The right-hand side is analytic in λ if $\lambda\neq 0$. Therefore we can write

$$C(\lambda^k)\,A^{(j-1)/k}u_j=\sum_{l=1}^{k}\Gamma_{jl}v_l \tag{5.11}$$

in a neighborhood of a pole λ_0 of $\mathbf{C}(\lambda)$, where $\Gamma_{jl}(\lambda)$ are operator-valued analytic functions regular in the same neighborhood. Solving for u_j, we obtain from (5.11),

$$u_j=A^{(1-j)/k}C^{-1}(\lambda^k)\sum_{l=1}^{k}\Gamma_{jl}v_l,$$

so that $\mathbf{C}^{-1}(\lambda)$ has the form

$$\begin{pmatrix} I & 0 & 0 & \dots & 0 \\ 0 & A^{-1/k} & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \dots & A^{-(k-1)/k}\end{pmatrix}\begin{pmatrix} C^{-1}(\lambda^k) & 0 & 0 & \dots & 0 \\ 0 & C^{-1}(\lambda^k) & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \dots & C^{-1}(\lambda^k)\end{pmatrix}$$

$$\cdot\begin{pmatrix}\Gamma_{11} & \Gamma_{12} & \dots & \Gamma_{1k} \\ \Gamma_{21} & \Gamma_{22} & \dots & \Gamma_{2k} \\ \cdot & \cdot & \cdots & \cdot \\ \Gamma_{k1} & \Gamma_{k2} & \dots & \Gamma_{kk}\end{pmatrix}\equiv \mathbf{EB\Gamma}.$$

$\mathbf{E}$ is constant and does not increase the order of the pole of $\mathbf{B\Gamma}$. $\mathbf{\Gamma}$ is analytic at λ_0. Therefore, if m is the order of the pole of $C^{-1}(\lambda^k)$ at λ_0, and n is the order of the pole of $\mathbf{C}^{-1}(\lambda)$ at λ_0, then $m\geqslant n$.

Next, from (5.10), (5.11), we see that

$$\Gamma_{jk}=\lambda^{j-1}A^{(k-1)/k}.$$

It follows that the entry (j,k) of $\mathbf{C}^{-1}(\lambda)$ is

$$\lambda^{j-1}A^{-(j-1)/k}C^{-1}(\lambda^k)\,A^{(k-1)/k}.$$

Since A is self-adjoint with zero null-space, the range of A^α (for any $\alpha>0$) is dense in H. It follows that the last operator has a pole of order m exactly. But no entry in $\mathbf{C}^{-1}(\lambda)$ can have a pole of order greater than n. Hence $m\leqslant n$. Since we have already proved above that $m\geqslant n$, we conclude that $m=n$.

Now let $(\mathbf{u}^0,\dots,\mathbf{u}^{n-1})$ be a packet of generalized eigenvectors of $\mathbf{C}(\lambda)$ at $\lambda=\lambda_0$ and denote the components of $\mathbf{u}^j$ by $u_1^j,\dots,u_k^j$. We have:

$$\sum_{j=0}^{s} \frac{1}{j!} \mathbf{C}^{(j)}(\lambda_0)\, \mathbf{u}^{s-j} = 0 \quad (0 \leqslant s \leqslant n-1). \tag{5.12}$$

We write
$$\mathbf{R}(\lambda) = \eta \lambda^{kp-(i-1)q} \mathbf{Q}(\lambda), \tag{5.13}$$

so that
$$\mathbf{C}(\lambda) = \lambda^q \mathbf{I} - A - \mathbf{R}(\lambda). \tag{5.14}$$

Then,
$$\mathbf{C}^{(j)}(\lambda) = \frac{q!}{(q-j)!} \lambda^{(q-j)} \mathbf{I} - \mathbf{R}^{(j)}(\lambda), \quad j \geqslant 1, \tag{5.15}$$

where, by definition, $m! = 0$ if $m < 0$. Substituting (5.15) into (5.12), we get

$$\mathbf{C}(\lambda_0)\, \mathbf{u}^s + \sum_{j=1}^{s} \left[\binom{q}{j} \lambda_0^{q-j} \mathbf{I} - \frac{1}{j!} \mathbf{R}^{(j)}(\lambda_0) \right] \mathbf{u}^{s-j} = 0 \quad (0 \leqslant s \leqslant n-1). \tag{5.16}$$

This means that

$$\begin{cases} \lambda_0^q u_1^0 - A^{1/k} u_2^0 = 0, \\ \cdots \\ \lambda_0^q u_{k-1}^0 - A^{1/k} u_k^0 = 0, \\ \lambda_0^q u_k^0 - A^{1/k} u_1^0 - A^{-(k-1)/k} (\lambda_0^{kp-(i-1)q} R(\lambda_0))\, A^{(i-1)/k} u_i^0 = 0 \quad (R(\lambda) = Q(\lambda^k)), \end{cases} \tag{5.17}$$

and, generally, for $1 \leqslant s \leqslant n-1$,

$$\begin{cases} \lambda_0^q u_1^s - A^{1/k} u_2^s + \sum_{j=1}^{s} \binom{q}{j} \lambda_0^{q-j} u_1^{s-j} = 0 \\ \cdots \\ \lambda_0^q u_{k-1}^s - A^{1/k} u_k^s + \sum_{j=1}^{s} \binom{q}{j} \lambda_0^{q-j} u_{k-1}^{s-j} = 0 \\ \lambda_0^q u_k^s - A^{1/k} u_1^s + \left[\sum_{j=1}^{s} \binom{q}{j} \lambda_0^{q-j} u_k^{s-j} - \sum_{j=0}^{s} \frac{1}{j!} A^{-(k-1)/k} [\lambda^{kp-(i-1)q} R(\lambda)]_{\lambda_0}^{(j)} A^{(i-1)/k} u_i^{s-j} \right] = 0. \end{cases} \tag{5.18}$$

The system (5.17) shows that

$$C(\lambda_0^k)\, u_1^0 = 0 \tag{5.19}$$

and, since $A^{(j-1)/k} u_j^0 = \lambda_0^{(j-1)q} u_1^0$, also

$$C(\lambda_0^k)\, A^{(j-1)/k} u_j^0 = 0. \tag{5.20}$$

We now turn to equations (5.18). They imply that

$$A^{1/k} u_j^s = \lambda_0^q u_{j-1}^s + \sum_{l=1}^{s} \binom{q}{l} \lambda_0^{q-l} u_{j-1}^{s-l} = \sum_{l=0}^{s} \binom{q}{l} \lambda_0^{q-l} u_{j-1}^{s-l}. \tag{5.21}$$

Therefore
$$A^{1/k} u_2^s = \lambda_0^q u_1^s + \sum_{l=1}^{s} \binom{q}{l} \lambda_0^{q-l} u_1^{s-l} = \sum_{l=0}^{s} \binom{q}{l} \lambda_0^{q-l} u_1^{s-l}. \tag{5.22}$$

In what follows we shall make use of the formula (see, for instance, [13])

$$\sum_{k=0}^{p} \binom{n}{k} \binom{m}{p-k} = \binom{n+m}{p} \quad (m \text{ an integer } \geqslant 1). \tag{5.23}$$

Using (5.21)–(5.23) we have:

$$\begin{aligned}
A^{2/k} u_3^s &= A^{1/k} \left[\lambda_0^q u_2^s + \sum_{l=1}^{s} \binom{q}{l} \lambda_0^{q-l} u_2^{s-l} \right] \\
&= \lambda_0^{2q} u_1^s + \sum_{l=1}^{s} \binom{q}{l} \lambda_0^{2q-l} u_1^{s-l} + \sum_{l=1}^{s} \binom{q}{l} \lambda_0^{q-l} \left[\lambda_0^q u_1^{s-l} + \sum_{m=l}^{s-l} \binom{q}{m} \lambda_0^{q-m} u_1^{s-(l+m)} \right] \\
&= \lambda_0^{2q} u_1^s + 2 \sum_{l=1}^{s} \binom{q}{l} \lambda_0^{2q-l} u_1^{s-l} + \sum_{l=1}^{s-1} \sum_{m=1}^{s-l} \binom{q}{l} \binom{q}{m} \lambda_0^{2q-(l+m)} u_1^{s-(l+m)} \\
&= \lambda_0^{2q} u_1^s + 2 \sum_{l=1}^{s} \binom{q}{l} \lambda_0^{2q-l} u_1^{s-l} + \sum_{l=2}^{s} \sum_{m=1}^{l-1} \binom{q}{m} \binom{q}{l-m} \lambda_0^{2q-l} u_1^{s-l} \\
&= \lambda_0^{2q} u_1^s + 2 \sum_{l=1}^{s} \binom{q}{l} \lambda_0^{2q-l} u_1^{s-l} + \sum_{l=2}^{s} \left[\binom{2q}{l} - 2 \binom{q}{l} \right] \lambda_0^{2q-l} u_1^{s-l} \\
&= \lambda_0^{2q} u_1^s + 2q \lambda_0^{2q-1} u_1^{s-1} + \sum_{l=2}^{s} \binom{2q}{l} \lambda_0^{2q-l} u_1^{s-l} \\
&= \lambda_0^{2q} u_1^s + \sum_{l=1}^{s} \binom{2q}{l} \lambda_0^{2q-l} u_1^{s-l}.
\end{aligned}$$

Thus

$$A^{2/k} u_3^s = \sum_{l=0}^{s} \binom{2q}{l} \lambda_0^{2q-l} u_1^{s-l}. \tag{5.24}$$

We can similarly proceed by induction to prove that

$$A^{j/k} u_{j+1}^s = \sum_{l=0}^{k} \binom{jq}{l} \lambda_0^{jq-l} u_1^{s-l}. \tag{5.25}$$

In fact, if (5.25) is true then we can write, by (5.21),

$$A^{(j+1)/k} u_{j+2}^s = A^{j/k} A^{1/k} u_{j+2}^s = A^{j/k} \left[\sum_{l=0}^{s} \binom{q}{l} \lambda_0^{q-l} u_{j+1}^{s-l} \right]$$

and evaluate the terms on the right by using (5.25) and (5.23). We thus find that (5.25) holds also with j replaced by $j+1$.

From the last equation in (5.18) and from (5.25), we get

$$\begin{aligned}
\lambda_0^q A^{(k-1)/k} u_k^s &= A u_1^s - \sum_{j=1}^{s} \binom{q}{j} \lambda_0^{q-j} A^{(k-1)/k} u_k^{s-j} - \sum_{j=0}^{s} \frac{1}{j!} [\lambda^{kp-(i-1)q} R(\lambda)]_{\lambda_0}^{(j)} A^{(i-1)/k} u_i^{s-j} \\
&= A u_1^s - \sum_{j=1}^{s} \binom{q}{j} \sum_{l=0}^{s-j} \binom{(k-1)q}{l} \lambda_0^{(k-1)q-l} u_1^{s-j-l} \\
&\quad + \sum_{j=0}^{s} \frac{1}{j!} [\lambda^{kp+(i-1)q} R(\lambda)]_{\lambda_0}^{(j)} \sum_{l=0}^{s-j} \binom{(i-1)q}{l} \lambda_0^{(i-1)q-l} u_1^{s-j-l} \\
&= A u_1^s - \sum_{l=1}^{s} \left[\binom{kq}{l} - \binom{(k-1)q}{l} \right] \lambda_0^{kq-l} u_1^{s-l} \\
&\quad + \sum_{j=0}^{s} \frac{1}{j!} [\lambda^{kp-(i-1)q} R(\lambda)]_{\lambda_0}^{(j)} \lambda_0^{(k-1)q+j-l} u_1^{s-l} \\
&= \sum_{l=0}^{s} \binom{(k-1)q}{l} \lambda_0^{kq-l} u_1^{s-l},
\end{aligned}$$

by (5.25). Abbreviating u_1^l to u^l, we then have

$$(\lambda_0^{kq} I - A) u^s + \sum_{l=1}^{s} \binom{kq}{l} \lambda_0^{kq-l} u^{s-l} - S(\lambda_0) = 0, \tag{5.26}$$

where
$$\begin{aligned}
S(\lambda) &= \sum_{l=0}^{s} \sum_{j=0}^{l} \frac{1}{j!} \binom{(i-1)q}{l-j} \lambda^{(i-1)q+j-l} [\lambda^{kp-(i-1)q} R(\lambda)]^{(j)} u^{s-l} \\
&= \sum_{l=0}^{s} \sum_{j=0}^{l} \sum_{m=0}^{j} \frac{1}{j!} \binom{(i-1)q}{l-j} \binom{j}{m} \lambda^{(i-1)q+j-l} R^{(j-m)}(\lambda) \frac{d^m}{d\lambda^m} [\lambda^{kp-(i-1)q}] u^{s-l} \\
&= \sum_{l=0}^{s} \sum_{j=0}^{l} \sum_{m=0}^{j} \binom{(i-1)q}{l-j} \frac{R^{(j-m)}}{m!(j-m)!} \frac{(kp-(i-1)q)!}{(kp-(i-1)q-m)!} \lambda^{kp+j-l-m} u^{s-l} \\
&= \sum_{l=0}^{s} \sum_{j=0}^{l} \sum_{m=0}^{j} \binom{(i-1)q}{l-j} \frac{R^{(m)}}{m!} \binom{kp-(i-1)q}{j-m} \lambda^{kp-l+m} u^{s-m} \\
&= \sum_{j=0}^{s} \sum_{m=0}^{l} \sum_{r=0}^{l-m} \binom{(i-1)q}{l-m-r} \binom{kp-(i-1)q}{r} \frac{1}{m!} \lambda^{kp-l+m} R^{(m)} u^{s-l} \\
&= \sum_{l=0}^{s} \sum_{m=0}^{l} \binom{kp}{l-m} \frac{1}{m!} \lambda^{kp-l+m} R^{(m)} u^{s-l} \\
&= \sum_{l=0}^{s} \frac{1}{l!} \frac{d^l}{d\lambda^l} (\lambda^{kp} R(\lambda)) u^{s-l}.
\end{aligned}$$

Combining this last result with (5.26) we find that, for $\lambda = \lambda_0$,

$$(\lambda^{kq} I - A) u^s + \sum_{l=1}^{s} \binom{kq}{l} \lambda^{kq-l} u^{s-l} + \sum_{l=0}^{s} \frac{1}{l!} \frac{d^l}{d\lambda^l} (\lambda^{kp} R(\lambda)) u^{s-l} = 0,$$

i.e.,
$$\sum_{l=0}^{s} \frac{1}{l!} \frac{d^l}{d\lambda^l} C(\lambda^k) u^{s-l} = 0 \quad (0 \leqslant s \leqslant n-1).$$

This shows that $(u_1^0, \ldots, u_1^{n-1})$ is a packet of generalized eigenvectors of $C(\lambda^k)$ at $\lambda = \lambda_0$. The proof of Lemma 5.1 is thereby completed.

Chapter 3. The case of general A

6. The completeness theorem

In this chapter we consider the eigenvalue problem (1.1), not making the assumption that A is self-adjoint. We shall establish a completeness theorem in case $q \geqslant 2, p = 1, h = q - 2$. In this case, (1.1) becomes

$$\lambda^q u = Au + \sum_{k=0}^{q-2} \lambda^{k+1} B_k u.$$

Setting $A_1 = A$, $A_{k+2} = B_k$, the last equation becomes

$$\lambda^q u = \sum_{k=1}^{q} \lambda^{k-1} A_k u. \tag{6.1}$$

The method we shall use is entirely different from the methods of the previous sections. We shall need the following assumption:

(H_1) A_1^{-1} exists and is a closed densely defined operator in H. Furthermore, the resolvent $R(\lambda; A_1^{-1}) = (\lambda I - A_1^{-1})^{-1}$ of A_1^{-1} exists for all λ, $-\infty < \lambda \leqslant 0$, and

$$\| R(\lambda; A_1^{-1}) \| \leqslant \frac{C}{1 + |\lambda|} \quad (-\infty < \lambda \leqslant 0). \tag{6.2}$$

Note that if A_1 is a bounded operator and A_1^{-1} exists, then A_1^{-1} is a closed operator. If A_1 is compact, then A_1^{-1} cannot be a bounded operator (since H is infinite dimensional).

Set $T = A_1^{-1}$. In view of (6.2), we can define the fractional powers $T^{-\theta}$ of T, for $0 < \theta < 1$, by (see Kato [9])

$$T^{-\theta} = \frac{\sin \pi\theta}{\pi} \int_0^\infty \lambda^{-\theta} (\lambda I + T)^{-1} d\lambda. \tag{6.3}$$

We have: $T^{-\theta_1} T^{-\theta_2} = T^{-(\theta_1 + \theta_2)}$. One further defines T^θ $(0 < \theta < 1)$ by $T^\theta = (T^{-\theta})^{-1}$. It is easy to show that

$$T^\theta u = \frac{\sin \pi\theta}{\pi} \int_0^\infty \lambda^{\theta - 1} T (\lambda I + T)^{-1} u d\lambda$$

if u belongs to the domain of T.

LEMMA 6.1. *Assume* (H_1). *If A_1 is compact then A_1^θ is also compact for* $0 < \theta < 1$.

Proof. $T^{-\theta}$ is clearly a bounded operator. Write

$$(\lambda I + T) T^{-1} = \lambda T^{-1} + I.$$

It follows that $(I+\lambda T^{-1})^{-1}$ exists and equals $T(\lambda I+T)^{-1}$. Hence by (6.2), $(I+\lambda T^{-1})^{-1}$ is a bounded operator; in fact,

$$\|(I+\lambda T^{-1})^{-1}\| \leqslant \|T(\lambda I+T)^{-1}\| \leqslant C.$$

Writing $(\lambda I+T)^{-1}=T^{-1}(\lambda T^{-1}+I)^{-1}$ and noting that T^{-1} is compact whereas $(\lambda T^{-1}+I)^{-1}$ is bounded, we conclude that $(\lambda I+T)^{-1}$ is compact. Since the integral

$$I_{\varepsilon N}\equiv\int_{\varepsilon}^{N}\lambda^{-\theta}(\lambda I+T)^{-1}\,d\lambda \quad (0<\varepsilon<N<\infty)$$

is a limit in the uniform topology of sums $\Sigma\lambda_i^{-\theta}(\lambda_i I+T)^{-1}\Delta\lambda_i$ of compact operators, it follows that $I_{\varepsilon N}$ is a compact operator. Since, finally, $I_{\varepsilon N}\to I_{0\infty}$ in the uniform topology, as $\varepsilon\to 0$, $N\to\infty$, the assertion follows.

Notations. Let A_1 satisfy (H_1). In this and in the following section we shall consistently write

$$B=A_1^{1/q}. \tag{6.4}$$

We set $H^q=H\times\ldots\times H$ (q times) and we shall write elements of H^q as columns of elements of H. In H^q we define an operator $\mathbf{A}$ as multiplication on the left by the matrix

$$\begin{pmatrix} 0 & B & 0 & \ldots\ 0 \\ 0 & 0 & B & \ldots\ 0 \\ 0 & 0 & 0 & \ldots\ 0 \\ \cdot & \cdot & \cdot & \ldots\ \cdot \\ 0 & 0 & 0 & \ldots\ B \\ B^{1-q}A_1 & B^{1-q}A_2B & B^{1-q}A_3B^2 & \ldots\ B^{1-q}A_qB^{q-1} \end{pmatrix}. \tag{6.5}$$

We shall denote this matrix by $\mathbf{A}$ also. We shall denote by $\sigma(\mathbf{A})$ and $\varrho(\mathbf{A})$ the spectrum and the resolvent sets of the operator $\mathbf{A}$. Finally, we shall denote by $\mathbf{I}$ the identity operator in H^q.

THEOREM 6.1. *Let $A_1, \ldots, A_q$ be compact operators and let A_1 satisfy* (H_1). *Suppose that the operators*

$$B^{1-q}A_k \text{ are bounded, } k=1, \ldots, q. \tag{6.6}$$

Then $\sigma(\mathbf{A})=\{0\}\cup\sigma(A_1; A_2, \ldots, A_q)$ and $\varrho(\mathbf{A})=\varrho(A_1; A_2, \ldots, A_q)$. Moreover, if $\lambda\in\sigma(A_1; A_2, \ldots, A_q)$ then an element $u\in H$ satisfies (6.1) *if and only if it is the first component of a vector $\mathbf{u}$ in H^q satisfying*

$$(\lambda\mathbf{I}-\mathbf{A})\mathbf{u}=0. \tag{6.7}$$

Proof. Let $\lambda \in \sigma(A_1; A_2, ..., A_q)$ and let u satisfy (6.1), $u \neq 0$. Write $u_1 = u$. Then we can rewrite (6.1) in the form

$$\lambda^q u_1 = B^{q-1} \sum_{k=1}^{q} \lambda^{k-1} B^{1-q} A_k u_1, \tag{6.8}$$

since $B^{1-q}A_k$ is a bounded operator. The last relation shows that u_1 is in the range of B^{q-1} and, consequently, also in the range of B^{k-1}. We can therefore define $u_2, ..., u_q$ by

$$\lambda^{k-1} u_1 = B^{k-1} u_k \quad (k = 2, ..., q). \tag{6.9}$$

It is then seen that $u_1, ..., u_q$ satisfy the equations

$$\begin{cases} \lambda u_1 = B u_2, \\ \lambda u_2 = B u_3, \\ \quad \vdots \\ \lambda u_{q-1} = B u_q, \\ \lambda u_q \;\; = \sum_{k=1}^{q} B^{1-q} A_k B^{k-1} u_k. \end{cases} \tag{6.10}$$

Therefore, if we write
$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_q \end{pmatrix},$$

we see that $\mathbf{u}$ satisfies (6.7) and $\lambda \in \sigma(\mathbf{A})$. Thus, $\sigma(A_1; A_2, ..., A_q) \subset \sigma(\mathbf{A})$.

Since (6.6) holds and B is compact (by Lemma 6.1) all the entries in $\mathbf{A}$ are compact, so that $\mathbf{A}$ is compact. Consequently, $0 \in \sigma(\mathbf{A})$.

Suppose now that $\lambda \in \sigma(\mathbf{A})$. We have to show that if $\lambda \neq 0$ then $\lambda \in \sigma(A_1; A_2, ..., A_q)$. Let $\mathbf{u}$ be a non-zero solution of (6.7). Then the equations (6.10) hold and $u_1 \neq 0$ (since otherwise $u_2 = ... = u_q = 0$ by (6.10), i.e., $\mathbf{u} = 0$). But now one immediately verifies that (6.10) implies (6.9), (6.8), and the latter equation clearly coincides with (6.1) with $u = u_1$.

Since $\sigma(\mathbf{A}) = \{0\} \cup \sigma(A_1; A_2, ..., A_q)$, the assertion $\varrho(\mathbf{A}) = \varrho(A_1; A_2, ..., A_q)$ follows by using Theorem 1.2. The last assertion of the theorem follows from the previous considerations.

We now note that the transformation of (6.7) into the system (6.9) is very similar to the transformation of linearization employed in Sec. 5, i.e., the transformation of (5.3) into the systems (5.4), (5.5). The only difference is that the matrix $\mathbf{Q}$ in (5.6) has just one non-zero entry, whereas in the present case it will have $q-1$ non-zero entries at the positions (q, i), $i = 2, ..., q$.

The proof of Lemma 5.1 can be extended to the present case since $R(A_1)$ (the range of A_1) is dense in H. In the proof that if $(\mathbf{u}^0, \ldots, \mathbf{u}^{n-1})$ is a packet of generalized eigenvectors of $\mathbf{C}(\lambda)$ at $\lambda = \lambda_0$ then $(u_1^0, \ldots, u_1^{n-1})$ is a packet of generalized eigenvectors of $C(\lambda^q)$ there are only minor modifications. We thus conclude:

LEMMA 6.2. *Let the assumptions of Theorem* 6.1 *hold. If* $(\mathbf{u}^0, \ldots, \mathbf{u}^{n-1})$ *is a packet of generalized eigenvectors of* $\lambda\mathbf{I} - \mathbf{A}$ ($\mathbf{A}$ *given by* (6.5)) *at* $\lambda_0 \neq 0$, *then the vector of the first components forms a packet of generalized eigenvectors of* $C(\lambda^q)$ *at* $\lambda = \lambda_0$, *where*

$$C(\lambda) = \lambda^q I - \sum_{k=1}^{q} \lambda^{k-1} A_k. \tag{6.11}$$

From Lemmas 6.2 and 2.2 we have:

COROLLARY. *If, for some* $R > 0$, $\mathrm{sp}_R(\mathbf{A}) = H^q$ *then* $\mathrm{sp}_{Rk}(C) = H$ *where* $C(\lambda)$ *is defined in* (6.11).

Definition. Let T be a compact operator in a Hilbert space X. Let $\{\mu_n\}$ be the sequence of eigenvalues of the positive self-adjoint compact operator $(T^*T)^{1/2}$. We shall write $T \in C_r$ (and say that T belongs to the class C_r) for some $r > 0$ if

$$\sum_{n=1}^{\infty} \mu_n^r < \infty.$$

In addition to the compactness of $A_1, \ldots, A_q$ and the condition (H_1), we shall need the following assumptions:

(H_2) There exist positive constants c_k, θ_k such that for all $u \in H$,

$$\|B^{-i} A_{k+1} B^k u\| \leq c_k \| B^{q\theta_k - i} u\| \quad (0 \leq i \leq q-1,\ 1 \leq k \leq q-1), \tag{6.12}$$

where, for each k, either (i) $\theta_k > 1$, or (ii) $\theta_k = 1$ and c_k is sufficiently small (more precisely, $c_k \Gamma \leq c_0$ where Γ is any bound on the norms of the operators $B, A_1, \ldots, A_q$ and c_0 is a constant depending only on q).

(H_3) For some $r > 0$, $B \in C_r$.

(Note that the compactness of B already follows from (H_1) and Lemma 1.1.)

(H_4) There exist differentiable arcs γ_j $(1 \leq j \leq s)$ initiating at the origin in $\mathbf{C}$ and terminating, say, on $|z| = 1$, such that these arcs do not intersect each other except at the origin, and such that the angle between each two neighboring arcs at the origin is less than π/r. Moreover,

$$\|(\lambda^q I - A_1)^{-1}\| = O(|\lambda|^{-q}) \tag{6.13}$$

as $\lambda \to 0$ along each of the arcs γ_j.

(H_5) For $u \in H$, $\lambda \in \gamma_j$,

$$\|A_1 u\| \leqslant c \|(\lambda^q I - A_1) u\| \quad (c \text{ constant}). \tag{6.14}$$

Since (see [10]) (H_1) implies that, for any $u \in H$, $0 < \theta < 1$,

$$\|A_1^\theta u\| \leqslant C \|A_1 u\|^\theta \|u\|^{1-\theta} \quad (C \text{ constant}),$$

it follows from (6.13), (6.14) that, for any $u \in H$,

$$\|\lambda^{q(1-\theta)} A_1^\theta u\| \leqslant c \|(\lambda^q I - A_1) u\| \quad (0 \leqslant \theta \leqslant 1, \ \lambda \in \gamma_j), \tag{6.15}$$

where c is a constant.

We can now state the following completeness theorem.

THEOREM 6.2. *Let $A_1, \ldots, A_q$ be compact operators satisfying the assumptions* (H_1) – (H_5). *Then the generalized eigenvectors of* (6.1) *are complete.*

The proof of this theorem is given in the following section.

7. Proof of Theorem 6.2

In this section, the letter c will be reserved for positive constants. The same letter c will often be used to denote different constants, even in the same formula.

We begin with two lemmas.

LEMMA 7.1. *Let the hypotheses of Theorem* 6.2 *hold. Then* $\mathbf{A} \in C_r$.

Proof. From the definition of the matrix $\mathbf{A}$ we obtain immediately that

$$\mathbf{A}^* = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & A_1^*(B^*)^{1-q} \\ B^* & 0 & 0 & \ldots & 0 & B^* A_2^* (B^*)^{1-q} \\ 0 & B^* & 0 & \ldots & 0 & (B^*)^2 A_3^* (B^*)^{1-q} \\ \cdot & \cdot & \cdot & \ldots & \cdot & \ldots \\ 0 & 0 & 0 & \ldots & B^* & (B^*)^{q-1} A_q^* (B^*)^{1-q} \end{pmatrix},$$

Define
$$D_k = B^{1-q} A_k B^{k-1} \quad (1 \leqslant k \leqslant q).$$

Note that $D_1 = I$. A direct computation shows that, since $A_1 = B^q$,

$$\mathbf{A}^* \mathbf{A} = B^* (\mathbf{I} + \mathbf{D}) B,$$

where
$$\mathbf{D} = \begin{pmatrix} 0 & D_1^* D_2 & D_1^* D_3 & \ldots & D_1^* D_q \\ D_2^* D_1 & D_2^* D_2 & D_2^* D_3 & \ldots & D_2^* D_q \\ D_3^* D_1 & D_3^* D_2 & D_3^* D_3 & \ldots & D_3^* D_q \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ D_q^* D_1 & D_q^* D_2 & D_q^* D_3 & \ldots & D_q^* D_q \end{pmatrix}.$$

Using (6.12) with $i=q-1$, we get

$$\|D_{k+1}u\| \leqslant c_k \|A_1^{\varrho_{k-1}}u\| \quad (1 \leqslant k \leqslant q-1),$$

while $D_1 = I$. Therefore all the D_k's are bounded operators. Consequently $\mathbf{D}$ is a bounded operator in H^q.

$\mathbf{A}$ is compact because of (6.12) with $i=q-1$ and the fact that B is compact. Denote the eigenvalues of B^*B by μ_n^2 and the eigenvalues of $\mathbf{A}^*\mathbf{A}$ by $\boldsymbol{\mu}_n^2$. Let M_{n-1} denote an arbitrary $(n-1)$-dimensional subspace of H^q. Then, by the well known minimax property of eigenvalues of selfadjoint compact operators,

$$\begin{aligned}\boldsymbol{\mu}_n^2 &= \min_{M_{n-1}} \max_{\mathbf{u} \in M_{n-1}^\perp} \frac{(\mathbf{A}^*\mathbf{A}\mathbf{u}, \mathbf{u})}{(\mathbf{u}, \mathbf{u})} = \min_{M_{n-1}} \max_{\mathbf{u} \in M_{n-1}^\perp} \frac{((\mathbf{I}+\mathbf{D})B\mathbf{u}, B\mathbf{u})}{(B\mathbf{u}, B\mathbf{u})} \cdot \frac{(B^*B\mathbf{u}, \mathbf{u})}{(\mathbf{u}, \mathbf{u})} \\ &\leqslant \|\mathbf{I}+\mathbf{D}\| \min_{M_{n-1}} \max_{\mathbf{u} \in M_{n-1}^\perp} \frac{(B^*B\mathbf{u}, \mathbf{u})}{(\mathbf{u}, \mathbf{u})} \leqslant c\hat{\mu}_n^2,\end{aligned}$$

where $\hat{\mu}_n^2$ are the eigenvalues of the operator $B^*B\mathbf{I}$. As is easily seen, $\hat{\mu}_{nq+i}^2 = \mu_{n+1}^2$ for $1 \leqslant i \leqslant q$. Hence, the last inequality and the assumption that $B \in C_r$ imply that $\mathbf{A} \in C_r$.

LEMMA 7.2. *Let the hypotheses of Theorem* 6.2 *hold. Then*

$$\|(\lambda \mathbf{I} - \mathbf{A})^{-1}\| = O(|\lambda|^{-q}) \tag{7.1}$$

as $\lambda \to 0$ *along each of the arcs* γ_j $(1 \leqslant j \leqslant s)$.

Proof. Define

$$\mathbf{B} = \begin{pmatrix} 0 & B & 0 & \dots & 0 \\ 0 & 0 & B & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & B \\ B & 0 & 0 & \dots & 0 \end{pmatrix}.$$

We first prove that (7.1) is valid if $\mathbf{A}$ is replaced by $\mathbf{B}$. Define

$$\mathbf{E} = \begin{pmatrix} \lambda^{q-1}I & \lambda^{q-2}B & \lambda^{q-3}B^2 & \dots & B^{q-1} \\ B^{q-1} & \lambda^{q-1}I & \lambda^{q-2}B & \dots & \lambda B^{q-2} \\ \lambda B^{q-2} & B^{q-1} & \lambda^{q-1}I & \dots & \lambda^2 B^{q-3} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \lambda^{q-2}B & \lambda^{q-3}B^2 & \lambda^{q-4}B^3 & \dots & \lambda^{q-1}I \end{pmatrix};$$

each row is obtained from the previous one by a cyclic permutation. A straightforward calculation shows that

$$(\lambda \mathbf{I} - \mathbf{B})^{-1} = (\lambda^q I - A_1)^{-1} \mathbf{E}. \tag{7.2}$$

Indeed, let

$$(\lambda \mathbf{I} - \mathbf{B}) \mathbf{u} = \mathbf{f},$$

where $\mathbf{f} = (f_1, \ldots, f_q)^*$. Then

$$\lambda u_j - B u_{j+1} = f_j \quad (1 \leqslant j \leqslant q-1), \tag{7.3}$$

$$\lambda u_q - B u_1 = f_q. \tag{7.4}$$

It easily follows that

$$\lambda^k u_1 = B^k u_{k+1} + \sum_{j=1}^{k} \lambda^{k-j} B^{j-1} f_j \quad (1 \leqslant k \leqslant q-1). \tag{7.5}$$

Applying B to the last equation in (7.5), and evaluating $\lambda^{q-1} B u_1$ from (7.4), we get

$$(\lambda^q I - A_1) u_q = \sum_{j=1}^{q-1} \lambda^{q-1-j} B^j f_j + \lambda^{q-1} f_q. \tag{7.6}$$

This gives u_q in terms of $\mathbf{f}$. Using (7.3) with $j = q-1, q-2, \ldots, 1$ we can also express $u_{q-1}, u_{q-2}, \ldots, u_1$ in terms of $\mathbf{f}$, and thereby derive (7.2).

Clearly, $\mathbf{E}$ remains bounded as $\lambda \to 0$. Therefore, by (H_4),

$$\|(\lambda \mathbf{I} - \mathbf{B})^{-1}\| = O(\|(\lambda^q I - A_1)^{-1}\|) = O(|\lambda|^{-q}) \quad \text{if } \lambda \in \gamma_j. \tag{7.7}$$

Next, let $\mathbf{C} = \mathbf{A} - \mathbf{B}$. Then

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & B^{1-q} A_2 B & B^{1-q} A_3 B^2 & \ldots & B^{1-q} A_q B^{q-1} \end{pmatrix}.$$

We have, by (7.2),

$$\mathbf{E}(\lambda \mathbf{I} - \mathbf{A}) = (\lambda^q I - A_1) \mathbf{I} - \mathbf{E}\mathbf{C}. \tag{7.8}$$

Also, if $\mathbf{u} = (u_1, u_2, \ldots, u_q)^*$ is in H^q,

$$\mathbf{E}\mathbf{C}\mathbf{u} = \begin{pmatrix} I \\ \lambda B^{-1} \\ \vdots \\ \lambda^{q-1} B^{-(q-1)} \end{pmatrix} \sum_{k=1}^{q-1} A_{k+1} B^k u_{k+1}.$$

Let $\mathbf{P}$ be the projection operator on the vectors in H^q with zero first component, i.e., $\mathbf{P}\mathbf{u} = (0, u_2, \ldots, u_q)^*$. Then

$$\|\mathbf{PECu}\| \leqslant c \sum_{i=1}^{q-1} \sum_{k=1}^{q-1} \|\lambda^i B^{-i} A_{k+1} B^k u_{k+1}\| \leqslant c \sum_{i=1}^{q-1} \sum_{k=1}^{q-1} c_k |\lambda|^i \|B^{q\theta_k - i} u_{k+1}\|$$

by (6.12). Using (6.15) with $\theta = \theta_k - i/q$, we then get (if $0 \leqslant \theta \leqslant 1$)

$$\|\mathbf{PECu}\| \leqslant c \sum_{i=1}^{q-1} \sum_{k=1}^{q-1} c_k |\lambda|^{q(\theta_k - 1)} \|(\lambda^q I - A_1) u_{k+1}\| \tag{7.9}$$

along the arcs γ_j. Note that $0 \leqslant \theta \leqslant 1$ when $\theta_k \leqslant 1 + 1/q$. If however $\theta_k > 1 + 1/q$ then all the assumptions of Theorem 6.2 continue to hold when θ_k is decreased so as to satisfy $\theta_k = 1 + 1/q$. Thus we can always assume $\theta_k \leqslant 1 + 1/q$, so that $0 \leqslant \theta \leqslant 1$.

To estimate each term on the right-hand side of (7.9) we take $|\lambda|$ sufficiently small if $\theta_k > 1$ and use the smallness of c_k if $\theta_k = 1$ (recall the assumption (H_2)). Thus we get

$$\|\mathbf{PECu}\| \leqslant \tfrac{1}{2} \|(\lambda^q I - A_1) \mathbf{Pu}\|$$

along the arcs γ_j, provided $|\lambda|$ is sufficiently small. Hence, (7.8) yields

$$\begin{aligned} \|\mathbf{PE}(\lambda \mathbf{I} - \mathbf{A}) \mathbf{u}\| &= \|(\lambda^q I - A_1) \mathbf{Pu} - \mathbf{PECu}\| \geqslant \|(\lambda^q I - A_1) \mathbf{Pu}\| - \|\mathbf{PECu}\| \\ &\geqslant \tfrac{1}{2} \|(\lambda^q I - A_1) \mathbf{Pu}\| \geqslant c |\lambda|^q \|\mathbf{Pu}\| \end{aligned} \tag{7.10}$$

as $\lambda \to 0$ along the arcs λ_j, where (H_4) has been used.

Let
$$(\lambda \mathbf{I} - \mathbf{A}) \mathbf{u} = \mathbf{f}. \tag{7.11}$$

An argument like that used in proving Theorem 6.1 shows that u_1, the first component of the solution $\mathbf{u}$ of (7.11), satisfies

$$(\lambda^q I - A_1 - \sum_{k=1}^{q-1} \lambda^k A_{k+1}) u_1 = \sum_{k=1}^{q} C_j(\lambda) f_j,$$

where $\mathbf{f} = (f_1, \dots, f_k)^*$ and the norms of the operators $C_j(\lambda)$ remain bounded as $\lambda \to 0$. Therefore,

$$\begin{aligned} \|\mathbf{f}\| \geqslant c \|\sum_{k=1}^{q} C_j(\lambda) f_j\| &\geqslant c \|(\lambda^q I - A_1) u_1\| - \sum_{k=1}^{q-1} \|\lambda^k A_{k+1} u_1\| \\ &\geqslant c \|(\lambda^q I - A_1) u_1\| - \sum_{k=1}^{q-1} c_k |\lambda|^k \|B^{q\theta_k - k} u_1\|, \end{aligned}$$

by (6.12) with $i = 0$. Hence, by (6.15),

$$\|\mathbf{f}\| \geqslant c \|(\lambda^q I - A_1) u_1\| \geqslant c |\lambda|^q \|u_1\|$$

along the arcs γ_j, provided $|\lambda|$ is sufficiently small.

From the last inequality and (7.10) we obtain, for $|\lambda|$ sufficiently small,

$$|\lambda|^q\|\mathbf{u}\| \leqslant c|\lambda|^q(\|u_1\| + \|\mathbf{Pu}\|) \leqslant c\|\mathbf{f}\| \quad \text{along the arcs } \gamma_j; \tag{7.12}$$

here (7.11) and the boundedness of $\|\mathbf{E}\|$ (as $\lambda \to 0$) have been used. Since $\mathbf{u} = (\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}$ the assertion (7.1) of Lemma 7.2 follows from (7.12).

We can now easily complete the proof of Theorem 7.2 with the aid of the following result (Theorem XI.9.29 of [4]):

LEMMA 7.3. *Let $T \in C_r$. Let $\gamma_1, \ldots, \gamma_s$ be differentiable arcs as in the hypothesis* (H_4). *Assume that for some positive integer N,*

$$\|(\lambda I - T)^{-1}\| = O(|\lambda|^{-N})$$

as $\lambda \to 0$ along the arcs γ_j. Then the subspace $\mathrm{sp}(T)$ *contains the range of T^N.*

In view of Lemmas 7.1, 7.2, we can apply Lemma 7.3 to the operator $T = \mathbf{A}$ in H^q, with $N = q$. We conclude that $\mathrm{sp}(\mathbf{A})$ contains $R(\mathbf{A}^q)$, the range of $\mathbf{A}^q$. Now the formula for $\mathbf{A}^*$ displayed in the proof of Lemma 7.1 shows that the null space $N(\mathbf{A}^*)$ of $\mathbf{A}^*$ is zero, whence it follows that $N((\mathbf{A}^*)^q) = \{0\}$. Hence $R(\mathbf{A}^q)$ is dense in H^q. Since $\mathrm{sp}(\mathbf{A})$ is a closed set which contains $R(\mathbf{A}^q)$, we deduce that $\mathrm{sp}(\mathbf{A}) = H^q$. Now use the corollary to Lemma 6.2.

Chapter 4. Some applications

8. Generalizations and applications of the results of Chapter 2

8.1. Generalizations. The results of Chapters 1, 2 can be extended to eigenvalue problems

$$\lambda^q u = Au + \sum_{k=0}^{\infty} \lambda^{p+k} B_k u, \tag{8.1}$$

assuming that

$$\sum_{k=0}^{\infty} \Lambda^k \|B_k\| < \infty \tag{8.2}$$

for some $\Lambda > 0$. The assertion of Theorem 1.2 now becomes:

$$\Omega_\Lambda \equiv \{\lambda; |\lambda| < \Lambda\} = \{0\} \cup \varrho_\Lambda(A; B_0, B_1, \ldots) \cup \sigma_\Lambda(A; B_0, B_1, \ldots),$$

where ϱ_Λ (σ_Λ) stands for the resolvent set (spectrum) in Ω_Λ, and the proof remains the same.

The assertion of Theorem 1.3 is valid for all λ in Ω_Λ. In extending the proof of that theorem, we now first choose n_0 such that

$$\Lambda^q \sum_{k=n_0+1}^{\infty} \Lambda^{p+k} \|B_k\| < \tfrac{1}{3}. \tag{8.3}$$

Next we decompose A, B_k $(1 \leqslant k \leqslant n_0)$ into sums $A = A^{(1)} + A^{(2)}$, $B_k^{(1)} + B_k^{(2)}$ respectively, where $A^{(1)}, B_k^{(1)}$ have finite-dimensional ranges and $A^{(2)}, B_k^{(2)}$ are such that

$$\Lambda^q \| A^{(2)} \| + \Lambda^q \sum_{k=0}^{n_0} \Lambda^{p+k} \| B_k^{(2)} \| < \tfrac{1}{3}.$$

We then can proceed similarly to the proof of Theorem 1.3, taking

$$B^{(1)}(\lambda) = A^{(1)} + \sum_{k=0}^{n_0} \lambda^{p+k} B_k^{(1)},$$

$$B^{(2)}(\lambda) = A^{(2)} + \sum_{k=0}^{n_0} \lambda^{p+k} B_k^{(2)} + \sum_{k=n_0+1}^{\infty} \lambda^{p+k} B_k.$$

We next turn to the results of Chap. 2. We introduce

$$C(\lambda) = \lambda^q I - A - \sum_{k=0}^{\infty} \lambda^{p+k} B_k \tag{8.4}$$

and assume, as before, that (8.2) holds. Then $C^{-1}(\lambda)$ is a regular analytic function in the resolvent set $\varrho_\Lambda(A;\, B_0, B_1, \ldots)$. Since the estimate (2.13) obviously generalizes to the present case, Lemma 2.6 also extends to the present case. Lemmas 3.1, 3.2 and Theorem 3.2 (with $R = \Lambda$) clearly also extend, but instead of (3.20) we now require that

$$\sum_{k=1}^{\infty} \beta_k \left(\frac{\| A \|}{1-\theta} \right)^{k-1} \leqslant \theta \tag{8.5}$$

for some $0 < \theta < 1$. Note that (8.2) implies that the series in (8.5) is convergent if $\Lambda > \| A \|$.

The results of Secs. 4, 5 also extend without difficulty to $C(\lambda)$ given by (8.4).

8.2. First application. Consider the differential operator

$$\begin{aligned} B(x, D_x, \lambda) \equiv B_0(x, D_x) + \eta \lambda B_1(x, D_x) + \ldots + \eta \lambda^{q-1} B_{q-1}(x, D_x) + \lambda^q \\ + \eta \lambda^{-1} B_{q+1}(x, D_x) + \ldots + \eta \lambda^{-h} B_{q+h}(x, D_x), \end{aligned} \tag{8.6}$$

where $B_j(x, D_x)$ is a linear differential operator of order s_j in a bounded domain Ω of R^n, and η is a fixed complex number. Assume that $s_0 = 2m$ and that $s_j < 2m$ for all $j \neq 0$. Assume also that $B_0(x, D_x)$ is elliptic, that, furthermore, its principal part $\tilde{B}_0(x, D_x)$ is such that $(-1)^m \tilde{B}_0(x, \xi)$ is never a negative number for real $\xi \neq 0$, and that the coefficients of all the operators B_j are continuous in $\bar{\Omega}$. Assume finally that the boundary $\partial\Omega$ of Ω is in class C^{2m}.

Consider the eigenvalue problem for the Dirichlet problem associated with (8.6), i.e.,

$$B(x, D_x, \lambda) u = 0 \text{ in } \Omega, \tag{8.7}$$

$$\frac{\partial^j u}{\partial \nu^j} = 0 \quad \text{on } \partial\Omega \quad (0 \leqslant j \leqslant m-1). \tag{8.8}$$

Denote by B_j the operator obtained by taking the closure of the operator $u \to B_j(x, D_x)u$ defined on the set of functions in $C^{2m}(\bar{\Omega})$ satisfying (8.8). Then, the domain of B_0 is $H^{2m}(\Omega) \cap H_0^m(\Omega)$ (see [0], where general elliptic boundary conditions are considered).

Our assumptions on $B_0(x, D_x)$ imply that $(\lambda I - B_0)^{-1}$ exists for all $\lambda \leqslant -k$, where k is some non-negative number, and that

$$\|(\lambda I - B_0)^{-1}\| \leqslant \frac{C}{1+|\lambda|} \quad \text{if } \lambda \leqslant -k. \tag{8.9}$$

For simplicity we shall assume that (8.9) holds with $k=0$.

Writing $v = B_0 u$ in the equation

$$B_0 u + \eta\lambda B_1 u + \ldots + \eta\lambda^{q-1} B_{q-1} u + \lambda^q u + \eta\lambda^{-1} B_{q+1} u + \ldots + \eta\lambda^{-h} B_{q+h} u = 0, \tag{8.10}$$

and replacing λ by $1/\lambda$, we obtain the equation

$$\lambda^q v = Av + \eta\lambda A_1 v + \ldots + \eta\lambda^{q-1} A_{q-1} v + \eta\lambda^{q+1} A_{q+1} v + \ldots + \eta\lambda^{q+h} A_{q+h} v, \tag{8.11}$$

where $$A = B_0^{-1},\ A_j = B_{q-j} B_0^{-1} \quad \text{for } 1 \leqslant j \leqslant q-1,\ q+1 \leqslant j \leqslant q+h. \tag{8.12}$$

We define the generalized eigenvectors of (8.10) to be the vectors $B_0^{-1}v$ where v is a generalized eigenvector of (8.11). An equivalent definition can be given by considering solutions of the equations (2.3) with $C(\lambda)u$ defined by the left-hand side of (8.10). It is easily seen that the generalized eigenvectors of (8.10) (corresponding to λ with $|\lambda| < R$) are complete if and only if the same is true of the generalized eigenvectors of (8.11) (corresponding to λ with $|\lambda| < R$).

We now recall the fact (see [6]) that

$$c\|u\|_{2m\varrho'} \leqslant \|B_0^{\varrho} u\| \leqslant C\|u\|_{2m\varrho''} \tag{8.13}$$

for any $0 < \varrho' < \varrho < \varrho'' < 1$, where c, C are positive constants. It follows that

$$A^{-1}A_j A^{\theta} \text{ is compact in } L^2(\Omega) \text{ if } s_{q-j} < 2m\theta \text{ for } 1 \leqslant j \leqslant q \text{ and}$$
$$\text{if } s_j < 2m\theta \text{ for } q+1 \leqslant j \leqslant q+h. \tag{8.14}$$

Assume now that $B_0(x, D_x)$ is formally self-adjoint. Then (see [7]) the eigenvalues λ_k of B_0 satisfy:

$$\lambda_k = ck^{2m/n}(1 + o(1)) \quad \text{as } k \to \infty. \tag{8.15}$$

It follows that A is in class C_r for any $r > n/2m$. Applying Theorem 5.1 we obtain the following result.

THEOREM 8.1. *Let the foregoing assumption on $B_0, \ldots, B_{q+h}$ hold. Suppose that $B_1 = \ldots = B_{p-1} = 0$, and that*

$$\frac{n}{2m} < \frac{p}{q} - \frac{s_j}{2m} \quad \textit{for all } j \neq 0. \tag{8.16}$$

If η is sufficiently small then the generalized eigenvectors of (8.7), (8.8) corresponding to the eigenvalues λ with $|\lambda| < R$ form a complete set in $L^2(\Omega)$, for any R sufficiently large.

In Theorem 8.1 we have assumed that B_0 satisfies (8.9) with $k=0$. This assumption can easily be removed if $q=1$. Indeed, making the substitution $\lambda \to \lambda + k$ in (8.11) (with $q=1$), we get an equation of the form

$$\lambda u = \tilde{A} u + \eta \sum_{j=2}^{1+h} \lambda^j \tilde{A}_j \quad \text{where } \tilde{A} = A + kI + \eta \tilde{A}.$$

If η is sufficiently small then Theorem 8.1 can be applied to the new equation.

We next note that in view of the generalizations given in § 8.1, Theorem 8.1 extends also to operators (8.6) where the coefficient of each operator $B_j(x, D_x)$ has the form $\eta\varphi(\lambda)$, where $\lambda^q \varphi(1/\lambda)$ is an analytic function in some disc $\{\lambda;\ |\lambda| < \Lambda\}$.

8.3. Second application. Let L be a one-to-one self-adjoint operator with compact inverse. Let M be another operator that will be looked upon as a perturbation of L. We shall derive the completeness of the generalized eigenvectors of $L + \eta M$, where η is a small complex parameter.

Let u be an eigenvector of $L + \eta M$ with the corresponding eigenvalue u. Then

$$\mu u = (L + \eta M) u. \tag{8.17}$$

It will be convenient to write $\mu = 1/\lambda$, so that (8.17) becomes

$$u = \lambda (L + \eta M) u. \tag{8.18}$$

Define
$$D(\lambda) = I - \lambda (L + \eta M). \tag{8.19}$$

As in Sec. 2, if $D^{-1}(\lambda)$ has a pole of order n at λ_0, then we define a packet of generalized eigenvectors $(u_0, \ldots, u_{n-1})$ of $D(\lambda)$ at λ_0 by

$$\begin{aligned} u_0 &= \lambda_0 (L + \eta M) u_0, \\ u_1 &= \lambda_0 (L + \eta M)(u_1 + u_0), \\ &\cdots \\ u_{n-1} &= \lambda_0 (L + \eta M)(u_{n-1} + u_{n-2}). \end{aligned} \tag{8.20}$$

The generalized eigenvectors are the components of the packet. They are also characterized as solutions of the equation

$$\left[(L+\eta M)-\frac{1}{\lambda_0}I\right]^n v=0.$$

Let

$$A=L^{-1},\quad B=-L^{-1}M$$

and define

$$C(\lambda)=\lambda I-A-\eta\lambda B.$$

Then $C(\lambda)=-A^{-1}D(\lambda)$ and it is easy to see that $(u_0, \ldots, u_{n-1})$ is a packet of generalized eigenvectors of $D(\lambda)$ if and only if it is a packet of generalized eigenvectors of $C(\lambda)$. By Theorem 5.1, if $A\in C_r$ for some r, $r<1+t$, and if $A^{-1}BA^{-t}=ML^t$ is a bounded operator, then $\mathrm{sp}_R(C)=H$ for all R sufficiently large, provided η is sufficiently small. (Note that B is compact if ML^t is bounded.) Setting $t=s-1$, we have:

THEOREM 8.2.(1) *Let L be a one-to-one self-adjoint operator with compact inverse in C_r, and assume that ML^{s-1} is a bounded operator for some $s>r$. Then the generalized eigenvectors of $D(\lambda)$ are complete provided η is sufficiently small.*

9. Applications of the results of Chapter 3

Consider the differential operator

$$B(x, D_x, \lambda)\equiv B_0(x, D_x)+\lambda B_1(x, D_x)+\ldots+\lambda^{q-1}B_{q-1}(x, D_x)+\lambda^q, \tag{9.1}$$

where $B_j(x, D_x)$ are linear differential operators of order s_j in a bounded domain Ω of R^n, and $s_0=2m$, $s_j<(2m/q)j$ if $1\leqslant j\leqslant q-1$. Assume also that $B_0(x, D_x)$ is elliptic, that B_0 and the B_j satisfy the same assumptions as in Sec. 8.2 and that $\partial\Omega$ is in class C^{2m}. Consider the eigenvalue problem (8.7), (8.8).

Agmon and Nirenberg have proved (see [1], Theorem 5.8″) a completeness theorem for the generalized eigenfunctions of the system obtained by linearizing a reduced weighted elliptic boundary value problem. We shall show how such a completeness theorem follows from Theorem 6.2 in the more special case of the Dirichlet boundary value problem. We should like to remark that our assumption $s_j<(2m/q)j$ is not essential, and that our method can actually be extended to the case $s_j\leqslant(2m/q)j$ under some restriction on the leading coefficients of $B_j(x, D_x)$. Furthermore, we actually get the completeness of the generalized eigenfunctions of the problem (8.7), (8.8) and not just of its linearized form.

We first transform the problem (8.7), (8.8) into the form (cf. (8.11))

(1) See Addendum at the end of this paper.

$$\lambda^q u = Au + \lambda A_1 u + \ldots + \lambda^{q-1} A_{q-1} u. \tag{9.2}$$

Now we have to verify the conditions of Theorem 6.2.

(H_2) follows from (8.13). (H_4) and (H_5) (for any $r>0$) follow from [0]. To prove (H_3), let $\tilde{A}$ be the operator defined similarly to A when $B_0(x, D_x)$ is the differential operator $(-\Delta)^m+1$. The eigenvalues of $\tilde{A}$ satisfy (8.15). Since $\tilde{A}$ is self-adjoint, it follows that, for any $0<\theta<1$, $\tilde{A}^{-\theta}$ is of class $C_{r\theta}$ for any $r>n/2m$. Using (8.13) we conclude that the operator $\Gamma=\tilde{A}^{-\theta}A^{\alpha}$ is a bounded operator if $\alpha>\theta$. We now recall (see [4]) that if B is in C_r and Γ is bounded, then ΓB is also in C_r. It follows that $\tilde{A}^{\alpha}=\tilde{A}^{\theta}(\tilde{A}^{-\theta}A^{\alpha})$ is in class C_r for any $r>n/2m\alpha$ (provided θ is properly chosen). This implies (H_3).

Eigenvalue problems of the form (8.7), (8.8) (and also problems with more general boundary conditions) occur in many physical applications. For examples, see [2].

Addendum. We have discovered several papers in the Russian literature dealing with completeness for equations as in Chapter 3; see [19], [20] and the references given there. These results are related to ours. They all assume roughly that A is normal, but the assumptions on the B_k are somewhat simpler than ours. The concept of completeness in these papers is not as direct as ours, since they do not have the result of Lemma 6.2.

As to Theorem 8.2, we have discovered that a more general result was proved in [19; p. 336], namely, that the assertion of Theorem 8.2 holds if L is one-to-one, self-adjoint with L^{-1} compact, ML^{-1} compact, $L^{-1}ML^{-1}$ in C_r for some $r<\infty$, and $\eta=1$. The proof, however, is based on the deep result of Lemma 7.3. We also wish to point out that the restriction on η made in Theorem 8.2 can be removed without difficulty (so that one can take $\eta=1$). Indeed, we consider $L+iaI$ (instead of L) with $a>0$ and note (by [19; Lemma 7.1]) that $\|M(L+iaI)^{-1}\|\leqslant\eta'$; $\eta'\to 0$ if $a\to\infty$. One now can check that the proof of Theorem 8.2 extends to the case where "η is small" is replaced by "η' is small".

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