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NONLINEAR ELLIPTIC EQUATIONS AND GAUSS MEASURE

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We prove existence and regularity results for weak solutions to nonlinear elliptic equations, whose propotype is:

$$\begin{cases} -\operatorname{div}\ (\varphi(x)|\nabla u|^{p-2}\nabla u) + b(x)\varphi(x)|\nabla u|^{p-1} = g\varphi - \operatorname{div}(f\varphi) & \text{in } \Omega \\ u = 0 & \text{on } \Omega, \end{cases}$$

where p is a real number $1 < p < +\infty$, $\varphi(x) = (2\pi)^{-\frac{n}{2}} \exp\ (-|x|^2/2)$ is the density of Gauss measure, the function g belongs to the weighted Zygmund space $L^{p'}(\log L)^{-\frac{1}{2}}(\varphi, \Omega)$, f belongs to the weighted Lebesgue space $L^{p'}(\varphi, \Omega)$ and the coefficient b belongs to the weighted Zygmund space $L^{\infty}(\log L)^{-\frac{1}{2}}(\varphi, \Omega)$.

1. Introduction.

In the present paper we prove existence and regularity results for weak solutions to the following nonlinear elliptic problem

$$(1.1)\quad \begin{cases} -\operatorname{div}(a(x,u,\nabla u))+H(x,\nabla u)+G(x,u)=g\varphi-\operatorname{div}(f\varphi) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\varphi(x) = (2\pi)^{-\frac{n}{2}} \exp\Big(-\frac{|x|^2}{2}\Big)$ is the density of Gauss measure $\gamma_n(dx) = \varphi(x)dx$, p is a real number $1 < p < +\infty$ and Ω is an

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open subset of $\mathbb{R}^n$ $(n \geq 2)$ with Gauss measure less than one. Moreover $a : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $H : \Omega \times \mathbb{R}^n \to \mathbb{R}$ and $G : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions such that

(i) $a(x, \eta, \xi)\xi \geq \varphi(x)|\xi|^p$;

(ii) $|a(x, \eta, \xi)| \leq c_1\varphi(x)\Big(|\eta|^{p-1} + |\xi|^{p-1} + k_1(x)\Big)$,

$c_1 > 0$, $k_1(x) \geq 0$ and $k_1(x) \in L^{p'}(\varphi, \Omega)$;

(iii) $\Big(a(x, \eta, \xi) - a(x, \eta, \overline{\xi})\Big)\Big(\xi - \overline{\xi}\Big) > 0$ if $\xi \neq \overline{\xi}$;

(iv) $|H(x, \xi)| \leq \varphi(x)\Big(b(x)|\xi|^{p-1} + k_2(x)\Big)$,

$b(x) \geq 0$, $k_2(x) \geq 0$, $b(x) \in L^\infty(\log L)^{-\frac{1}{2}}(\varphi, \Omega)$ and $k_2(x) \in L^{p'}(\varphi, \Omega)$;

(v) $|G(x, \eta)| \leq c_3\varphi(x)\Big(|\eta|^{p-1} + k_3(x)\Big)$,

$c_3 > 0$, $k_3(x) \geq 0$ and $k_3(x) \in L^{p'}(\varphi, \Omega)$;

(vi) $G(x, \eta)\eta \geq 0$;

(vii) $g \in L^{p'}(\log L)^{-\frac{1}{2}}(\varphi, \Omega)$;

(viii) $|f| \in L^{p'}(\varphi, \Omega)$,

for a.e. $x \in \Omega$, for any $\eta \in \mathbb{R}$ and $\xi, \overline{\xi} \in \mathbb{R}^n$.

We are interested in studying both existence and regularity of weak solutions to the problem (1.1) belonging to the weighted Sobolev space $W_0^{1,p}(\varphi, \Omega)$.

We just observe that when Ω is bounded, problem (1.1) is uniformly elliptic, then existence results can be found in [17], [18] and [19].

There are three main difficulties in studying problem (1.1). The first one is due to the operator -div $(a(x, u, \nabla u))$ which is not uniformly elliptic. The second difficulty is due to the fact that the domain Ω can be unbounded and finally, the third difficulty is due to the presence of the lower order term $H(x, u)$ which produce a lack of coerciveness when $\|b\|_{L^\infty(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}$ is not sufficiently small.

Let us describe how we overcome such difficulties in studying existence. The main step is to find some apriori estimate for $W_0^{1,p}(\varphi, \Omega)$-norm of solutions to problem (1.1). These estimates are obtained by adapting the classical techniques due to Talenti (see [32]) which is based on the use of classical isoperimetric inequality and Schwarz symmetrization (see also [32], [34], [33], [1], [2], [3] and [4]). The presence of the function $\varphi(x)$ that appear in condition (i) and the fact that Ω can be unbounded, bring us to use the isoperimetric inequality with respect to the Gauss measure and to use the notion of rearrangement with respect to the Gauss measure.

In this paper we link also the summability of u with the summability of data. When Ω is bounded, regularity results for solution to linear and nonlinear degenerate equations are well known (see [30], [36], [2], [3], [10], [9], [13], [14]). In our case, by logarithmic Sobolev imbedding theorem, we have that if $u \in W_0^{1,p}(\varphi, \Omega)$ is a solution to problem (1.1), then u belongs to the Lorentz-Zygmund space $L^p(\log L)^{\frac{1}{2}}(\varphi, \Omega)$ (see section 2 for the definition); we show how the summability of u improves by improving the summability of the data in the Lorentz-Zygmund spaces $L^{a,q}(\log L)^{\alpha}(\varphi, \Omega)$. Finally we prove a pointwise comparison result (see section 4).

In order to obtain this results we prove a pointwise comparison (see section 4),

$$u^\star(x) \le w(x), \tag{1.2}$$

where $w(x)$ is solution to the following problem:

$$\begin{cases} -\left(\varphi|w_{x_1}|^{p-2}w_{x_1}\right)_{x_1} - B(\Phi(x_1))\varphi|w_{x_1}|^{p-2}w_{x_1} - k_2^\star\varphi = & \text{in } \Omega^\star \\ \qquad = g^\star\varphi - (F(\Phi(x_1))\varphi) & \\ w = 0 & \text{on } \partial\Omega^\star. \end{cases} \tag{1.3}$$

Here $\Omega^\star$ is the half-space $\{x \in \mathbb{R}^n : x_1 > \lambda\}$, with $\lambda \in \mathbb{R}$ such that $\gamma_n(\Omega^\star) = \gamma_n(\Omega)$, $g^\star(x)$ and $k_2^\star(x)$ are the rearrangements with respect to Gauss measure of the functions $g(x)$ and $k_2(x)$, $F(x)$ and $B(x)$ are functions built on the level sets of u and $\Phi(\tau) = \gamma_1(\tau, +\infty)$.

Comparison and regularity results for solution to linear degenerate equations, using rearrangement with respect to Gauss measure, are proved in [7], [20] and in [21]. Parabolic case has been studied in [15] and the eigenvalues problem is contained in [8].

2. Notation and preliminary results.

2.1. Gauss measure and rearrangements.

Let Ω be an open set of $\mathbb{R}^n$, we denote by γ_n the n-dimensional normalized Gauss measure on $\mathbb{R}^n$ defined as

$$\gamma_n(dx) = \varphi(x)dx = (2\pi)^{-\frac{n}{2}} \exp\left(-\frac{|x|^2}{2}\right) dx, \qquad x \in \mathbb{R}^n.$$

Observe that (see [25])

$$\lim_{t\to 0^+;1^-} (2\pi)^{-\frac{1}{2}} \frac{\exp\left(-\frac{\Phi^{-1}(t)^2}{2}\right)}{t\left(2\log\frac{1}{t}\right)^{\frac{1}{2}}} = 1 \tag{2.1}$$

holds, where $\Phi(\tau) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_\tau^{+\infty} \exp\left(-\frac{t^2}{2}\right) dt,\ \tau \in \mathbb{R} \cup \{-\infty, +\infty\}$.

One of the main tools to prove our results is the isoperimetric inequality with respect to Gauss measure (see [12], [22]) which states that among all $(n-1)-$rectificable subsets E of $\mathbb{R}^n$ with fixed Gauss measure, the half-spaces archive the smallest perimeter with respect to Gauss measure, that is

$$P(E) = \int_{\partial E} \varphi(x)\mathcal{H}_{n-1}(dx) \geq P(H),$$

where H is the half-space such that $\gamma_n(H) = \gamma_n(E)$ and $\mathcal{H}_{n-1}(x)$ is the $(n-1)-$dimensional Hausdorff measure.

Now we give the notion of some equimeasurable rearrangements. If u is a measurable function, we denote by

- u^* the usual decreasing rearrangement of u with respect to Lebesgue measure[1], i.e.

$$u^*(s) = \inf\{t \geq 0 : |\{x \in \Omega : |u| > t\}| \leq s\} \qquad s \in]0, 1];$$

- $u^\otimes$ the decreasing rearrangement of u with respect to Gauss measure, i.e.

$$u^\otimes(s) = \inf\{t \geq 0 : \mu(t) \leq s\} \qquad s \in]0, 1],$$

  where $\mu(t) = \gamma_n(\{x \in \Omega : |u| > t\})$ is the distribution function of u;

- $u^\star$ the rearrangement with respect to Gauss measure of u, i.e.

$$u^\star(x) = u^\otimes(\Phi(x_1)) \qquad x \in \Omega^\star,$$

[1] We denote by $|D|$ the n-dimentional Lebesgue measure of a subset $D \subset \mathbb{R}^n$.

where $\Omega^\star = \{x = (x_1, ..., x_n) \in \mathbb{R}^n : x_1 > \lambda\}$ is the half-space such that $\gamma_n(\Omega^\star) = \gamma_n(\Omega)$.

For these rearrangements a Hardy-Littlewood inequality and Polya-Szëgo principle (see [35]) hold.

Let us introduce the following notation of pseudo-rearrangement firstly introduced in [2].

To this end let u be a measurable function in Ω, $f \in L^p(\varphi, \Omega)$ with $1 \le p \le +\infty$, $f \ge 0$ and $\Omega^\otimes = (0, \gamma_n(\Omega))$. We will say that a function $\tilde{f}_u : \Omega^\otimes \to \mathbb{R}$ is a Gauss pseudo-rearrangement of f with respect to u if there exists a family $\mathcal{E}(u) = \{E(s)\}_{s\in\Omega^\otimes}$ of measurable subsets of Ω such that

$$\gamma_n(E(s)) = s,$$

$$s_1 \le s_2 \quad \Rightarrow \quad E(s_1) \subseteq E(s_2)$$

$$E(s) = \{x \in \Omega : |u(x)| > u^\otimes(s)\} \quad \text{if} \quad \exists t \in \mathbb{R}, s = \mu(t)$$

and

$$\tilde{f}_u(s) = \frac{d}{ds} \int_{E(s)} f(x)\varphi(x)\, dx \quad \text{for a.e. } s \in \Omega^\otimes. \tag{2.2}$$

For general results about the properties of rearrangement and pseudo-rearrangement we refer for instance to [16], [31] and [23].

2.2. Some inequalities.

We often will use the following Hardy inequalities with fixed weight (see [5]):

Proposition 2.1. *Let ψ be a nonnegative measurable function on $(0, 1)$. Suppose $r > 0$ and $-\infty < \alpha < +\infty$. If $1 \le q < \infty$, then the inequalities*

$$\Big(\int_0^1 \Big(t^{-r}(1-\log t)^\alpha \int_0^t \psi(s)ds\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} \le c\Big(\int_0^1 (t^{1-r}(1-\log t)^\alpha \psi(t))^q \frac{dt}{t}\Big)^{\frac{1}{q}} \tag{2.3}$$

and

$$\Big(\int_0^1 \Big(t^{r}(1-\log t)^\alpha \int_t^1 \psi(s)ds\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} \le c\ \Big(\int_0^1 (t^{1+r}(1-\log t)^\alpha \psi(t))^q \frac{dt}{t}\Big)^{\frac{1}{q}} \tag{2.4}$$

hold; while, for $q=\infty$ it holds that

$$(2.5)\quad \sup_{0<t<1}\Big(t^{-r}(1-\log t)^{\alpha}\big(\int_0^t \psi(s)ds\big)\Big)\le c\sup_{0<t<1}\left(t^{1-r}(1-\log t)^{\alpha}\psi(t)\right)$$

and

$$(2.6)\quad \sup_{0<t<1}\Big(t^{r}(1-\log t)^{\alpha}\Big(\int_t^1 \psi(s)ds\Big)\Big)\le c\sup_{0<t<1}\left(t^{1+r}(1-\log t)^{\alpha}\psi(t)\right).$$

Suppose $r=0$. If $1\le q<\infty$ and $\frac{1}{q}+\alpha<0$ then the inequality

$$(2.7)\quad \Big(\int_0^1\Big((1-\log t)^{\alpha}\int_t^1 \psi(s)ds\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\le c\Big(\int_0^1 (t(1-\log t)^{\alpha+1}\psi(t))^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$

holds, while for $q=\infty$ and $\alpha<0$ it holds that

$$(2.8)\quad \sup_{0<t<1}\Big((1-\log t)^{\alpha}\int_t^1 \psi(s)\ ds\Big)\le c\sup_{0<t<1}\left(t(1-\log t)^{\alpha+1}\psi(t)\right).$$

In all cases, the constants $c=c(r,q,\alpha)$ are independent by ψ.

2.3. Lorentz-Zygmund spaces.

In this section we recall a few properties of Lorentz-Zygmund spaces. Let u be any measurable function in Ω for $0<q,p\le\infty$ and $-\infty<\alpha<+\infty$, we denote[2]

$$(2.9)\quad ||u||_{L^{p,q}(\log L)^{\alpha}(\varphi,\Omega)}= \begin{cases}\left(\int_0^{\gamma_n(\Omega)}[t^{\frac{1}{p}}(1-\log t)^{\alpha}u^{\otimes}(t)]^q\frac{dt}{t}\right)^{\frac{1}{q}} & \text{if } 0<q<\infty,\\ \sup\limits_{t\in(0,\gamma_n(\Omega))}[t^{\frac{1}{p}}(1-\log t)^{\alpha}u^{\otimes}(t)] & \text{if } q=\infty.\end{cases}$$

A measurable function u belongs to the Lorentz-Zygmund space $L^{p,q}(\log L)^{\alpha}(\varphi,\Omega)$ if $||u||^p_{L^{p,q}(\log L)^{\alpha}(\varphi,\Omega)}<\infty$.

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[2] We will use the following 'arithmetic' convention:

$$\frac{s}{\infty}=0,\quad \text{for } s>0.$$

For the definition of the classical Lorentz-Zygmund space and for more properties and details we refer to [29], [6] and [5]. For brevity the quasinorm (2.9) will be denoted by $||u||_{p,q;\alpha}$.

Recall also that the weighted Sobolev space $W_0^{1,p}(\varphi, \Omega)$ is the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{W_0^{1,p}(\varphi,\Omega)} = \left(\int_\Omega |\nabla u(x)|^p \, d\gamma_n(x) \right)^{\frac{1}{p}}.$$

Moreover if $\gamma_n(\Omega) < 1$ a Poincarè type inequality holds (see [20] and [15], where the case $p = 2$ has been considered).

Finally, we recall an inequality (see [21]) which will be useful in the following.

Proposition 2.2. *Let* $f \in L^{p,q}(\log L)^\alpha(\varphi, \Omega)$ *with* $1 \leq \sigma < p \leq \infty, \sigma \leq q \leq \infty$ *and* $-\infty < \alpha < +\infty$ *and* $F^\sigma = (\widetilde{f^\sigma})_u$*, then* $F \in L^{p,q}(\log L)^\alpha(\Omega^\otimes)$ *and*

$$\|F\|_{L^{p,q}(\log L)^\alpha(\Omega^\otimes)} \leq C\|f\|_{L^{p,q}(\log L)^\alpha(\varphi,\Omega)}.$$

3. Existence result.

In this section we prove an existence result for weak solutions to problem (1.1). First of all we recall the definition of weak solution to problem (1.1).

Definition 3.1. We say that $u \in W_0^{1,p}(\varphi, \Omega)$ is a weak solution to problem (1.1), if

$$\int_\Omega a(x, u, \nabla u)\nabla\psi dx + \int_\Omega H(x, \nabla u)\psi dx + \int_\Omega G(x, u)\psi dx = \tag{3.1}$$

$$= \int_\Omega (g\psi\varphi + f\nabla\psi\varphi)dx, \qquad \forall\psi \in W_0^{1,p}(\varphi, \Omega).$$

If $\gamma_n(\Omega) < 1$, under our hypotheses it is easy to verify that all terms in (3.1) are well defined.

Now we state our existence result that will be proved in Section 3.2.

Theorem 3.2. *Let us assume (i)-(viii), $\gamma_n(\Omega) < 1$ and one of the following conditions holds true:*

(a) $\|b\|_{L^\infty(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}$ *is sufficiently small,*

(b) $b \in L^{\infty,r}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)$ *with* $2 < r < \infty$.

Then there exists at least a weak solution u to problem (1.1).

3.1. Apriori estimate for a solution to problem (1.1).

In this subsection we prove some apriori estimate for solution to problem (1.1).

Lemma 3.1. *Let u be a weak solution to problem* (1.1). *Under the assumptions of Theorem* 3.2, *then the following estimate holds true*

$$\|u\|_{W_0^{1,p}(\varphi,\Omega)} \le C_1\|f\|^{p-1}_{L^{p'}(\varphi,\Omega)} + C_2\|g\|^{p-1}_{L^{p'}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)} + \quad (3.2)$$
$$+ C_3\|k_2\|^{p'}_{L^{p'}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)},$$

where C_1, C_2 and C_3 are constants which depend only on p, $\gamma_n(\Omega)$ and $\|b\|_{L^\infty(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}$.

Proof. Proceeding in a standard way (see [34] and [7], [21] to) we obtain

$$-\frac{d}{dt}\int_{|u|>t}|\nabla u|^p\varphi(x)dx \le \int_t^{+\infty}\Big(-\frac{d}{ds}\int_{|u|>s}b^p(x)\varphi(x)dx\Big)^{\frac{1}{p}}\times$$
$$\times\Big(-\frac{d}{ds}\int_{|u|>s}|\nabla u|^p\varphi(x)dx\Big)^{\frac{1}{p'}}ds+ \quad (3.3)$$
$$+\Big(-\frac{d}{dt}\int_{|u|>t}|f|^{p'}\varphi(x)\,dx\Big)^{\frac{1}{p'}}\Big(-\frac{d}{dt}\int_{|u|>t}|\nabla u|^p\varphi(x)\,dx\Big)^{\frac{1}{p}}+$$
$$+\int_{|u|>t}k_2(x)\text{sign } u\ \varphi(x)dx + \int_{|u|>t}g(x)\text{sign } u\ \varphi(x)\,dx.$$

On the other hand, coarea formula (see [24]), isoperimetric inequality with respect to the Gauss measure and Hölder inequality imply

$$1 \le (2\pi)^{\frac{1}{2}}\exp\Big(\frac{\Phi^{-1}(\mu(t))^2}{2}\Big)(-\mu'(t))^{\frac{1}{p'}}\Big(-\frac{d}{dt}\int_{|u|>t}|\nabla u|^p\varphi(x)\,dx\Big)^{\frac{1}{p}}. \quad (3.4)$$

In (3.3), applying (3.4) and Hardy-Littlewood inequality by definitions of pseudorearrangement, we have

$$\Big(-\frac{d}{dt}\int\limits_{|u|>t}|\nabla u|^p\varphi(x)dx\Big)^{\frac{1}{p'}}\leq$$

$$\leq F(\mu(t))(-\mu'(t))^{\frac{1}{p'}}+(2\pi)^{\frac{1}{2}}\exp\Big(\frac{\Phi^{-1}(\mu(t))^2}{2}t\Big)(-\mu'(t))^{\frac{1}{p'}}\times$$

$$\int_t^{+\infty}B(\mu(s))(-\mu'(s))^{\frac{1}{p}}\Big(-\frac{d}{ds}\int_{|u|>s}|\nabla u|^p\varphi(x)dx\Big)^{\frac{1}{p'}}ds+$$

$$+(2\pi)^{\frac{1}{2}}\exp\Big(\frac{\Phi^{-1}(\mu(t))^2}{2}\Big)(-\mu'(t))^{\frac{1}{p'}}\Big(\int_0^{\mu(t)}(k_2^{\circ *}(s)+g^{\otimes}(s))ds\Big).$$

where F and B are functions such that $F^{p'}=(\widetilde{f^{p'}})_u$ and $B^p=(\widetilde{b^p})_u$.

If (a) or (b) hold we can use a slight modification of Gronwall lemma (see [28] for the proof of the classical one and [21] for the precise statement) obtaining

$$\Big(-\frac{d}{dt}\int\limits_{|u|>t}|\nabla u|^p\varphi(x)dx\Big)^{\frac{1}{p'}}\leq F(\mu(t))(-\mu'(t))^{\frac{1}{p'}}+ \tag{3.5}$$

$$(2\pi)^{\frac{1}{2}}\exp\Big(\frac{\Phi^{-1}(\mu(t))^2}{2}\Big)(-\mu'(t))^{\frac{1}{p'}}\int\limits_0^{\mu(t)}(k_2^{\otimes}(s)+g^{\otimes}(s))ds+$$

$$(2\pi)^{\frac{1}{2}}(-\mu'(t))^{\frac{1}{p'}}\exp\Big(\frac{\Phi^{-1}(\mu(t))^2}{2}\Big)\int\limits_t^{+\infty}B(\mu(\tau))(-\mu'(\tau))\times$$

$$\times\Big[F(\mu(\tau))+(2\pi)^{\frac{1}{2}}\exp\Big(\frac{\Phi^{-1}(\mu(\tau))^2}{2}\Big)\Big(\int\limits_0^{\mu(\tau)}(k_2^{\otimes}(s)+g^{\otimes}(s))ds\Big)\Big]\times$$

$$\times\exp\Big[(2\pi)^{\frac{1}{2}}\int\limits_t^{\tau}B(\mu(r))\times\exp\Big(\frac{\Phi^{-1}(\mu(r))^2}{2}\Big)(-\mu'\ (r))dr\Big]d\tau,$$

that is

$$(3.6) \qquad \int_\Omega |\nabla u|^p \varphi(x)dx \le$$

$$\le c\Big\{ \int_0^{\gamma_n(\Omega)} (F(t))^{p'} + (2\pi)^{\frac{p'}{2}} \exp\Big(\frac{\Phi^{-1}(t)^2}{2}p'\Big)\Big(\int_0^t k_2^{\otimes}(s) + g^{\otimes}(s)ds\Big)^{p'}$$

$$+ (2\pi)^{\frac{p'}{2}} \exp\Big(\frac{\Phi^{-1}(t)^2}{2}p'\Big)\Big(\int_0^t B(\tau)\Big[F(\tau) + (2\pi)^{\frac12}\exp\Big(\frac{\Phi^{-1}(\tau)^2}{2}\Big)\times$$

$$\int_0^\tau (k_2^{\otimes}(s) + g^{\otimes}(s))ds\Big] \exp\Big[(2\pi)^{\frac12}\int_\tau^t B(r)\exp\Big(\frac{\Phi^{-1}(r)^2}{2}\Big)dr\Big]d\tau\Big)^{p'} dt\Big\}.$$

We observe that, if (a) is true, integrating by parts and using Proposition 2.2 we have

$$(3.7) \quad \int_s^t \frac{B(r)}{r(1-\log r)^{\frac12}}dr \le C\|B\|_{L^\infty(\log L)^{-\frac12}(\Omega^{\otimes})} + C\int_s^t \frac{\int_0^r B(\tau)d\tau}{r^2(1-\log r)^{\frac12}}dr$$
$$\le C\|b\|_{L^\infty(\log L)^{-\frac12}(\varphi,\Omega)} + \|b\|_{L^\infty(\log L)^{-\frac12}(\varphi,\Omega)} \log\Big(\frac{t}{s}\Big),$$

for some positive constant C.

If (b) hold, Hölder and Young inequalities give

$$(3.8) \quad \int_s^t \frac{B(r)}{r(1-\log r)^{\frac12}}dr \le C(\varepsilon)\int_0^{t-s}\Big(\frac{B^*(r)}{(1-\log r)^{\frac12}}\Big)^a \frac{dr}{r} + \varepsilon\int_s^t \frac1r dr$$
$$\le C(\varepsilon)\|b\|^a_{L^{\infty,r}(\log L)^{-\frac12}(\varphi,\Omega)} + \varepsilon\log\Big(\frac{t}{s}\Big),$$

where ε can be arbitrary small and $C(\varepsilon)$ is a suitable constant depending on ε.

From now on c will denote a positive constant depending only on $p, \gamma_n(\Omega)$ and $\|b\|_{L^\infty(\log L)^{-\frac12}(\varphi,\Omega)}$, which may vary from line to line.

Now we evaluate the right-hand side of (3.6) by using (2.1) and (3.7) in case (a) or (3.8) in case (b):

$$\int_{\Omega} |\nabla u|^p \varphi(x)dx \leq \tag{3.9}$$

$$\leq \int_0^{\gamma_n(\Omega)} F^{p'}(t)dt + c\int_0^{\gamma_n(\Omega)} \frac{1}{t^{p'}(1-\log t)^{\frac{p'}{2}}}\Big(\int_0^t (k_2^{\otimes}(s)+g^{\otimes}(s))ds\Big)^{p'} +$$

$$+c\int_0^{\gamma_n(\Omega)} \frac{1}{t^{p'}(1-\log t)^{\frac{p'}{2}}}\Big(\int_0^t B(\tau)F(\tau)\Big(\frac{t}{\tau}\Big)^{\beta} d\tau\Big)^{p'} dt$$

$$+c\int_0^{\gamma_n(\Omega)} \frac{1}{t^{p'}(1-\log t)^{\frac{p'}{2}}}\Big(\int_0^t \frac{(2\pi)^{\frac{1}{2}}B(\tau)}{\tau(1-\log\tau)^{\frac{1}{2}}}\times$$

$$\times \int_0^{\tau} (k_2^{\otimes}(s)+g^{\otimes}(s))ds \left(\frac{t}{\tau}\right)^{\beta} d\tau\Big)^{p'} dt \ ,$$

where $\beta = c\varepsilon$ if we use (3.7) and $\beta = c\|b\|_{L^{\infty}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}$ if we use (3.8).

Using (2.3) for a sufficiently small β, Hardy-Littlewood inequality and Proposition 2.2 the two last integrals in the right-hand side of (3.9) become

$$\int_0^{\gamma_n(\Omega)} \frac{1}{t^{p'}(1-\log t)^{\frac{p'}{2}}}\Big(\int_0^t B(\tau)F(\tau)\Big(\frac{t}{\tau}\Big)^{\beta} d\tau\Big)^{p'} dt$$

$$\leq c\int_0^{\gamma_n(\Omega)} \frac{B^{*p'}(t)F^{*p'}(t)}{(1-\log t)^{\frac{p'}{2}}}\, dt \leq c\|b\|^{p'}_{L^{\infty}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)} \|f\|^{p'}_{L^{p'}(\varphi,\Omega)}$$

and

$$\int_0^{\gamma_n(\Omega)} \frac{1}{t^{p'}(1-\log t)^{\frac{p'}{2}}}\Big(\int_0^t \frac{(2\pi)^{\frac{1}{2}}B(\tau)}{\tau(1-\log\tau)^{\frac{1}{2}}} \times$$

$$\times \int_0^{\tau} (k_2^{\otimes}(s)+g^{\otimes}(s))ds\Big(\frac{t}{\tau}\Big)^{\beta} d\tau\Big)^{p'} dt \ \leq$$

$$\leq c\int_0^{\gamma_n(\Omega)} \frac{B^{*p'}(t)}{t^{p'}(1-\log t)^{p'}}\Big(\int_0^t (k_2^{\otimes}(s)+g^{\otimes}(s))ds\Big)^{p'} dt$$

$$\leq c\|b\|^{p'}_{L^{\infty}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}\Big(\|g\|^{p'}_{L^{p'}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)} + \|k_2\|^{p'}_{L^{p'}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}\Big).$$

Therefore

$$\int_\Omega |\nabla u|^p \varphi(x)dx \le c\Big(\|f\|^{p'}_{L^{p'}(\varphi,\Omega)} + \|g\|^{p'}_{L^{p'}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)} + \|k_2\|^{p'}_{L^{p'}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}\Big)$$

that is (3.2). □

3.2. Proof of Theorem 3.2.

In order to prove Theorem 3.2 we firstly approximate the data of problem (1.1) and we consider the sequence of the corresponding approximated problems. We prove that weak solutions to such problems exists by adapting the classical method due to Leray-Lions (see Appendix 1 below). Then we observe that the apriori estimates given by Lemma 3.1 hold true for the solutions of the approximated problems. This allows us to pass to the limit in the approximated problem and therefore to obtain a solution to problem (1.1).

For the sake of simplicity we give the proof with $G \equiv 0$ and $|f| = 0$ in problem (1.1) .

Denote by $T_h : \mathbb{R} \to \mathbb{R}$ the usual truncation at level $h > 0$, that is

$$T_h(s) = \begin{cases} s & \text{if } |s| \le h, \\ \dfrac{s}{|s|}h & \text{if } |s| > h. \end{cases}$$

Let us consider the approximated problem

$$\begin{cases} -\text{div}(a(x, u_h, \nabla u_h)) + H_h(x, \nabla u_h)\varphi = g\varphi & \text{in } \Omega \\ u_h \ = \ 0 & \text{on } \partial\Omega, \end{cases} \tag{3.10}$$

where

$$H_h(x,\xi) = T_h\Big(\tilde{H}(x,\xi)\Big) \quad \text{and} \quad \tilde{H}(x,\xi) = \frac{H(x,\xi)}{\varphi(x)}.$$

Moreover we observe that

$$|H_h(x,\xi)|\varphi(x) \le h\varphi(x)$$

for a.e. $x \in \Omega, \forall\xi \in \mathbb{R}^n$.

Problem (3.10) has at least a weak solution by Theorem 6.1 (in the Appendix 1 below). Moreover it follows from Lemma 3.1 that the sequence $(u_h)_h$ is bounded in $W_0^{1,p}(\varphi,\Omega)$, hence there exist a function

$u \in W_0^{1,p}(\varphi, \Omega)$ and a subsequence, still denoted by $(u_h)_h$ such that

$$\begin{cases} u_h \rightharpoonup u & \text{weakly in } W_0^{1,p}(\varphi, \Omega), \\ u_h \to u & \text{strongly in } L^p(\varphi, \Omega), \\ u_h \to u & \text{a.e. in } \Omega. \end{cases} \tag{3.11}$$

We observe that, using (iv), (v) and estimate (3.2), the function $H_h(x, \nabla u_h)$ is uniformly bounded in $L^1(\varphi, \Omega)$.

Using Theorem 7.1 (in the Appendix 2 below) we conclude that, for some subsequence, still denoted by $(u_h)_h$, we have

$$\nabla u_h \to \nabla u \quad \text{a.e. in } \Omega. \tag{3.12}$$

Recalling that $a(x, \eta, \xi)$ and $H(x, \xi)$ are Carathéodory functions, from (3.12) and (3.11) we have

$$\begin{cases} \tilde{a}(x, u_h, \nabla u_h) \to \tilde{a}(x, u, \nabla u) & \text{a.e. in } \Omega, \\ H_h(x, \nabla u_h) \to \tilde{H}(x, \nabla u) & \text{a.e. in } \Omega, \end{cases}$$

where

$$\tilde{a}(x, \eta, \xi) = \frac{a(x, \eta, \xi)}{\varphi(x)}. \tag{3.13}$$

Using (ii), (iv) and Hölder inequality for any fixed $s \in [1, p')$ and for any γ_n-measurable subset E, it follows

$$\int_E \left|\tilde{a}(x, u_h, \nabla u_h)\right|^s \varphi dx \leq$$

$$\leq c_1 \Big(\int_E |u_h|^{s(p-1)} \varphi dx + \int_E |\nabla u_h|^{s(p-1)} \varphi dx + \int_E k_1^s(x) \varphi dx \Big)$$

$$\leq c_1 \Big(\int_E |u_h|^p \varphi \; dx \Big)^{\frac{s}{p'}} \gamma_n(E)^{1-\frac{s}{p'}} + c_1 \Big(\int_E |\nabla u_h|^p \varphi \; dx \Big)^{\frac{s}{p'}} \gamma_n(E)^{1-\frac{s}{p'}}$$

$$+ c_1 \int_E k_1^s(x) \varphi \; dx$$

and

$$\int_E |H_h(x, \nabla u_h)| \varphi \; dx \leq$$

$$\Big(\int_E |b(x)|^p \varphi \; dx \Big)^{\frac{1}{p}} \Big(\int_E |\nabla u_h|^p \varphi \; dx \Big)^{\frac{1}{p'}} + \int_E k_2(x) \varphi \; dx.$$

By Vitali theorem, we can conclude that

$$(3.14)\quad \begin{cases} \tilde{a}(x,u_h,\nabla u_h) \to \tilde{a}(x,u,\nabla u) & \text{strongly in } L^s(\varphi,\Omega),\ s\in[1,p'[,\\ H_h(x,\nabla u_h) \to \tilde{H}(x,\nabla u) & \text{strongly in } L^1(\varphi,\Omega). \end{cases}$$

Using (3.14) it is possible to pass to the limit in

$$\int_\Omega a(x,u_h,\nabla u_h)\nabla\psi\,dx + \int_\Omega H_h(x,\nabla u_h)\psi\varphi\,dx = \int_\Omega g\psi\varphi\,dx \quad \forall\psi\in\mathcal{D}(\Omega)$$

and we obtain

$$\int_\Omega a(x,u,\nabla u)\nabla\psi\,dx + \int_\Omega H(x,\nabla u)\psi\,dx = \int_\Omega g\psi\varphi\,dx \quad \forall\psi\in\mathcal{D}(\Omega).$$

This conclude the proof. □

4. Comparison result.

In this section we compare the solution to problem (1.1) with the solution to problem (1.3) which is defined in an half-space and has coefficients depending on one variable. The comparison give also a pointwise estimate of $u(x)$ in terms of the data since the solution to (1.3) can be explicitly written.

Theorem 4.1. *Let $u\in W_0^{1,p}(\varphi,\Omega)$ be solution to* (1.1) *with the assumptions (i)-(viii), $\gamma_n(\Omega)<1$ and one of the following conditions holds true:*

(a) $\|b\|_{L^\infty(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}$ *is sufficiently small,*

(b) $b\in L^{\infty,r}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)$ *with* $2<r<\infty$.

Then we have

$$(4.1)\qquad u^\star(x_1)\le w(x) \qquad for\ a.e.\ x\in\Omega^\star$$

and

$$(4.2)\qquad \int_\Omega |\nabla u|^q\varphi(x)dx \le \int_{\Omega^\star}|\nabla w|^q\varphi(x)dx \qquad for\ all\ \ 0<q\le p$$

where

$$w(x)=\int_\lambda^{x_1}\exp\Big(-\frac{\tau^2}{2}\Big)\Big\{\exp\Big(\frac{\tau^2}{2}\frac{p}{p'}\Big)F(\Phi(\tau))+$$

$$+\exp\Big(\frac{\tau^2}{2}p\Big)\int_\tau^{+\infty}(g^\star(\sigma)+k_2^\star(\sigma))\exp\Big(\int_\tau^\sigma B(\Phi(r))dr-\frac{\sigma^2}{2}\Big)\,d\sigma+$$

$$+\exp\Big(\frac{\tau^2}{2}p\Big)\int_\tau^{+\infty}F(\Phi(\sigma))B(\Phi(\sigma))\exp\Big(\int_\tau^\sigma B(\Phi(r))dr-\frac{\sigma^2}{2}\Big)d\sigma\Big\}^{\frac{p'}{p}}\,d\tau$$

is the solution to the problem (1.3) *and* λ *is such that* $\gamma_n(\Omega^\star)=\gamma_n(\Omega)$.

Proof. Arguing as in the proof of Lemma 3.1, by (3.5) and raising to the $\frac{p'}{p}$ power, relation (3.4) can be rewrite as

$$1\le(2\pi)^{\frac{p'}{2}}(-\mu'(t))\Big\{(2\pi)^{-\frac12}\exp\Big(\frac{\Phi^{-1}(\mu(t))^2}{2}\frac{p}{p'}\Big)F(\mu(t))+$$
$$+\exp\Big(\frac{\Phi^{-1}(\mu(t))^2}{2}p\Big)\int_0^{\mu(t)}(g^\otimes(s)+$$
$$k_2^\otimes(s))ds+\exp\Big(\frac{\Phi^{-1}(\mu(t))^2}{2}p\Big)\int_t^\infty(F(\mu(\tau))B(\mu(\tau))+$$
$$+(2\pi)^{\frac12}B(\mu(\tau))\exp\Big(\frac{\Phi^{-1}(\mu(\tau))^2}{2}\Big)\int_0^{\mu(\tau)}(g^\otimes(s)+k_2^\otimes(s))ds\Big)(-\mu'(\tau))\times$$
$$\times\exp\Big[(2\pi)^{\frac12}\int_t^\tau B(\mu(r))\exp\Big(\frac{\Phi^{-1}(\mu(r))^2}{2}\Big)(-\mu'(r))dr\Big]d\tau\Big\}^{\frac{p'}{p}}.\tag{4.3}$$

Integrating between 0 and t, (4.3) becomes

$$t\le\int_{\mu(t)}^{\gamma_n(\Omega)}(2\pi)^{\frac{p'}{2}}\Big\{(2\pi)^{-\frac12}\exp\Big[\frac{\Phi^{-1}(\sigma)^2}{2}\frac{p}{p'}\Big]F(\sigma)+$$
$$+\exp\Big[\frac{\Phi^{-1}(\sigma)^2}{2}p\Big]\int_0^\sigma(g^\otimes(s)+k_2^\otimes(s))ds+\exp\Big[\frac{\Phi^{-1}(\sigma)^2}{2}p\Big]\times$$
$$\times\int_0^\sigma\exp\Big[(2\pi)^{\frac12}\int_r^\sigma B(\tau)\exp\Big(\frac{\Phi^{-1}(\tau)^2}{2}\Big)d\tau\Big]\times$$
$$\times\Big(F(r)B(r)+(2\pi)^{\frac12}B(r)\exp\Big[\frac{\Phi^{-1}(r)^2}{2}\Big]\int_0^r(g^\otimes(s)+k_2^\otimes(s))ds\Big)dr\Big\}^{\frac{p'}{p}}\,d\sigma.$$

Now putting $\mu(t)=s$, $s=\Phi(x_1)$ and observing that

$$\int_\tau^{+\infty}B(\Phi(\sigma))\exp\Big[\int_\tau^\sigma B(\Phi(r))dr\Big]\int_\sigma^{+\infty}(g^\star(r)+k_2^\star(r))\exp\Big(-\frac{r^2}{2}\Big)dr\;d\sigma=$$

$$= -\int_\tau^{+\infty} (g^\star(\sigma) + k_2^\star(\sigma)) \exp\Big(-\frac{\sigma^2}{2}\Big) d\sigma +$$
$$+ \int_\tau^{+\infty} (k_2^\star(\sigma) + g^\star(\sigma)) \exp\Big(\int_\tau^\sigma B(\Phi(r))dr - \frac{\sigma^2}{2}\Big) d\sigma$$

we have

$$u^\star(x) \le \int_\lambda^{x_1} \exp\Big(-\frac{\tau^2}{2}\Big)\Big\{\exp\Big(\frac{\tau^2}{2}\frac{p}{p'}\Big)F(\Phi(\tau)) + \tag{4.4}$$
$$+\exp\Big(\frac{\tau^2}{2}p\Big)\int_\tau^{+\infty} F(\Phi(\sigma))B(\Phi(\sigma))\exp\Big(\int_\tau^\sigma B(\Phi(r))dr - \frac{\sigma^2}{2}\Big)d\sigma +$$
$$+ \exp\Big(\frac{\tau^2}{2}p\Big)\int_\tau^{+\infty}(g^\star(\sigma)+k_2^\star(\sigma))\exp\Big(\int_\tau^\sigma B(\Phi(r))dr - \frac{\sigma^2}{2}\Big)d\sigma\Big\}^{\frac{p'}{p}} d\tau,$$

where λ is such that $\gamma_n(\Omega^\star) = \gamma_n(\Omega)$ and $\Omega^\star = \{x = (x_1, ..., x_n) \in \mathbb{R}^n : x_1 > \lambda\}$. To complete the proof we observe that the right-hand side of (4.4) is the solution to problem (1.3).

Let us prove now (4.2). Using Hölder inequality and (3.5) we have

$$-\frac{d}{dt}\int_{|u|>t} |\nabla u|^q \varphi(x)dx \le (-\mu'(t))^{1-\frac{q}{p}}\Big(-\frac{d}{dt}\int_{|u|>t}|\nabla u|^p\varphi(x)dx\Big)^{\frac{q}{p}} \le$$
$$\le (-\mu'(t))\Big\{F(\mu(t)) + (2\pi)^{\frac12}\exp\Big(\frac{\Phi^{-1}(\mu(t))^2}{2}\Big)\times$$
$$\times \int_0^{\mu(t)} (g^\otimes + k_2^\otimes)ds + (2\pi)^{\frac12}\exp\Big(\frac{\Phi^{-1}(\mu(t))^2}{2}\Big)\times$$
$$\times \int_t^\infty \Big(F(\mu(\tau)) + (2\pi)^{\frac12}\exp\Big(\frac{\Phi^{-1}(\mu(\tau))^2}{2}\Big)\int_0^{\mu(\tau)}(g^\otimes + k_2^\otimes)ds\Big)\times$$
$$\times B(\mu(\tau))(-\mu'(\tau))\times$$
$$\times \exp\Big[(2\pi)^{\frac12}\int_t^\tau B(\mu(r))\exp\Big(\frac{\Phi^{-1}(\mu(r))^2}{2}\Big)(-\mu'(r))dr\Big]d\tau\Big\}^{\frac{q}{p-1}}.$$

Integrating between 0 and $+\infty$ the last inequality becomes

$$\int_\Omega |\nabla u|^q\varphi(x)dx \le \int_0^{\gamma_n(\Omega)}\Big\{F(s) + (2\pi)^{\frac12}\exp\Big(\frac{\Phi^{-1}(s)^2}{2}\Big)\times$$

$$\times\Big(\int_0^s (g^{\otimes}(\tau)+k_2^{\otimes}(\tau))d\tau\Big)+(2\pi)^{\frac{1}{2}}\exp\Big(\frac{\Phi^{-1}(s)^2}{2}\Big)\times$$

$$\times\int_0^s\Big(F(\tau)+(2\pi)^{\frac{1}{2}}\exp\Big(\frac{\Phi^{-1}(\tau)^2}{2}\Big)\int_0^\tau (g^{\otimes}(\sigma)+k_2^{\otimes}(\sigma))d\sigma\Big)\times$$

$$\times B(\tau)\exp\Big[(2\pi)^{\frac{1}{2}}\int_\tau^s B(\sigma)\exp\Big(\frac{\Phi^{-1}(\sigma)^2}{2}\Big)d\sigma\Big]d\tau\Big\}^{\frac{q}{p-1}}ds,$$

that is (4.2). □

5. Regularity results.

In this section we give an estimates of a solution u to the problem in suitable Lorentz-Zygmund spaces, which are obtained evaluating the norm of the function w defined by (1.2). The estimate give the link between the summability of the data f and g and the summability of u.

We will consider separately cases $|f|\equiv 0$ and $g\equiv 0$.

Theorem 5.1. *Under the assumptions of Theorem* 4.1 *when* $|f|\equiv 0$ *the following results hold:*

(1) *if* $g\in L^{\frac{a}{p-1},\frac{q}{p-1}}(\log L)^{\alpha(p-1)-\frac{p}{2}}(\varphi,\Omega)$ *then* $u\in L^{a,q}(\log L)^{\alpha}(\varphi,\Omega)$, *where*

$$\begin{cases} a=p,\\ 1\le q\le p,\\ \alpha(p-1)-\frac{p}{2}+\frac{1}{2}\ge 0,\end{cases}\qquad or\qquad \begin{cases} a=p,\\ p<q\le\infty,\\ \alpha(p-1)-\frac{p}{2}+\frac{p-1}{q}>\frac{1}{2}-\frac{1}{p},\end{cases}$$

or

$$p<a<\infty,\ 1\le q\le\infty\ \text{and}\ -\infty<\alpha<+\infty.$$

Moreover

$$\|u\|_{L^{a,q}(\log L)^{\alpha}(\varphi,\Omega)}\le C_1\ \|g\|^{\frac{1}{p-1}}_{L^{\frac{a}{p-1},\frac{q}{p-1}}(\log L)^{\alpha(p-1)-\frac{p}{2}}(\varphi,\Omega)}+C_2 \tag{5.1}$$

holds;

(2) *if* $g\in L^{\infty,\frac{q}{p-1}}(\log L)^{\alpha(p-1)+\frac{p}{2}-1}(\varphi,\Omega)$ *then* $u\in L^{\infty,q}(\log L)^{\alpha}(\varphi,\Omega)$,

where

$$\begin{cases} 1 \le q < \infty, \\ \alpha < -\frac{1}{q}, \\ \frac{p-1}{q} + \alpha(p-1) < -\frac{p}{2} + 1, \end{cases} \qquad or \qquad \begin{cases} q = \infty, \\ \alpha < 0, \\ \alpha(p-1) < -\frac{p}{2} + 1, \end{cases}$$

and

$$(5.2) \qquad \|u\|_{L^{\infty,q}(\log L)^{\alpha}(\varphi,\Omega)} \le C_3 \ \|g\|^{\frac{1}{p-1}}_{L^{\infty,\frac{q}{p-1}}(\log L)^{\alpha(p-1)+\frac{p}{2}-1}(\varphi,\Omega)} + C_4.$$

The constants C_1, C_2, C_3 *and* C_4 *depend on* p, a, q, $\alpha, \gamma_n(\Omega)$, $\|b\|_{L^{\infty}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}$ *and* $\|k_2\|_{L^{p'}(\varphi,\Omega)}$.

Proof. Let $w \in W_0^{1,p}(\varphi, \Omega^{\star})$ be a weak solution to the problem (1.3) with $F \equiv 0$. Using (2.1) and (3.7) in the case (a) or (3.8) in the case (b), it follows that

$$(5.3) \quad w^{\otimes}(t) \le c \int_t^{\gamma_n(\Omega)} \frac{1}{\sigma^{p'}(1-\log\sigma)^{\frac{p'}{2}}} \left(\int_0^{\sigma} \left(\frac{\sigma}{s}\right)^{\beta} (g^{\otimes}(s) + k_2^{\otimes}(s))ds \right)^{\frac{p'}{p}} d\sigma,$$

where $\beta = c\|b\|_{L^{\infty}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}$ if we use (3.7) and $\beta = c\varepsilon$ if we use (3.8) and c is a positive constant which depends only by p, $\gamma_n(\Omega)$ and $\|b\|_{L^{\infty}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}$ and which may vary from line to line.

We prove (5.1) when $1 \le q < \infty$. By definition (2.9), (5.3), inequality (2.4) and (2.3) for a sufficiently small β we have

$$(5.4) \qquad ||w||^q_{a,q;\alpha} \le C_1 \|g\|^{\frac{1}{p-1}}_{\frac{a}{p-1},\frac{q}{p-1},\alpha(p-1)-\frac{p}{2}} + C_2.$$

Therefore, (4.1) and (5.4) give (5.1).

When $q = \infty$, the inequality (5.1) follows by analogous arguments as before with (2.4) replaced by (2.6) and (2.3) replaced by (2.5).

If $1 \le q < \infty$ inequality (5.2) follows by the above arguments using (2.7), when $\alpha < -\frac{1}{q}$, and (2.3).

When $q = \infty$ we can use (2.8) with $\alpha < 0$ and (2.5). □

Theorem 5.2. *Under the assumptions of Theorem* 4.1 *when* $g \equiv 0$ *the following results hold:*

(1) *if* $|f| \in L^{\frac{a}{p-1},\frac{q}{p-1}}(\log L)^{(p-1)(\alpha-\frac{1}{2})}(\varphi,\Omega)$ *then* $u \in L^{a,q}(\log L)^{\alpha}(\varphi,\Omega)$, *where*

$$p < a < \infty, \ \ p \le q \le \infty \ \textit{and} \ -\infty < \alpha < +\infty;$$

and

$$\|u\|_{L^{a,q}(\log L)^{\alpha}(\varphi,\Omega)} \le C_1 \ \|f\|^{\frac{1}{p-1}}_{L^{\frac{a}{p-1},\frac{q}{p-1}}(\log L)^{(p-1)(\alpha-\frac{1}{2})}(\varphi,\Omega)} + C_2 \tag{5.5}$$

holds;

(2) *if* $|f| \in L^{\infty,\frac{q}{p-1}}(\log L)^{(p-1)(\alpha+\frac{1}{2})}(\varphi,\Omega)$ *then* $u \in L^{\infty,q}(\log L)^{\alpha}(\varphi,\Omega)$, *where*

$$\begin{cases} p \le q < \infty, \\ \alpha < -\frac{1}{q}, \\ (p-1)(\alpha+\frac{1}{2}+\frac{1}{q}) < 0, \end{cases} \quad \textit{or} \quad \begin{cases} q = \infty, \\ \alpha < 0, \\ (p-1)(\alpha+\frac{1}{2}) \le 0; \end{cases}$$

and

$$\|u\|_{L^{\infty,q}(\log L)^{\alpha}(\varphi,\Omega)} \le C_3 \ \|f\|^{\frac{1}{p-1}}_{L^{\infty,\frac{q}{p-1}}(\log L)^{(p-1)(\alpha+\frac{1}{2})}(\varphi,\Omega)} + C_4. \tag{5.6}$$

The constants C_1, C_2, C_3 *and* C_4 *depends on* $p, a, q, \alpha, \gamma_n\ (\Omega)$, $\|b\|_{L^{\infty}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}$ *and* $\|k_2\|_{L^{p'}(\varphi,\Omega)}$.

Proof. We first recall that the solution to the problem (1.1) with $g = 0$ is

$$w^{\otimes}(t) = (2\pi)^{\frac{p'}{2}} \int_t^{\gamma_n(\Omega)} \Big\{ (2\pi)^{-\frac{1}{2}} \exp\Big[\frac{\Phi^{-1}(\sigma)^2}{2}\frac{p}{p'}\Big] F(\sigma) + \tag{5.7}$$

$$+\exp\Big[\frac{\Phi^{-1}(\sigma)^2}{2} p\Big] \int_0^{\sigma} \exp\Big[(2\pi)^{\frac{1}{2}} \int_r^{\sigma} B(\tau) \exp\Big(\frac{\Phi^{-1}(\tau)^2}{2}\Big) d\tau\Big] B(r) F(r) dr +$$

$$+\exp\Big(\frac{\Phi^{-1}(\sigma)^2}{2} p\Big) \int_0^{\sigma} \exp\Big[(2\pi)^{\frac{1}{2}} \int_r^{\sigma} B(\tau) \exp\Big(\frac{\Phi^{-1}(\tau)^2}{2}\Big) d\tau\Big] k_2^{\otimes}(r) dr \Big\}^{\frac{p'}{p}} d\sigma.$$

We will denote by c a positive constant which depends only by p, $\gamma_n(\Omega)$ and $\|b\|_{L^{\infty}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}$ and which may vary from line to line.

Applying (2.1) and (3.7) in case (*a*), or (3.8) in case (*b*), it follows that

$$w^{\otimes}(t) \le c(w_1(t) + w_2(t) + w_3(t))$$

with

$$w_1(t) = \int_t^{\gamma_n(\Omega)} \frac{1}{\sigma(1-\log\sigma)^{\frac{1}{2}}}(F(\sigma))^{\frac{p'}{p}} d\sigma,$$
$$w_2(t) = \int_t^{\gamma_n(\Omega)} \frac{1}{\sigma^{p'}(1-\log\sigma)^{\frac{p'}{2}}}\Big(\int_0^{\sigma}\Big(\frac{\sigma}{r}\Big)^{\beta} B(r)F(r)dr\Big)^{\frac{p'}{p}} d\sigma$$

and

$$w_3(t) = \int_t^{\gamma_n(\Omega)} \frac{1}{\sigma^{p'}(1-\log\sigma)^{\frac{p'}{2}}}\Big(\int_0^{\sigma}\Big(\frac{\sigma}{r}\Big)^{\beta} k_2^{\otimes}(r)dr\Big)^{\frac{p'}{p}} d\sigma,$$

where $\beta = c\|b\|$ if we use (3.7) and $\beta = c\varepsilon$ if we use (3.8).

We first prove (5.5) when $p \le q < \infty$. The argument used in the proof of Theorem 5.1 shows that

$$\|w_3\|_{a,q;\alpha} \le c\|k_2\|_{L^{p'}(\varphi,\Omega)}^{\frac{1}{p-1}}, \tag{5.8}$$

and

$$\|w_2\|_{a,q;\alpha}^{q} \le c\|b\|_{L^{\infty}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}^{\frac{q}{p-1}} \|f\|_{L^{\frac{a}{p-1},\frac{q}{p-1}}(\log L)^{(p-1)(\alpha-\frac{1}{2})}(\varphi,\Omega)}^{\frac{q}{p-1}}. \tag{5.9}$$

Let us observe that integrating by parts, using Hardy-Littlewood inequality and Proposition 2.2 we get:

$$w_1(t) \le c\frac{||f||_{L^{\frac{1}{p-1}}(\varphi,\Omega)}^{p-1}}{\gamma_n(\Omega)(1-\log\gamma_n(\Omega))^{\frac{1}{2}}} + c\int_t^{\gamma_n(\Omega)} \frac{\int_0^{\sigma}(F^*(s))^{\frac{1}{p-1}}ds}{\sigma^2(1-\log\sigma)^{\frac{1}{2}}}d\sigma \tag{5.10}$$

By (5.10), using (2.4), (2.3) and Proposition 2.2, we have

$$\|w_1\|_{a,q;\alpha} \leq c\|f\|^{\frac{1}{p-1}}_{L^{\frac{1}{p-1}}(\varphi,\Omega)} \Big(\int_0^{\gamma_n(\Omega)} t^{\frac{q}{a}}(1-\log t)^{\alpha q}\frac{dt}{t}\Big)^{\frac{1}{q}} + \tag{5.11}$$

$$+c\Big(\int_0^{\gamma_n(\Omega)} t^{\frac{q}{a}}(1-\log t)^{\alpha q}\Big(\int_t^{\gamma_n(\Omega)} \frac{\int_0^\sigma (F^*(s))^{\frac{1}{p-1}}ds}{\sigma^2(1-\log\sigma)^{\frac{1}{2}}}d\sigma\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$

$$\leq c\|f\|^{\frac{1}{p-1}}_{L^{\frac{a}{p-1},\frac{q}{p-1}}(\log L)^{(p-1)(\alpha-\frac{1}{2})}(\varphi,\Omega)} +$$

$$+c\Big(\int_0^{\gamma_n(\Omega)} t^{\frac{q}{a}-q}(1-\log t)^{\alpha q-\frac{q}{2}}\Big(\int_0^t (F^*(\sigma))^{\frac{1}{p-1}}d\sigma\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$

$$\leq c\|f\|^{\frac{1}{p-1}}_{L^{\frac{a}{p-1},\frac{q}{p-1}}(\log L)^{(p-1)(\alpha-\frac{1}{2})}(\varphi,\Omega)}.$$

By (4.1), the assert follows from (5.11), (5.9) and (5.8).

Estimate (5.5) when $q=\infty$ can be handled in the same way, we have just to replace (2.4) with (2.6) and (2.3) with (2.5) under the same conditions.

Estimate (5.6) can be obtained with similar arguments. □

Remark 5.3. Arguing as in Theorem 5.1 and Theorem 5.2 analogous regularity result can be obtain for problem (1.1).

Remark 5.4. In the proof of Lemma 3.1 if we replace (*iv*) with the assumption

(iv)' $|H(x,\xi)| \leq (b(x)|\xi|^\sigma + k_2(x))\varphi(x)$
a.e. $x\in\Omega$, $\forall\xi\in R^n, \sigma\in[0,p-1[$, $b(x)\in L^\infty(\log L)^{-\frac{1}{2}}(\varphi,\Omega)$ and $k_2(x)\in L^{p'}(\varphi,\Omega)$,

we can show an estimate like (3.2) without smallness hypotheses on $\|b\|_{L^\infty(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}$ or $b\in L^{\infty,r}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)$ with $2<r<\infty$, i.e.

$$\|u\|_{W_0^{1,p}(\varphi,\Omega)} \leq C_1\|f\|^{p-1}_{L^{p'}(\varphi,\Omega)} + C_2\|g\|^{p-1}_{L^{p'}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)} + C_3\|k_2\|^{p'}_{L^{p'}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)} + C_4, \tag{5.12}$$

where C_1, C_2, C_3 and C_4 are nonnegative constants which depend only on $p, \gamma_n(\Omega), \|b\|_{L^\infty(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}$.

In this case by (5.12) existence result can be proved with a slight modifications of the proof of Theorem 3.2 and comparison and regularity results can be obtained observing that

$$|\nabla u|^\sigma \leq (|\nabla u|^{p-1} + 1).$$

Remark 5.5. In the framework of this paper under assumptions (i)-(iii) and (v)-$(viii)$ we can prove existence, comparison and regularity results for the following problem

$$(5.13) \quad \begin{cases} -\mathrm{div}(a(x,u,\nabla u)) - \mathrm{div}(K(x,u)) + G(x,u) = \\ \qquad\qquad = g\varphi - \mathrm{div}(f\varphi) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $K : \Omega \times R \to R^n$ is a Carathéodory function such that

(ix) $|K(x,\eta)| \leq (d(x)|\eta|^{p-1} + k_4(x))\varphi(x)$

a.e. $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $d(x), k_4(x) \geq 0$, $d(x) \in L^\infty(\log L)^{-\frac{1}{2}}(\varphi,\Omega)$ and $k_4(x) \in L^{p'}(\varphi,\Omega)$.

6. Appendix 1: Existence result for coercive equations.

In this Appendix we enunciate an existence result for nonlinear coercive elliptic equations that can be proved by adapting the classical methods due to Leray-Lions (see [26]).

Let us consider the problem

$$(6.1) \quad \begin{cases} -\mathrm{div}(a(x,u,\nabla u)) + H(x,\nabla u) + G(x,u) = g\varphi - \mathrm{div}(f\varphi) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, and $G : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions which satisfy the assumptions (i)-(iii) and (v)-(vi) respectively. Moreover we assume that $H : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function which satisfy (instead of the more general assumption (iv))

(iv)" $|H(x,\xi)| \leq \varphi(x)(c_1|\xi|^{p-1} + k_2(x))$,
$c_1 \geq 0$, $k_2(x) \geq 0$, $k_2(x) \in L^{p'}(\varphi,\Omega)$, a.e. $x \in \Omega$, $\forall \xi \in \mathbb{R}^n$.

Finally we assume that f satisfy the assumption (vii) while g satisfy $(viii)$.

Observe that there exists $Lu \in W^{-1,p'}(\varphi, \Omega)$ such that

$$\langle Lu, w\rangle = \int_\Omega a(x, u, \nabla u)\nabla w dx + \int_\Omega H(x, \nabla u) w dx + \int_\Omega G(x, u) w dx, \quad \forall w \in W_0^{1,p}(\varphi, \Omega)$$

and

$$(6.2) \quad Lu = -\text{div} \quad a(x, u, \nabla u) + H(x, \nabla u) + G(x, u), \quad \forall u \in D(\Omega).$$

Theorem 6.1. *Let us assume (i)-(iii), (iv)" and (v). Let L be defined by (6.2) and*

$$\frac{< Lv, v >}{\|v\|_{W_0^{1,p}(\varphi,\Omega)}} \to \infty, \quad if \quad \|v\|_{W_0^{1,p}(\varphi,\Omega)} \to \infty.$$

Then for every $T \in W^{-1,p'}(\varphi, \Omega)$[3] *it exists $u \in W_0^{1,p}(\varphi, \Omega)$ such that*

$$Lu = T.$$

Proof. By the classical theory on pseudomonotone and coercive operator, it is sufficient to prove that the operator L is bounded and there exists an operator

$$\overline{L} : W_0^{1,p}(\varphi, \Omega) \times W_0^{1,p}(\varphi, \Omega) \to W^{-1,p'}(\varphi, \Omega)$$

such that

(1) $Lu = \overline{L}(u, u)$;

(2) the map $v \to \overline{L}(u, v)$ $\forall u \in W_0^{1,p}(\varphi, \Omega)$ is bounded, hemicontinous[4] and

$$(6.3) \quad < \overline{L}(u, u) - \overline{L}(u, v), u - v > \geq 0;$$

(3) the map $u \to \overline{L}(u, v)$ $\forall v \in W_0^{1,p}(\varphi, \Omega)$ is bounded and hemicontinous;

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[3] We denote by $W^{-1,p'}(\varphi, \Omega)$ the dual space of $W_0^{1,p}(\varphi, \Omega)$.

[4] The application $T : v \in W_0^{1,p}(\varphi, \Omega) \to Tv \in W^{-1,p'}(\varphi, \Omega)$ is hemicontinuous if the map

$$\lambda \in \mathbb{R} \to \langle T(v_1 + \lambda v_2), w\rangle \in \mathbb{R}$$

is continuous for any v_1, v_2 and $w \in W_0^{1,p}(\varphi, \Omega)$.

(4) if $u_k \rightharpoonup u$ in $W_0^{1,p}(\varphi,\Omega)$ and $< \overline{L}(u_k,u_k)-\overline{L}(u_k,u), u_k-u >\to 0$, then for every $v \in W_0^{1,p}(\varphi,\Omega)$

$$\overline{L}(u_k,v) \rightharpoonup \overline{L}(u,v) \quad \text{in} \quad W^{-1,p'}(\varphi,\Omega);$$

(5) if $u_k \rightharpoonup u$ in $W_0^{1,p}(\varphi,\Omega)$ and $\overline{L}(u_k,v) \to \Psi$ in $W^{-1,p'}(\varphi,\Omega)$ then

$$< \overline{L}(u_k,v), u_k > \to < \Psi, u > .$$

We adopt the notation of Section 3.2 and, for sake of simplicity, we denote

$$\overline{a}(u,v,w) = \int_\Omega a(x,u,\nabla v)\nabla w \; dx,$$

$$\overline{H}(u,w) = \int_\Omega H(x,\nabla u)w \; dx,$$

$$\overline{G}(u,w) = \int_\Omega G(x,u)w \; dx,$$

for any $u, v, w \in W_0^{1,p}(\varphi,\Omega)$.

The map $w \to \overline{a}(u,v,w)+\overline{H}(u,w)+\overline{G}(u,w)$ $\forall u, v \in W_0^{1,p}(\varphi,\Omega)$ is linear and bounded, therefore there exists $\overline{L} : W_0^{1,p}(\varphi,\Omega)\times W_0^{1,p}(\varphi,\Omega) \to W^{-1,p'}(\varphi,\Omega)$ such that

$$\langle \overline{L}(u,v), w\rangle = \overline{a}(u,v,w) + \overline{H}(u,w) + \overline{G}(u,w),$$

for any $u, v, w \in W_0^{1,p}(\varphi,\Omega)$.

We begin by proving (2). Using (iii), it follows immediately (6.3). By continuity of the function $a(x,\eta,\xi)$ in the variable ξ, we obtain

$$\tilde{a}(x,u,\nabla(v_1+\lambda v_2)) \to \tilde{a}(x,u,\nabla v_1) \quad \text{in } L^{p'}(\varphi,\Omega)-\text{weakly}.$$

Therefore

$$\overline{a}(u,v_1+\lambda v_2,w) \to \overline{a}(u,v_1,w) \text{ if } \lambda \to 0 \quad \forall u, v_1, v_2, w \in W_0^{1,p}(\varphi,\Omega).$$

Similarly it follows (3). Indeed by continuity of the functions $a(x,\eta,\xi)$ and $G(x,\eta)$ in the variable η and of the function $H(x,\xi)$ in the variable ξ, we obtain

$$\tilde{a}(x,u_1+\lambda u_2,\nabla v) \to \tilde{a}(x,u_1,\nabla v) \text{ in } L^{p'}(\varphi,\Omega)-\text{weakly},$$

$$\tilde{H}(x,\nabla u_1+\lambda\nabla u_2) \to \tilde{H}(x,\nabla u_1) \text{ in } L^{p'}(\varphi,\Omega)-\text{weakly},$$

$$\tilde{G}(x,u_1+\lambda u_2) \to \tilde{G}(x,u_1) \text{ in } L^{p'}(\varphi,\Omega)-\text{weakly}.$$

Therefore if $\lambda \to 0$ it follows that

$$\overline{a}(u_1 + \lambda u_2, v, w) + \overline{H}(\nabla u_1 + \lambda \nabla u_2, w) + \overline{G}(u_1 + \lambda u_2, w) \to$$
$$\overline{a}(u_1, v, w) + \overline{H}(\nabla u_1, w) + \overline{G}(u_1, w),$$

for any $u_1, u_2, v, w \in W_0^{1,p}(\varphi, \Omega)$.

Now to prove (4), let us suppose that $u_k \rightharpoonup u$ in $W_0^{1,p}(\varphi, \Omega)$ and

$$< \overline{L}(u_k, u_k) - \overline{L}(u_k, u), u_k - u > \to 0.$$

It follows that

$$(6.4) \quad \begin{cases} u_k \to u & \text{a.e. in } \Omega \\ \exists \rho \in L^p(\varphi, \Omega) : |u_k(x)| \le \rho(x) \quad \forall k \in \mathbb{N} & \text{a.e. in } \Omega \\ \|u_k\|_{W_0^{1,p}(\varphi,\Omega)} \le L, & \text{for some } L > 0. \end{cases}$$

Moreover using Lemma 2.2, Chapter 2 of [26] with the obviously modification we have

$$(6.5) \qquad \nabla u_k(x) \to \nabla u(x) \text{ a.e. in } \Omega.$$

Therefore by continuity of the functions $a(x, \eta, \xi)$ and $G(x, \eta)$ in the variable η and the function $H(x, \xi)$ in the variable ξ, by (6.5) and (6.4), we get

$$(6.6) \qquad \tilde{a}(x, u_k, \nabla v) \to \tilde{a}(x, u, \nabla v) \text{ a.e. in } \Omega,$$

$$(6.7) \qquad \tilde{H}(x, \nabla u_k) \to \tilde{H}(x, \nabla u) \text{ a.e. in } \Omega,$$

$$(6.8) \qquad \tilde{G}(x, u_k) \to \tilde{G}(x, u) \text{ a.e. in } \Omega.$$

Furthermore by (ii), (iv)”, (v) and (6.4) the functions $\tilde{a}(x, u_k, \nabla v)$, $\tilde{H}(x, \nabla u_k)$ and $\tilde{G}(x, u_k)$ belongs to $L^{p'}(\varphi, \Omega)$. Thus by (6.6), (5.7) and (6.8) we get (see Lemma 1.3 of [26])

$$\tilde{a}(x, u_k, \nabla v) \to \tilde{a}(x, u, \nabla v) \text{ in } L^{p'}(\varphi, \Omega)\text{−weakly},$$
$$\tilde{H}(x, \nabla u_k) \to \tilde{H}(x, \nabla u) \text{ in } L^{p'}(\varphi, \Omega)\text{−weakly},$$
$$\tilde{G}(x, u_k) \to \tilde{G}(x, u) \text{ in } L^{p'}(\varphi, \Omega)\text{−weakly},$$

therefore

$$\overline{a}(u_k, v, w) \to \overline{a}(u, v, w), \ \forall w \in W_0^{1,p}(\varphi, \Omega)$$

$$\overline{H}(u_k, w) \to \overline{H}(u, w), \ \forall w \in W_0^{1,p}(\varphi, \Omega)$$

$$\overline{G}(u_k, w) \to \overline{G}(u, w), \ \forall w \in W_0^{1,p}(\varphi, \Omega)$$

that is

$$\overline{L}(u_k, v) \rightharpoonup \overline{L}(u, v) \quad \text{in} \quad W^{-1,p'}(\varphi, \Omega).$$

Finally to prove (5) let us suppose that $u_k \rightharpoonup u$ in $W_0^{1,p}(\varphi, \Omega)$ and $\overline{L}(u_k, v) \to \Psi$ in $W^{-1,p'}(\varphi, \Omega)$. Using (ii) and (6.4), we obtain

$$\left|\tilde{a}(x, u_k, \nabla v)\right| \leq c\left[|u_k|^{p-1} + |\nabla v|^{p-1} + |k_1|\right] \leq$$

$$\leq c\left[|\rho|^{p-1} + |\nabla v|^{p-1} + |k_1|\right] = z(x),$$

with $z \in L^{p'}(\varphi, \Omega)$. By dominate convergence theorem we have

$$\tilde{a}(x, u_k, \nabla v) \to \tilde{a}\ (x, u, \nabla v) \ \text{in} \ L^{p'}(\varphi, \Omega),$$

hence

$$\overline{a}(u_k, v, u_k) \to \overline{a}(u, v, u). \tag{6.9}$$

Using (iv)", (v) and (6.4) we get

$$\left|\overline{H}(u_k, u_k - u) + \overline{G}(u_k, u_k - u)\right| \leq c\|u_k - u\|_{L^p(\varphi,\Omega)},$$

and therefore

$$\overline{H}(u_k, u_k - u) + \overline{G}(u_k, u_k - u) \to 0. \tag{6.10}$$

Moreover

$$\overline{H}(u_k, u) + \overline{G}(u_k, u) =< \overline{L}(u_k, v), u > -\overline{a}(u_k, v, u) \to$$

$$< \Psi, u > -\overline{a}(u, v, u). \tag{6.11}$$

From (6.10) and (6.11) we deduce that

$$\overline{H}(u_k, u_k) + \overline{G}(u_k, u_k) \to < \Psi, u > -\overline{a}(u, v, u). \tag{6.12}$$

By (6.9) and (6.12) it follows that

$$< \overline{L}(u_k, v), u_k >= \overline{a}(u_k, v, u_k) - \overline{H}(u_k, u_k) + \overline{G}(u_k, u_k) \to < \Psi, u > .$$

This complete the proof. □

7. Appendix 2: An extension of Theorem 2.1 of [11].

The following theorem is an extension of Theorem 2.1 of [11]:

Theorem 7.1. *Assume that the hypothesis (i)-(iii) hold. Let* $(f_h)_h \subset L^{p'}(\varphi, \Omega)$, $(g_h)_h \subset L^1(\varphi, \Omega)$ *and let* u_h *be a weak solution to equations*

$$\begin{cases} -\operatorname{div}(a(x, u_h, \nabla u_h)) = g_h\varphi - \operatorname{div}(f_h\varphi) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

such that

(7.1) $u_h \rightharpoonup u$ *weakly in* $W_0^{1,p}(\varphi,\Omega)$, *strongly in* $L^p(\varphi,\Omega)$ *and a.e. in* Ω,

(7.2) $f_h \to f$ *strongly in* $\left(L^{p'}(\varphi, \Omega)\right)^n$,

and

$$\|g_h\|_{L^1(\varphi,\Omega)} \leq M, \tag{7.3}$$

where M *is a constant which does not depend on* h.

Then

$$\nabla u_h \to \nabla u \qquad a.e. \text{ in } \Omega.$$

Proof. Using $v_h = T_\varrho(u_h - u) \in W_0^{1,p}(\varphi, \Omega)$ as test function, we obtain

$$\begin{aligned} &\int_\Omega [a(x, u_h, \nabla u_h) - a(x, u_h, \nabla u)]\nabla T_\varrho(u_h - u)\, dx = \\ &= -\int_\Omega a(x, u_h, \nabla u)\nabla T_\varrho(u_h - u)\, dx + \int_\Omega f_h\, \nabla T_\varrho(u_h - u)\varphi dx \\ &\qquad + \int_\Omega g_h T_\varrho(u_h - u)\varphi dx. \end{aligned} \tag{7.4}$$

By (7.1), it follows that

$$T_\varrho u_h \rightharpoonup T_\varrho u \text{ weakly in } W_0^{1,p}(\varphi, \Omega),$$

which implies that for ϱ fixed, the first two term of the right-hand side of (7.4) tend to 0 when h tends to infinity. On the other hand by (7.3)

$$\left| \int_\Omega g_h T_\varrho(u_h - u)\varphi dx \right| \leq M\rho.$$

This proves that, for ϱ fixed,

$$\limsup_{h\to\infty} \int_\Omega [a(x, u_h, \nabla u_h) - a(x, u_h, \nabla u)]\nabla T_\varrho(u_h - u)\, dx \leq M\rho. \tag{7.5}$$

Now using (3.13) we put

$$z_h(x) = \big[\tilde{a}(x, u_h, \nabla u_h) - \tilde{a}(x, u_h, \nabla u)\big](\nabla u_h - \nabla u).$$

It is easy to adapt the arguments contained in [11] in order to show that $z_h(x)^\theta \to 0$ strongly in $L^1(\varphi, \Omega)$ with $0 < \theta < 1$, then there exists a subsequence, still denoted by $(z_h)_h$, such that

$$(7.6) \qquad z_h(x) \to 0 \quad \forall x \in \Omega - \Omega_0 \quad \text{with} \quad \gamma_n(\Omega_0) = 0.$$

The claim now follows by a slight modification of Lemma 2.2, Chapter 2-[26]. □

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