Nonlinear Elliptic Equations, Rearrangements of Functions and Orlicz Spaces (*).

GIORGIO TALENTI (Firenze)

Summary. – *See Introduction.*

1. – Introduction.

In this paper we take into account a class of nonlinear elliptic second-order partial differential equations in divergence form. Typically we deal with equations of the following form

$$-\sum_{i=1}^{m} \frac{\partial}{\partial x_i} a_i(x, u, Du) = H(x, u) \tag{1.1}$$

where $x = (x_1, \ldots, x_m)$ is a point in m-dimensional euclidean space R^m, u is the relevant solution and Du its gradient, a_i and H are given real-valued functions of the specified arguments. The main hypotheses we assume are the following:

(i) the coefficients a_i are measurable in $R^m \times R^1 \times R^m$ and a function $A(r)$ exists such that

$$A(r) \text{ is convex in } 0 \leqslant r < +\infty, \tag{1.2a}$$

$$A(r)/r \to 0 \quad \text{if } r \to 0, \tag{1.2b}$$

$$\sum_{i=1}^{m} a_i(x, u, \xi)\, \xi_i \geqslant A(|\xi|) \quad \text{for all } (x, u, \xi). \tag{1.2c}$$

Note that $A(r)$ and $A(r)/r$ are positive and increasing by (1.2*a*) (1.2*b*). Although no additional assumption on $A(r)$ is really needed for our purposes, we shall assume for convenience that $A(r)$ is strictly increasing and twice continuously differentiable.

(ii) The right-hand side H is measurable in $R^m \times R^1$ and

$$\big(H(x, u) - H(x, 0)\big)\, u \leqslant 0 \quad \text{for all } (x, u). \quad \blacksquare \tag{1.3a}$$

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Let us give some examples. The first example is taken from the classic calculus of variations. Let $a(x, Du)$ and $b(x, u)$ depend smoothly on the last arguments; then the Euler equation of the integral $\int\big(a(x, Du) + b(x, u)\big)\,dx$ has the form (1.1) with $a_i = \partial a/\partial u_{x_i}$, $H = -\,\partial b/\partial u$. In particular the Euler equation of the following integral

$$\int\Big\{\int_0^{|Du|}\frac{A(r)}{r}\,dr - \int_0^u H(x, t)\,dt\Big\}dx$$

is

$$-\sum_{i=1}^{m}\frac{\partial}{\partial x_i}\left(\frac{A(|Du|)}{|Du|^2}\,u_{x_i}\right) = H(x, u)$$

and verifies the above hypotheses (i)(ii) provided $A(r)$ satisfies (1.2*a*)(1.2*b*) and $H(x, u)$ is a decreasing function of u.

The second example might be a linear equation with merely measurable coefficients, and having the following form

$$-\sum_{i,k=1}^{m}\frac{\partial}{\partial x_i}\{a_{ik}(x)u_{x_k}\} + c(x)u = f(x)\,.$$

In this case hypothesis (i) is just the usual ellipticity condition

$$\sum_{i,k=1}^{m} a_{ik}(x)\,\xi_i\xi_k \geqslant (\text{a positive constant})\sum_{i=1}^{m}\xi_i^2$$

and hypothesis (ii) is the same as the positiveness of $c(x)$.

The capillary equation with positive gravity

$$\sum_{i=1}^{m}\frac{\partial}{\partial x_i}\frac{u_{x_i}}{\sqrt{1 + |Du|^2}} = u$$

verifies conditions (i) (ii); the same is done by the equation of (non parametric) surfaces with prescribed mean curvature

$$\sum_{i=1}^{m}\frac{\partial}{\partial x_i}\frac{u_{x_i}}{\sqrt{1 + |Du|^2}} = -\,H(x, u)\,,$$

provided the curvature $-\,H(x, u)$ is an increasing function of height u. The same holds for

$$-\sum_{i=1}^{m}\frac{\partial}{\partial x_i}\frac{u_{x_i}}{1 + |Du|} = H(x, u)$$

the Euler equation of the integral $\int\Big(|Du| - \ln\,(1 + |Du|) - \int_0^u H(x, t)\,dt\Big)\,dx$. In these examples the most appropriate choice of $A(r)$ is: $A(r) = r^2/(1 + r)$.

The following equations

$$-\sum_{i,k=1}^{m}\frac{\partial}{\partial x_i}\{a_{ik}(x)|Du|^{p-2}u_{x_k}\} = \text{a lower order term}$$

$$-\sum_{i,k=1}^{m}\frac{\partial}{\partial x_i}\left\{a_{ik}(x)\frac{\exp(|Du|)-1}{|Du|}u_{x_k}\right\} = \dots$$

$$-\sum_{i,k=1}^{m}\frac{\partial}{\partial x_i}\left\{a_{ik}(x)\frac{\ln(1+|Du|)}{|Du|}u_{x_k}\right\} = \dots$$

where $p>1$ and $a_{ik}(x)$ are measurable functions, belong to our class if the right-hand side is a function $H(x, u)$ as in (ii) and the lowest eigenvalue of the symmetric part of the matrix $(a_{ik}(x))$ is bounded away from zero, e.g. $\sum_{i,k=1}^{m} a_{ik}(x)\xi_i\xi_k \geqslant \xi_1^2 + \dots + \xi_m^2$. The appropriate $A(r)$ are respectively: r^p, $r(e^r-1)$, $r\ln(1+r)$. ■

We take into account solutions of *Dirichlet* boundary value problems. Thus a basic ingredient is

$$G = \text{an open subset of } R^m\,, \tag{1.4}$$

and the solutions we are concerned with are real-valued functions verifying the equation (1.1) in G and the condition

$$u = g \quad \text{on the boundary } \partial G \text{ of } G\,, \tag{1.5a}$$

where g is some given function.

To avoid unnecessary complications, we shall consider only domains G with finite measure. Furthermore we assume that the boundary datum g is a *bounded* function. From our point of view the latter seems to be an unpleasant restriction, but we are unable to remove it at present. In fact in our treatement of Dirichlet problems we need to know that the level sets $\{x\colon |u(x)|>t\}$ of the relevant solutions do not meet the boundary « much » if t is large enough. As a matter of fact, we shall focus our attention mainly on the $g=0$ case.

We emphasize that no smoothness of ground domain G, or of coefficients a_i or of right-hand side H, is required. ■

We take into account *weak* solutions. Roughly speaking, a weak solution in an open set G to equation (1.1) is a function u, endowed with first derivatives in the domain G, such that

$$\int_G \sum_{i=1}^{m} a_i(x, u, Du)\varphi_{x_i}dx = \int_G H(x, u)\varphi\, dx \tag{1.6a}$$

for every φ from a suitable collection of test functions. A weak solution u in G verifies the boundary condition (1.5a) if the difference $u-g$ is one of the above test functions. How to choose such test functions? To avoid contradictions, this collection should not be too big: in particular, the boundary behaviour of test functions must be suitably restricted of course. At the same time, this same collection should be broad enough to draw significant conclusions about solutions. In particular, the following are appropriate requirements: (*a*) every compactly supported function, from the function class of the relevant solutions, is a test function; (*b*) the set of test functions is linear and closed. Accordingly, the set of test functions is defined to be the smallest set enjoying the above listed properties; or, more precisely, the closure of $C_0^1(G)$ with respect to the topology of *the linear hull* of the function class of relevant solutions. Here $C_0^1(G)$ stands for the collection of continuously differentiable functions vanishing outside a compact subset of G.

This setting is suggested by a standard argument of the calculus of variations. In fact, care is taken so that the weak solutions (in the above sense) of the Euler equation of any integral $\int f(x, u, Du)\,dx$ (with a smooth f) are exactly the critical points, *i.e.* the functions annihilating the Fréchet derivative of the integral.

We investigate solutions from the so-called Orlicz-Sobolev spaces. Specifically, we take weak solutions u, whose distributional derivatives in the domain G are functions from *the convex Orlicz class*

$$\left\{f \text{ measurable}: \int_G A\big(|f(x)|\big)\,dx < \infty\right\}, \tag{1.7}$$

namely

$$\int_G A\big(|Du(x)|\big)\,dx < \infty\,. \tag{1.8}$$

Here $A(r)$ is the weight appearing in (1.2). Hence

(1.6*b*) the class of test functions = the closure, called $W_0^{1,A}(G)$, of $C_0^1(G)$ with respect to the Sobolev-Orlicz space $W^{1,A}(G)$;

in other words, equation (1.6*a*) is to hold for every φ from (1.6*b*) if u is a solution of interest. Furthermore our assumptions on boundary data read

$$g \in L^\infty(G) \cap W^{1,A}(G)\,. \tag{1.5b}$$

Let us explain some notations. As is well-known (see LUXEMBURG [10] and WEISS [17]), the function class (1.7) is linear if and only if $A(r)$ satisfies (in addition to conditions (1.2*a*) (1.2*b*)) the special Orlicz's Δ_2-condition: $A(2r) = O\big(A(r)\big)$ as $r \to \infty$. The linear hull of (1.7) is the Orlicz space $L^A(G) = \{f$ measurable: a constant λ exists such that $\int_G A\big(|f(x)/\lambda|\big)\,dx < \infty\}$, a complete Banach space under the norm $\|f\| = \inf\{\lambda > 0: \int_G A\big(|f(x)/\lambda|\big)\,dx \leq 1\}$. The Orlicz-Sobolev space $W^{1,A}(G)$ can be conveniently defined as the collection of those members of $L^A(G)$ whose distributional first derivatives are in $L^A(G)$ too. Note that $W_0^{1,A}(G)$, previously defined, is the same thing as the completion of $C_0^1(G)$ under the norm $u \to \| |Du| \|$; in fact, if u is a Lipschitz continuous function vanishing outside a set of finite measure, the following Poincaré-type inequality holds

$$\int_{R^m} A\left(\left(\frac{1}{C_m}\text{meas. sprt. } u\right)^{-1/m} |u|\right) dx \leq \int_{R^m} A(|Du|)\, dx \tag{1.9}$$

where $C_m = \pi^{m/2}/\Gamma(1 + m/2)$ is the measure of m-dimensional unit ball and sprt u denotes the support of u.

The properties of the considered solutions could be more accurately described by making the following remarks. The first derivatives of these solutions are integrable over the whole of G; in fact (1.8) holds, and Jensen's inequality for convex functions gives

$$\int_G |Du|\, dx \leq (\text{meas. } G) A^{-1}\left(\frac{1}{\text{meas. } G}\int_G A(|Du|)\, dx\right),$$

A^{-1} being the inverse function of A. The solutions themselves are in the Orlicz space $L^A(G)$: this follows from the boundary condition (1.5), more precisely from the boundedness of g and from the inequality (1.9) applied to test functions. Additional informations about our solutions can be obtained from the imbedding theorems of DONALDSON and TRUDINGER [5] (see also ADAMS [1]), provided the weight $A(r)$ is suitably well-behaved.

Obviously the sole hypothesis (1.2) is not enough to ensure that the Orlicz-Sobolev space $W^{1,A}(G)$, or the subset (1.8) of it, is the appropriate space for « all » weak solutions to equation (1.1). In other words, some extra assumptions on the growth of coefficients a_i and right-hand side H are needed to decide if *any* member u, satisfying (1.8), of the space $W^{1,A}(G)$ is a weak solution in the domain G to equation (1.1) or not. In fact, as shown by the very definition of weak solution, such a decision demands that the operators $u \to \sum_{i=1}^{m} (\partial/\partial x_i)\, a_i(x, u, Du)$ and $u \to H(x, u)$ map the subset (1.8) of $W^{1,A}(G)$ into the Banach adjoint of the test functions class $W_0^{1,A}(G)$. Clearly this point is crucial in investigations about the existence and uniqueness of weak solutions, see for instance DONALDSON [19] or GOSSEZ [21] [22]; it could be elucidated with the aid of theorems on the action of the so-called Nemyckii's operators on Orlicz spaces, see e.g. section 17 of KRASNOSEL'SKII-RUTICKII [9]. We do not want to discuss this matter in detail, since in the present paper we *assume* the existence of weak solutions from the specified class and we seek a priori bounds for these solutions.

We emphasize that *no investigation about the existence* (or uniqueness) *of solutions is made in this paper*, and that we focus our attention on *a priori estimates* only. We emphasize that our results depend only on ellipticity condition (1.2) and do not depend on the growth of coefficients or on the geometry of the ground domain. For what concerns the existence of solutions to boundary value problems for nonliner second order elliptic equations, we refer to SERRIN [43]; see also GILBARG [42], SERRIN [44], STAMPACCHIA [45].

Let us notice that

$$\left[\sum_{i=1}^{m} a_i(x, u, \xi)^2\right]^{\frac{1}{2}} \leqslant (\text{Const.})\tilde{A}^{-1}(A(|\xi|)) \tag{1.10a}$$

is a sufficient (although somewhat crude) condition for the distribution $\sum_{i=1}^{m} (\partial/\partial x_i)\cdot$ $\cdot a_i(x, u, Du)$ to be a bounded linear functional on the test functions space $W_0^{1,A}(G)$, u being a member of $W^{1,A}(G)$ verifying (1.8). Here $\tilde{A}^{-1}$ is the inverse of $\tilde{A}$ and the latter is the Young-conjugate of A, namely

$$\tilde{A}(s) = \max\{rs - A(r) : 0 \leqslant r < +\infty\} = \int_0^s \sup\{r \geqslant 0 : A'(r) \leqslant t\}\, dt\,. \tag{1.10b}$$

In fact Cauchy-Schwartz inequality for m-vectors and Young's inequality $rs \leqslant A(r) + \tilde{A}(s)$ give

$$\left|\int_G \sum_{i=1}^{m} a_i(x, u, Du)\varphi_{x_i} dx\right| \leqslant (\text{Const.})\lambda \left[\int_G A(|Du|)\, dx + \int_G A(|D\varphi|/\lambda)\, dx\right]$$

for every positive number λ, hence

$$\left|\int_G \sum_{i=1}^{m} a_i(x, u, Du)\varphi_{x_i} dx\right| \leqslant (\text{Const.}) \|\, |D\varphi|\, \| \left[1 + \int_G A(|Du|)\, dx\right]$$

just by the definition of Orlicz norm.

It can be easily seen that all sample equations we have displayed above satisfy condition (1.10), provided the coefficients formerly called $a_{ik}(x)$ are bounded. This assertion derives from the straightforward inequality:

$$\tilde{A}(A(r)/r) \leqslant A(r)\,. \quad \blacksquare$$

The present paper is a continuation of an earlier paper [46]. In [46] we showed that some a priori estimates of solutions to Dirichlet boundary value problems for linear elliptic equations can be stated in the form of isoperimetric theorems: we proved in fact that some norms (the L^∞-norm, L^p-norms or Lorentz norms, as well as some L^q-norm of the gradient) of the solutions attain their greatest value when the ground domain is a ball, the differential operator is a numerical multiple of the Laplacian and the right-hand side is spherically symmetric. As a by-product, we were able to obtain the sharp form of some estimates for solutions to linear problems. Here we present a similar and more general approach to a priori estimates.

Roughly, this approach can be described as follows: (i) Take a set of boundary value problems including the problem under consideration. Such a set could be defined by the information we have (or we are willing to use) on the data at our disposal. Here use is made of the following information only: form (1.1) of the differential equation, ellipticity condition (1.2), the measure of ground domain (1.4), the boundedness of Dirichlet datum g in (1.5). As far as right-hand side H is concerned, we use monotonicity condition (1.3*a*) and the distribution function of $H(\cdot, 0)$, namely the measure of the level sets $\{x\colon |H(x, 0)| > t\}$; the constraints on this distribution function being

$$(1.3b) \qquad H(\cdot, 0) \text{ is integrable}$$

and

$$(1.3c) \qquad \begin{cases} \text{either } \lim\limits_{r\to\infty} A(r)/r = \infty \text{ or} \\ \sup\limits_{s>0} s^{-1+1/m} \sup\left\{\int\limits_E |H(x, 0)|dx\colon \text{meas. } E = s\right\} < mC_m^{1/m} \lim\limits_{r\to\infty} A(r)/r\,, \end{cases}$$

a bound for the norm of $H(\cdot, 0)$ in the Lorentz space $L(m, \infty)$ (= weak L^m space; see section 3 for relevant definitions).

Hence we consider the set of of all problems (1.1) (1.2) (1.3) (1.4) (1.5) such that: the convex weight $A(r)$, the measure of G and the supremum of $|g|$ are fixed; $H(\cdot, 0)$ is equidistributed with a fixed integrable function which satisfies a condition analogous to (1.3*c*).

The meaning of (1.3*c*) will be clear presently. As Hölder's inequality shows, a sufficient condition for the inequality in (1.3*c*) to be true is

$$\left(\int\limits_{R^m} |H(x, 0)|^m dx\right)^{1/m} < mC_m^{1/m} \lim_{r\to\infty} A(r)/r\,.$$

Notice also that $mC_m^{1/m}$ is the isoperimetric constant, that is the constant in the classic isoperimetric inequality: $(\text{meas. } E)^{1-1/m} \leqslant mC_m^{1/m} H_{m-1}(\partial E)$, $E =$ any smooth bounded subset of R^m, $H_{m-1} =$ the $(m-1)$-dimensional measure.

(ii) Determine the worst problem in the above mentioned set, namely the problem which has the largest solution. Here we are able to allow any Luxemburg-Zaanen norm to evaluate the magnitude of solutions; moreover, in the case of a homogeneous boundary condition, we allow $\int M(|Du|)\, dx$ for the same purpose, where $M(r)$ is either $A(r)$ or a suitable weight « weaker » than $A(r)$. Following KALLMAN-ROTA [8], we call Luxemburg-Zaanen the norms on the linear spaces of locally integrable real-valued functions introduced in [12]; the essential feature of these norms is: $\|u\| \leqslant \|v\|$ if $|u| \leqslant |v|$ and $\|u\| = \|v\|$ if u is equidistributed with v.

The main result of this paper can be stated in this way: *the worst problem is the simplest one.* In fact let u be a solution from the Orlicz-Sobolev class (1.8) of (1.1) (1.2) (1.3) (1.4) (1.5) and let the norms and the weight $M(r)$ as before; *we prove*

$$\|u\| \leqslant \|v\| , \tag{1.11a}$$

as well as

$$\int M(|Du|)\, dx \leqslant \int M(|Dv|)\, dx \tag{1.11b}$$

in the case of a homogeneous boundary condition, *if v is the solution to the following problem*

$$\begin{cases} -\sum_{i=1}^{m} \dfrac{\partial}{\partial x_i}\left(\dfrac{A(|Dv|)}{|Dv|^2}\, v_{x_i}\right) = f(|x|) & \text{in the ball } |x| < a\,, \\ v(x) = b & \text{on the boundary } |x| = a\,. \end{cases} \tag{1.12}$$

Here $f(|\cdot|)$ is the positive spherically symmetric function equidistributed with $H(\cdot, 0)$; namely the (unique) positive function whose level sets are balls (centered at the origin) with the same measure as the level sets of $H(\cdot, 0)$. In the terminology of Hardy-Littlewood, this $f(|\cdot|)$ is called the spherically symmetric rearrangement of $H(\cdot, 0)$. Radius a is such that $C_m a^m (=$ measure of the ball $|x| < a) =$ meas. G, and $b = \sup |g|$.

A discussion of problem (1.12) is very simple. First of all, the solution of (1.12) is unique in the Orlicz-Sobolev class (1.8). For differential equation (1.12a) is the Euler equation of the integral

$$J(v) = \int_{|x|<a} \big(j(|Dv|) - f(|x|)\, v\big)\, dx\,,$$

where $j(r) = \int_0^r (A(t)/t)\, dt$. As $A(r)/r$ increases strictly according to our assumptions, $j(r)$ is strictly convex and verifies: $j(r) \leqslant A(r)$. Hence J is well-defined on the Orlicz-Sobolev class (1.8) (or so is its nonlinear part at least) and strictly convex. The asserted uniqueness follows. On the other hand, we claim that the solution to (1.12) (from the specified class) must be spherically symmetric, i.e. a function of $|x|$ only. In fact, the solution of (1.12) gives the minimum of J because of the proved convexity; moreover, Jensen's inequality for convex functions easily shows that J decreases under some symmetrization, i.e.

$$J(v) \geqslant J(w)$$

if v and w are connected by

$$w(x) = \frac{1}{mC_m} \int_{|\xi|=1} v(|x|\xi)\, H_{m-1}(d\xi)\,.$$

Finally, an explicit representation formula can be written for spherically symmetric solutions to (1.12). In fact, for such solutions equation (1.12a) becomes the following ordinary differential equation

$$-\frac{\partial}{\partial|x|}\left\{|x|^{m-1}B\left(\frac{\partial v}{\partial|x|}\right)\operatorname{sgn}\frac{\partial v}{\partial|x|}\right\} = |x|^{m-1}f(|x|)\,,$$

where

$$B(r) = A(r)/r\,.$$

By integrating and disregarding the solutions which are singular at $x = 0$, we obtain the claimed representation formula

$$v(x) = b + \int_{|x|}^{a} B^{-1}\left(r^{1-m}\int_0^r s^{m-1}f(s)\,ds\right)dr\,, \tag{1.13}$$

if the positiveness of f and boundary condition (1.12b) are taken into account.

Now we are in a position to explain condition (1.3c). It is apparent from (1.13) and the previous discussion that problem (1.12) has a solution only if the value of $r^{1-m}\int_0^r s^{m-1}f(s)\,ds$ lies in $\left[0, \lim_{r\to\infty} A(r)/r\right[$, = the range of B, for any r. As $f(|\cdot|)$ is the spherically symmetric rearrangement of $H(\cdot, 0)$, a theorem of Hardy-Littlewood yields

$$s^{-1+1/m}\sup\left\{\int_E |H(x,0)|\,dx\colon \text{meas. } E = s\right\} = mC_m^{1/m}r^{1-m}\int_0^r t^{m-1}f(t)\,dt \qquad (s = C_m r^m)\,,$$

the well-known formula which supports the very definition of Lorentz spaces. Thus (1.3c) is a necessary condition for « maximizing » problem (1.12) to have a solution. ■

The underlying idea of our proofs consists of the following two steps: (i) derive a differential inequality for the distribution function of the solution under estimation; (ii) read this differential inequality in terms of rearrangements of functions.

Step (i) seems to be strongly based on the divergence structure of the differential equation. If a great smoothness of solutions is taken for granted, so that pathologies in the geometry of level sets are prevented by Sard's theorem, step (i) starts by taking integrals of both sides of the differential equation over level sets of the solution and goes ahead essentially along the same lines of [46, section 3]. In the case of non-smooth weak solutions, more sophisticated arguments are needed.

Step (ii) requires tools which will be mentioned at be beginning of the next section.

2. – Main result.

Let us recall some basic definitions from the theory of rearrangements of functions in the sense of Hardy and Littlewood. We refer to [46, section 2] for some details and comments, and to HARDY-LITTLEWOOD-PÓLYA [26, chapter 10], PÓLYA-

SZEGÖ [30, chapter 7], HUNT [27], O'NEIL [15] [28], O'NEIL-WEISS [29], SPERNER [31] [32] and [33] for further developements.

Let u be a real-valued measurable function, defined in a measurable subset G of R^m. The distribution function of u is

$$\mu(t) = \text{meas}\,\{x \in G\colon |u(x)| > t\}\,. \tag{2.1}$$

The decreasing rearrangement of u into $[0, +\infty]$ is the smallest decreasing function u^* from $[0, +\infty]$ into $[0, +\infty]$ such that $u^*(\mu(t)) \geqslant t$ for every t. Equivalently

$$u^*(s) = \inf\,\{t \geqslant 0 : \mu(t) < s\}\,. \tag{2.2}$$

The spherically symmetric rearrangement of u is defined by

$$u^\star(x) = u^*(C_m|x|^m)\,, \tag{2.3}$$

where

$$C_m = \pi^{m/2}/\Gamma(1 + m/2) \tag{2.4}$$

is the measure of m-dimensional unit ball. A basic and easy theorem tell us that u, u^* and $u^\star$ are equidistributed, i.e.

$$\text{meas}\,\{x \in G\colon |u(x)| > t\} = \text{lenght of } \{s \geqslant 0 : u^*(s) > t\} = \text{meas}\,\{x \in R^m\colon u^\star(x) > t\}\,. \tag{2.5}$$

Moreover a theorem of Hardy-Littlewood [26, thm 378] gives

$$\int_G |uv|\,dx \leqslant \int_0^\infty u^*(s)\,v^*(s)\,ds = \int_{R^m} u^\star v^\star\,dx\,, \tag{2.6a}$$

in particular

$$\int_E |u|\,dx \leqslant \int_0^{\text{meas.}\,E} u^*(s)\,ds\,, \tag{2.6b}$$

whenever u, v are real-valued measurable functions defined in the domain G and E is a measurable subset of G. ∎

Now we state and prove our main theorem.

THEOREM 1. – *List of assumptions*:

(i) *ellipticity condition* (1.2) *and conditions* (1.3) = (1.3*a*) + (1.3*b*) + (1.3*c*) *on the right-hand side.*

(ii) *ground domain* (1.4) *has finite measure and boundary datum* g *is a bounded function. More precisely,* g *satisfies conditions* (1.5*b*).

(iii) u *is a weak solution to Dirichlet problem* (1.1)(1.4)(1.5) *from Orlicz-Sobolev class* (1.8), *i.e.* $\int_G A(|Du|)\,dx < \infty$.

(iv) $G^\star = \{x \in R^m \colon |x| < C_m^{-1/m}(\text{meas. } G)^{-1/m}\}$ *is the ball with the same measure as domain* G *and* v *is the function defined in* $G^\star$ *by*

$$v(x) = \sup|g| + \int_{C_m|x|^m}^{\text{meas. } G} B^{-1}\left(\frac{r^{-1+1/m}}{mC_m^{1/m}}\int_0^r H(\cdot, 0)^*(s)ds\right)\frac{r^{-1+1/m}}{mC_m^{1/m}}dr\,, \tag{2.7}$$

where $B(r) = A(r)/r$.

List of conclusions:

(v) $u^\star(x) \leqslant v(x)$ *for every* x *in* $G^\star$.

(vi) $\int_{\{x\in G:|u(x)|>\sup|g|\}} M(|Du|)\,dx \leqslant \int_{G^\star} M(|Dv|)\,dx$, *where* M *is any twice continuously differentiable function such that*

$$\begin{cases} M'(r) > 0 \quad \textit{and} \quad M''(r)/M'(r) \leqslant A''(r)/A'(r) \quad \textit{if } 0 < r < \infty, \\ M(0+) = 0\,. \end{cases} \tag{2.8}$$

LEMMA 1. – Assume hypotheses (i)(ii)(iii) of theorem 1. Then the restriction of the decreasing function

$$t \to \int_{|u(x)|>t} A(|Du|)\,dx \tag{2.9}$$

to the interval $\sup|g| < t < +\infty$ is Lipschitz continuous, and the inequality

$$0 \leqslant -\frac{d}{dt}\int_{|u(x)|>t} A(|Du|)\,dx \leqslant \int_0^{\mu(t)} H(\cdot, 0)^*(s)\,ds \tag{2.10}$$

holds for almost every (a.e.) $t > \sup|g|$. Here and henceforth $\int_{|u(x)|>t} \dots dx$ is a short form notation for $\int_{\{x\in G:|u(x)|>t\}} \dots dx$; $\mu(t)$ is the distribution function (2.1) and $H(\cdot, 0)^*(s)$ is the value at point s of the decreasing rearrangement of $H(\cdot, 0)$ into $[0, +\infty]$.

PROOF. – According to hypothesis (iii), we are assuming that u belongs to Orlicz-Sobolev space $W^{1,A}(G)$ and that equation (1.6*a*) holds for every φ from the space

$W_0^{1,A}(G)$ defined in (1.6*b*). We choose test functions φ so defined:

$$\varphi(x) = \begin{cases} u(x) - t & \text{if } x \text{ is such that } u(x) > t \\ u(x) + t & \text{if } x \text{ is such that } u(x) < -t \\ 0 & \text{otherwise,} \end{cases} \tag{2.11}$$

where t is any number greater than $\sup|g|$. Lemma 8.31 of [1] tell us that these φ are in $W^{1,A}(G)$ and that the derivatives are obtained by the usual chain rule; as t is larger than the boundary values of u, it is easily seen that such φ actually are in $W_0^{1,A}(G)$. If we insert these test functions into (1.6*a*), we see that the decreasing function Φ defined by

$$\Phi(t) = \int_{|u(x)|>t} \sum_{i=1}^{m} a_i(x, u, Du) u_{x_i} dx \tag{2.12}$$

satisfies

$$\Phi(t) = \int_{|u(x)|>t} H(x, u) \big(|u| - t\big) \operatorname{sgn} u \, dx \qquad \textit{for every } t > \sup|g|. \tag{2.13}$$

Let $h > 0$ and $t > \sup|g|$; from (2.13) we get

$$-\Phi(t+h) + \Phi(t) = h \int_{|u(x)|>t} H(x, u) \operatorname{sgn} u \, dx - h \int_{t<|u(x)|\leq t+h} H(x, u) \operatorname{sgn} u \left(1 - \frac{|u| - t}{h}\right) dx\,,$$

hence

$$-\frac{1}{h}\big(\Phi(t+h) - \Phi(t)\big) \leq \int_{|u(x)|>t} |H(x, 0)|\, dx + \int_{t<|u(x)|\leq t+h} |H(x, 0)|\, dx$$

because of hypothesis (1.3*a*); thus according to hypothesis (1.3*b*)

$$0 \leq -\Phi(t+h) + \Phi(t) \leq 2h \int_G |H(x, 0)|\, dx\,, \tag{2.14a}$$

and

$$-\Phi'(t) \leq \int_{|u(x)|>t} |H(x, 0)|\, dx \tag{2.14b}$$

if t is any point of continuity for the distribution function of u.

As is easily seen, (2.12) and (2.14) imply the asserted conclusions via ellipticity condition (1.2). The following consequence of Hardy-Littlewood theorem (2.6):

$$\int_{|u(x)|>t} |H(x, 0)|\, dx \leq \int_0^{\mu(t)} H(\cdot, 0)^*(s)\, ds \tag{2.15}$$

must also be used.

LEMMA 2. – Let $0 \leqslant r \to A(r)$ be a positive increasing convex function such that $A(r)/r \to 0$ if $r \to 0$. Let G be any open subset of R^m with finite measure, and u a function from Orlicz-Sobolev space $W^{1,A}(G)$ such that: (i) $\int_G A(|Du|)\,dx < \infty$ (ii) u agrees on the boundary of G with a bounded function g. Then

$$-\mu'(t) \geqslant m C_m^{1/m} \mu(t)^{1-1/m} C\left(\frac{-1}{m C_m^{1/m} \mu(t)^{1-1/m}} \frac{d}{dt} \int_{|u(x)|>t} A(|Du|)\,dx\right) \tag{2.16}$$

for a.e. $t > \sup |g|$. Here μ is the distribution function of u, and C is defined by

$$\frac{1}{C(s)} = \sup\{r \geqslant 0 \colon A(r)/r \leqslant s\}. \tag{2.17}$$

PROOF. – For the sake of simplicity, we prove this lemma under a supplementary hypothesis on $A(r)$, i.e. $A(r)$ is twice continuously differentiable and strictly increasing. From the latter condition we easily infer that

$$B(r) = A(r)/r \tag{2.18a}$$

increases strictly from $B(0) = 0$ to $B(+\infty) = \lim_{r \to +\infty} A(r)/r$ as r increases from 0 to $+\infty$. Thus

$$C(s) = 1/B^{-1}(s) \quad \text{if } 0 < s < \lim_{r \to +\infty} A(r)/r, \quad = 0 \text{ otherwise}. \tag{2.18b}$$

Straightforward computations give the formulas

$$C'(s) = \frac{-1}{r^2 B'(r)} \qquad C''(s) = \frac{A''(r)}{(rB'(r))^3}, \tag{2.19}$$

where r and s are connected by $s = B(r)$. Consequently, C is a *decreasing convex* function in half-line $0 < s < +\infty$.

Jensen's inequality for convex functions gives

$$C\left(\frac{\int_{t<|u(x)|\leqslant t+h} A(|Du|)\,dx}{\int_{t<|u(x)|\leqslant t+h} |Du|\,dx}\right) = \tag{2.20}$$

$$C\left(\frac{\int B(|Du|)\,|Du|\,dx}{\int |Du|\,dx}\right) \leqslant \frac{\int C(B(|Du|))\,|Du|\,dx}{\int |Du|\,dx} = \frac{(-\mu(t+h)+\mu(t))}{\int_{t<|u(x)|\leqslant t+h} |Du|\,dx}.$$

Hence we obtain the inequality

$$C\left(\frac{(d/dt)\int_{|u(x)|>t} A(|Du|)\,dx}{(d/dt)\int_{|u(x)|>t} |Du|\,dx}\right) \leqslant \frac{\mu'(t)}{(d/dt)\int_{|u(x)|>t} |Du|\,dx} \tag{2.21}$$

for almost every t.

We can prove

$$(2.22)\qquad -\frac{d}{dt}\int\limits_{|u(x)|>t}|Du|\,dx \geqslant mC_m^{1/m}\mu(t)^{1-1/m} \qquad \textit{for a.e. } t > \sup|g|.$$

Clearly, (2.21) (2.22) and the monotonicity of C imply the lemma.

Inequality (2.22) is an easy consequence of Fleming-Rishel formula [39] and of the isoperimetric theorem. Fleming-Rishel formula reads

$$(2.23a)\qquad \text{total variation of } \varphi = \int\limits_0^{+\infty} P\{x\in R^m\colon |\varphi(x)|>\lambda\}\,d\lambda,$$

provided φ is integrable over R^m and the left-hand side is finite; here

$$(2.23b)\qquad \text{tot. var. } \varphi = \sup\left\{\int\limits_{R^m}\varphi\,\operatorname{div} v\,dx\colon v\in\big(C_0^1(R^m)\big)^m,\ \max|v|\leqslant 1\right\}$$

and P stands for perimeter in the sense of De Giorgi, namely $P(E)$ is the total variation of the characteristic function of E. As is easy to see, the total variation of an integrable function φ, endowed with integrable first derivatives, is $\int\limits_{R^m}|D\varphi|\,dx$; the perimeter of a smooth open subset of R^m agrees with the $(m-1)$-dimensional measure of the boundary. Let us apply these formulas to the following function:

$$(2.24)\qquad \varphi(x) = \begin{cases} |u(x)|-t & \text{if } x \text{ is such that } |u(x)|>t\\ 0 & \text{if } x\notin G \text{ or if } |u(x)|\leqslant t,\end{cases}$$

where t is any number greater than $\sup|g|$. As in the proof of lemma 1, we see that φ is in the Orlicz-Sobolev space $W_0^{1,A}(G)$. Hence φ is in the (usual) Sobolev space $W_0^{1,1}(G)$, since G has finite measure. Thus we obtain

$$(2.25)\qquad \int\limits_{|u(x)|>t}|Du|\,dx = \int\limits_t^{+\infty}P\{x\in G\colon |u(x)|>\lambda\}\,d\lambda \qquad \textit{for } t>\sup|g|.$$

By taking derivatives, and using the following case of De Giorgi's isoperimetric theorem [37] [38]

$$(2.26)\qquad P\{x\in G\colon |u(x)|>t\}\geqslant mC_m^{1/m}\mu(t)^{1-1/m},$$

we obtain the wanted inequality (2.22).

PROOF OF THEOREM 1. – From lemma 1 and lemma 2 we deduce the following differential inequality for the distribution function of u:

$$(2.27)\qquad 1\leqslant \frac{-\mu'(t)}{mC_m^{1/m}\mu(t)^{1-1/m}}\,B^{-1}\left(\frac{\int_0^{\mu(t)}H(\cdot,0)^*(s)\,ds}{mC_m^{1/m}\mu(t)^{1-1/m}}\right) \qquad \text{where } t>\sup|g|.$$

In this deduction we have to bear in mind formulas (2.18) and hypothesis (1.3c), as well as equation

$$\int_0^s H(\cdot,0)^*(t)\,dt = \sup\left\{\int_E |H(x,0)|dx\colon \text{meas. } E = s\right\} \tag{2.28}$$

which is an easy corollary of Hardy-Littlewood theorem (2.6b).

Note that the right-hand side of (2.27) is the derivative of an increasing function of t. Hence by integrating with respect to t we have

$$t \leqslant \sup|g| + \int_{\mu(t)}^{\text{meas. } G} B^{-1}\left(\frac{\int_0^r H(\cdot,0)^*(s)\,ds}{mC_m^{1/m} r^{1-1/m}}\right)\frac{dr}{mC_m^{1/m} r^{1-1/m}}\,. \tag{2.29}$$

Obviously (2.29) holds for every $t > 0$. By the very definition (2.2) of decreasing rearrangement, (2.29) can be rewritten in the following manner

$$u^*(s) \leqslant \sup|g| + \int_{s}^{\text{meas. } G} B^{-1}\left(\frac{\int_0^r H(\cdot,0)^*(s)\,ds}{mC_m^{1/m} r^{1-1/m}}\right)\frac{dr}{mC_m^{1/m} r^{1-1/m}}\,. \tag{2.30}$$

The (2.30) agrees with assertion (v) of theorem 1, because of the definition (2.3) of spherically symmetric rearrangement.

Now we proceed with the proof of assertion (vi). We consider at first the simplest case, namely the case where $M(r) = A(r)$ and g, the boundary datum, $= 0$. Under these circumstances, solution u is in space $W_0^{1,A}(G)$, that is u itself is a test function. Thus equation (1.6a) with $\varphi = u$ gives

$$\int_G \sum_{i=1}^m a_i(x,u,Du)\,u_{x_i}\,dx = \int_G H(x,u)\,u\,dx\,. \tag{2.31}$$

The proof continues according to the following scheme (numbers mark assertions, brackets contain motivations):

$$\int_G A(|Du|)\,dx$$

(2.32) $\leqslant \int_G H(x,u)\,u\,dx$ (eq. (2.31) and ellipticity condtn. (1.2))

(2.33) $\leqslant \int_G H(x,0)\,u\,dx$ (monotonicity assumption (1.3a))

(2.34) $\leqslant \int_0^{\text{meas. } G} H(\cdot,0)^*(s)\,u^*(s)\,ds$ (Hardy-Littlewood thm (2.6a))

$$\leqslant \int_0^{\text{meas.}\,G} H(\cdot,0)^*(s)\left\{\int_s^{\text{meas.}\,G} B^{-1}\left(\frac{\int_0^r H(\cdot,0)^*(t)\,dt}{mC_m^{1/m}r^{1-1/m}}\right)\frac{dr}{mC_m^{1/m}r^{1-1/m}}\right\}ds \tag{2.35}$$

(inequality (2.30) with $g=0$)

$$= \int_0^{\text{meas.}\,G} \frac{\int_0^s H(\cdot,0)^*(t)\,dt}{mC_m^{1/m}s^{1-1/m}}\,B^{-1}\left(\frac{\int_0^s H(\cdot,0)^*(t)\,dt}{mC_m^{1/m}s^{1-1/m}}\right)ds \tag{2.36}$$

(integration by parts)

$$= \int_{G^\star} A(|Dv|)\,dx \tag{2.37}$$

the last step being a consequence of the obvious equations $sB^{-1}(s) = A(B^{-1}(s))$ and of the following formula

$$|Dv(x)| = B^{-1}\left(\frac{1}{mC_m|x|^{m-1}}\int_0^{C_m|x|^m} H(\cdot,0)^*(s)\,ds\right) \tag{2.38}$$

for the gradient of function (2.7).

A full proof of assertion (vi) is as follows. A straightforward argument shows that the function K, defined on the range of M by

$$K(s) = A(M^{-1}(s))\,, \tag{2.39a}$$

is *convex*. Then Jensen's inequality gives

$$K\left(\frac{\int_{t<|u(x)|\leqslant t+h} M(|Du|)\,dx}{-\mu(t+h)+\mu(t)}\right) \leqslant \frac{\int_{t<|u(x)|\leqslant t+h} A(|Du|)\,dx}{-\mu(t+h)+\mu(t)}\,. \tag{2.39b}$$

The proof continues in the following way:

$$M^{-1}\left(\frac{1}{\mu'(t)}\frac{d}{dt}\int_{|u(x)|>t} M(|Du|)\,dx\right)$$

$$\leqslant A^{-1}\left(\frac{1}{\mu'(t)}\frac{d}{dt}\int_{|u(x)|>t} A(|Du|)\,dx\right) \tag{2.40}$$

(inequality (2.39))

$$\leqslant\left\{C\left(\frac{-1}{mC_m^{1/m}\mu(t)^{1-1/m}}\frac{d}{dt}\int_{|u(x)|>t} A(|Du|)\,dx\right)\right\}^{-1} \tag{2.41}$$

($t>\sup|g|$ and lemma 2)

$$(2.42)\qquad \leqslant B^{-1}\left(\frac{1}{mC_m^{1/m}\mu(t)^{1-1/m}}\int_0^{\mu(t)} H(\cdot,0)^*(s)\,ds\right)$$

($t > \sup|g|$, lemma 1, hypothesis (1.3c) and eq. (2.28), eq. (2.18));

$$\int_{G^\star} M(|Dv|)\,dx$$

$$(2.43)\qquad = \int_0^{\text{meas.}\,G} M\left(B^{-1}\left(\frac{1}{mC_m^{1/m}r^{1-1/m}}\int_0^r H(\cdot,0)^*(s)\,ds\right)\right)dr$$

(formula (2.38))

$$(2.44)\qquad \geqslant \int_0^{+\infty} M\left(B^{-1}\left(\frac{1}{mC_m^{1/m}\mu(t)^{1-1/m}}\int_0^{\mu(t)} H(\cdot,0)^*(s)\,ds\right)\right)(-\mu'(t))\,dt$$

(monotonicity of $\mu(t)$)

$$(2.45)\qquad \geqslant \int_{\sup|g|}^{+\infty} dt\left(-\frac{d}{dt}\int_{|u(x)|>t} M(|Du|)\,dx\right)$$

(inequalities (2.40)(2.41)(2.42))

$$= \int_{|u(x)|>\sup|g|} M(|Du|)\,dx\,.$$

In the last step we use the absolute continuity of

$$(2.46)\qquad t \to \int_{|u(x)|>t} M(|Du|)\,dx\,.$$

As could be proved, this absolute continuity depends on hypothesis $M(0+) = 0$ and on the fact that either $\{x \in G\colon |u(x)| = t\}$ has m-dimensional measure zero or $|Du|$ vanishes almost everywhere on the same set.

Finally, it is worth-while to remark that the hypotheses made on $M(r)$ imply

$$(2.47)\qquad M(r) = O(A(r)) \quad \text{if } r \to +\infty.$$

The proof is complete.

3. – Bounded solutions.

In this section we sketch some easy consequences of theorem 1.

THEOREM 2. – *Assume hypotheses* (i) (ii) (iii) *of theorem* 1. *Suppose that* $H(\cdot, 0)$ *actually belongs to Lorentz space* $L(m, \infty)$. *Then* u *is bounded, and the following ine-*

quality holds

$$\sup|u| \leqslant \sup|g| + \left(\frac{1}{C_m}\,\text{meas.}\, G\right)^{1/m} B^{-1}\left(\frac{1}{mC_m^{1/m}}\|H(\cdot\,,0)\|_{m,\infty}\right), \tag{3.1}$$

where $B(r) = A(r)/r$.

Note that (3.1) contains in particular a weak form of the maximum principle. Of course all through this paper *sup* stands for essential supremum. ■

For convenience of the reader, we recall the definition of Lorentz spaces $L(p, q)$. We refer to HUNT [27] and O'NEIL [15] [28] for details. If $1 \leqslant p,\ q < \infty$, $L(p, q)$ is the collection of all measurable real-valued functions f on R^m such that the following norm

$$\|f\|_{p,q} = \left[\int_0^\infty (s^{1/p}\bar{f}(s))^q \frac{ds}{s}\right]^{1/q} \tag{3.2a}$$

is finite. If $0 < p \leqslant \infty$, the appropriate norm on $L(p, \infty)$ is

$$\|f\|_{p,\infty} = \sup_{s>0} s^{1/p}\bar{f}(s)\,. \tag{3.2b}$$

Here $\bar{f}$ is the Hardy-Littlewood maximal function associated with the decreasing rearrangement of f into $[0, \infty]$, namely

$$\bar{f}(s) = \frac{1}{s}\int_0^s f^*(t)\,dt\,. \tag{3.3a}$$

The following alternative representation is easily deduced from Hardy-Littlewood theorem (2.6)

$$\bar{f}(s) = \frac{1}{s}\sup\left\{\int_E |f(x)|\,dx\colon \text{meas.}\, E = s\right\}. \tag{3.3b}$$

Lorentz spaces $L(p, p)$ (with $1 < p < \infty$) agree with the usual Lebesgue $L^p(R^m)$ spaces. In fact the monotonicity of f^* gives: $\bar{f}(s) \geqslant f^*(s)$; while Hardy's inequality gives: $\int_0^\infty \bar{f}(s)^p\,ds \leqslant (p/(p-1))^p \int_0^\infty f^*(s)^p\,ds$. Lorentz spaces $L(p, \infty)$ (with $1 \leqslant p < \infty$) are also called weak L^p spaces; they contain $L^p(R^m)$, since Hölder's inequality and formulas (3.3) give: $\bar{f}(s) \leqslant s^{-1/p} \times \left(\int_0^\infty f^*(t)^p dt\right)^{1/p}$. By putting together these two remarks we have the inclusion $L(p, p) \subset L(p, q)$ (with $1 < p < q \leqslant \infty$), since $\|f\|_{p,q} \leqslant \|f\|_{p,p}^{p/q}\|f\|_{p,\infty}^{1-p/q}$ by a trivial argument. Notice that $L(1, \infty) = L^1(R^m)$ with equality between the relevant norms, as (3.2b) and (3.3) show. ■

PROOF OF THEOREM 2. – From assertion (v) of theorem 1 and equations (2.5) we get: $\sup|u|\leqslant \sup v$. From formulas (2.7) and (3.2b) we get

$$\sup v = v(0) \leqslant \sup|g| + \int_0^{\text{meas.}G} B^{-1}\left(\frac{1}{mC_m^{1/m}}\|H(\cdot,0)\|_{m,\infty}\right)\frac{r^{-1+1/m}}{mC_m^{1/m}}\,dr =$$

the right-hand side of (3.1). ∎

EXAMPLE. – Any solution u of the following problem

$$(3.4a)\qquad \begin{cases} -\sum\limits_{i,k=1}^{m}\dfrac{\partial}{\partial x_i}\left\{a_{ik}(x)\dfrac{\ln(1+|Du|)}{|Du|}u_{x_k}\right\} = H(x,u) & \text{in } G\subset R^m \\ u=0 & \text{on the boundary } \partial G \end{cases}$$

is bounded and verifies

$$(3.5)\qquad \sup|u|\leqslant\left(\frac{1}{C_m}\,\text{meas. }G\right)^{1/m}\left\{\exp\left(\frac{1}{mC_m^{1/m}}\|H(\cdot,0)\|_{m,\infty}\right)-1\right\},$$

provided

$$(3.4b)\qquad \begin{cases} \sum\limits_{i,k=1}^{m} a_{ik}(x)\,\xi_i\xi_k \geqslant \sum\limits_{i=1}^{m}\xi_i^2 \\ H(x,\cdot)\quad \text{is decreasing and } H(\cdot,0)\in L(m,\infty) \\ \text{meas. } G<\infty, \end{cases}$$

and provided u is sought in the following Orlicz-Sobolev class

$$(3.4c)\qquad \int_G |Du|\ln(1+|Du|)\,dx<\infty. \quad ∎$$

THEOREM 3. – *Assume*

$$(3.6)\qquad \int^{\infty}(r/A(r))^k\,dr<\infty \qquad \textit{for some } k>0$$

and

$$(3.7)\qquad H(\cdot,0)\in L(p,\infty) \qquad \textit{for } p=mk/(k+1).$$

Moreover assume ellipticity condition (1.2) *and monotonicity hypothesis* (1.3a) *as well as hypotheses* (ii) (iii) *of theorem* 1. *Then u is bounded. In addition, if the full integral* $\int_0^\infty (r/A(r))^k\,dr$ *converges, the following inequality holds*

$$(3.8)\qquad \sup|u|\leqslant \sup|g| + m^{-k}C_m^{-(1+k)/m}\|H(\cdot,0)\|_{p,\infty}^k\int_0^\infty (r/A(r))^k\,dr.$$

PROOF. – Let us limit ourselves to the proof of (3.8). As in the proof of thm. 2, we have only to show that $v(0)$, the value at the origin of function (2.7), is bounded by the right-hand side of (3.8). From (3.2*b*) and (3.3*a*) we get

$$v(0)\leqslant \sup|g| + \int_0^{\text{meas.}\,G} B^{-1}\left(\frac{r^{1/m-1/p}}{mC_m^{1/m}}\|H(\cdot,0)\|_{p,\infty}\right)\frac{r^{-1+1/m}}{mC_m^{1/m}}\,dr\,,$$

hence a change of variables in the integral gives

$$v(0)\leqslant \sup|g| + m^{-k}C_m^{-(1+k)/m}\|H(\cdot,0)\|_{p,\infty}^k\, k\int_0^\infty t^{-k-1}B^{-1}(t)\,dt\,.$$

Since

$$k\int_0^\infty t^{-k-1}B^{-1}(t)\,dt = \int_0^\infty \big(r/A(r)\big)^k dr\,,$$

we conclude as planned. Note that in this proof we have used this fact: $B(r) = A(r)/r \to +\infty$ if $r\to+\infty$, a consequence of (3.6). ■

EXAMPLE. – Any solution u of the following problem

$$\text{(3.10}a\text{)}\qquad \begin{cases} -\displaystyle\sum_{i,k=1}^m \frac{\partial}{\partial x_i}\left\{a_{ik}(x)\frac{\exp(|Du|)-1}{|Du|}u_{x_k}\right\} = H(x,u) & \text{in } G\subset R^m\\ u=0 & \text{on the boundary } \partial G\end{cases}$$

verifies

$$\text{(3.11)}\qquad \sup|u|\leqslant \frac{\pi}{C_m^{1/m}\sin(\pi/(m-1))}\left[\int_G |H(x,0)|\,dx\right]^{1/(m-1)},$$

provided

$$\text{(3.10}b\text{)}\qquad \begin{cases} \displaystyle\sum_{i,k=1}^m a_{ik}(x)\,\xi_i\xi_k \geqslant \sum_{i=1}^m \xi_i^2\\ H(x,\cdot)\quad \text{is decreasing and } H(\cdot,0)\in L^1(G)\\ \text{meas.}\,G<\infty\,,\end{cases}$$

and provided

$$\text{(3.10}c\text{)}\qquad \int_G |Du|[\exp(|Du|)-1]\,dx<\infty\,.$$

4. – Some applications of theorem 1.

4.1. – In this subsection we derive some a priori estimates of solutions to the following Dirichlet problem:

$$(4.1)\qquad \begin{cases} -\sum\limits_{i,k=1}^{m} \dfrac{\partial}{\partial x_i}\{a_{ik}(x)|Du|^{\lambda-2}u_{x_k}\} = H(x, u) & \text{in } G \\ u = 0 & \text{on the boundary } \partial G . \end{cases}$$

List of assumptions: λ is any number > 1; a_{ik} are measurable functions such that $\sum\limits_{i,k=1}^{m} a_{ik}(x)\,\xi_i\xi_k \geq \sum\limits_{i=1}^{m} \xi_i^2$ for all x and ξ; H is a measurable function such that $H(x, \cdot)$ is decreasing for all x and $H(\cdot, 0)$ belongs to some Lorentz space $L(p, k)$; G is any open subset of R^m with finite measure.

Incidentally, the existence and uniqueness of solutions to problem (4.1) can be set (at least if the nonlinearity of the right-hand is not too severe) in the framework of monotone operators on reflexive Banach spaces, see e.g. BROWDER [35] [36], LERAY-LIONS [41]. ■

Let u be a solution to problem (4.1) belonging to the (usual) Sobolev space $W_0^{1,\lambda}(G)$. Theorem 1 gives

$$(4.2a)\qquad u^\star(x) \leq v(x) \quad \text{for every } x \text{ in } G^\star$$

$$(4.2b)\qquad \int_G |Du|^q\,dx \leq \int_{G^\star} |Dv|^q\,dx \quad \text{if } 0 < q \leq \lambda,$$

where $G^\star$ is the ball with the same measure as G and

$$(4.3)\qquad v(x) = (mC_m^{1/m})^{-\lambda/(\lambda-1)} \int_{C_m|x|^m}^{\text{meas.}\,G} r^{-1+\lambda/(m(\lambda-1))}\left(\frac{1}{r}\int_0^r H(\cdot, 0)^*(s)\,ds\right)^{\lambda/(\lambda-1)} dr ,$$

a solution to the following problem

$$(4.4)\qquad \begin{cases} -\sum\limits_{i=1}^{m} \dfrac{\partial}{\partial x_i}\{|Dv|^{\lambda-2}v_{x_i}\} = H(\cdot, 0)^\star(x) & \text{in } G^\star \\ v = 0 & \text{on } \partial G^\star . \end{cases}$$

From (4.3) the following formulas are easily drawn:

$$(4.5a)\qquad \sup v = (mC_m^{1/m})^{-\lambda/(\lambda-1)} \int_0^{\text{meas.}\,G} r^{-1+\lambda/(m(\lambda-1))}\left(\frac{1}{r}\int_0^r H(\cdot, 0)^*(s)\,ds\right)^{\lambda/(\lambda-1)} dr ,$$

$$\bar{v}(s) = (mC_m^{1/m})^{-\lambda/(\lambda-1)} \int_0^{\text{meas. } G} \frac{r^{\lambda/(m(\lambda-1))}}{\max(r, s)} \left(\frac{1}{r}\int_0^r H(\cdot, 0)^*(t)\,dt\right)^{\lambda/(\lambda-1)} dr\,, \tag{4.5b}$$

$$\int_{G^*} |Dv|^q\,dx = \int_0^{\text{meas. } G} \frac{r^{1/m}}{mC_m^{1/m}} \left(\frac{1}{r}\int_0^r H(\cdot, 0)^*(s)\,ds\right)^{q/(\lambda-1)} dr\,. \tag{4.5c}$$

In (4.5*b*) $\bar{v}$ indicates the Hardy-Littlewood maximal function associated with the decreasing rearrangement v^* of v, see (3.3).

The right-hand side of formulas (4.5) can be conveniently estimated in terms of Lorentz norms of $H(\cdot, 0)$, via definition (3.2) and standard technical tools (such as Hölder's inequality, or theorem 319 of [26]). Thus from (4.2) (4.5) we obtain the following results:

(i) $\sup |u| \leqslant K(\text{meas. } G)^{(\lambda p - m)/(m(\lambda-1)p)} \|H(\cdot, 0)\|_{p,k}^{1/(\lambda-1)}\,.$

Here $p > m/\lambda$, $k > 1/(\lambda - 1)$ and K is the following constant

$$K = (mC_m^{1/m})^{-\lambda/(\lambda-1)} \left[\frac{mp(\lambda-1)}{\lambda p - m}\left(1 - \frac{1}{k(\lambda-1)}\right)\right]^{1-1/(k(\lambda-1))}.$$

(ii) $\|u\|_{q,k(\lambda-1)} \leqslant K \|H(\cdot, 0)\|_{p,k}^{1/(\lambda-1)}\,.$

Here

$$\frac{m}{\lambda} > p > \frac{m}{\lambda + (\lambda-1)m}\,, \qquad k \geqslant \frac{1}{\lambda - 1}\,, \qquad q = \frac{mp(\lambda-1)}{m - \lambda p}$$

and

$$K = (mC_m^{1/m})^{-\lambda/(\lambda-1)} \frac{q^2}{q-1}\,.$$

(iii) $\left[\int_G |Du|^q\,dx\right]^{1/q} \leqslant K(\text{meas. } G)^{1/q + (p-m)/(mp(\lambda-1))} \|H(\cdot, 0)\|_{p,k}^{1/(\lambda-1)}\,.$

Here

$$0 < q \leqslant \lambda\,, \qquad p > \frac{mq}{m(\lambda-1) + q}\,, \qquad k > \frac{q}{\lambda - 1}$$

and

$$K = (mC_m^{1/m})^{-1/(\lambda-1)} \left[\left(1 + \frac{q(p-m)}{mp(\lambda-1)}\right)\frac{k(\lambda-1)}{k(\lambda-1) - q}\right]^{-1/q + 1/(k(\lambda-1))}.$$

4.2. – In this subsection we derive some a priori estimates for nonparametric surfaces with prescribed mean curvature. An exhaustive treatement of boundary value problems for such surfaces can be found in GIUSTI [40]. Here we consider Dirichlet problems only, and for the sake of simplicity we limit ourselves to a homogeneous boundary condition. Thus we consider the following problem:

$$(4.6)\qquad \begin{cases} \sum_{i=1}^{m} \dfrac{\partial}{\partial x_i} \dfrac{u_{x_i}}{\sqrt{1+|Du|^2}} = H(x,u) & \text{in } G \\ u = 0 & \text{on the boundary } \partial G\,. \end{cases}$$

List of assumptions:

(i) G is any open subset of R^m with finite measure.

(ii) H is a measurable function such that

$$(4.7a)\qquad \big(H(x,u) - H(x,0)\big)\,u \geqslant 0 \quad \text{for all } x \text{ and } u\,,$$

$$(4.7b)\qquad \|H(\cdot,0)\|_{m,\infty} < mC_m^{1/m} (= \text{the isoperimetric constant})\,.$$

(iii) u is a solution to problem (4.6) from the following Orlicz-Sobolev class:

$$(4.8)\qquad \int_G |Du|^2(1+|Du|^2)^{-\frac{1}{2}}\,dx < \infty\,.$$

REMARKS. – The norm in (4.7*b*) is that of weak L^m spaces, as in (3.2*b*) (3.3*b*). Incidentally, assumptions (4.7) are closely related to those of BOMBIERI-GIUSTI [34]. Of course, assumptions (4.7) can be replaced by the following more stringent ones:

$$H(x,\cdot)\ \text{increases}\,, \qquad \Big(\int_{R^m} |H(x,0)|^m dx\Big)^{1/m} < mC_m^{1/m}\,.$$

The Orlicz-Sobolev class (4.8), coupled with boundary condition (4.6*b*), is the same thing as the usual Sobolev space $W_0^{1,1}(G)$. For G has finite measure and

$$r \geqslant r^2(1+r^2)^{-\frac{1}{2}} \geqslant sr - s^2(1+\sqrt{1-s})^{-2} \qquad (0 \leqslant s \leqslant 1,$$

since $r^2(1+r^2)^{-\frac{1}{2}}$ is greater than

$$(4.9)\qquad A(r) = r^2/(1+r)\,,$$

which is a convex increasing function on $0 \leqslant r < \infty$, whose Young-conjugate is $\tilde{A}(s) = s^2(1+\sqrt{1-s})^2$. ∎

As is easily seen, problem (4.6) (supported by the specified hypotheses) can be imbedded in a class of boundary value problems of the type (1.1) (1.2) (1.3) (1.4) (1.5), provided ellipticity weight $A(r)$ is chosen as in (4.9). Theorem 1 tell us that the « maximizing » problem in such a class is the following:

$$\left\{\begin{array}{ll} -\sum_{i=1}^{m}\frac{\partial}{\partial x_i}\frac{v_{x_i}}{1+|Dv|}=H(\cdot\,,0)^\star(x) & \text{in } G^\star \\ v=0 & \text{on } \partial G^\star\,, \end{array}\right. \tag{4.10}$$

where stars have the customary meaning. The explicit representation formula

$$v(x)=\int\limits_{C_m|x|^m}^{\text{meas. }G}\frac{(r^{-1+1/m}/mC_m^{1/m})\int_0^r H(\cdot\,,0)^*(s)\,ds}{1-(r^{-1+1/m}/mC_m^{1/m})\int_0^r H(\cdot\,,0)^*(s)\,ds}\,\frac{r^{-1+1/m}}{mC_m^{1/m}}\,dr\,, \tag{4.11}$$

and hypothesis (4.7*b*), show that the (appropriate) solution v to (4.10) is « dominated » by the numerical multiple

$$Cw(x)\,,\qquad \frac{1}{C}=1-\frac{1}{mC_m^{1/m}}\|H(\cdot\,,0)\|_{m,\infty} \tag{4.12}$$

of the function w defined by

$$w(x)=\int\limits_{C_m|x|^m}^{\text{meas. }G}dr\,\frac{r^{-2+2/m}}{m^2C_m^{2/m}}\int_0^r H(\cdot\,,0)^*(s)\,ds\,. \tag{4.13}$$

This argument more precisely put and an inspection to formula (4.13) give the following result. *Let u be a solution to problem* (4.6) *satisfying* (4.8); *let w be the* (Lipschitz continuous) *solution to the following linear problem*:

$$\left\{\begin{array}{ll} -\Delta w=H(\cdot\,,0)^\star(x) & \text{in } G^\star \\ w=0 & \text{on } \partial G^\star\,. \end{array}\right. \tag{4.14}$$

The following inequality holds:

$$u^\star(x)\leqslant Cw(x)\qquad \textit{for every } x \textit{ in } G^\star, \tag{4.15}$$

where C is the constant defined by (4.12). *Moreover, estimates of the gradient of u can be derived, for example*:

$$\int_G\left(\frac{|Du|^2}{1+|Du|}\right)^q dx\leqslant C^q\int_{G^\star}|Dw|^{2q}dx\qquad \textit{if } 0<q\leqslant 1. \tag{4.16}$$

As a by-product of this theorem, we quote the estimate written below. This estimate comes from straightforward manipulations of formula (4.13).

$$\sup|u|\leqslant K\ (\text{meas. }G)^{2/m-1/p}\frac{\|H(\cdot\,,0)\|_{p,k}}{1-(1/mC_m^{1/m})\|H(\cdot\,,0)\|_{m,\infty}}\,,$$

where u is a solution to problem (4.6), $p > m/2$, $k \geqslant 1$ and

$$\frac{1}{K} = m^2 C_m^{2/m} \left[\frac{k}{k-1}\left(\frac{2}{m} - \frac{1}{p}\right)\right]^{1-1/k}$$

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