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NONLINEAR EQUATIONS AT RESONANCE AND GENERALIZED EIGENVALUE PROBLEMS

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1. INTRODUCTION

IN A 1967 PAPER, Loud [25] obtained sharp nonresonance conditions for the second order differential equation

$$-x'' = g(x) + h(t). \tag{1}$$

More precisely, assuming g to be an odd function of class C^1 and h to be a continuous T-periodic function, h being even and odd-harmonic, he proved the existence (and uniqueness) of a T-periodic solution to (1), whenever the range of the derivative of g does not interact with the set of eigenvalues $\{(2\pi i/T)^2, i = 0, 1, \dots\}$ of the differential operator, i.e. for some $n \in \mathbb{N}$,

$$\left(\frac{2\pi n}{T}\right)^2 < a \le g'(x) \le b < \left(\frac{2\pi(n+1)^2}{T}\right), \qquad (x \in \mathbb{R}). \tag{2}$$

That theorem was generalized by Leach [24], who obtained the same existence condition, without assuming g to be odd and without hypothesis on h other than the continuity and the T-periodicity. Similar sharp existence results for equations whose nonlinear part has linear growth had already been given by Dolph [13] for elliptic equations.

In recent years, much effort has been devoted to generalize the above existence result. We would like to distinguish between two different branches along which the theory developed.

First of all, it was shown by Mawhin [26] that condition (2) can be weakened to

$$\left(\frac{2\pi n}{T}\right)^2 < \liminf_{|x|\to\infty} \frac{g(x)}{x} \le \limsup_{|x|\to\infty} \frac{g(x)}{x} < \left(\frac{2\pi(n+1)}{T}\right)^2, \tag{3}$$

still ensuring the existence of a solution to equation (1).

On their part, Lazer and Leach [23] considered the resonance situation when equality can replace one of the strict inequalities in (3). In this case, besides a restriction on the growth of g, they introduced some conditions at resonance which are nowadays known as Landesman–Lazer conditions, after the generalization operated in [21]. More precisely, assuming $(g(x) - (2\pi n)/T)^2 x$ to be bounded, it was proved in [23] that a sufficient condition for the existence of a solution to (1) is the following:

$$\frac{T}{\pi}\left\{\liminf_{x\to+\infty}\left(g(x) - \left(\frac{2\pi n}{T}\right)^2 x\right) - \limsup_{x\to-\infty}\left(g(x) - \left(\frac{2\pi n}{T}\right)^2 x\right)\right\}$$
$$> \left\{\left(\int_0^T h(t)\sin\left(\frac{2\pi n}{T}\right)t\,dt\right)^2 + \left(\int_0^T h(t)\cos\left(\frac{2\pi n}{T}\right)t\,dt\right)^2\right\}^{1/2}.$$

It can also be shown that in some cases, the above condition is also necessary for the existence of a solution to equation (1).

These one-sided resonance situations have been extensively studied in the sequel (cf. [12, 16, 19, 27]), and abstract versions were provided in [5, 8, 11, 30].

In a recent paper [14], the authors considered even the situation where double resonance can arise, i.e. when both strict inequalities in (3) can be replaced by equalities. They proved the existence of T-periodic solutions under Landesman–Lazer type conditions at both sides. Other types of conditions at resonance have been considered, e.g. in [1, 31, 32].

The other direction followed to generalize the result of Loud was introduced by Lazer [22] and Ahmad [2]. They considered an M-dimensional system of the form

$$-x'' = \operatorname{grad} G(x) + h(t), \tag{4}$$

and replaced condition (2) by a corresponding one involving the Hessian matrix $H(x)$ of $G(x)$. They introduced two symmetric matrices A and B with respective eigenvalues $\alpha_1 \leq \cdots \leq \alpha_n$ and $\beta_1 \leq \cdots \leq \beta_n$, such that

$$\begin{cases} A \leq H(x) \leq B & (x \in \mathbb{R}^M) \\ \displaystyle\bigcup_{i \in \{1,\ldots,n\}} [\alpha_i, \beta_i] \cap \left\{ \left(\frac{2\pi k}{T}\right)^2 \mid k = 0, 1, \ldots \right\} = \varnothing. \end{cases} \tag{5}$$

Under the above conditions, they were able to prove the existence (and uniqueness) of a T-periodic solution to (4). This kind of approach has been generalized in various ways (cf. [3, 7, 9, 17, 29, 33]), and abstract versions were given in [4, 6, 10].

In this paper we try to combine the two different approaches mentioned above. We provide abstract results and give applications to systems of ordinary and partial differential equations with nonlinearities which satisfy weaker versions of condition (5), where resonance can occur at one or both sides of the intervals $[\alpha_i, \beta_i]$. In other words, some of the α_i and/or β_i can belong to the spectrum of the differential operator. In particular, we provide an abstract formulation for the result in [14].

In Section 2, we introduce our abstract setting and show the relation between different types of assumptions. In particular, we deal with a generalized eigenvalue problem $Lx = \lambda Sx$, where L and S are self-adjoint operators, and introduce a change of variable which permits this problem to be treated as a usual eigenvalue problem for an associated operator $\tilde{L}$. The use of generalized eigenvalue problems has been introduced into the study of nonresonance problems in [15] and the reduction of the generalized eigenvalue problem to a usual one can be found in Kato [20].

Section 3 is devoted to the study of the "simple resonance" situation from an abstract point of view, while in Section 4 we deal with "double resonance".

In Sections 5 and 6, we give some applications of the abstract results of Sections 3 and 4, respectively. In Section 5, we introduce generalized Landesman–Lazer conditions for systems of elliptic equations, improving some previous results by Brezis and Nirenberg [8]. We also prove an existence result for a periodic problem for which the Landesman–Lazer condition cannot be written. In Section 6, we prove the existence of solutions in a double resonance situation for a Dirichlet problem.

Finally, in Section 7 we compare two of our abstract theorems when applied to a two-dimensional periodic problem.

2. NOTATIONS AND MAIN ASSUMPTIONS

Let H be a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and corresponding norm $\|\cdot\|$. We are interested in finding solutions of the equation

$$Lx = Nx, \tag{6}$$

where $L\colon \mathrm{dom}(L) \subset H \to H$ is a linear self-adjoint operator having closed range and a compact resolvent, and $N\colon H \to H$ is continuous and maps bounded sets into bounded sets.

It is well known that, with these assumptions, L is a Fredholm operator of index zero whose spectrum is made of eigenvalues with finite multiplicities, and that L has a compact right inverse.

We are given a continuous linear self-adjoint operator $S\colon H \to H$ which is positive and invertible. We will denote by S^{-1} the inverse of S, and by $S^{1/2}$ and $S^{-1/2}$ the square root operators of S and S^{-1}, respectively.

We consider the following assumption on the operator L.

(L1) There exists $\alpha > 1$ such that

$$Lx = \lambda Sx, \qquad \lambda \in]0, \alpha[\Rightarrow x = 0.$$

Some equivalent formulations of assumption (L1) are given below; we will use the notation $\sigma(T)$ for the spectrum of an operator $T\colon \mathrm{dom}(T) \subset H \to H$.

LEMMA 1. Condition (L1) is verified if and only if, for some $\alpha > 1$, one of the following propositions is true.

(L2) $\sigma(S^{-1/2}LS^{-1/2}) \cap]0, \alpha[= \varnothing$.

(L3) $(L - (\alpha/2)S)$ is invertible and $\|(S^{-1/2}(L - (\alpha/2)S)S^{-1/2})^{-1}\| \leq (2/\alpha)$.

(L4) $\langle S^{-1}Lx, Lx\rangle \geq \alpha\langle Lx, x\rangle$, for every $x \in \mathrm{dom}(L)$.

Proof. By the change of variable $x = S^{-1/2}u$, the equation $Lx = \lambda Sx$ can be transformed into $S^{-1/2}LS^{-1/2}u = \lambda u$. It can be seen that $S^{-1/2}LS^{-1/2}$ has a compact resolvent, so that its spectrum is made of eigenvalues, and it is then easy to see that (L1) is equivalent to (L2). On the other hand, (L2) is clearly equivalent to saying that $(S^{-1/2}LS^{-1/2} - (\alpha/2)I)$ is invertible and that $\mathrm{dist}(\alpha/2, \sigma(S^{-1/2}LS^{-1/2})) \geq \alpha/2$. The equivalence of (L2) and (L3) then follows from the fact that, for an invertible self-adjoint operator T, one has $\|T^{-1}\| = [\mathrm{dist}(0, \sigma(T))]^{-1}$. We can write (L3) as follows:

$$\left\| S^{-1/2}\left(L - \frac{\alpha}{2}S\right)S^{-1/2}u \right\| \geq \frac{\alpha}{2}\|u\| \qquad (u \in H).$$

Replacing u by $S^{1/2}x$ and squaring both sides, one easily verifies that (L3) is equivalent to (L4). ■

Our main assumption on N is the following.

(N1) There exist $\beta < (\alpha - 1)/\alpha$ and $\varphi\colon H \to \mathbb{R}^+$ such that $\varphi(x) = o(\|x\|^2)$ as $\|x\| \to \infty$ and

$$\langle Nx, x\rangle \geq \langle S^{-1}Nx, Nx\rangle - \beta\langle S^{-1}Lx, Lx\rangle - \varphi(x),$$

for every $x \in H$.

In the next lemma we give a sufficient condition for (N1) to be satisfied. For self-adjoint operators $A, B: H \to H$, we will write $A \leq B$ when $B - A$ is positive semidefinite, i.e. when $\langle (B - A)u, u\rangle \geq 0$, for every $u \in H$.

LEMMA 2. A sufficient condition for (N1) to hold (with $\beta = 0$) is the following.

(N2) There exist $\Gamma: H \to \mathcal{L}_s(H)$ (the space of linear self-adjoint operators in H) and $R: H \to H$ such that

$$0 \leq \Gamma x \leq S \ (x \in H), \qquad \|Rx\| = o(\|x\|) \qquad \text{as } \|x\| \to \infty$$

and

$$Nx = (\Gamma x)x + Rx.$$

Proof. Fix $x \in H$ and set $T = 2\Gamma x - S$. Then,

$$|\langle Tu, u\rangle| \leq \langle Su, u\rangle \qquad (u \in H).$$

By taking $v = S^{1/2}u$, one can conclude that

$$\|S^{-1/2}TS^{-1/2}\| = \sup_{v \neq 0} \frac{|\langle S^{-1/2}TS^{-1/2}v, v\rangle|}{\|v\|^2} \leq 1,$$

since $S^{-1/2}TS^{-1/2}$ is self-adjoint. Hence, we have

$$\|S^{-1/2}TS^{-1/2}v\| \leq \|v\| \qquad (v \in H)$$

or, with $u = S^{-1/2}v$,

$$\|S^{-1/2}(2\Gamma x - S)u\| \leq \|S^{1/2}u\| \qquad (u \in H).$$

In particular, the above relation holds for $u = x$. So we get

$$\|S^{-1/2}(2Nx - Sx)\| \leq \|S^{1/2}x\| + 2\|S^{-1/2}Rx\|.$$

Squaring both sides, we obtain

$$\langle Nx, x\rangle \geq \langle S^{-1}Nx, Nx\rangle - [\|S^{-1/2}Rx\|^2 + \|S^{-1/2}Rx\|\,\|S^{1/2}x\|].$$

Setting $\varphi(x) = \|S^{-1/2}Rx\|^2 + \|S^{-1/2}Rx\|\,\|S^{1/2}x\|$, one has that $\varphi(x) = o(\|x\|^2)$ as $\|x\| \to \infty$, and the proof is complete. ■

Remarks. By making the change of variable $u = S^{1/2}x$, equation (6) becomes of the form

$$\tilde{L}u = \tilde{N}u, \tag{7}$$

with $\tilde{L} = S^{-1/2}LS^{-1/2}$, $\tilde{N} = S^{-1/2}NS^{-1/2}$. Condition (L1) and the inequality in (N1) are then equivalent, respectively, to:

(L5) $$\left\|\tilde{L}u - \frac{\alpha}{2}u\right\| \geq \left\|\frac{\alpha}{2}u\right\|,$$

(N3) $$\|\tilde{N}u - \tfrac{1}{2}u\|^2 \leq \|\tfrac{1}{2}u\|^2 + \beta\|\tilde{L}u\|^2 + \tilde{\varphi}(u),$$

where $\tilde{\varphi}(u) = \varphi(S^{-1/2}u)$. One can see that such kind of assumptions generalize the setting in [8, Chapter 3].

3. EXISTENCE RESULTS

In this section we prove two existence results for the general equation (6). For the first two results, we will assume that hypotheses (L1) and (N1) hold and we will give some extra conditions in order to avoid resonance. Notice that the conditions (L1) and (N1) alone are not sufficient to guarantee the existence of a solution to (6) if $\ker L \neq \varnothing$. This can be seen, for example, by taking Nx to be a constant vector which is not orthogonal to $\ker L$.

We will denote by $P_0 : H \to H$ the orthogonal projection onto $\ker L$.

THEOREM 1. Assume (L1) and (N1). If $\ker L \neq \{0\}$ suppose moreover that, for all $x \in \mathrm{dom}(L)$ with sufficiently large norm which verify the inequality $\langle S^{-1}Lx, Lx\rangle \leq ((\alpha - 1)/\alpha - \beta)^{-1}\varphi(x)$, one has, either

(R1) $$\langle Nx, P_0 x\rangle > 0,$$

or

(R2) $$\langle Nx, x\rangle - \alpha^{-1}\left(\frac{\alpha - 1}{\alpha} - \beta\right)^{-1}\varphi(x) > 0.$$

Then equation (6) has a solution.

THEOREM 2. Assume (L1) and (N1). If $\ker L \neq \{0\}$, suppose that there exist positive constants C_1, C_2 such that

$$\|P_0 Nx - P_0 Ny\| \leq C_1\|x - y\| + C_2 \tag{8}$$

for every $x, y \in H$. If moreover

(R3) $$\liminf_{\substack{\|u\| \to \infty \\ u \in \ker L}} \frac{\langle Nu, u\rangle}{\|u\|^2} > 0,$$

then equation (6) has a solution.

Proof of theorem 1 *and theorem* 2. We will apply a continuation theorem by Mawhin [28, theorem IV.13]. To this aim, we are going to prove the existence of a constant $R > 0$ such that, for any solution x of the equation

$$Lx = \lambda Nx \tag{9$_\lambda$}$$

with $\lambda \in]0, 1[$, we have $\|x\| < R$; moreover, we need to prove that the Brouwer degree

$$\deg(P_0 N, B(0, R) \cap \ker L, 0) \tag{10}$$

is well defined and different from 0.

By contradiction, let us suppose that there exist sequences (x_n), (λ_n) such that x_n is a solution of $(9)_{\lambda_n}$, $\lambda_n \in]0, 1[$ and $\|x_n\| \to \infty$. From $(9)_{\lambda_n}$, using (L4) and (N1), we deduce the inequalities

$$\langle Nx_n, x_n\rangle = \frac{1}{\lambda_n}\langle Lx_n, x_n\rangle \leq \frac{1}{\alpha\lambda_n}\langle S^{-1}Lx_n, Lx_n\rangle, \tag{11}$$

$$\langle Nx_n, x_n\rangle \geq \left(\frac{1}{\lambda_n^2} - \beta\right)\langle S^{-1}Lx_n, Lx_n\rangle - \varphi(x_n). \tag{12}$$

Combining (11) with (12), one gets

$$\lambda_n\alpha\left(\frac{1}{\lambda_n} - \lambda_n\beta - \frac{1}{\alpha}\right)\langle Nx_n, x_n\rangle \le \left(\frac{1}{\lambda_n} - \lambda_n\beta - \frac{1}{\alpha}\right)\langle S^{-1}Lx_n, Lx_n\rangle \le \lambda_n\varphi(x_n) \tag{13}$$

from which results in particular that

$$\left(\frac{\alpha - 1}{\alpha} - \beta\right)\langle S^{-1}Lx_n, Lx_n\rangle \le \varphi(x_n). \tag{14}$$

Moreover, x_n, being a solution of $(9)_{\lambda_n}$, must satisfy the conditions

$$P_0Nx_n = 0, \tag{15}$$

$$\langle Nx_n, P_0x_n\rangle = 0, \tag{16}$$

since $\operatorname{Im} L$ is orthogonal to $\ker L$. On the other hand, from (13) we also have

$$\alpha\left(\frac{\alpha - 1}{\alpha} - \beta\right)\langle Nx_n, x_n\rangle \le \varphi(x_n). \tag{17}$$

Now, since x_n verifies (14), the conditions (16) and (17) give a contradiction either with (R1) or with (R2); so, we have an *a priori* bound for the solutions of $(9)_\lambda$ when we are in the situation of theorem 1.

In the situation of theorem 2, we first notice that (14) implies that $\|(I - P_0)x_n\| = o(\|x_n\|)$ for $n \to \infty$, as it results from the fact that L has a continuous right inverse. Using (15) and (8), we see that

$$\frac{\|P_0NP_0x_n\|}{\|P_0x_n\|} \le C_1\frac{\|(I - P_0)x_n\|}{\|P_0x_n\|} + \frac{C_2}{\|P_0x_n\|} \to 0 \qquad \text{for } n \to \infty.$$

This contradicts (R3). Therefore, we also have an *a priori* bound for the solutions of $(9)_\lambda$ in this case.

It remains to prove that the degree (10) is well defined and different from 0. This is fairly straightforward, since our assumptions all imply that $\langle P_0Nu, u\rangle > 0$ for every $u \in \ker L$ with sufficiently large norm.

The conclusion then follows from Mawhin's theorem. ■

Remarks. Theorem 1 can be seen to generalize some results by Brezis and Nirenberg [8, Chapter 3]. They deal with the case where S is a multiple of the identity; it can be seen that their hypotheses, in theorem III.2′, are a particular case of (R2). We have seen, however, in the remarks at the end of Section 2, that one can always reduce to the case where S is the identity by a suitable change of variable. On the other hand, condition (R1) permits the recovery of the Landesman–Lazer conditions when dealing with applications to partial differential equations, without the need of further assumptions as in [8]. This will be made clear in the next section.

Theorem 2 deals with an essentially nonresonant situation. By this we mean that under the same assumptions one can conclude that for any $z \in H$ the equation

$$Lx = Nx + z$$

has a solution. On the contrary, theorem 1 would require further assumptions upon z in order to be applied.

As can be seen from the proof, one can generalize condition (R3) of theorem 2 by replacing it, for example, by:

(R4) for every $u \in \ker L$, with sufficiently large norm,

$$\langle P_0 Nu, u\rangle > -\|P_0 Nu\|\,\|u\|$$

and

$$\liminf_{\substack{\|u\|\to\infty \\ u\in\ker L}} \frac{\|P_0 Nu\|}{\|u\|} > 0.$$

We now state a corollary of the proof of theorem 1.

Corollary 1. Assume the following.

(L1′) $$Lx = \lambda Sx, \qquad \lambda \in]0, 1] \Rightarrow x = 0;$$

(N1′) there exist positive constants c_1 and c_2 such that

$$\langle Nx, x\rangle \geq \langle S^{-1}Nx, Nx\rangle - c_1\|x\| - c_2,$$

for every $x \in H$.

(R1′) If $\ker L \neq \{0\}$, assume that for every sequence (x_n) such that

$$\|x_n\| \to \infty \qquad \text{and} \qquad \|Lx_n\| = O(\|x_n\|^{1/2}) \qquad \text{as } n \to \infty,$$

one has

$$\liminf_{n\to\infty} \langle Nx_n, P_0 x_n\rangle > 0.$$

Then, equation (6) has a solution.

Notice that (L1′) is equivalent to (L1), since the spectrum of an operator is closed. On the other hand, if the sequence (x_n) verifies (14) with $\varphi(x) = c_1\|x\| - c_2$, one clearly has $\|Lx_n\| = O(\|x_n\|^{1/2})$ for $n \to \infty$. This allows hypothesis (R1) to be replaced by (R1′).

4. DOUBLE RESONANCE: AN ABSTRACT APPROACH

In this section we will weaken assumption (L1) on L in order to permit both $\ker L$ and $\ker(S - L)$ to be nontrivial. We will use the following

(L1″) $$Lx = \lambda Sx, \qquad \lambda \in]0, 1[\Rightarrow x = 0.$$

It is straightforward to obtain equivalent characterizations as in lemma 1: just set $\alpha = 1$. In particular, (L1″) is equivalent to

(L4″) $$\langle S^{-1}Lx, Lx\rangle \geq \langle Lx, x\rangle \qquad \text{for every } x \in \operatorname{dom}(L).$$

The assumption on N will be a stronger version of condition (N2) of lemma 2, as shown in the following.

(N2′) There exist $\Gamma: H \to \mathcal{L}_s(H)$ (the space of linear self-adjoint operators in H) and $R: H \to H$ such that

$$0 \leq \Gamma x \leq S \qquad (x \in H),$$

R has bounded image and

$$Nx = (\Gamma x)x + Rx.$$

We will introduce P_0, P_1, projections on $\ker L$ and $\ker(S - L)$, respectively. Our main result is the following theorem.

THEOREM 3. Assume (L1″) and (N2′). Suppose moreover

(N2″) there exists $\varepsilon > 0$ such that, for every $x \in H$, either

$$\langle (\Gamma x)u, u\rangle \geq \varepsilon \langle Su, u\rangle \qquad (u \in \ker L \oplus \ker(S - L)),$$

or

$$\langle (\Gamma x)u, u\rangle \leq (1 - \varepsilon)\langle Su, u\rangle \qquad (u \in \ker L \oplus \ker(S - L));$$

(R1″) for every sequence (x_n) such that $\|x_n\| \to \infty$ and $\|Lx_n\| = O(\|x_n\|^{1/2})$ for $n \to \infty$, one has

$$\liminf_{n\to\infty} \langle Nx_n, P_0 x_n\rangle > 0;$$

(R2″) for every sequence (y_n) such that $\|y_n\| \to \infty$ and $\|(S - L)y_n\| = O(\|y_n\|^{1/2})$ for $n \to \infty$, one has

$$\liminf_{n\to\infty} \langle Sy_n - Ny_n, P_1 y_n\rangle > 0.$$

Then, equation (6) has a solution.

In order to prove theorem 3, we need the following result.

LEMMA 3. If (L1″) holds, one can split the space H as $H = [\ker L \oplus \ker(S - L)] \oplus \tilde{H}$, in such a way that, writing $x \in H$ as $x = \bar{x} + \tilde{x}$, with $\bar{x} \in [\ker L \oplus \ker(S - L)]$ and $\tilde{x} \in \tilde{H}$, one has, for all $x \in \mathrm{dom}(L)$,

$$\langle S^{-1}Lx, Lx\rangle - \langle Lx, x\rangle \geq \delta\|\tilde{x}\|^2, \tag{18}$$

for some $\delta > 0$.

Proof. It can be shown that, since L has a compact resolvent, the same is true for $\tilde{L} = S^{-1/2}LS^{-1/2}$. Let (u_i) be the eigenvectors of $\tilde{L}$ with corresponding eigenvalues (λ_i), i.e. $Lu_i = \lambda_i Su_i$. Then the (u_i) form a basis for H, and we can define $\tilde{H}$ as the subspace of H generated by the vectors u_i for which $\lambda_i \notin \{0, 1\}$. The result then follows from standard Fourier analysis. ■

Proof of theorem 3. By the Leray-Schauder degree theory, using a homotopy argument, it is sufficient to prove that, for $\lambda \in]0, 1[$, there is an *a priori* bound for the solutions of

$$Lx = (1 - \lambda)\tfrac{1}{2}Sx + \lambda Nx. \tag{19$_\lambda$}$$

Assume by contradiction that this is not true. Then there are sequences (λ_n), (x_n) such that $\lambda_n \in]0, 1[$, $\|x_n\| \to \infty$ and x_n is a solution of $(19)_{\lambda_n}$. From $(19)_{\lambda_n}$ one has

$$\langle Nx_n, x_n\rangle \leq \frac{1}{\lambda_n}\langle Lx_n, x_n\rangle - \frac{1 - \lambda_n}{2\lambda_n}\langle Sx_n, x_n\rangle \leq \frac{1}{\lambda_n}\langle Lx_n, x_n\rangle. \tag{20}$$

On the other hand, (N2′) implies (N1) with $\beta = 0$ and $\varphi(x) = O(\|x\|)$ for $\|x\| \to \infty$ (see the proof of lemma 2). Since, by (L4″),

$$\begin{aligned}\langle S^{-1}Nx_n, Nx_n\rangle &= \frac{1}{\lambda_n^2}\langle S^{-1}Lx_n, Lx_n\rangle + \left(\frac{1 - \lambda_n}{2\lambda_n}\right)^2\langle Sx_n, x_n\rangle - \frac{1 - \lambda_n}{\lambda_n^2}\langle Lx_n, x_n\rangle \\ &\geq \frac{1}{\lambda_n}\langle S^{-1}Lx_n, Lx_n\rangle,\end{aligned} \tag{21}$$

we have, from (N1) and (21),

$$\langle Nx_n, x_n\rangle \geq \frac{1}{\lambda_n}\langle S^{-1}Lx_n, Lx_n\rangle - \varphi(x_n). \tag{22}$$

Combining (20) with (22), one has

$$\langle S^{-1}Lx_n, Lx_n\rangle - \langle Lx_n, x_n\rangle \leq \varphi(x_n). \tag{23}$$

By lemma 3, we then have that

$$\|\tilde{x}_n\| = O(\|x_n\|^{1/2}) \qquad \text{as } n \to \infty. \tag{24}$$

Now, because of condition (N2″) we can split the sequence x_n in two subsequences, x_n' and x_n'', satisfying respectively

$$\langle(\Gamma x_n')u, u\rangle \geq \varepsilon\langle Su, u\rangle \qquad (u \in \ker L \oplus \ker(S - L)), \tag{25}$$

and

$$\langle(\Gamma x_n'')u, u\rangle \leq (1 - \varepsilon)\langle Su, u\rangle \qquad (u \in \ker L \oplus \ker(S - L)). \tag{26}$$

We want to show that both situations lead to a contradiction, proving the theorem. The two cases being symmetrical, we only consider one of the two, e.g. (26). Working as in lemma 2, and using the fact that $0 \leq \Gamma x_n''$, one gets

$$\|S^{-1/2}(2\Gamma x_n'' - (1 - \varepsilon)S)u\| \leq (1 - \varepsilon)\|S^{1/2}u\| \tag{27}$$

for every $u \in \ker L \oplus \ker(S - L)$ (just replace S by $(1 - \varepsilon)S$). In particular, (27) holds for $u = \bar{x}_n''$. Then, for a certain $c > 0$,

$$\|S^{-1/2}(2\Gamma x_n'' - (1 - \varepsilon)S)x_n''\| \leq (1 - \varepsilon)\|S^{1/2}x_n''\| + c\|\tilde{x}_n''\|,$$

and so

$$\|S^{-1/2}(2Nx_n'' - (1 - \varepsilon)Sx_n'')\| \leq (1 - \varepsilon)\|S^{1/2}x_n''\| + c\|\tilde{x}_n''\| + 2\|S^{-1/2}Rx_n''\|. \tag{28}$$

Because of (24), it is then easy to define a map $\Psi\colon H \to \mathbb{R}^+$ such that $\Psi(x) = O(\|x\|)$ as $\|x\| \to \infty$ and, from (28),

$$\langle Nx_n'', x_n''\rangle \geq \frac{1}{1 - \varepsilon}\langle S^{-1}Nx_n'', Nx_n''\rangle - \Psi(x_n''). \tag{29}$$

Since (x_n'') is a subsequence of (x_n), it satisfies (20), which, combined with (29), (21) and (L4″), gives

$$\frac{\varepsilon}{1 - \varepsilon}\langle S^{-1}Lx_n'', Lx_n''\rangle \leq \Psi(x_n''),$$

and so

$$\|Lx_n''\| = O(\|x_n''\|^{1/2}) \qquad \text{as } n \to \infty. \tag{30}$$

On the other hand, x_n'' being a solution of $(19)_{\lambda_n}$, we obtain

$$(1 - \lambda_n)\tfrac{1}{2}\langle Sx_n'', P_0x_n''\rangle + \lambda_n\langle Nx_n'', P_0x_n''\rangle = 0.$$

From (30), we have that $\|(I - P_0)x_n''\| = O(\|x_n''\|^{1/2})$ as $n \to \infty$, and, S being positive definite, it is easy to see that

$$\limsup_{n\to\infty}\langle Nx_n'', P_0x_n''\rangle = -\liminf_{n\to\infty}\frac{1 - \lambda_n}{2\lambda_n}\langle Sx_n'', P_0x_n''\rangle \leq 0.$$

This, combined with (30), gives a contradiction with (R1″). ■

5. SIMPLE RESONANCE IN SYSTEMS OF DIFFERENTIAL EQUATIONS

As a model of application of theorem 1, let us consider the following Dirichlet problem for a system of elliptic partial differential equations

$$\text{(D)} \quad \begin{cases} \Delta u + Au + g(x, u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, Ω is bounded regular domain in $\mathbb{R}^N$, the Laplacian Δ acts on functions having their values in $\mathbb{R}^M$, A is a symmetric $M \times M$ matrix and $g: \Omega \times \mathbb{R}^M \to \mathbb{R}^M$ satisfies the Carathéodory conditions. Let B be another symmetric $M \times M$ matrix such that $B - A$ is positive definite; we denote by $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_M$ and $\beta_1 \le \beta_2 \le \cdots \le \beta_M$ the eigenvalues of A and B respectively. By a monotonicity property of the eigenvalues (see, for instance, [18, p. 182]), we have that $\alpha_i < \beta_i$ $(i = 1, \dots, M)$. We will assume that

$$\text{(L6)} \qquad \bigcup_{i=1}^{n}]\alpha_i, \beta_i] \cap \sigma_D(-\Delta) = \varnothing,$$

where $\sigma_D(-\Delta)$ is the spectrum of the Laplace operator $(-\Delta)$, under the Dirichlet boundary conditions. Suppose moreover that g satisfies the following condition:

(N4) there exist $c \in L^2(\Omega; \mathbb{R})$, $d \in L^1(\Omega; \mathbb{R})$ such that, for a.e. $x \in \Omega$, for all $u \in \mathbb{R}^M$,

$$(g(x, u), u) \ge ((B - A)^{-1} g(x, u), g(x, u)) - c(x)|u| - d(x).$$

Let us introduce the abstract setting. Let $H = L^2(\Omega; \mathbb{R}^M)$, $\operatorname{dom} L = H^2(\Omega; \mathbb{R}^M) \cap H_0^1(\Omega; \mathbb{R}^M)$, $Lu = -\Delta u - Au$, $(Nu)(x) = g(x, u(x))$ and $(Su)(x) = (B - A)u(x)$. Then we have the following lemma.

LEMMA 4. Condition (L6) is equivalent to (L1′), while (N4) implies (N1′).

Proof. Integrating the inequality in (N4) immediately proves the second part of the statement, with $c_1 = \|c\|_{L^2}$ and $c_2 = \|d\|_{L^1}$. In order to show the equivalence of (L6) and (L1′), fix $\lambda \in]0, 1]$ and define $D = \lambda B + (1 - \lambda)A$. Let $\{d_i \mid i = 1, \dots, M\}$ be a basis in $\mathbb{R}^M$ made of eigenvectors of D, d_i being associated to an eigenvalue δ_i and the δ_i being arranged in increasing order: $\delta_1 \le \delta_2 \le \cdots \le \delta_M$. By the monotonicity property mentioned above, we have $\alpha_i < \delta_i \le \beta_i$ for $i = 1, \dots, M$. For $u \in H$, we can write $u(t) = \sum_{i=1}^{M} u_i(t)\, d_i$, so that

$$Lu - \lambda Su = \sum_{i=1}^{M} (-\Delta u_i - \delta_i u_i) d_i.$$

It is clear that u is a nontrivial solution of $Lu = \lambda Su$ if and only if $\delta_i \in \sigma_D(-\Delta)$. But, this is precisely forbidden by (L6). Hence, (L6) is equivalent to the fact that, for $\lambda \in]0, 1]$, $Lu = \lambda Su$ implies $u = 0$. ■

We shall denote by E one of the following Banach spaces of functions, in which $H^2(\Omega; \mathbb{R}^M)$ is compactly imbedded:

$$\begin{aligned} &\text{if } N = 1, && E = C^1(\Omega; \mathbb{R}^M); \\ &\text{if } N = 2, && E = W^{1,p}(\Omega; \mathbb{R}^M) && (1 \le p < \infty); \\ &\text{if } N \ge 3, && E = W^{1,p}(\Omega; \mathbb{R}^M) && \left(p < \frac{2N}{N-2}\right). \end{aligned}$$

It is known that the operator L, considered as an operator acting on E, has a compact resolvent. We will write $v_n \twoheadrightarrow u$ whenever $v_n \overset{E}{\rightarrow} u$ and $v_n(x) \to u(x)$ a.e. on Ω. Let us also introduce the orthogonal projector $P_0: H \to H$ onto $\ker L$.

THEOREM 4. Assume that (L6) and (N4) hold. Moreover, assume that there exists a number $k_1 > 0$ and a function $k_2 \in L^2(\Omega; \mathbb{R})$ such that, for a.e. $x \in \mathbb{R}$, for all $v, w \in \mathbb{R}^M$,

$$(g(x, v), w) \geq -k_1|w - v|^2 - k_2(x)|w|. \tag{31}$$

If, for every $u \in \ker L\backslash\{0\}$,

$$\text{(LL)} \qquad \int_\Omega \liminf_{\substack{v \twoheadrightarrow u \\ \mu \to +\infty}} (g(x, \mu v(x)), P_0 v(x))\, dx > 0,$$

then problem (D) has a solution.

Remarks. (1) In the line of lemma 2, we can observe that the hypothesis (N4) can be replaced by

$$\text{(N5)} \qquad g(x, u) = \Gamma(x, u)u + h(x, u),$$

where, for every $(x, u) \in \Omega \times \mathbb{R}^M$, $\Gamma(x, u)$ is a symmetric matrix such that $0 \leq \Gamma(x, u) \leq B - A$, while h is a bounded function.

Moreover, if (N5) holds, it is immediate to find $k_1 > 0$ and $k_2 \in L^2(\Omega; \mathbb{R})$ such that (31) is satisfied as well.

(2) When $M = 1$, condition (LL) reduces to the classical Landesman–Lazer condition, as will be shown below. Notice first that, in the case $M = 1$, condition (31) is satisfied if there exists $h \in L^2(\Omega; \mathbb{R})$ such that, for a.e. $x \in \Omega$ and for all $u \in \mathbb{R}^M$,

$$(\operatorname{sgn} u)g(x, u) \geq -h(x), \tag{32}$$

and if g is growing at most linearly in u. This can be seen by considering separately in (31) the cases where v and w have the same sign or opposite signs.

If g satisfies (32), we can define $g_+(x) = \liminf_{r \to +\infty} g(x, r)$ and $g^-(x) = \limsup_{r \to -\infty} g(x, r)$. Then, for $u \in \ker L\backslash\{0\}$, we have

$$\liminf_{\substack{v \twoheadrightarrow u \\ \mu \to +\infty}} g(x, \mu v(x))P_0 v(x) = \begin{cases} g_+(x)u(x) & \text{if } u(x) > 0 \\ g^-(x)u(x) & \text{if } u(x) < 0. \end{cases}$$

So, condition (LL) becomes

$$\int_{u>0} g_+(x)u(x)\, dx + \int_{u<0} g^-(x)u(x)\, dx > 0,$$

which is the usual Landesman–Lazer condition.

As far as the hypotheses (L6) and (N4) are concerned, they are obtained by assuming that $A = \lambda_n \in \sigma_D(-\Delta)$, $]A, B] \cap \sigma_D(-\Delta) = \emptyset$ and

$$0 \leq \liminf_{|u| \to \infty} \frac{g(x, u)}{u} \leq \limsup_{|u| \to \infty} \frac{g(x, u)}{u} \leq B - A. \tag{33}$$

Using (32) and (33), it can be shown that g can be split up into $g(x, u) = \gamma(x, u)u + h(x, u)$, where

$$0 \le \gamma(x, u) \le B - A \qquad \text{and} \qquad |h(x, u)| \le \hat{h}(x),$$

with $\hat{h} \in L^2(\Omega; \mathbb{R})$ (see Fabry and Fonda [14]). It is then easy to check that (N4) holds.

(3) Results similar to theorem 4 have been obtained by Brezis and Nirenberg [8]. However, in the scalar case, they can deduce the usual Landesman–Lazer condition only on a supplementary condition, namely that $g^-(x) \le g_+(x)$ a.e. on Ω, or that g has sublinear growth. This supplementary assumption was avoided in [5, 12]. However, those results did not follow from an abstract theorem as the one we have, but rather from some semi-abstract one.

(4) Notice that, in case g turns out to be bounded, condition (LL) of theorem 4 becomes equivalent to

$$\int_\Omega \liminf_{\substack{v \twoheadrightarrow u \\ \mu \to +\infty}} (g(x, \mu v(x)), u(x))\, \mathrm{d}x > 0.$$

Proof of theorem 4. We will apply corollary 1. Lemma 3 shows that (L1′) holds, as well as (N1′). It then remains to prove that (R1′) is satisfied. Suppose, by contradiction, that there exists a sequence (u_n) such that $\|u_n\| \to \infty$, $\|Lu_n\| = O(\|u_n\|^{1/2})$ and (for a subsequence)

$$\lim_{n \to \infty} \langle Nu_n, P_0 u_n \rangle \le 0.$$

Applying the right inverse $K: \operatorname{Im} L \to E$ of L, which is compact, we can extract a subsequence, still denoted by (u_n), such that $(I - P_0)u_n/\|u_n\|^{1/2}$ converges strongly in E. In particular, $(I - P_0)u_n/\|u_n\|$ converges to 0. Since $\ker L$ is finite dimensional, it can be assumed that $P_0 u_n/\|u_n\|$ and, consequently, $v_n := u_n/\|u_n\|$ converges strongly in E to some function $u \in \ker L \setminus \{0\}$. Going if necessary to a further subsequence, we will also have that $v_n(x)$ converges pointwisely to $u(x)$, a.e. in Ω, so that $v_n \twoheadrightarrow u$. Letting $\mu_n = \|u_n\|$, we have, by the hypothesis made on the u_n,

$$\liminf_{n \to \infty} \frac{1}{\|u_n\|} \langle Nu_n, P_0 u_n \rangle = \liminf_{n \to \infty} \int_\Omega (g(x, \mu_n v_n(x)), P_0 v_n(x))\, \mathrm{d}x \le 0.$$

Now, we want to apply Fatou's lemma. By (31), we have

$$(g(x, \mu_n v_n(x)), P_0 v_n(x)) \ge -\frac{k_1}{\mu_n} |(I - P_0)u_n(x)|^2 - k_2(x)|P_0 v_n(x)|.$$

Since $(I - P_0)u_n/\|u_n\|^{1/2}$ converges strongly in E, and hence in $L^2(\Omega; \mathbb{R}^M)$, we can find a function $\eta_1 \in L^2(\Omega; \mathbb{R})$ such that, for a subsequence,

$$\left| \frac{(I - P_0)u_n(x)}{\|u_n\|^{1/2}} \right| \le \eta_1(x) \qquad \text{for a.e. } x \in \Omega.$$

On the other hand, since $P_0 v_n$ converges to u, there exists a function $\eta_2 \in L^1(\Omega; \mathbb{R})$ such that

$$|k_2(x) P_0 v_n(x)| \le \eta_2(x).$$

We can then conclude that there exists a function $\gamma \in L^1(\Omega; \mathbb{R})$ such that, for a.e. $x \in \Omega$,

$$(g(x, \mu_n v_n(x)), P_0 v_n(x)) \ge \gamma(x).$$

This allows us to apply Fatou's lemma, which gives

$$\int_\Omega \liminf_{n\to\infty}(g(x, \mu_n v_n(x)), P_0 v_n(x))\,dx \le 0,$$

contradicting hypothesis (LL). ■

We will now give an application of theorem 2. We consider the periodic problem

$$(\mathrm{P})\quad \begin{cases} u'' + Au + g(x, u) = 0 \\ u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0. \end{cases}$$

As before, A and B will be two symmetric $M \times M$ matrices with eigenvalues $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_M$ and $\beta_1 \le \beta_2 \le \cdots \le \beta_M$, respectively, and we will have $\alpha_1 < \beta_i$ for $i = 1, \ldots, M$. In analogy with condition (L6), we assume

$$(\mathrm{L7})\qquad \bigcup_{i=1}^{M} \,]\alpha_i, \beta_i] \cap \sigma_P(-u'') = \emptyset,$$

where $\sigma_P(-u'')$ denotes the spectrum of the operator $u \to -u''$, under the periodic boundary conditions. The function $g\colon [0, 2\pi] \times \mathbb{R}^M \to \mathbb{R}^M$ is supposed to satisfy Carathéodory conditions and (N4). In the following theorem, we denote by $|u|$ the Euclidian norm of $u \in \mathbb{R}^M$.

THEOREM 5. Assume that the conditions (L7) and (N4) hold (with $\Omega = (0, 2\pi)$). Suppose that there exist functions $c_1 \in L^2((0, 2\pi); \mathbb{R})$ and $c_2 \in L^2((0, 2\pi); \mathbb{R})$ such that, for a.e. $x \in (0, 2\pi)$, for all $u, v \in \mathbb{R}^M$,

$$|g(x, u) - g(x, v)| \le c_1(x)|u - v| + c_2(x). \tag{34}$$

Suppose moreover that $g(x, u) = g_1(x, u) + g_2(u)$, where g_2 is an even function and

$$\liminf_{|u|\to\infty} \frac{(g_1(x, u), u)}{|u|^2} \ge a > 0. \tag{35}$$

If there exists a positive integer m such that each α_i which belongs to $\sigma_P(-u'')$ is the square of an odd multiple of m, then problem (P) has a solution.

Proof. We will apply theorem 2 with $H = L^2((0, 2\pi); \mathbb{R}^M)$,

$$\operatorname{dom} L = \{u \in H^2((0, 2\pi); \mathbb{R}^M) \mid u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0\},$$

$Lu = -u'' - Au$, $(Nu)(x) = g(x, u(x))$ and $(Su)(x) = (B - A)u(x)$. It is easy to see that the arguments of lemma 3 also apply to this case, so that (L1′) and (N1′) hold. Consequently, (L1) is satisfied for some $\alpha > 1$, and (N1) for $\beta = 0$. Let $(N_1u)(x) = g_1(x, u(x))$ and $(N_2u)(x) = g_2(u(x))$. From (35), it is easy to deduce that

$$\liminf_{\|u\|\to\infty} \frac{\langle N_1 u, u\rangle}{\|u\|^2} \ge a > 0.$$

On the other hand, by the hypothesis on those eigenvalues α_i of A which belong to $\sigma_P(-x'')$, the components of any element $u \in \ker L$ are linear combinations of $\cos(k_i mx)$ and $\sin(k_i mx)$, when the k_i are odd numbers. It follows from this observation that u is of period $2\pi/m$ and that $u((x - \pi)/m) = -u(x/m)$, for all $x \in \mathbb{R}$. Let us now evaluate $\langle N_2 u, u\rangle$. We have:

$$\langle N_2 u, u\rangle = \int_0^{2\pi} (g_2(u(x)), u(x))\, \mathrm{d}x$$

or, using the $2\pi/m$-periodicity of u,

$$\langle N_2 u, u\rangle = \int_0^{2\pi} \left(g_2\left(u\left(\frac{s}{m}\right)\right), u\left(\frac{s}{m}\right)\right) \mathrm{d}s.$$

Splitting the integral into two parts, we can write

$$\langle N_2 u, u\rangle = \int_0^{\pi} \left(g_2\left(u\left(\frac{s}{m}\right)\right), u\left(\frac{s}{m}\right)\right) \mathrm{d}s + \int_{\pi}^{2\pi} \left(g_2\left(u\left(\frac{s}{m}\right)\right), u\left(\frac{s}{m}\right)\right) \mathrm{d}s;$$

the change of variable $s' = s - \pi$, the fact that $u((s - \pi)/m) = -u(s/m)$, and that g_2 is even, then show that

$$\langle N_2 u, u\rangle = 0.$$

This, combined with the result obtained above for N_1, shows that condition (R3) holds. The result then follows from theorem 2. ■

It is easy to see that, in the above theorem, the condition (N4) could be weakened by replacing $c(x)|u| + d(x)$ by $\varphi(u)$, where $\varphi(u) = o(u^2)$ for $|u| \to \infty$. Under that generalization, we see that theorem 5 can be applied, with $M = 1$, in cases where the function $(\operatorname{sgn} u)g(x, u)$ is not bounded below. For instance, the result holds in the scalar case for $A = n^2$, n odd, $0 < \eta < n + 1/2$, $e \in L^2((0, 2\pi); \mathbb{R})$, c arbitrary and

$$g(x, u) = \eta u - c|u|^{-1/2}u + \eta|u| \sin(\log(1 + |u|)) + e(x).$$

This shows that there exist functions g verifying the hypotheses of theorem 5, but not Landesman–Lazer conditions; indeed, those conditions can only be written when $(\operatorname{sgn} u)g(x, u)$ is bounded below.

6. DOUBLE RESONANCE FOR A SCALAR PROBLEM

In this section we will give an application of theorem 3 of Section 4 to the following scalar Dirichlet problem:

$$(\mathrm{D}') \begin{cases} \Delta u + g(x, u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, Ω is a bounded regular domain of $\mathbb{R}^N$ and $g: \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that, for every $r \geq 0$,

$$\sup_{|u| \leq r} |g(\cdot, u)| \in L^2(\Omega; \mathbb{R}).$$

Let $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ be the eigenvalues (which can be multiple) of $(-\Delta)$ with Dirichlet boundary conditions, and assume that, for some $k \geq 1$,

$$\lambda_k \leq \liminf_{|u|\to\infty} \frac{g(x, u)}{u} \leq \limsup_{|u|\to\infty} \frac{g(x, u)}{u} \leq \lambda_{k+1},$$

uniformly in x. Assume moreover that there exists $h \in L^2(\Omega; \mathbb{R})$ such that

$$\operatorname{sgn}(u)[g(x, u) - \lambda_k u] \geq -h(x)$$

and

$$\operatorname{sgn}(u)[\lambda_{k+1} u - g(x, u)] \geq -h(x)$$

for every $u \in \mathbb{R}$ and almost every $x \in \Omega$.

THEOREM 6. Assume the above setting. If, moreover, the following condition holds:
(LL) for every $v \in \ker(-\Delta - \lambda_k)\backslash\{0\}$,

$$\int_{v>0} \liminf_{u\to+\infty}[g(x, u) - \lambda_k u]v(x)\,dx + \int_{v<0} \limsup_{u\to-\infty}[g(x, u) - \lambda_k u]v(x)\,dx > 0,$$

for every $w \in \ker(-\Delta - \lambda_{k+1})\backslash\{0\}$,

$$\int_{w>0} \liminf_{u\to+\infty}[\lambda_{k+1} u - g(x, u)]w(x)\,dx + \int_{w<0} \limsup_{u\to-\infty}[\lambda_{k+1} u - g(x, u)]w(x)\,dx > 0,$$

then problem (D′) has a solution.

Proof. We will apply theorem 3. It can be shown as in [14] that it is possible to write $g(x, u)$ as

$$g(x, u) = \gamma(x, u)u + r(x, u),$$

where $\lambda_k \leq \gamma(x, u) \leq \lambda_{k+1}$ and $r(x, u)$ is bounded by an L^2-function. Defining $H = L^2(\Omega; \mathbb{R})$, $\operatorname{dom}(L) = H^2(\Omega) \cap H_0^1(\Omega)$, $Lu = -\Delta u - \lambda_k u$, $Nu = g(\cdot, u(\cdot)) - \lambda_k u(\cdot)$, $S = (\lambda_{k+1} - \lambda_k)I$ and $(\Gamma u)v = [\gamma(\cdot, u(\cdot)) - \lambda_k]v(\cdot)$, it is easy to show that (L1″) and (N2′) hold. By contradiction, suppose that (N2″) is not verified. Then, there exist sequences (u_n) in $L^2(\Omega; \mathbb{R})$ and (v_n), (w_n) in $\ker L \oplus \ker(S - L)$ such that

$$\int_\Omega \left(\gamma(x, u_n(x)) - \lambda_k - \frac{1}{n}(\lambda_{k+1} - \lambda_k)\right)v_n^2(x)\,dx \leq 0 \tag{36}$$

and

$$\int_\Omega \left(\lambda_{k+1} - \gamma(x, u_n(x)) - \frac{1}{n}(\lambda_{k+1} - \lambda_k)\right)w_n^2(x)\,dx \leq 0. \tag{37}$$

Moreover, (v_n) and (w_n) can be normalized in $L^2(\Omega; \mathbb{R})$. Then, since $\ker L \oplus \ker(S - L)$ is finite dimensional, we can assume that, for subsequences, v_n converges to some $v \neq 0$ and w_n converges to some $w \neq 0$ in, e.g. $H^2(\Omega) \subset L^{2N/(N-2)}(\Omega; \mathbb{R})$. On the other hand, we can suppose that $\gamma(\cdot, u_n(\cdot))$ converges weakly to some $\tilde{\gamma}$ in $L^{N/2}(\Omega; \mathbb{R})$, and, by the fact that the convex set $\{p \in L^{N/2}(\Omega; \mathbb{R}): \lambda_k \leq p(x) \leq \lambda_{k+1}$ a.e. on $\Omega\}$ is weakly closed, one has $\lambda_k \leq \tilde{\gamma}(x) \leq \lambda_{k+1}$ a.e. on Ω. But then, passing to the limit and using the fact that the eigenfunctions of the Laplacian can be equal to zero only on sets of zero measure, one deduces from (36) that $\tilde{\gamma}(x) = \lambda_k$ for a.e. $x \in \Omega$, and from (37) that $\tilde{\gamma}(x) = \lambda_{k+1}$ for a.e. $x \in \Omega$, which is a contradiction. So (N2″) is proved to hold. One can finally show as in the proof of theorem 4 that (LL) implies both (R1″) and (R2″). The conclusion follows. ■

It is easy to see that a result similar to theorem 6 can be proved by the use of theorem 3 for a periodic boundary value problem for a scalar ordinary differential equation. Thus, we have been able to provide an abstract approach to the result in [14].

7. A TWO-DIMENSIONAL PERIODIC PROBLEM AT RESONANCE

In this section we want to compare theorems 1 and 3 when applied to the following periodic problem in $\mathbb{R}^2$:

$$(\mathrm{P}')\quad \begin{cases} x'' + n^2 x + g(t, x) = 0 \\ x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0. \end{cases}$$

Here $x = (x_1, x_2) \in \mathbb{R}^2$, $g = [0, 2\pi] \times \mathbb{R}^2 \to \mathbb{R}^2$ is a Carathéodory function whose components are of the form

$$g_i(t, x) = \gamma_i(t, x)x_i + r_i(t, x) \qquad (i = 1, 2),$$

where

$$0 \le \gamma_i(t, x) \le \gamma < 2n + 1$$

and $r_i(t, x)$ is bounded by an L^2-function. The following is a consequence of theorem 3.

Theorem 7. Assume that, for an $\bar{\varepsilon} > 0$,

$$\gamma_1(t, x) + \gamma_2(t, x) \ge \bar{\varepsilon}, \tag{38}$$

for all $x \in \mathbb{R}^2$ and a.e. $t \in [0, 2\pi]$. Suppose moreover that, for any function v: $[0, 2\pi] \to \mathbb{R}$ of the form $v(t) = a \cos nt + b \sin nt$, one has

$$\int_{v>0} \liminf_{\substack{x_1 \to +\infty \\ x_2/x_1 \to 0}} g_1(t, x_1, x_2)v(t)\,\mathrm{d}t + \int_{v<0} \limsup_{\substack{x_1 \to -\infty \\ x_2/x_1 \to 0}} g_1(t, x_1, x_2)v(t)\,\mathrm{d}t > 0, \tag{39}$$

and

$$\int_{v>0} \liminf_{\substack{x_2 \to +\infty \\ x_1/x_2 \to 0}} g_2(t, x_1, x_2)v(t)\,\mathrm{d}t + \int_{v<0} \limsup_{\substack{x_2 \to -\infty \\ x_1/x_2 \to 0}} g_2(t, x_1, x_2)v(t)\,\mathrm{d}t > 0. \tag{40}$$

Then, problem (P′) has a solution.

Proof. Let $H = L^2([0, 2\pi]; \mathbb{R}^2)$,

$$\mathrm{dom}(L) = \{x \in H^2([0, 2\pi]; \mathbb{R}^2) \colon x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0\}.$$

Choose $\gamma' \in]\gamma, 2n + 1[$ and define:

$$Lx = (-x_1'' - n^2 x_1, x_2'' + (n^2 + \gamma')x_2),$$

$$Nx = (g_1(\cdot, x_1(\cdot), x_2(\cdot)), \gamma' x_2(\cdot) - g_2(\cdot, x_1(\cdot), x_2(\cdot))),$$

$$(\Gamma x)u = (\gamma_1(\cdot, x_1(\cdot), x_2(\cdot))u_1(\cdot), [\gamma' - \gamma_2(\cdot, x_1(\cdot), x_2(\cdot))]u_2(\cdot)),$$

$$S = \gamma' I.$$

It is not difficult to verify the conditions (L1″) and (N2′). Moreover, we have

$$\langle(\Gamma x)u, u\rangle = \int_0^{2\pi} \gamma_1(t, x_1(t), x_2(t))u_1^2(t)\,\mathrm{d}t + \int_0^{2\pi} [\gamma' - \gamma_2(t, x_1(t), x_2(t))]u_2^2(t)\,\mathrm{d}t.$$

We notice that each element of $\ker L$ is of the form $(a\cos nt + b\sin nt, 0)$, whereas each element of $\ker(S - L)$ is of the form $(0, a\cos nt + b\sin nt)$. Hence, condition (N2″) is verified, in particular, if there exists $\varepsilon' > 0$ such that, for every $x \in L^2([0, 2\pi]; \mathbb{R}^2)$, either

$$\int_0^{2\pi} \gamma_1(t, x_1(t), x_2(t))u^2(t)\,\mathrm{d}t \geq \varepsilon' \int_0^{2\pi} u^2(t)\,\mathrm{d}t \tag{41}$$

for every u of the form $u(t) = a\cos nt + b\sin nt$, or

$$\int_0^{2\pi} \gamma_2(t, x_1(t), x_2(t))u^2(t)\,\mathrm{d}t \geq \varepsilon' \int_0^{2\pi} u^2(t)\,\mathrm{d}t$$

for every u of the same form. That one of the two branches of the alternative holds follows from (38). In fact, defining the intervals

$$I_1 = \{t \in [0, 2\pi]: \gamma_1(t, x_1(t), x_2(t)) \geq \bar{\varepsilon}/2\},$$

$$I_2 = \{t \in [0, 2\pi]: \gamma_2(t, x_1(t), x_2(t)) \geq \bar{\varepsilon}/2\},$$

one has from (38) that $I_1 \cup I_2 = [0, 2\pi]$, and so one of the two intervals has the measure of at least π. If, for instance, $\operatorname{meas}(I_1) \geq \pi$, it is easy to find an $\varepsilon' > 0$ such that (41) holds.

Once condition (N2″) is proved, one proceeds as in the proof of theorem 4 to show that (39) implies (R1″) and (40) implies (R2″). For that purpose, notice that if for a sequence (x^m), $\|Lx^m\| = O(\|x^m\|^{1/2})$ for $m \to \infty$, one has that $x_2^m/\|x_1^m\|_{L^2}$ converges to zero in $L^2([0, 2\pi]; \mathbb{R})$ and, for a subsequence, the convergence also holds pointwise. Moreover, going if necessary to a subsequence, $x_1^m/\|x_1^m\|_{L^2}$ will converge pointwise to some function of the form $a\cos nt + b\sin nt$. This implies that $x_2^m(t)/x_1^m(t)$ will converge to zero as $n \to \infty$, for a.e. $t \in]0, 2\pi[$. In this way, one can see why the limits in (39) can be taken for a ratio x_2/x_1 going to 0. ■

We simply state now the result one obtains by applying directly theorem 1 to problem (P′), taking $Lx = -x'' - n^2x$, $Nx = g(\cdot, x(\cdot))$, $S = \gamma I$. In this context, we drop condition (38), but on the other hand reinforce conditions (39) and (40), and obtain the following theorem.

THEOREM 8. Assume that, for any function v: $[0, 2\pi] \to \mathbb{R}$ of the form $v(t) = a\cos nt + b\sin nt$, one has

$$\int_{v>0} \liminf_{\substack{x_1 \to +\infty \\ x_2 \in \mathbb{R}}} g_1(t, x_1, x_2)v(t)\,\mathrm{d}t + \int_{v<0} \limsup_{\substack{x_1 \to -\infty \\ x_2 \in \mathbb{R}}} g_1(t, x_1, x_2)v(t)\,\mathrm{d}t > 0, \tag{42}$$

and

$$\int_{v>0} \liminf_{\substack{x_2 \to +\infty \\ x_1 \in \mathbb{R}}} g_2(t, x_1, x_2)v(t)\,\mathrm{d}t + \int_{v<0} \limsup_{\substack{x_2 \to -\infty \\ x_1 \in \mathbb{R}}} g_2(t, x_1, x_2)v(t)\,\mathrm{d}t > 0. \tag{43}$$

Then problem (P′) has a solution.

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