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Nonlinear Evolution Equations Generated from the Bäcklund Transformation for the Boussinesq Equation

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A Bäcklund transformation for the Boussinesq equation is given in the bilinear form. It is shown that the Bäcklund transformation generates an important class of nonlinear evolution equations exhibiting N-soliton solutions. They are a modified Boussinesq equation, a higher order water wave equation introduced by Kaup and a coupled equation whose N-soliton solution reduces to that of the nonlinear Schrödinger equation with normal dispersion. The relation between the Bäcklund transformation and the inverse scattering method is also discussed.

§ 1. Introduction

In recent years it has been shown that a broad class of nonlinear evolution equations has multi-soliton solutions which describe collisions of pulse-like waves.[1] One of the analytical methods to obtain the solutions is the Bäcklund transformation which has its origin in the study of simultaneous differential equations arising in differential geometry.[2] Seeger, Donth and Kochendörfer,[3] and later, Lamb[4] applied the Bäcklund transformation to find multi-soliton solutions of the sine Gordon equation. Wahlquist and Estabrook[5] extended the transformation to the Korteweg-de Vries equation and obtained a hierarchy of solutions which is a family of multi-solitons. After their work the Bäcklund transformations for several nonlinear equations have been found subsequently.[6)~10] It has also been pointed out by many authors that the Bäcklund transformation is closely related to the inverse scattering scheme.[5),6),9)~11]

A new form of Bäcklund transformation was proposed by one of the authors (R.H.).[12] It is written in terms of the transformed variables and new differential operators. An advantage of the form is that the transformation equations are linear with respect to each dependent variable. Examining the Bäcklund transformation, we have found that some new nonlinear evolution equations are generated from it by suitable transformations of the dependent variables.[13)~15] The generated equations have N-soliton solutions under certain conditions. One of the results is that a modified Korteweg-de Vries equation is generated from the Bäcklund transformation for the Korteweg-de Vries equation. Another is that the Bäcklund transformation for the Toda equation generates a nonlinear network equation which reduces, in the special cases, to the nonlinear self-dual network equation, the equation describing a Volterra system and a discrete Korteweg-de Vries equation. These results suggest not only close relationship among several already-known nonlinear evolution equations but also a possibility of extending the class of nonlinear equations exhibiting N-soliton solutions.

In this paper, we consider the following wave equation:

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-3\frac{\partial^2 u^2}{\partial x^2}-\frac{\partial^4 u}{\partial x^4}=0\,, \tag{1·1}$$

which was first presented by Boussinesq[16] to describe the motions of long waves in shallow water (for this reason, we call Eq. (1·1) the Boussinesq equation). This equation is also a continuum approximation to equations for one-dimensional nonlinear lattices.[17] An N-soliton solution of Eq. (1·1) has been obtained by the dependent variable transformation method.[18] It has also been shown that the inverse scattering scheme may be applied to Eq. (1·1).[19),20]

In § 2 we rewrite the Boussinesq equation in bilinear forms through a variable transformation. Then we give a Bäcklund transformation for the resultant equation. It is shown in § 3 that an important class of nonlinear equations can be produced from the Bäcklund transformation. New variables are introduced for this purpose and the Bäcklund transformation is rewritten to express nonlinear evolution equations. They are a modified Boussinesq equation, a higher order water wave equation introduced by Kaup and a coupled equation whose N-soliton solution reduces to that of the nonlinear Schrödinger equation with normal dispersion. It is also shown that the inverse scattering scheme for the Boussinesq equation is reproduced from the Bäcklund transformation. In § 4 we show one- and two-soliton solutions of the equations generated from the Bäcklund transformation using a kind of perturbational technique. Finally in § 5 we show an N-soliton solution of the equations.

§ 2. Bäcklund transformation for the Boussinesq equation

In order to find a Bäcklund transformation for Eq. (1·1), we rewrite Eq. (1·1) in a bilinear form with a dependent variable transformation. The bilinear form was first used to obtain an N-soliton solution for the Boussinesq equation.[18]

Let f be defined by

$$u=2\frac{\partial^2}{\partial x^2}(\log f)\,. \tag{2·1}$$

Substituting Eq. (2·1) into Eq. (1·1) and integrating twice with respect to x, we have

$$(D_t^2-D_x^2-D_x^4)f\cdot f=0\,, \tag{2·2}$$

where we impose a boundary condition $u=0$ at $|x|=\infty$. The operators D_t and D_x in Eq. (2·2) are defined by

$$D_t^n D_x^m a(t,x)\cdot b(t,x)$$
$$\equiv\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m a(t,x)\,b(t',x')\Big|_{\substack{t=t'\\x=x'}}. \qquad (2\cdot3)$$

These operators are convenient to deal with soliton problems. Some properties of them are listed in previous papers.[12),14),21)]

We propose a Bäcklund transformation for Eq. (2·2). It is written as

$$(D_t+aD_x^2)f\cdot f'=0\,, \qquad (2\cdot4)$$

$$(aD_tD_x+D_x+D_x^3)f\cdot f'=0\,, \qquad (2\cdot5)$$

where a is a parameter introduced for the following discussion. This Bäcklund transformation relates pairs of solutions of the Boussinesq equation, that is, if f is some solution of Eq. (2·2), f' defined by Eqs. (2·4) and (2·5) could be another solution of the same equation. We see that the transformation equations (2·4) and (2·5) are linear with respect to f or f'.

Now we verify that if f is any solution of Eq. (2·2), f' defined by Eqs. (2·4) and (2·5) satisfies Eq. (2·2). It is easily shown (see Ref. 14)) that the following identities hold for the operators D_t and D_x:

$$(D_x^2 f\cdot f)f'f'-ff(D_x^2 f'\cdot f')=2D_x(D_x f\cdot f')\cdot ff'\,, \qquad (2\cdot6)$$

$$(D_x^4 f\cdot f)f'f'-ff(D_x^4 f'\cdot f')$$
$$=2D_x(D_x^3 f\cdot f')\cdot ff'-6D_x(D_x^2 f\cdot f')\cdot(D_x f\cdot f')\,, \qquad (2\cdot7)$$

$$D_t(D_x^2 f\cdot f')\cdot ff'=D_x[(D_xD_t f\cdot f')\cdot ff'+(D_t f\cdot f')\cdot(D_x f\cdot f')]. \qquad (2\cdot8)$$

Making use of Eqs. (2·6) and (2·7), we have

$$P\equiv[(D_t^2-D_x^2-D_x^4)f\cdot f]f'f'-ff[(D_t^2-D_x^2-D_x^4)f'\cdot f']$$
$$=2\{D_t(D_t f\cdot f')\cdot ff'-D_x(D_x f\cdot f'+D_x^3 f\cdot f')\cdot ff'$$
$$+3D_x(D_x^2 f\cdot f')\cdot(D_x f\cdot f')\}.$$

Substituting Eqs. (2·4) and (2·5) into the above equation and noticing the identity (2·8), we obtain

$$P=-2D_x\{(aD_t-3D_x^2)f\cdot f'\}\cdot(D_x f\cdot f')\,,$$

which vanishes for $a^2=-3$ by virtue of Eq. (2·4). Thus we have proved that Eqs. (2·4) and (2·5) with $a^2=-3$ constitute a Bäcklund transformation for Eq. (2·2).

§ 3. Equations generated from the Bäcklund transformation

The Bäcklund transformation obtained in § 2 is written in the bilinear forms. We have developed a method of finding N-soliton solutions of the equations which can be written in bilinear forms.[14)] Thus it seems quite reasonable to expect that

Eqs. (2·4) and (2·5) are the bilinear forms of a certain equation and the original equation has an N-soliton solution. In fact, several nonlinear equations are generated from Eqs. (2·4) and (2·5) by suitable dependent variable transformations. As is shown in the following section, Eqs. (2·4) and (2·5) yield N-soliton solutions for $a^2=-3$ or 1. So we show here that for these values of a, Eqs. (2·4) and (2·5) produce some nonlinear evolution equations depending on the types of dependent variable transformations. It is to be noted that Eqs. (2·4) and (2·5) are not the Bäcklund transformation for the Boussinesq equation in the case of $a^2=1$.

First we introduce the following new variables:

$$\phi=\log(f'/f), \tag{3·1}$$

$$\rho=\log(ff'). \tag{3·2}$$

All terms of Eqs. (2·4) and (2·5) can be written by these variables.[14] Then Eqs. (2·4) and (2·5) reduce to[6]

$$\phi_t-a[\rho_{xx}+(\phi_x)^2]=0\,, \tag{3·3}$$

$$a(\rho_{xt}+\phi_x\phi_t)-\phi_x-\phi_{xxx}-3\phi_x\rho_{xx}-(\phi_x)^3=0\,, \tag{3·4}$$

where suffixes denote partial differentiations.

3-1 *A modified Boussinesq equation*

For $a=\sqrt{3}\,i$, eliminating ρ in Eqs. (3·3) and (3·4) and introducing a new variable v by $v=i\phi$, we have

$$v_{tt}-v_{xx}-6v_x^2v_{xx}+2\sqrt{3}\,v_{xx}v_t-v_{xxxx}=0\,. \tag{3·5}$$

Equation (3·5) may be considered as a modified Boussinesq equation which corresponds to the modified Korteweg-de Vries equation generated from the Bäcklund transformation for the Korteweg-de Vries equation.[14),15]

3-2 *Miura's transformation*

We show that one of the Bäcklund transformations, Eq. (2·4), reduces to a nonlinear transformation between Eqs. (3·5) and (1·1), which corresponds to the Miura transformation which transforms a solution of the modified Korteweg-de Vries equation to that of the Korteweg-de Vries equation.[22]

We have identities (see Ref. 14))

$$(D_x^2f'\cdot f)/f'f=[\log(f'f)]_{xx}+\{[\log(f'/f)]_x\}^2$$
$$=2(\log f)_{xx}+[\log(f'/f)]_{xx}+\{[\log(f'/f)]_x\}^2\,, \tag{3·6}$$

$$(D_tf'\cdot f)/f'f=[\log(f'/f)]_t\,. \tag{3·7}$$

Substituting Eqs. (3·6) and (3·7) into Eq. (2·4) and using the relations $u=(2\log f)_{xx}$ and $v=i\log(f'/f)$, we obtain

$$u=iv_{xx}+v_x^2-ia^{-1}v_t\,, \tag{3·8}$$

which is a kind of Miura transformation.

For $a=i\sqrt{3}$, we find that u defined by Eq. (3·8) satisfies the Boussinesq equation if v satisfies the modified Boussinesq equation (3·5). In fact, substitution of Eq. (3·8) into Eq. (1·1) gives

$$u_{tt}-u_{xx}-3(u^2)_{xx}-u_{xxxx}$$
$$=\left(i\frac{\partial^2}{\partial x^2}-ia^{-1}\frac{\partial}{\partial t}+2\frac{\partial}{\partial x}v_x\right)(v_{tt}-v_{xx}-6v_x{}^2v_{xx}+2\sqrt{3}v_{xx}v_t-v_{xxxx}). \quad (3\cdot 9)$$

3-3 *Higher order water wave equation*

For $a=1$, Eqs. (3·3) and (3·4) are rewritten as

$$\pi=\Phi_t+\epsilon(\Phi_x)^2/2, \quad (3\cdot 10)$$

$$\pi_t=\Phi_{xx}+\beta^2\Phi_{xxxx}-\epsilon(\Phi_x\pi)_x, \quad (3\cdot 11)$$

by the transformations

$$\left.\begin{aligned}&\rho_{xx}\to-\epsilon\pi/2,\\&\phi\to-\epsilon\Phi/2\beta,\\&x\to x/\beta,\\&t\to t/\beta.\end{aligned}\right\} \quad (3\cdot 12)$$

A set of Eqs. (3·10) and (3·11) is a higher order water wave equation introduced by Kaup.[23] He found that this equation is derivable from the shallow water wave equation and solvable exactly by the inverse scattering method.

3-4 *Inverse scattering scheme for the Boussinesq equation*

Another choice for the new variables is

$$\psi=f'/f, \quad (3\cdot 13)$$

$$w=2(\log f)_x. \quad (3\cdot 14)$$

By the use of these variables, Eqs. (2·4) and (2·5) are expressed as

$$\psi_t-a(\psi_{xx}+w_x\psi)=0, \quad (3\cdot 15)$$

$$a(\psi_{xt}+w_t\psi)-\psi_x-\psi_{xxx}-3w_x\psi_x=0. \quad (3\cdot 16)$$

For $a=i\sqrt{3}$, Eqs. (3·15) and (3·16) reduce to the inverse scattering scheme for the Boussinesq equation discovered by Zakharov.[19] In fact, if we choose the operators L and A as

$$L=4\frac{\partial^3}{\partial x^3}+3\left(w_x\frac{\partial}{\partial x}+\frac{\partial}{\partial x}w_x\right)+\frac{\partial}{\partial x}-\sqrt{3}iw_t, \quad (3\cdot 17)$$

$$A=\sqrt{3}i\left(\frac{\partial^2}{\partial x^2}+w_x\right), \quad (3\cdot 18)$$

Eqs. (3·16) and (3·15) with $a=i\sqrt{3}$ are written as

$$L\psi=0\,, \tag{3·19}$$

$$\psi_t=A\psi\,, \tag{3·20}$$

respectively, and the equation

$$L_t=AL-LA \tag{3·21}$$

reduces to the Boussinesq equation (1·1) noticing that $u=w_x$.

It is noted that in our case Eq. (3·17) contains no parameters and so it does not form an eigenvalue problem.

3-5 *Relation to nonlinear Schrödinger equation with normal dispersion*

Next we consider the case $a=1$. Then we obtain from Eqs. (3·15) and (3·16)

$$i\psi_t-\psi_{xx}-iw_x\psi=0\,, \tag{3·22}$$

$$iw_t+w_{xx}-i(1-2iw_x)\psi_x/\psi=0\,, \tag{3·23}$$

where the independent variables are transformed as $t\to it$ and $x\to ix$. Equations (3·22) and (3·23) are a coupled equation whose N-soliton solution reduces to that of the nonlinear Schrödinger equation with normal dispersion,[21),24)]

$$i\psi_t-\psi_{xx}+\tfrac{1}{2}(|\psi|^2-1)\psi=0\,. \tag{3·24}$$

When we impose a conjecture

$$1-2iw_x=|\psi|^2\,, \tag{3·25}$$

Eq. (3·22) reduces to Eq. (3·24), and Eq. (3·23) to

$$i(|\psi|^2)_t-(\psi_{xx}\psi^*-\psi\psi^*_{xx})=0\,,$$

which are also an alternative expression of Eq. (3·24). Here the asterisk denotes complex conjugate. The conjecture (3·25) is confirmed by considering the bilinear form of Eq. (3·24). Equation (3·24) is transformed into a couple of bilinear equations

$$(iD_t-D_x^2)f'\cdot f=0\,, \tag{3·26}$$

$$f^2-2D_x^2f\cdot f=f'f'^*\,, \tag{3·27}$$

through a variable transformation, $\psi=f'/f$ with real f.[21)] The former equation is equivalent to Eq. (3·22) and the latter to Eq. (3·25). Thus, considering the procedure to obtain N-soliton solutions, we see that a coupled equation (3·22) and (3·23) and Eq. (3·24) have the same N-soliton solution as that for ψ.

§ 4. One- and two-soliton solutions

In the preceding section, we have shown that several nonlinear evolution equations are generated from the Bäcklund transformation for the Boussinesq equation. Reversing the procedure generating those equations, we find that they can be reduced to the bilinear forms, Eqs. (2·4) and (2·5), by suitable variable transformations. Here we obtain one- and two-soliton solutions of the generated equations using a kind of perturbational technique on Eqs. (2·4) and (2·5).

We expand f and f' in Eqs. (2·4) and (2·5) as power series in a parameter ϵ,

$$\left.\begin{aligned} f&=1+\epsilon f_1+\epsilon^2 f_2+\cdots,\\ f'&=1+\epsilon f_1'+\epsilon^2 f_2'+\cdots. \end{aligned}\right\} \qquad (4\cdot 1)$$

Substituting Eq. (4·1) into Eqs. (2·4) and (2·5) and equating the terms with the same powers in ϵ, we have in the order of ϵ

$$\left.\begin{aligned} &(f_1-f_1')_t+a(f_1+f_1')_{xx}=0\,,\\ &a(f_1+f_1')_{xt}+(f_1-f_1')_x+(f_1-f_1')_{xxx}=0\,, \end{aligned}\right\} \qquad (4\cdot 2)$$

and, in the order of ϵ^2,

$$\left.\begin{aligned} &(f_2-f_2')_t+a(f_2+f_2')_{xx}=-(D_t+aD_x^2)f_1\cdot f_1'\,,\\ &a(f_2+f_2')_t+(f_2-f_2')_x+(f_2-f_2')_{xxx}=-(aD_tD_x+D_x+D_x^3)f_1\cdot f_1' \end{aligned}\right\} \qquad (4\cdot 3)$$

and so on. Equation (4·2) is linear and can be easily solved. Equations in the higher order of ϵ are linear equations with the known inhomogeneous terms. We can find particular solutions solving these equations successively.

The simplest nontrivial solution is obtained by choosing the following set of starting solution of Eq. (4·2):

$$f_1=\exp\eta\,, \qquad (4\cdot 4)$$

$$f_1'=\exp(\eta+\theta), \qquad (4\cdot 5)$$

$$\eta=Px-\Omega t+\eta^0\,, \qquad (4\cdot 6)$$

where P and η^0 are arbitrary constants. Substitution of Eqs. (4·4)~(4·6) into Eq. (4·2) gives the dispersion relation, $\Omega^2=P^2(1+P^2)$, and the phase factor, $\exp\theta=(\Omega-aP^2)/(\Omega+aP^2)$. For this starting solution all higher order terms in Eq. (4·1) can be taken to be zero. Thus we have a solution for Eqs. (2·4) and (2·5),

$$f=1+\exp\eta\,, \qquad (4\cdot 7)$$

$$f'=1+\exp(\eta+\theta), \qquad (4\cdot 8)$$

which gives one-soliton solution of the equations generated from Eqs. (2·4) and (2·5). We note that this solution satisfies Eqs. (2·4) and (2·5) for arbitrary a.

In order to obtain a two-soliton solution, we start with the following solution of Eq. (4·2):

$$f_1=\exp\eta_1+\exp\eta_2\,, \qquad (4\cdot 9)$$

$$f_1' = \exp(\eta_1+\theta_1)+\exp(\eta_2+\theta_2), \tag{4·10}$$

where

$$\eta_i = P_i x - \Omega_i t + \eta_i^0, \tag{4·11}$$

$$\Omega_i^2 = P_i^2(1+P_i^2), \tag{4·12}$$

$$\exp\theta_i = (\Omega_i - aP_i^2)/(\Omega_i + aP_i^2), \tag{4·13}$$

with P_i and η_i^0 being arbitrary constants. Substituting Eqs. (4·9) and (4·10) into the right-hand side of Eq. (4·3) and solving the resulting equation, we find the terms in the order of ϵ^2 in Eq. (4·1). They are given by

$$f_2 = \exp(\eta_1+\eta_2+A_{12}), \tag{4·14}$$

$$f_2' = \exp(\eta_1+\eta_2+\theta_1+\theta_2+A_{12}), \tag{4·15}$$

$$\exp A_{12} = \frac{(P_1\Omega_2-P_2\Omega_1)^2 - a^2P_1^2P_2^2(P_1-P_2)^2}{(P_1\Omega_2-P_2\Omega_1)^2 - a^2P_1^2P_2^2(P_1+P_2)^2}, \tag{4·16a}$$

$$\exp A_{12} = \frac{(P_1-P_2)[P_2\{(a^2-1)P_1+2P_2\}\Omega_1 + P_1\{-(a^2-1)P_2-2P_1\}\Omega_2]}{(P_1+P_2)[P_2\{(a^2-1)P_1-2P_2\}\Omega_1 + P_1\{(a^2-1)P_2-2P_1\}\Omega_2]}. \tag{4·16b}$$

Equations (4·16a) and (4·16b) are compatible only for $a^2=-3$ or 1. All terms higher than ϵ^2 in Eq. (4·1) can be taken to be zero for this solution and we obtain a solution associated with a two-soliton solution

$$f = 1+\exp\eta_1+\exp\eta_2+\exp(\eta_1+\eta_2+A_{12}), \tag{4·17}$$

$$f' = 1+\exp(\eta_1+\theta_1)+\exp(\eta_2+\theta_2)+\exp(\eta_1+\eta_2+\theta_1+\theta_2+A_{12}). \tag{4·18}$$

§ 5. *N*-soliton solution

The form of two-soliton solutions (4·17) and (4·18) suggests to us a form of possible N-soliton solution. Here we show a priori a solution associated with the N-soliton solution and prove, by mathematical induction, that it satisfies Eqs. (2·4) and (2·5). We shall show in the following that the N-soliton solution, as well as the two-soliton solution, satisfies Eqs. (2·4) and (2·5) only when $a^2=-3$ or 1.

The solution is expressed as

$$f = \sum_{\mu=0,1} \exp\Big[\sum_{i<j}^{(N)} A_{ij}\mu_i\mu_j + \sum_{i=1}^{N}\mu_i\eta_i\Big], \tag{5·1}$$

$$f' = \sum_{\mu=0,1} \exp\Big[\sum_{i<j}^{(N)} A_{ij}\mu_i\mu_j + \sum_{i=1}^{N}\mu_i(\eta_i+\theta_i)\Big], \tag{5·2}$$

where

$$\eta_i = P_i x - \Omega_i t + \eta_i^0, \tag{5·3}$$

$$\Omega_i = \epsilon_i P_i\sqrt{1+P_i^2} \quad (\epsilon_i = \pm 1), \tag{5·4}$$

$$\exp\theta_i=(\Omega_i-aP_i^2)/(\Omega_i+aP_i^2), \tag{5·5}$$

$$\exp A_{ij}=\frac{(P_i\Omega_j-P_j\Omega_i)^2-a^2P_i^2P_j^2(P_i-P_j)^2}{(P_i\Omega_j-P_j\Omega_i)^2-a^2P_i^2P_j^2(P_i+P_j)^2}\,. \tag{5·6}$$

Here P_i and η_i^0 are constants, $\sum_{\mu=0,1}$ is the summation over all possible combinations of $\mu_1=0,\ 1,\ \mu_2=0,\ 1,\ \cdots,\ \mu_N=0,\ 1$, and $\sum_{i<j}^{(N)}$ is the summation over all possible pairs chosen from N elements.

Substituting Eqs. (5·1) and (5·2) into Eq. (2·4), we obtain

$$\sum_{\mu=0,1}\sum_{\nu=0,1}[-\sum_{i=1}^{N}(\mu_i-\nu_i)\Omega_i+a\{\sum_{i=1}^{N}(\mu_i-\nu_i)P_i\}^2]$$
$$\times\exp[\sum_{i>j}^{(N)}A_{ij}(\mu_i\mu_j+\nu_i\nu_j)+\sum_{i=1}^{N}(\mu_i+\nu_i)\eta_i+\sum_{i=1}^{N}\nu_i\theta_i]=0\,. \tag{5·7}$$

Let the coefficient of the factor $\exp(\sum_{i=1}^{n}\eta_i+\sum_{i=n+1}^{m}2\eta_i)$ in Eq. (5·7) be $D_1(1,\ 2,\ \cdots,n;n+1,n+2,\ \cdots,m)$. Then we have

$$D_1=\sum_{\mu=0,1}\sum_{\nu=0,1}\mathrm{cond}(\mu,\nu)[-\sum_{i=1}^{n}(\mu_i-\nu_i)\Omega_i+a\{\sum_{i=1}^{n}(\mu_i-\nu_i)P_i\}^2]$$
$$\times\exp[\sum_{i=1}^{m}\nu_i\theta_i+\sum_{i<j}^{(N)}A_{ij}(\mu_i\mu_j+\nu_i\nu_j)], \tag{5·8}$$

where cond (μ,ν) implies that the summation over μ and ν should be performed under the conditions

$$\left.\begin{array}{lll}\mu_i+\nu_i=1 & \text{for} & i=1,2,\cdots,n\,,\\ \mu_i=\nu_i=1 & \text{for} & i=n+1,n+2,\cdots,m\,,\\ \mu_i=\nu_i=0 & \text{for} & i=m+1,m+2,\cdots,N\,.\end{array}\right\}$$

Substituting Eqs. (5·4)~(5·6) into Eq. (5·8) and introducing $\sigma_i=\mu_i-\nu_i$, we find

$$D_1(1,2,\cdots,n;n+1,n+2,\cdots,m)=\mathrm{const}\,\widehat{D}_1(P_1,P_2,\cdots,P_n), \tag{5·9}$$

where

$$\widehat{D}_1=\sum_{\sigma=\pm1}[-\sum_{i=1}^{n}\epsilon_i\sigma_iP_iv_i+a(\sum_{i=1}^{n}\sigma_iP_i)^2]\prod_{i=1}^{n}(\epsilon_iv_i+a\sigma_iP_i)$$
$$\times\prod_{i<j}^{(n)}[-(\epsilon_iv_i-\epsilon_jv_j)^2+a^2(\sigma_iP_i-\sigma_jP_j)^2], \tag{5·10}$$

where $v_i=\sqrt{1+P_i^2}$ and $\prod_{i<j}^{(n)}$ indicates the product of all possible combinations of the n elements. Thus, Eqs. (5·1) and (5·2) would be a solution of Eq. (2·4) if the following identity holds:

$$D_1(P_1,P_2,\cdots,P_n)=0 \ \text{for}\ n=1,2,\cdots,N. \tag{5·11}$$

Similarly we can show that Eqs. (5·1) and (5·2) would be a solution of Eq. (2·5) if

$$\widehat{D}_2(p_1, p_2, \cdots, p_n)$$
$$= \sum_{\sigma=\pm 1} (\sum_{i=1}^{n} \sigma_i p_i) [-a \sum_{i=1}^{n} \epsilon_i \sigma_i P_i v_i + 1 + (\sum_{i=1}^{n} \sigma_i P_i)^2]$$
$$\times \prod_{i=1}^{n} (\epsilon_i v_i + a\sigma_i P_i) \prod_{i<j}^{(n)} [-(\epsilon_i v_i - \epsilon_j v_j)^2 + a^2(\sigma_i P_i - \sigma_j P_j)^2] \quad (5\cdot 12)$$

holds for $n=1, 2, \cdots, N$.

The identities (5·11) and (5·12) can be proved by mathematical induction used in the previous papers (they are cited in Ref. 14)). $\widehat{D}_1$ and $\widehat{D}_2$ are symmetric and even functions of $P_1, P_2, \cdots, P_n$ and hence, if we consider $\widehat{D}_1$ and $\widehat{D}_2$ to be functions of $\epsilon_1 v_1$, $\epsilon_2 v_2$, $\cdots$, $\epsilon_n v_n$, they become polynomials of variables $\widehat{v}_1, \widehat{v}_2, \cdots, \widehat{v}_n$ where $\widehat{v}_i = \epsilon_i v_i$. $\widehat{D}_j$ for $j=1, 2$ has the following properties:

(i) $\widehat{D}_j$ are symmetric polynomials of $\widehat{v}_1, \widehat{v}_2, \cdots, \widehat{v}_n$,

(ii) $$\widehat{D}_j|_{\widehat{v}_1=\pm 1} = \pm \prod_{i=2}^{(n)} [-(1\mp\widehat{v}_i)^2 + a^2 P_i^2] \widehat{D}_j(\widehat{v}_2, \widehat{v}_3, \cdots, \widehat{v}_n), \quad (5\cdot 13)$$

(iii) $$\widehat{D}_j|_{\widehat{v}_1=\widehat{v}_2} = 4a^2 P_1^2 (\widehat{v}_1^2 - a^2 P_1^2) \prod_{i=3}^{n} [\{-(\widehat{v}_1 - \widehat{v}_i)^2 + a^2(P_1 - P_i)^2\}$$
$$\times \{-(\widehat{v}_1 - \widehat{v}_i)^2 + a^2(P_1 + P_i)^2\}] \widehat{D}_j(\widehat{v}_3, \widehat{v}_4, \cdots, \widehat{v}_n). \quad (5\cdot 14)$$

The identities (5·11) and (5·12) are easily verified for $n=1$. It is also shown that the identities hold for $n=2$ under the condition $a^2=-3$ or 1. Moreover, by a tedious but straightforward calculation, we can prove that the identity (5·12) holds for $n=3$ under the same value of a for $n=2$. Now we assume that the identities hold for $n-1$ and $n-2$. Then, relying on properties (i), (ii) and (iii), we see that $\widehat{D}_1$ and $\widehat{D}_2$ can be factored by a polynomial

$$\prod_{i<j}^{(n)} (\widehat{v}_i - \widehat{v}_j)^2 \prod_{i=1}^{n} (\widehat{v}_i^2 - 1)$$

of degree $n(n-1)+2n$. On the other hand, we find that $\widehat{D}_1$ is a polynomial of $\widehat{v}_1$, $\widehat{v}_2$, $\cdots$, $\widehat{v}_n$ of degree $n(n-1)+n+2$ and $\widehat{D}_2$ of degree $n(n-1)+n+3$ at most. Hence $\widehat{D}_1$ and $\widehat{D}_2$ must vanish for n. Thus we have proved that Eqs. (5·1) and (5·2) satisfy Eqs. (2·4) and (2·5) under the condition $a^2=-3$ or 1.

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