Nonlinear Expectations,
Noninear Evaluations
and Risk Measures

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Chapter 1

INTRODUCTION

1.1 Searching the mechanism of evaluations of risky assets

We are interested in the following problem: let X_t $t \in [0, T]$ be an $\mathbf{R}^d$–valued process, Y a random value depending on the trajectory of X. Assume that, at each fixed time $t \leq T$, the information available to an agent (an individual, a firm, or even a market) is the trajectory of X before t. Thus at time T, the random value of $Y(\omega)$ will become known to this agent. The question is: how this agent evaluates Y at the time t? If this Y is traded in a financial market, it is called a derivative, i.e., a contract whose outcome depends on the evolution of the underlying process X. This output of this evaluation can be the maximum value the agent can be accepted to buy or minimum value to sell. It depends his economic situation, his risk aversion and utility preference. In many situation this individual evaluation may be very different from the actual market price.

Examples of derivatives are futures and option contracts based on the underlying asset X such as commodities, stocks indexes, interest rates, exchange rates; or on individual stocks; or on mortgage backed securities. Here the term derivative is in general sense, i.e., it may be a positive or a negative number.

The well–known Black & Scholes option pricing theory (1973) has made the most significant contribution, over the last 30 years, of the model of evaluation by a financial market of the derivatives.

One of the important limitations of Black–Scholes-Merton approach is that it is heavily based on the assumption that the statistic behavior of the stochastic process X is exogenously specified. The fact that the Black–Scholes pricing of Y is independent of the preference of the involved individuals are also frequently argued. On the other hand, in the situation where Y is not traded, thus the main arguments of BS model, i.e., replicating and arbitrage–free, are no longer viable, the evaluation of Y is often preference–dependent.

In this paper the evaluation of Y will be treated in a new viewpoint. We will introduce an evaluation operator $\mathcal{E}_{t,T}[Y]$ to define the evaluated value of Y of the agent at time t. This operator $\mathcal{E}_{t,T}[\cdot]$ assigns an $(X_s)_{0 \leq s \leq T}$–dependent random variable Y to an $(X_s)_{0 \leq s \leq t}$–dependent one $\mathcal{E}_{t,T}[Y]$. Although this value $\mathcal{E}_{t,T}[Y]$ is very complicated and is different from one agent to anther, we can still find some axiomatic conditions to describe the mathematical mechanics of this operator. In

many situation, the evaluation of the discounted Y is treated as a filtration consistent
nonlinear expectation. In more general situation it is a filtration consistent nonlinear
evaluation. In these notes we will prove that in many situations In some situations,
this evaluation coincides with a g–expectation, or more general, g-evaluation, which
is the solution of a 1–dimensional backward stochastic differential equation (BSDE)
with a given function g as its generator.

1.2   Axiomatic Assumptions for evaluations of derivatives

1.2.1   General situations

We give a more mathematical formulation to the above described evaluation problem.
Let $X = (X_t)_{t \geq 0}$ be a d–dimensional process, it may be the prices of stocks in a
financial market, the rates of exchanges, the rates of local and global inflations etc.
For simplification, we assume that X is a continuous process, i.e., the realization
trajectory of X is in the space of $\mathbf{R}^d$–valued continuous processes starting from
$X_0 = x \in \mathbf{R}^d$, i.e., $X \in \mathbf{W}_x^d := C_x(0, \infty; \mathbf{R}^d)$. We assume that at each time $t \geq 0$,
the information for of an agent (a firm, a group of people, a financial market) is the
history of X during the time interval $[0, t]$. Namely, his actual filtration is

$$\mathcal{F}_t^X = \sigma\{X_s; s \leq t\}.$$

We denote the set of all real valued $\mathcal{F}_t^X$–measurable random variables by $m\mathcal{F}_t^X$. Under
this notation an X–underlying derivative Y, with maturity $T \in [0, \infty)$, is an $\mathcal{F}_T^X$–
measurable random variable, i.e., $Y \in m\mathcal{F}_T^X$. We will find the law of evaluation of Y
at each time $t \in [0, T]$. We denote this evaluated value by $\mathcal{E}_{t,T}[Y]$. It is reasonable to
assume that $\mathcal{E}_{t,T}[Y]$ is $\mathcal{F}_t^X$–measurable. In other words

$$\mathcal{E}_{t,T}[Y] : m\mathcal{F}_T^X \longrightarrow m\mathcal{F}_t^X.$$

In particular

$$\mathcal{E}_{0,T}[Y] : m\mathcal{F}_T^X \longrightarrow \mathbf{R}.$$

We will make the following **Axiomatic Conditions** for $(\mathcal{E}_{t,T}[\cdot])_{0 \leq t \leq T < \infty}$:

(i) **Monotonicity**: $\mathcal{E}_{t,T}[Y] \geq \mathcal{E}_{t,T}[Y']$, if $Y \geq Y'$.

(ii) $\mathcal{E}_{t,t}[Y] = Y$, if $Y \in m\mathcal{F}_t^X$. Particularly $\mathcal{E}_{0,0}[c] = c$.

(iii) "**Zero–one law**": for each $t \leq T$, $\mathcal{E}_{t,T}[1_A Y] = 1_A \mathcal{E}_{t,T}[Y]$, $\forall A \in \mathcal{F}_t^X$.

(iv) **Time consistent**: $\mathcal{E}_{s,t}[\mathcal{E}_{t,T}[Y]] = \mathcal{E}_{s,T}[Y]$, if $s \leq t \leq T$.

Remark 1 *Conditions (i) and (ii) are obvious. The meaning of condition (iii) is: at
time t, the agent knows whether $X_{\cdot \wedge t}$ is in A. If $X_{\cdot \wedge t}$ is in A, then the value $\mathcal{E}_{t,T}[1_A Y]$*

is the same as $\mathcal{E}_{t,T}[Y]$ since will have the same right as to have a derivative Y with the maturity T. Otherwise $1_A Y$ is zero thus it costs nothing.

Remark 2 *Condition (iv) means that $\mathcal{E}_{t,T}[Y]$ can be also treated as a derivative with the maturity t. At a time $s \leq t$, the price $\mathcal{E}_{s,t}[\mathcal{E}_{t,T}[Y]]$ of this derivative is the same as the price of the original derivative Y with maturity T, i.e., $\mathcal{E}_{s,T}[Y]$.*

1.2.2   Nonlinear evaluations

In many situations we assume furthermore, instead of **(ii)** in the Axiomatic Assumption, that
   (ii') $\mathcal{E}_{t,T}[Y] = Y$, if $Y \in m\mathcal{F}_t^X$. Particularly $\mathcal{E}_{0,T}[c] = c$.

Remark 3 *Condition (ii') implies that the market has a zero–interesting rate, i.e., $r_t \equiv 0$. We observe that this assumption is not too strong since, in the case $r_t \not\equiv 0$, we can define the following discounted evaluation*

$$\mathcal{E}_{t,T}'[Y] := \mathcal{E}_{t,T}[Y \exp(-\int_t^T r_s ds)],$$

This $\mathcal{E}_{t,T}'[\cdot]$ satisfies (ii'). In some more complicated situation we should consider a "nonlinear discount effect".

   In this case it is clear that, for each $0 \leq t \leq T \leq T' < \infty$ and $Y \in L^2(\Omega, \mathcal{F}_T^X, P; \mathbf{R})$, we have

$$\mathcal{E}_{t,T'}[Y] = \mathcal{E}_{t,T}[\mathcal{E}_{T,T'}[Y]] = \mathcal{E}_{t,T}[Y].$$

We then can set $\mathcal{E}[Y|\mathcal{F}_t^X] := \mathcal{E}_{t,T}[Y] = \lim_{T'\to\infty} \mathcal{E}_{t,T'}[Y]$. $\mathcal{E}[Y] := \mathcal{E}[Y|\mathcal{F}_0^X]$.

$$\mathcal{E}[Y|\mathcal{F}_t^X] \quad : \quad m\mathcal{F}_T^X \to m\mathcal{F}_t^X,$$
$$\mathcal{E}[Y] \quad : \quad m\mathcal{F}_T^X \to \mathbf{R}.$$

$\mathcal{E}[Y|\mathcal{F}_t^X]$ is the agent's price at time t of the derivative $Y \in m\mathcal{F}_T^X$ with any maturity $T' \geq T$.
   By the Axiomatic Assumptions, we have

(i) Monotonicity: $\mathcal{E}[Y|\mathcal{F}_t^X] \geq \mathcal{E}[Z|\mathcal{F}_t^X]$, if $Y \geq Z$.

(ii') Constant–preserving: $\mathcal{E}[Y|\mathcal{F}_t^X] = Y$, if $Y \in m\mathcal{F}_t^X$.
(iii) "Zero–one law": for each $t \leq T$, $\mathcal{E}[1_A Y|\mathcal{F}_t^X] = 1_A \mathcal{E}[Y|\mathcal{F}_t^X], \quad \forall A \in \mathcal{F}_t^X$.
(iv) Time consistent: $\mathcal{E}_s[\mathcal{E}[Y|\mathcal{F}_t^X]|\mathcal{F}_s^X] = \mathcal{E}[Y|\mathcal{F}_s^X]$, if $s \leq t \leq T$.

In particular, the functional $\mathcal{E}[\cdot]$ is a nonlinear expectation, i.e., it satisfies

(a) Monotonicity: $\mathcal{E}[Y] \geq \mathcal{E}[Z]$, if $Y \geq Z$.

(b) Constant–preserving: $\mathcal{E}[c] = c$.

From (iii) and (iv) we have, each $0 \leq T < \infty$ and $Y \in m\mathcal{F}_T^X$,

$$\mathcal{E}[1_A\mathcal{E}[Y|\mathcal{F}_t^X]] = \mathcal{E}[1_AY], \ \forall A \in \mathcal{F}_t^X. \tag{1.1}$$

We recall that this is just the classical definition of the conditional expectation given $\mathcal{F}_t^X$. In the next section we will prove that in nonlinear situation we can also derive all the Axiomatic Conditions (i), (ii'), (iii) and (iv) by this definition provided $\mathcal{E}$ is strictly monotone. In this case we call $\mathcal{E}[\cdot]$ an $\mathcal{F}_t^X$–**consistent nonlinear expectation**.

In this case the $(\mathcal{E}_{t,T}[\cdot])_{0\leq t \leq T<\infty}$ is no more a family of nonlinear expectations. But we will see that it is a "stochastic backward semigroup".

Remark 4 *From the above reasoning it is clear that the axiomatic Conditions (i)–(iv) are also applied in many other situations to measuring a risky values Y in a dynamical situation. In fact, an advantage is that they are also workable in the situation where the risky value Y does not exchanged in a market. In fact the results of many decisions are not exchangeable. For example, it is applicable to an individual or a group's evaluating of a derivative Y. In some situation an agent can not have all information $\mathcal{F}_t^X$, but our method is applied to the situation partially observation, i.e., with a smaller filtration $\mathcal{G}_t \subset \mathcal{F}_t$, $t \geq 0$.*

Remark 5 *It is clear that formulation of an $\mathcal{F}^X$–consistent evaluation does not need to introduce an a priori probability space. But in this notes we will be within the frmework Brownian motion filtration. For more general situation, see [P2003].*

1.3   Organization of the notes

In the next chapter, we will give the formulations of filtration consistent evaluations and expectations under the Brownian motion's framework. Then in Chapter 3, we present BSDE theory and introduce a large sort of filtration consistent nonlinear evaluations and expectations, i.e., g–evaluations and g–expectations. We also present a nonlinear decomposition theorem of Doob–Meyer's type, under these g–expectations and $g$evaluations. Chapter 4 is devoted to prove that the notion of g–expectations is large enough to represent all regular $\mathcal{F}_t$–consistent nonlinear expectations. This result permit us to find the simple mechanism, i.e., the function g, of the above apparently very abstract evaluations. We also provide a simple method to test and then find the function g. In Chapter 5, we present some basic method to solve numerically BSDE such as g–expectations and g–evaluations.

Chapter 2

BROWNIAN FILTRATION CONSISTENT EVALUATIONS AND EXPECTATIONS

In these notes, we will study the above evaluation problem within the following standard framework. Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped a filtration $(\mathcal{F}_t)_{t \geq 0}$, $(B_t)_{t \geq 0}$ be a standard d-dimensional Brownian motion defined on this space. We assume that $(\mathcal{F}_t)$ is the natural filtration of B:

$$\mathcal{F}_t = \sigma\{\{B_s; \, 0 \leq s \leq t\} \cup \mathcal{N}\}, \quad \mathcal{F}_\infty^0 := \bigcup_{t > 0} \mathcal{F}_t.$$

where $\mathcal{N}$ is the collection of P–null sets in Ω. A vector valued stochastic process $X_t = X(\omega, t)$ is said to be $\mathcal{F}_t$-adapted $((\mathcal{F}_t)_{0 \leq t < \infty}$ -adapted), if for each $t \in [0, \infty)$, $(X_t(\cdot))$ is an $\mathcal{F}_t$–measurable random variable. Heuristically, $(\mathcal{F}_t)$ represents our information before time t. Thus the meaning that X is $(\mathcal{F}_t)$-adapted process is that at the current time t_0, we know all trajectory of X_t for $t \leq t_0$, The actual risk of X is on its behavior after t_0. All processes discussed in this notes are $\mathcal{F}_t$–adapted.

We list notations of this notes

- $L^p(\Omega, \mathcal{F}_T, P; \mathbf{R}^m)$: the set of $\mathbf{R}^m$–valued $\mathcal{F}_t$–measurable random variables such that $E[|\xi|^p] < \infty$ $(p \geq 1)$;

- $L^p(\Omega, \mathcal{F}_T, P;) = L^p(\Omega, \mathcal{F}_T, P; \mathbf{R})$

- $L^p(\Omega, \mathcal{F}_\infty^0, P; \mathbf{R}^m) = \bigcup_{t > 0} L^p(\Omega, \mathcal{F}_t, P; \mathbf{R}^m)$;

- $L_{\mathcal{F}}^p(0, T; \mathbf{R}^m)$: the set of all $\mathbf{R}^m$–valued and $\mathcal{F}_t$–adapted processes such that $E \int_0^T |v_s|^p ds < \infty$.

- $\mathcal{M}(0, T; \mathbf{R}^m) = L_{\mathcal{F}}^p(0, T; \mathbf{R}^m)$, $\mathcal{M}(0, T) = \mathcal{M}(0, T; \mathbf{R})$.

2.1 $\mathcal{F}$–consistent nonlinear expectations

Under this language we can give the precise meaning of the above discussed filtration consistent nonlinear expectations and evaluations. We will see that this notion can be introduced in a classical way.

Definition 1 *A nonlinear expectation is a functional:*

$$\mathcal{E}[\cdot] : L^2(\Omega, \mathcal{F}^0_\infty, P) \longmapsto R$$

satisfying the following properties:
  (i) Strict monotonicity:

$$\text{if} \quad Y_1 \geq Y_2 \quad a.s., \quad \text{then} \quad \mathcal{E}[Y_1] \geq \mathcal{E}[Y_2];$$
$$\text{if} \quad Y_1 \geq Y_2 \quad a.s., \quad \mathcal{E}[Y_1] = \mathcal{E}[Y_2] \quad \Longleftrightarrow \quad Y_1 = Y_2 \quad a.s.$$

  (ii) preserving of constants:

$$\mathcal{E}[c] = c, \quad \text{for each constant } c.$$

Lemma 2 *Let $t \leq T$ and $\eta_1, \eta_2 \in L^2(\Omega, \mathcal{F}_t, P)$. If*

$$\mathcal{E}[\eta_1 1_A] = \mathcal{E}[\eta_2 1_A], \quad \forall A \in \mathcal{F}_t,$$

then

$$\eta_2 = \eta_1, \quad a.s. \tag{2.1}$$

Proof.   We choose $A = \{\eta_1 \geq \eta_2\} \in \mathcal{F}_t$. Since $(\eta_1 - \eta_2)1_A \geq 0$ and $\mathcal{E}[\eta_1 1_A] = \mathcal{E}[\eta_2 1_A]$, it follows that $\eta_1 1_A = \eta_2 1_A$ a.s.. Thus $\eta_2 \geq \eta_1$ a.s. With the same argument we can prove that $\eta_1 \geq \eta_2$ a.s. It follows that (2.1) holds. The proof is complete. ∎

Definition 3 *For the given filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$, a nonlinear expectation is called $\mathcal{F}$-expectation if for each $Y \in L^2(\Omega, \mathcal{F}_T, P)$ and for each $t \in [0, T]$ there exists a random variable $\eta \in L^2(\Omega, \mathcal{F}_t, P)$, such that*

$$\mathcal{E}[Y 1_A] = \mathcal{E}[\eta 1_A], \quad \forall A \in \mathcal{F}_t. \tag{2.2}$$

From Lemma 2 above, such an η is uniquely defined. We denote it by $\eta = \mathcal{E}[Y|\mathcal{F}_t]$. $\mathcal{E}[Y|\mathcal{F}_t]$ is called the conditional $\mathcal{F}$-expectation of Y under $\mathcal{F}_t$. It is characterized by

$$\mathcal{E}[Y 1_A] = \mathcal{E}[\mathcal{E}[Y|\mathcal{F}_t] 1_A], \quad \forall A \in \mathcal{F}_t. \tag{2.3}$$

Lemma 4 *We have, for each $0 \leq s \leq t \leq T$,*

$$\mathcal{E}[\mathcal{E}[Y|\mathcal{F}_t]|\mathcal{F}_s] = \mathcal{E}[Y|\mathcal{F}_s] \quad a.s. \tag{2.4}$$

In particular,

$$\mathcal{E}[\mathcal{E}[Y|\mathcal{F}_t]] = \mathcal{E}[Y]. \tag{2.5}$$

Proof.   For each $A \in \mathcal{F}_s$ we have $A \in \mathcal{F}_t$. Thus

$$
\begin{aligned}
\mathcal{E}[\mathcal{E}[\mathcal{E}[Y|\mathcal{F}_t]|\mathcal{F}_s]1_A] &= \mathcal{E}[\mathcal{E}[Y|\mathcal{F}_t]1_A] \\
&= \mathcal{E}[Y1_A] \\
&= \mathcal{E}[\mathcal{E}[Y|\mathcal{F}_s]1_A]
\end{aligned}
$$

It follows from Lemma 2 that (2.4) holds.

(2.5) follows then easily from the fact that $\mathcal{F}_0$ is the trivial σ-algebra (since $B_0 = 0$). ∎

Lemma 5 *We have a.s.*

$$\mathcal{E}[Y1_A|\mathcal{F}_t] = \mathcal{E}[Y|\mathcal{F}_t]1_A, \quad \forall A \in \mathcal{F}_t. \tag{2.6}$$

Proof.   For each $B \in \mathcal{F}_t$, we have

$$
\begin{aligned}
\mathcal{E}[\mathcal{E}[Y1_A|\mathcal{F}_t]1_B] &= \mathcal{E}[Y1_A1_B] \\
&= \mathcal{E}[\mathcal{E}[Y|\mathcal{F}_t]1_{A\cap B}] \\
&= \mathcal{E}[[\mathcal{E}[Y|\mathcal{F}_t]1_A]1_B].
\end{aligned}
$$

∎

We then can conclude

Proposition 6 **??***Let $\mathcal{E}[\cdot]$ be defined in Definition 1. If for each $0 \le t \le T < \infty$ and $Y \in L^2(\Omega, \mathcal{F}_T, P)$, there exists a $\mathcal{E}[Y|\mathcal{F}_t] \in L^2(\Omega, \mathcal{F}_t, P)$ satisfying relation 3, then $(\mathcal{E}[Y|\mathcal{F}_t])_{0\le t<\infty}$ satisfies Axiomatic Assumptions (i), (ii'), (iii) and (iv) listed in subsection 2.2.*

    (i) Monotonicity: $\mathcal{E}[Y|\mathcal{F}_t] \ge \mathcal{E}[Z|\mathcal{F}_t]$, a.s., if $Y \ge Z$, a.s.;
    (ii') Constant–preserving: $\mathcal{E}[Y|\mathcal{F}_t] = Y$, if $Y \in L^2(\Omega, \mathcal{F}_t, P; \mathbf{R})$
    (iii) "Zero–one law": for each t, $\mathcal{E}[1_AY|\mathcal{F}_t] = 1_A\mathcal{E}[Y|\mathcal{F}_t]$, $\forall A \in \mathcal{F}_t$.
    (iv) Time consistent: $\mathcal{E}_s[\mathcal{E}[Y|\mathcal{F}_t]|\mathcal{F}_s] = \mathcal{E}[Y|\mathcal{F}_s]$, if $s \le t \le T$.

Lemma 7 *For any $Y, \zeta \in L^2(\Omega, \mathcal{F}_T, P)$ and for each $t \in [0,T]$ and $A \in \mathcal{F}_t$ we have*

$$\mathcal{E}[Y1_A + \zeta 1_{A^C}|\mathcal{F}_t] = \mathcal{E}[Y|\mathcal{F}_t]1_A + \mathcal{E}[\zeta|\mathcal{F}_t]1_{A^C}$$

Proof.   According to Lemma4. above,

$$
\begin{aligned}
\mathcal{E}[Y1_A + \zeta 1_{A^C}|\mathcal{F}_t] &= \mathcal{E}[Y1_A + \zeta 1_{A^C}|\mathcal{F}_t]1_A + \mathcal{E}[Y1_A + \zeta 1_{A^C}|\mathcal{F}_t]1_{A^C} \\
&= \mathcal{E}[(Y1_A + \zeta 1_{A^C})1_A|\mathcal{F}_t] + \mathcal{E}[(Y1_A + \zeta 1_{A^C})1_{A^C}|\mathcal{F}_t] \\
&= \mathcal{E}[Y1_A|\mathcal{F}_t] + \mathcal{E}[\zeta 1_{A^C}|\mathcal{F}_t] \\
&= \mathcal{E}[Y|\mathcal{F}_t]1_A + \mathcal{E}[\zeta|\mathcal{F}_t]1_{A^C}.
\end{aligned}
$$

∎

Lemma 8 *For any X, $Y \in L^2(\Omega, \mathcal{F}_T, P)$, if $X \leq Y$ a.s., then we have for each $t \in [0, T]$,*

$$\mathcal{E}[X|\mathcal{F}_t] \leq \mathcal{E}[Y|\mathcal{F}_t] \quad a.s.$$

In this case, if for some $t \in [0, T)$, one has $\mathcal{E}[X|\mathcal{F}_t] = \mathcal{E}[Y|\mathcal{F}_t]$ a.s., then $X = Y$, a.s.

Proof.   Define $X_t = \mathcal{E}[X|\mathcal{F}_t]$ and $Y_t = \mathcal{E}[Y|\mathcal{F}_t]$, and let $A \in \mathcal{F}_t$. Because of the monotonicity of $\mathcal{E}$, we have

$$\mathcal{E}[X_t 1_A] = \mathcal{E}[X 1_A] \leq \mathcal{E}[Y 1_A] = \mathcal{E}[Y_t 1_A].$$

Now, take $A = \{X_t > Y_t\}$. If $P(A) > 0$, the strict monotonicity of $\mathcal{E}$ implies that

$$\mathcal{E}[X_t 1_A] > \mathcal{E}[Y_t 1_A].$$

Comparing the two above inequalities, we conclude that $P(A) = 0$.

Now if for some $t \in [0, T)$, one has $\mathcal{E}[X|\mathcal{F}_t] = \mathcal{E}[Y|\mathcal{F}_t]$, then $\mathcal{E}[X] = \mathcal{E}[Y]$. It follows from the strict monotonicity of $\mathcal{E}[\cdot]$ that $X = Y$, a.s.. ∎

2.2 $\mathcal{F}$-consistent nonlinear evaluations

Similarly as in the above zero–interest rate situation, in the case where the evaluation does not preserve constants, we can also introduce $\mathcal{F}$–consistent nonlinear evaluations the evaluation operators $(\mathcal{E}_{t,T}[\cdot])_{0 \leq t \leq T < \infty}$ satisfy Axiomatic Conditions (i), (ii), (iii) and (iv) listed subsection 1.2.1. The approach is also quite similar to the last section.

Definition 9 *An evaluation with maturity $0 \leq T < \infty$, is a nonlinear, functional:*

$$\mathcal{E}_{0,T}[\cdot] : L^2(\Omega, \mathcal{F}_T, P) \longmapsto R$$

which satisfies the following strict monotonicity properties.

$$if \quad Y_1 \geq Y_2 \quad a.s., \quad then \quad \mathcal{E}_{0,T}[Y_1] \geq \mathcal{E}_{0,T}[Y_2];$$
$$Y_1 \geq Y_2 \quad a.s., \ and \ \mathcal{E}_{0,T}[Y_1] = \mathcal{E}_{0,T}[Y_2] \quad \Longleftrightarrow \quad Y_1 = Y_2 \quad a.s.$$

Lemma 10 *For each $t < \infty$ and $\eta_1, \eta_2 \in L^2(\Omega, \mathcal{F}_t, P)$. If*

$$\mathcal{E}_{0,t}[\eta_1 1_A] = \mathcal{E}_{0,t}[\eta_2 1_A], \quad \forall A \in \mathcal{F}_t,$$

then

$$\eta_2 = \eta_1, \quad a.s. \tag{2.7}$$

Proof.   We choose $A = \{\eta_1 \geq \eta_2\} \in \mathcal{F}_t$. Since $(\eta_1 - \eta_2)1_A \geq 0$ and $\mathcal{E}_{0,t}[\eta_1 1_A] = \mathcal{E}_{0,t}[\eta_2 1_A]$, it follows by the strict monotonicity that $\eta_1 1_A = \eta_2 1_A$ a.s.. Thus $\eta_2 \geq \eta_1$ a.s. With the same argument we can prove that $\eta_1 \geq \eta_2$ a.s. It follows that (2.7) holds. The proof is complete. ∎

We can also have $\mathcal{F}_t$–consistent evaluation operators

Definition 11 *For the given filtration* $(\mathcal{F}_t)_{0 \leq t \leq T}$*, a nonlinear evaluation is called* $\mathcal{F}_t$*-consistent evaluation* $0 \leq t \leq T < \infty$ *and* $Y \in L^2(\Omega, \mathcal{F}_T, P)$ *there exists a random variable* $\eta \in L^2(\Omega, \mathcal{F}_t, P)$*, such that*

$$\mathcal{E}_{0,T}[Y 1_A] = \mathcal{E}_{0,t}[\eta 1_A], \quad \forall A \in \mathcal{F}_t.$$

From Lemma 10 above, such η is uniquely defined. We denote it by $\eta = \mathcal{E}_{t,T}[Y]$. $\mathcal{E}_{t,T}[Y]$ is called the evaluated value of Y with maturity T at the time t.

$$\mathcal{E}_{0,T}[Y 1_A] = \mathcal{E}_{0,t}[\mathcal{E}_{t,T}[Y] 1_A], \quad \forall A \in \mathcal{F}_t. \tag{2.8}$$

We can prove the semigroup property:

Lemma 12 *For each* $0 \leq s \leq t \leq T$ *and* $Y \in L^2(\Omega, \mathcal{F}_T, P; \mathbf{R})$*, we have*

$$\mathcal{E}_{s,t}[\mathcal{E}_{t,T}[Y]] = \mathcal{E}_{s,T}[Y] \quad a.s. \tag{2.9}$$

In particular,

$$\mathcal{E}_{0,t}[\mathcal{E}_{t,T}[Y]] = \mathcal{E}_{0,T}[Y] \quad a.s.. \tag{2.10}$$

Proof.   For each $A \in \mathcal{F}_s$ we have $A \in \mathcal{F}_t$. Thus, by $\mathcal{F}_t$–consistence,

$$
\begin{aligned}
\mathcal{E}_{0,s}[\mathcal{E}_{s,t}[\mathcal{E}_{t,T}[Y]] 1_A] &= \mathcal{E}_{0,t}[\mathcal{E}_{t,T}[Y] 1_A] \\
&= \mathcal{E}_{0,T}[Y 1_A] \\
&= \mathcal{E}_{0,s}[\mathcal{E}_{s,T}[Y] 1_A]
\end{aligned}
$$

It follows from Lemma 10 that (2.9) holds.

(2.10) follows then easily from the fact that $\mathcal{F}_0$ is the trivial σ-algebra (since $B_0 = 0$). ∎

The zero–one law is also holds:

Lemma 13 *For each* $0 \leq t \leq T$*,* $Y \in L^2(\Omega, \mathcal{F}_T, P; \mathbf{R})$ *and* $A \in \mathcal{F}_t$*, we have*

$$\mathcal{E}_{t,T}[Y 1_A] = \mathcal{E}_{t,T}[Y] 1_A, \quad a.s.. \tag{2.11}$$

Proof.   For each $0 \leq t \leq T$ and $B \in \mathcal{F}_t$, we have

$$
\begin{aligned}
\mathcal{E}_{0,t}[\mathcal{E}_{t,T}[Y1_A]1_B] &= \mathcal{E}_{0,T}[Y1_A1_B] \\
&= \mathcal{E}_{0,t}[\mathcal{E}_{t,T}[Y]1_{A\cap B}] \\
&= \mathcal{E}_{0,t}[[\mathcal{E}_{t,T}[Y]1_A]1_B].
\end{aligned}
$$

■For each $0 \leq t < \infty$, and $\eta \in L^2(\Omega, \mathcal{F}_t, P; \mathbf{R})$, we have

Lemma 14

$$
\mathcal{E}_{t,t}[\eta] = \eta, \ \ a.s..
$$

Proof.   By the definition of $\mathcal{E}_{t,T}[\cdot]$, we have

$$
\mathcal{E}_{0,t}[\mathcal{E}_{t,t}[\eta]1_A] = \mathcal{E}_{0,t}[\eta 1_A], \ \ \forall A \in L^2(\Omega, \mathcal{F}_t, P).
$$

We then can conclude

Proposition 15 ??_Let $\mathcal{E}[\cdot]$ be defined in Definition 9. If for each $0 \leq t \leq T < \infty$ and $Y \in L^2(\Omega, \mathcal{F}_T, P)$, there exists a $\mathcal{E}_{t,T}[Y|\mathcal{F}_t] \in L^2(\Omega, \mathcal{F}_t, P)$ satisfying relation 11, then $(\mathcal{E}[Y|\mathcal{F}_t])_{0 \leq t < \infty}$ satisfies Axiomatic Assumptions (i), (ii), (iii) and (iv) listed in subsection 2.2. i.e.,_

(i) Monotonicity: $\mathcal{E}_{t,T}[Y] \geq \mathcal{E}_{t,T}[Y']$,a.s., if $Y \geq Y'$, a.s.;

(ii) $\mathcal{E}_{t,t}[Y] = Y$,if $Y \in L^2(\Omega, \mathcal{F}_T, P)$. Particularly $\mathcal{E}_{0,0}[c] = c$.

(iii) "Zero–one law": for each $t \leq T$, $\mathcal{E}_{t,T}[1_A Y] = 1_A \mathcal{E}_{t,T}[Y], \ \ \forall A \in \mathcal{F}_t^X$.

(iv) Time consistent: $\mathcal{E}_{s,t}[\mathcal{E}_{t,T}[Y]] = \mathcal{E}_{s,T}[Y]$, if $s \leq t \leq T$.

We also have the following properties

Lemma 16 _For each $0 \leq t \leq T < \infty$, X, $Y \in L^2(\Omega, \mathcal{F}_T, P)$ and $A \in \mathcal{F}_t$, we have_

$$
\mathcal{E}_{t,T}[X1_A + Y1_{A^C}] = \mathcal{E}_{t,T}[X]1_A + \mathcal{E}_{t,T}[Y]1_{A^C}
$$

Proof.   According to Lemma 13,

$$
\begin{aligned}
\mathcal{E}_{t,T}[X1_A + Y1_{A^C}] &= \mathcal{E}_{t,T}[X1_A + Y1_{A^C}]1_A + \mathcal{E}_{t,T}[X1_A + Y1_{A^C}]1_{A^C} \\
&= \mathcal{E}_{t,T}[(X1_A + Y1_{A^C})1_A] + \mathcal{E}_{t,T}[(X1_A + Y1_{A^C})1_{A^C}] \\
&= \mathcal{E}_{t,T}[X1_A] + \mathcal{E}_{t,T}[Y1_{A^C}] \\
&= \mathcal{E}_{t,T}[X]1_A + \mathcal{E}_{t,T}[Y]1_{A^C}.
\end{aligned}
$$

∎

The following monotonicity of $\mathcal{E}_{t,T}[\cdot]$ is important.

Lemma 17 *For each $0 \leq t \leq T < \infty$ and $X,\ Y \in L^2(\Omega, \mathcal{F}_T, P)$ such that $X \leq Y$ a.s., we have,*

$$\mathcal{E}_{t,T}[X] \leq \mathcal{E}_{t,T}[Y] \quad a.s.$$

If $X \leq Y$ a.s. and, for some $t \in [0,T]$, one has $\mathcal{E}_{t,T}[X] = \mathcal{E}_{t,T}[Y]$ a.s., then $X = Y$, a.s.

Proof.   Define $X_t = \mathcal{E}_{t,T}[X]$ and $Y_t = \mathcal{E}_{t,T}[Y]$, and let $A \in \mathcal{F}_t$. Because of the monotonicity of $\mathcal{E}_{0,t}[\cdot]$, we have

$$\mathcal{E}_{0,t}[X_t 1_A] = \mathcal{E}_{0,T}[X1_A] \leq \mathcal{E}_{0,T}[Y1_A] = \mathcal{E}_{0,t}[Y_t 1_A].$$

Now, we take $A = \{X_t > Y_t\}$. If $P(A) > 0$, the strict monotonicity of $\mathcal{E}$ implies that

$$\mathcal{E}_{0,t}[X_t 1_A] > \mathcal{E}_{0,t}[Y_t 1_A].$$

It contradicts the first inequalities. We then conclude that $P(A) = 0$.

Now if $X \leq Y$ a.s. and, for some $t \in [0,T]$, $\mathcal{E}_{t,T}[X] = \mathcal{E}_{t,T}[Y]$, then $\mathcal{E}_{0,T}[X] = \mathcal{E}_{0,T}[Y]$. It follows from the strict monotonicity of $\mathcal{E}[\cdot]$ that $X = Y$, a.s..

∎

Chapter 3
BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS: G–EVALUATIONS AND G–EXPECTATIONS

3.1   BSDE: existence, uniqueness and basic estimates

We will see that 1–dimensional BSDE is a very tool to study filtration consistent evaluations and and expectations. In this case, we have a very simple mechanism, i.e., the generator g of the BSDE, that entirely deterimines the related evaluation or expectation operator. We call them g–evaluations and g–expectations.

Remark 6 *The condition that $(\mathcal{F}_t)$ is generated by a Brownian motion can be largely extended. But in order to adapt a wider audience, we will limited ourself within this typical situation. Readers interested in a more general situation can see [B], [], ... For infinite time horizon, see [], [], ...*

The norm and scalar product of the Euclid space $\mathbf{R}^n$ are respectively denoted by $< \cdot, \cdot >$ and $| \cdot |$. For a given $(\mathcal{F}_t)$-stopping time τ, we denote by $\mathcal{M}(0, \tau; \mathbf{R}^n)$ the space of all $(\mathcal{F}_t)$-adapted and $\mathbf{R}^n$-valued processes satisfying

$$\mathbf{E} \int_0^\tau |v_t|^2 dt < \infty.$$

It is a Hilbert space.

In this paper we consider the following form of BSDE:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s. \tag{3.1}$$

The setting of our problem is somewhat unusual: to find a pair of $\mathcal{F}_t$-adapted processes $(Y_t, Z_t) \in \mathcal{M}(0, T; \mathbf{R}^m \times \mathbf{R}^{m \times d})$ satisfying BSDE (3.1).

Remark 7 *The solution Y is an ordinary Itô's process:*

$$Y_t = Y_0 - \int_0^t g(s, Y_s, Z_s)ds + \int_0^t Z_s dB_s,$$

To prove the existence and uniqueness of BSDE (3.1) we first consider a very simple case: g is a real valued process that is independent of the variable (y, z). We have

Lemma 18   *For a fixed $\xi \in L^2(\omega, \mathcal{F}_T, P; \mathbf{R})$ and $g_0(\cdot)$ satisfying*

$$\mathbf{E}(\int_0^T |g_0(t)|dt)^2 < \infty$$

there exists a unique pair of processes $(y_t, z_t) \in \mathcal{M}(0, T; \mathbf{R}^{1+d})$, satisfies the following BSDE

$$y_t = \xi + \int_t^T g_0(s)ds - \int_t^T z_s dB_s. \tag{3.2}$$

If $g_0(\cdot) \in \mathcal{M}(0, T; \mathbf{R})$, then we have the following basic estimate:

$$|y_t|^2 + \mathbf{E}^{\mathcal{F}_t}\int_t^T [\frac{\beta}{2}|y_s|^2 + |z_s|^2]e^{\beta(s-t)}ds \tag{3.3}$$

$$\leq \mathbf{E}^{\mathcal{F}_t}|\xi|^2 e^{\beta eta(T-t)} + \frac{2}{\beta}\mathbf{E}^{\mathcal{F}_t}\int_t^T |g_0(s)|^2 e^{\beta(s-t)}ds$$

In particular

$$|y_0|^2 + \mathbf{E}\int_0^T [\frac{\beta}{2}|y_s|^2 + |z_s|^2]e^{\beta s}ds \tag{3.4}$$

$$\leq \mathbf{E}|\xi|^2 e^{\beta T} + \frac{2}{\beta}\mathbf{E}\int_0^T |g_0(s)|^2 e^{\beta s}ds,$$

where β is an arbitrary constant.

Proof.   We define

$$M_t = \mathbf{E}^{\mathcal{F}_t}[\xi + \int_0^T g_0(s)ds].$$

M is a square integrable $(\mathcal{F}_t)$-martingale. By representation theorem of Brownian martingale by Itô's integral, there exists a unique adapted process $(z_t) \in \mathcal{M}(0, T; \mathbf{R}^d)$ such that

$$M_t = M_0 + \int_0^t z_s dB_s.$$

Thus

$$M_t = M_T - \int_t^T z_s dB_s.$$

We denote

$$y_t = M_t - \int_0^t g_0(s)ds = M_T - \int_0^t g_0(s)ds - \int_t^T z_s ds.$$

Since $M_T = \xi + \int_0^T g_0(s)ds$, we have immediately (3.2).

The uniqueness is a simple corollary of the estimates (3.3) or (3.4). We only need to prove these two estimates. To prove (3.3), we first consider the case where ξ and $g_0(\cdot)$ are both bounded case. In this case

$$y_t = \mathbf{E}^{\mathcal{F}_t}\left[\xi + \int_t^T g_0(s)ds\right]$$

Thus y is also bounded. We then apply Itô's formula to $|y_s|^2 e^{\beta(s-t)}$ for $s \in [t, T]$:

$$|y_t|^2 + \int_t^T [\beta|y_s|^2 + |z_s|^2]e^{\beta s}ds$$
$$= |\xi|^2 e^{\beta T} + \int_t^T 2y_s g_0(s)e^{\beta s}ds - \int_t^T e^{\beta s}2y_s z_s dB_s.$$

Since y_t is $(\mathcal{F}_t)$-measurable, we take $(\mathcal{F}_t)$ conditional expectation on both sides of the above relation

$$|y_t|^2 + \mathbf{E}^{\mathcal{F}_t} \int_t^T [\beta|y_s|^2 + |z_s|^2]e^{\beta(s-t)}ds$$
$$= \mathbf{E}^{\mathcal{F}_t}|\xi|^2 e^{\beta(T-t)} + \mathbf{E}^{\mathcal{F}_t} \int_t^T 2y_s g_0(s)e^{\beta(s-t)}ds$$
$$\leq \mathbf{E}^{\mathcal{F}_t}|\xi|^2 e^{\beta(T-t)} + \mathbf{E}^{\mathcal{F}_t} \int_t^T [\frac{\beta}{2}|y_s|^2 + \frac{2}{\beta}|g_0(s)|^2]e^{\beta(s-t)}ds.$$

From this it follows (3.3) and (3.4).

We now consider the case where ξ and $g_0(\cdot)$ are possibly unbounded. We set

$$\xi^n := (\xi \wedge n) \vee (-n), \qquad g_0^n(s) := (g_0(s) \wedge n) \vee (-n)$$

and

$$y_t^n := \xi^n + \int_t^T g_0^n(s)ds - \int_t^T z_s^n dB_s.$$

We observe that, for each positive integers n and k, ξ^n, ξ^k, g_0^n as well as g_0^k are all bounded. We thus have

$$|y_t^n|^2 + \mathbf{E}^{\mathcal{F}_t} \int_t^T [\frac{\beta}{2}|y_s^n|^2 + |z_s^n|^2]e^{\beta(s-t)}ds$$
$$\leq \mathbf{E}^{\mathcal{F}_t}|\xi^n|^2 e^{\beta(T-t)} + \frac{2}{\beta}\mathbf{E}^{\mathcal{F}_t} \int_t^T |g_0^n(s)|^2 e^{\beta(s-t)}ds,$$

and

$$\mathbf{E} \int_0^T [\frac{\beta}{2}|y_s^n - y_s^k|^2 + |z_s^n - z_s^k|^2]e^{\beta s}ds$$
$$\leq \mathbf{E}|\xi^n - \xi^k|^2 e^{\beta T} + \frac{2}{\beta}\mathbf{E} \int_0^T |g_0^n(s) - g_0^k(s)|^2 e^{\beta s}ds.$$

The second inequality implies that both $\{y^n\}$ and $\{z^n\}$ are Cauchy sequences in $\mathcal{M}(0, T)$. Thus (3.3) is proved when we let n tends to ∞. $\blacksquare$

With the above basic estimates we can consider the general case of BSDE (3.1). We assume that

$$g = g(\omega, t, y, z) : \omega \times [0, T] \times \mathbf{R}^m \times \mathbf{R}^{m \times d} \to \mathbf{R}^m$$

satisfies the following conditions: for each $(y, z) \in \mathbf{R}^m \times \mathbf{R}^{m \times d}$, $g(\cdot, y, z)$ is $\mathbf{R}^m$–valued and $(\mathcal{F}_t)$-adapted process satisfying

$$\int_0^T |g(\cdot, 0, 0)|ds \in L^2(\Omega, \mathcal{F}_T, P). \tag{3.5}$$

we assume g satisfy Lipschitz condition in (y, z): for each $y, y' \in \mathbf{R}^m$ and $z, z' \in \mathbf{R}^{m \times d}$

$$|g(t, y, z) - g(t, y', z')| \leq C(|y - y'| + |z - z'|). \tag{3.6}$$

The following is the basic theorem of BSDE: the existence and uniqueness.

Theorem 19 *Assume that g satisfies (3.5) and (3.6). Then for any given terminal condition $\xi \in L^2(\omega, \mathcal{F}_T, P; \mathbf{R}^m)$, BSDE (3.1) has a unique solution, i.e., there exists a unique pair of $\mathcal{F}_t$-adapted processes $(Y, Z) \in \mathcal{M}(0, T; \mathbf{R}^m \times \mathbf{R}^{m \times d})$ satisfying (3.1).*

Proof.   In the basic estimate (3.3) we fix $\beta = 8(1 + C^2)$, where C is the Lipschitz constant of g in (y, z). Related to β, we introduce a norm in Hilbert space $\mathcal{M}(0, T; \mathbf{R}^n)$:

$$\|v(\cdot)\|_\beta \equiv \{\mathbf{E} \int_0^T |v_s|^2 e^{\beta s} ds\}^{\frac{1}{2}}$$

Clearly this norm is equivalent to the original one of $\mathcal{M}(0, T; \mathbf{R}^n)$. But this norm is more convenient to construct a contract mapping in order to apply the fixed point theorem. We thus set

$$Y_t = \xi + \int_t^T g(s, y_s, z_s) ds - \int_t^T Z_s dB_s$$

We define a mapping

$$I[(y., z.)] := (Y., Z.) : \mathcal{M}(0, T; \mathbf{R}^m \times \mathbf{R}^{m \times d}) \to \mathcal{M}(0, T; \mathbf{R}^m \times \mathbf{R}^{m \times d}).$$

We need to prove that I is a contract mapping under the norm $\|\cdot\|_\beta$.
For any two elements (y, z) and (y', z') in $\mathcal{M}(0, T; \mathbf{R}^m \times \mathbf{R}^{m \times d})$ we set

$$(Y, Z) = I[(y, z)], \qquad (Y', Z') = I[(y', z')],$$

and denote their differences by $(\hat{y}, \hat{z}) = (y - y', z - z')$, $(\hat{Y}, \hat{Z}) = (Y - Y', Z - Z')$. By the basic estimate (3.4) we have

$$\mathbf{E} \int_0^T (\frac{\beta}{2}|\hat{Y}_s|^2 + |\hat{Z}_s|^2) e^{\beta s} ds \le \frac{2}{\beta} \mathbf{E} \int_0^T |g(s, y_s, z_s) - g(s, y'_s z'_s)|^2 e^{\beta s} ds,$$

Since g satisfy Lipschitz condition

$$\mathbf{E} \int_0^T [\frac{\beta}{2}|\hat{Y}_s|^2 + |\hat{Z}_s|^2] e^{\beta s} ds \le \frac{4C^2}{\beta} \mathbf{E} \int_0^T [|\hat{y}_s|^2 + |\hat{z}_s|^2] e^{\beta s} ds,$$

We let $\beta = 8(1 + C^2)$

$$\mathbf{E} \int_0^T [|\hat{Y}_s|^2 + |\hat{Z}_s|^2] e^{\beta s} ds \le \frac{1}{2} \mathbf{E} \int_0^T [|\hat{y}_s|^2 + |\hat{z}_s|^2] e^{\beta s} ds,$$

or

$$\|(\hat{Y}, \hat{Z})\|_\beta \le \frac{1}{\sqrt{2}} \|(\hat{y}, \hat{z})\|_\beta.$$

Thus I is a strict contract mapping of $\mathcal{M}(0, T; \mathbf{R}^m \times \mathbf{R}^{m \times d})$. It follows by the fixed point theorem that BSDE (3.1) has a unique solution which is the fixed point of I. ∎

The basic estimates (3.3) and (3.4) can also be applied to prove the continuous dependence theorem of BSDE (3.1) with respect parameters. Let (Y^1, Z^1) and (Y^1, Z^2) be respectively the solution of the following two BSDEs:

$$Y_t^1 = \xi^1 + \int_t^T [g(s, Y_s^1, Z_s^1) + \varphi^1{}_s] ds - \int_t^T Z_s^1 dB_s. \tag{3.7}$$

$$Y_t^2 = \xi^2 + \int_t^T [g(s, Y_s^2, Z_s^2) + \varphi_s^2]ds - \int_t^T Z_s^2 dB_s. \tag{3.8}$$

Here the terminal condition ξ^1 and ξ^2 are given elements in $L^2(\omega, \mathcal{F}_T, P; \mathbf{R}^m)$ and φ^1 and φ^2 are two given processes in $\mathcal{M}(0, T; \mathbf{R}^m)$. Let g be the same as in Theorem19. Analogue to the previous method, using Itô's formula applied to $|Y_s^1 - Y_s^2|^2 e^{\beta(s-t)}$ in the interval $[t, T]$, we can obtain the following theorem.

Theorem 20 *The difference of the solutions of BSDE (3.7) and (3.8)satisfies*

$$|Y_t^1 - Y_t^2|^2 + \frac{1}{2}\mathbf{E}^{\mathcal{F}_t} \int_t^T [|Y_s^1 - Y_s^2|^2 + |Z_s^1 - Z_s^2|^2]e^{\beta(s-t)}ds \tag{3.9}$$

$$\leq \mathbf{E}^{\mathcal{F}_t}|\xi^1 - \xi^2|^2 e^{\beta(T-t)} + \mathbf{E}^{\mathcal{F}_t} \int_t^T |\varphi_s^1 - \varphi_s^2|e^{\beta(s-t)}ds,$$

where $\beta = 16(1 + C^2)$.

For a fixed $t_0 \in [0, T]$, we denote

$$\mathcal{F}_t^{t_0} = \sigma\{(B_s - B_{t_0}; t_0 \leq s \leq t) \cup \mathcal{N}\}, \quad t \in [t_0, T].$$

The following is a simple corollary of the uniqueness of BSDE (3.1).

Proposition 21 *We still assume that g satisfies Assumptions (3.5) and (3.6). If moreover, for a fixed $t_0 \in [0, T]$ and for each $(y, z) \in \mathbf{R}^m \times \mathbf{R}^{m \times d}$, the process $g(\cdot, y, z)$ is $(\mathcal{F}_t^{t_0})$-adapted on the interval $[t_0, T]$ and $\xi \in L^2(\Omega, \mathcal{F}_T^{t_0}, P; \mathbf{R}^m)$. Then the solution $(Y., Z.)$ of BSDE (3.1) is also $(\mathcal{F}_t^{t_0})$-adapted on $[t_0, T]$. In particular, Y_{t_0} and Z_{t_0} are deterministic.*

Proof. Let $(Y'., Z'.)$ be the solution of $(\mathcal{F}_t^{t_0})$-adapted solution, on the interval $[t_0, T]$ of the BSDE

$$Y_t' = \xi + \int_t^T g(s, Y_s', Z_s')ds - \int_t^T Z_s' dB_s^0,$$

where we denote $B_t^0 \equiv B_t - B_{t_0}$. Observe that $(B_s^0)_{t_0 \leq s \leq T}$ is an $(\mathcal{F}_t^{t_0})$-Brownian motion on the interval $[t_0, T]$. But on the other hand the same processes $(Y_t', Z_t')_{t_0 \leq s \leq T}$ is also $(\mathcal{F}_t)$-adapted and

$$\int_t^T Z_s' dB_s = \int_t^T Z_s' dB_s^0, \quad t \in [t_0, T].$$

Thus from the uniqueness result of Theorem 19, The solution (Y, Z) of BSDE (3.1) coincides with (Y', Z') on $[t_0, T]$. Thus (Y, Z) is $(\mathcal{F}_t^{t_0})$-adapted. ∎

Remark 8 *A special situation of BSDE (3.1) is when ξ is deterministic and $g(t, y, z)$ is a deterministic function of (t, y, z). In this case the solution of BSDE (3.1) is simply $(Y., Z.) \equiv (Y_0(\cdot), 0)$, where $Y_0(\cdot)$ is the solution of the following ordinary differential equation defined on $[0, T]$:*

$$-\dot{Y}_0(t) = g(t, Y_0(t), 0), \quad Y_0(T) = \xi.$$

3.2   1–Dimensional BSDE

In this section we limited ourselves to 1–dimensional case of BSDE, i.e., g, and thus y, is real valued ($m = 1$). The importance of this situation is that many filtration–consistent nonlinear evaluations and nonlinear expectations are generated by this BSDE. The present state of art of mathematical finance corresponds mostly to $m = 1$. It also covers most parabolic and elliptic PDEs, linear or nonlinear. In fact $m > 1$ corresponds systems of PDEs of the above types. We first present an important property: **Comparison Theorem** of BSDE. It corresponds the comparison theorem parabolic PDE theory.

3.2.1   Comparison Theorem

We will present this comparison theorem in the case where the solution Y is possibly a RCLL (right continuous with eft limit) process. i.e., P-almost all of its paths of $Y.(\omega)$ are right continuous with left limit. An RCLL process $(A_t(\omega))$ is called an increasing process if P-almost all of its paths are non-decreasing with $A_0(\omega) = 0$.

We first consider the following problem: to find a solution $(Y, Z) \in \mathcal{M}(0, T; \mathbf{R}^{1+d})$ of the following BSDE

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds + (V_T - V_t) - \int_t^T Z_s dB_s, \tag{3.10}$$

where (V_t) is a given RCLL process satisfying

$$V. \in \mathcal{M}(0, T) \text{ and } \mathbf{E}\sup_{t \leq T}|V|^2 < \infty. \tag{3.11}$$

The following is a simple corollary of Theorem 19.

Proposition 22   *We assume (3.5), (3.6) and (3.11). Then for each $\xi \in L^2(\omega, \mathcal{F}_T, P)$, there exists a unique solution $(Y_t, Z_t) \in \mathcal{M}(0, T; \mathbf{R}^{1+d})$ of BSDE (3.10). Moreover $(Y + V)$ is a continuous process. We also have the following estimate:*

$$\mathbf{E}[\sup_{0 \leq t \leq T}|Y_t|^2] < \infty. \tag{3.12}$$

Proof.   The case $V_t \equiv 0$ corresponds Theorem 19. For a general situation we let $\bar{Y}_t := Y_t + V_t$. The above BSDE becomes the standard case:

$$\bar{Y}_t = \xi + V_T + \int_t^T g(s, \bar{Y}_s - V_s, Z_s)ds - \int_t^T Z_s dB_s.$$

Estimate (3.12) is derived by

$$\mathbf{E}\sup_{0 \leq t \leq T}|\int_0^t Z_s dB_s|^2 < \infty, \quad \mathbf{E}\int_0^T |g(s, Y_s, Z_s)|^2 ds < \infty.$$

∎

For a given a random variable

$$\hat{\xi} \in L^2(\Omega, \mathcal{F}_T, P), \tag{3.13}$$

Let $(\hat{Y}, \hat{Z}) \in \mathcal{M}(0, T; \mathbf{R}^{1+d})$ be the solution of the following BSDE

$$\hat{Y}_t = \hat{\xi} + \int_t^T g(s, \hat{Y}_s, \hat{Z}_s)ds + (V_T - V_t) - \int_t^T \hat{Z}_s dB_s. \tag{3.14}$$

It is easy to prove that the difference $(Y - \hat{Y}, Z - \hat{Y})$ satisfies exactly the same estimate (3.9) given in Theorem 20. Using Buckholder–Davis–Gundy inequality, we then derive the following estimate.

Proposition 23 *We assume (3.5), (3.6) and (3.11). Then the difference of the solutions of BDSE (3.10) and (3.14) satisfies*

$$\mathbf{E}[\sup_{0 \leq t \leq T} |Y_t - \hat{Y}_t|^2] + \mathbf{E} \int_0^T |Z_s - \hat{Z}_s|^2 ds \leq CE|\xi - \hat{\xi}|^2. \tag{3.15}$$

The following Comparison Theorem was firstly obtained in [P5], for the case where g is a C^1 function of (y, z) with bounded derivatives. Then, in [EPQ], under present Lipschitz condition. Strict Comparison Theorem was obtained in [P10]. The present proof is from [EPQ].

Theorem 24 *(Comparison Theorem) We make the same assumption as in Proposition 3.1. Let $(\bar{Y}, \bar{Z})$ be the solution of the following simple BSDE*

$$\bar{Y}_t = \bar{\xi} + \int_t^T \bar{g}_s + V_T - V_t - \int_t^T \bar{Z}_s dB_s. \tag{3.16}$$

where $(\bar{g}_t)$, $(\bar{V}_t) \in \mathcal{M}(0, T; \mathbf{R})$ and $\bar{\xi} \in L^2(\Omega, \mathcal{F}_T, P; \mathbf{R})$ are given such that

$$\xi \geq \bar{\xi}, \quad g(\bar{Y}_t, \bar{Z}_t, t) \geq \bar{g}_t, \ a.s., \ a.e., \tag{3.17}$$

and such that $\hat{V} = V - \bar{V}$ is an increasing process. We then have

$$Y_t \geq \bar{Y}_t, \quad a.e., \ a.s.. \tag{3.18}$$

We also have Strict Comparison Theorem: under the above conditions

$$Y_0 = \bar{Y}_0 \iff \xi = \bar{\xi}, , g(s, \bar{Y}_s, \bar{Z}_s) \equiv \bar{g}_s, \ and \ V_s \equiv \bar{V}_s. \tag{3.19}$$

Sketch of the Proof. For We only consider the case $d = 1$ (i.e., B is a 1-dimensional Brownian motion) and prove the case $t = 0$. The general situation is left to the reader as an exercise. We set $\hat{g}_s = g(s, \bar{Y}_s, \bar{Z}_s) - \bar{g}_s$ and

$$\hat{Y} = Y - \bar{Y}, \ \hat{Z} = Z - \bar{Z}, \ \hat{\xi} = \xi - \bar{\xi}.$$

The pair $(\hat{Y}, \hat{Z})$ can be regarded as the solution of the following linear BSDE:

$$\begin{cases} -d\hat{Y}_s = & (a_s\hat{Y}_s + b_s\hat{Z}_s + \hat{g}_s)ds + d\hat{V}_s - \hat{Z}_s dB_s, \\ \hat{Y}_T = & \hat{\xi}, \end{cases}$$

where

$$a_s := \begin{cases} \frac{g(s,Y_s,Z_s)-g(s,\bar{Y}_s,Z_s)}{Y_s-\bar{Y}_s}, & \text{if } Y_s \neq \bar{Y}_s, \\ 0, & \text{if } Y_s = \bar{Y}_s, \end{cases}$$

$$b_s := \begin{cases} \frac{g(s,\bar{Y}_s,Z_s)-g(s,\bar{Y}_s,\bar{Z}_s)}{Z_s-\bar{Z}_s}, & \text{if } Z_s \neq \bar{Z}_s, \\ 0, & \text{if } Z_s = \bar{Z}_s. \end{cases}$$

Since g satisfies Lipschitz condition, thus $|a_s| \leq C$ and $|b_s| \leq C$. We set

$$Q_t := \exp\left[\int_0^t b_s dB_s - \frac{1}{2}\int_0^t |b_s|^2 ds + \int_0^t a_s ds\right].$$

We apply Itô's formula to $Q_t\hat{Y}_t$ on the interval $[0, T]$ and then take expectation:

$$\hat{Y}_0 = \mathbf{E}[\hat{Y}_T Q_T + \int_0^T Q_t\hat{g}_t dt + \int_0^T Q_t d\hat{V}_t] \geq 0.$$

From this we have $Y_0 \geq \bar{Y}_0$. This method also applies to the case $t > 0$. The strict comparison is due to the fact that

$$\mathbf{E}[\hat{Y}_T Q_T + \int_0^T Q_t\hat{g}_t dt + \int_0^T Q_t d\hat{V}_t] = 0$$

is equivalent to $\hat{Y} = 0$ $\hat{g}_t \equiv 0$ and $\hat{V}_T = 0$. ∎

Remark 9   *In many situations Comparison Theorem is applied to compare the following type of two BSDEs:*

$$Y_t^1 = \xi^1 + \int_t^T [g(s, Y_s^1, Z_s^1) + c_s^1]ds - \int_t^T Z_s^1 dB_s, \qquad (3.20)$$

and

$$Y_t^2 = \xi^2 + \int_t^T [g(s, Y_s^2, Z_s^2) + c_s^2]ds - \int_t^T Z_s^2 dB_s, \qquad (3.21)$$

where $c^1(\cdot), c^2(\cdot) \in \mathcal{M}(0, T, \mathbf{R})$. In this case if we have

$$c_s^1 \geq c_s^2, \ a.s., \ a.e., \ \xi^1 \geq \xi^2, \ a.s..$$

Then it is easy to apply Theorem 24 to derive $Y_t^1 \geq Y_t^2$, a.s., a.e..

Example 25   *We consider a special case of BSDE (3.20) with $g(s, 0, 0) \equiv 0$. In this case if $c_s^2 \equiv 0$ and $\xi^2 = 0$, then the unique solution of BSDE (3.21) is $(Y_s^2, Z_s^2) \equiv 0$. It then follows from Remark 3.4 that if ξ^1 and $c^1(\cdot)$ are both non negative, then the solution Y^1 of (3.20) is also non negative. In this case we have also, by strict comparison,*

$$y_0^1 = 0 \Longleftrightarrow c_s^1 \equiv 0 \ \ and \ \ \xi^1 = 0.$$

An interpretation in finance is: If an investor want obtain an opportunity of non negative return, i.e., $\xi^1 \geq 0$, then he must invest present time i.e., $y_0^1 \geq 0$. If $\xi \geq 0$, a.s. and $\mathbf{E}\xi^1 > 0$, then his investment has to be positive: $y_0^1 > 0$.

Example 26   *We assume that $g(s, 0, 0) \equiv 0$ and $\xi \geq 0$ with $\mathbf{E}[\xi] > 0$. consider the following BSDE parameterized by $\lambda \in (0, \infty)$:*

$$Y_t^\lambda = \lambda \xi + \int_t^T g(s, Y_s^\lambda, Z_s^\lambda) ds - \int_t^T Z_s^\lambda dB_s.$$

We can prove that

$$\lim_{\lambda \uparrow \infty} Y_0^\lambda = +\infty.$$

In fact we compare its solution with the one of the following BSDE

$$\bar{Y}_t^\lambda = \lambda \xi + \int_t^T C(-|\bar{Y}_s^\lambda| - |\bar{Z}_s^\lambda|) ds - \int_t^T \bar{Z}_s^\lambda dB_s,$$

where $C > 0$ is the Lipschitz constant of g with respect to (y, z). By Comparison Theorem, we have

   (i) $Y_0^\lambda \geq \bar{Y}_0^\lambda$, for each $\lambda > 0$;
   (ii) $\bar{Y}_0^1 > 0$, when $\lambda = 1$
      We also observe that for each $\lambda \geq 0$, we have $\bar{Y}_t^\lambda \equiv \lambda \bar{Y}_t^1$ and $\bar{Z}_t^\lambda \equiv \lambda \bar{Z}_t^1$. From this and (i), (ii) it follows that

$$Y_0^\lambda \geq \bar{Y}_0^\lambda = \lambda \bar{Y}_0^1 \uparrow \infty.$$

Exercise 3.2.1   *Prove that Y_0^λ is also bounded by:*

$$Y_0^\lambda \leq \lambda \hat{Y}_0,$$

where $\hat{Y}_0$ is a constant.

3.2.2   Stochastic monotone semigroups and g–evaluaions

We now discuss the backward semigroup property of the solution Y of a BSDE. We do not fix $T > 0$, we set $\mathcal{F}^0_\infty := \bigcup_{T>0} \mathcal{F}_T$. It is clear that, for each $\xi \in^2 (\Omega, \mathcal{F}^0_\infty, P; \mathbf{R})$, here exists a non negative $T < \infty$, such that, $\xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbf{R})$. The function g is defined as follows

$$g(\omega, t, y, z) : \Omega \times [0, \infty) \times R \times R^d \longmapsto R.$$

We assume that

$$
\begin{cases}
\text{(i)} & g(\cdot, y, z) \in \mathcal{M}([0, \infty); \mathbf{R}), \quad \text{for each } y \in R, \ z \in R^d; \\
\text{(ii)} & \exists \mu, \ \nu \geq 0 \quad \text{such that} \quad \forall y_1, \ y_2 \in R, \quad z_1, z_2 \in R^d, \\
& |g(t, y_1, z_1) - g(t, y_2, z_2)| \leq \nu|y_1 - y_2| + \mu|z_1 - z_2|; \\
\text{(iii)} & g(\cdot, y, z)|_{y=0, \, z=0} \equiv 0; \\
\text{(iii')} & g(\cdot, y, 0) \equiv 0, \ \forall y \in \mathbf{R}; \\
\text{(iii'')} & g \text{ is independent of } y \in \mathbf{R}, \text{ and } g(\cdot, z)|_{z=0} \equiv 0.
\end{cases}
\tag{3.22}
$$

We introduce the following definition: Let $\xi \in L^2(\Omega, \mathcal{F}^0_\infty, P; \mathbf{R})$ and let $T \geq 0$ be such that $\xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbf{R})$. We consider the following BSDE defined on the interval $[0, T]$

$$y_t = \xi + \int_t^T g(s, y_s, z_s) ds - \int_t^T z_s dB_s, \quad t \in [0, T]. \tag{3.23}$$

Definition 27 *We define, for each $0 \leq t \leq T < \infty$ and $\xi \in L^2(\Omega, \mathcal{F}_T, P; R)$*

$$\mathcal{E}^g_{t,T}[\xi] := y_t: \ . \tag{3.24}$$

We call $\mathcal{E}^g_{t,T}[\xi] : L^2(\Omega, \mathcal{F}_T, P; R) \to L^2(\Omega, \mathcal{F}_t, P; R)$ the g–evaluation of ξ at the time t.

Remark 10 *$t \leq T$ can be also two uniformly bounded $\mathcal{F}_t$–stopping times.*

Theorem 28 *Let the function g satisfies (i)–(iii) of (3.22). Then the g–evaluation $\mathcal{E}^g_{t,T}[\cdot]$ defined in (3.24) satisfies all (i)–(iv) of Axiomatic Assumptions listed in Proposition ?? for $\mathcal{F}_t$–consistent nonlinear evaluation operators. Furthermore, we have*

$$\lim_{s \uparrow t} \mathcal{E}^g_{s,t}[\eta] = \eta, \quad \forall \eta \in L^2(\Omega, \mathcal{F}_t, P; R)$$

and, for each $Y_1, \ Y_2 \in L^2(\Omega, \mathcal{F}_T, P; R)$

$$\mathcal{E}^{-\mu,-\nu}_{t,T}[Y_1 - Y_2] \leq \mathcal{E}^g_{t,T}[Y_1] - \mathcal{E}^g_{t,T}[Y_2] \leq \mathcal{E}^{\mu,\nu}_{t,T}[Y_1 - Y_2],$$

Here $\mathcal{E}^{\mu,\nu}_{t,T}[\cdot]$ (resp. $\mathcal{E}^{-\mu,-\nu}_{t,T}[\cdot]$) stands for the $\mathcal{E}^g_{t,T}[\cdot]$ with $g = \nu|y| + \mu|z|$ (resp. $g = -\nu|y| - \mu|z|$).

Proof. (i) is directly from Comparison Theorem. (ii) is obvious. As for (iii), we multiply BSDE (3.23) by 1_A on the interval $[t, T]$. Since $g(s, 0, 0) \equiv 0$, we have

$$
\begin{aligned}
y_s 1_A &= Y 1_A + \int_s^T 1_A g(r, y_r, z_r) dr - \int_s^T 1_A z_r dB_r \\
&= Y 1_A + \int_s^T g(r, 1_A y_r, 1_A z_r) dr - \int_s^T 1_A z_r dB_r.
\end{aligned}
$$

This implies that $(1_A y_s, 1_A z_s)_{s \in [t, T]}$ is the solution this BSDE with terminal condition $Y 1_A$. Thus

$$
1_A \mathcal{E}_{s,T}^g[Y] = \mathcal{E}_{s,T}^g[1_A Y], \quad s \in [t, T].
$$

In particular, we have (iii). (iv) simply follows from the uniqueness of BSDE, i.e., for each $s \le t \le T$, we have

$$
\mathcal{E}_{s,T}^g[Y] = \mathcal{E}_{s,t}^g[y_t] = \mathcal{E}_{s,t}^g[\mathcal{E}_{t,T}^g[Y]]. \tag{3.25}
$$

(iv) is due to the continuity of y_s with respect to s. (iv) is the direct consequence of the following proposition. ∎

Proposition 29 *We assume that g_1 and g_2 satisfy (i)–(ii) of assumption (3.22). If g_1 is dominated by g_2 in the following sense*

$$
g_1(t, y, z) - g_1(t, y', z') \le g_2(t, y - y', z - z'), \quad \forall y, y' \in \mathbf{R}, \ \forall z, z' \in \mathbf{R}^d, \tag{3.26}
$$

then $\mathcal{E}^{g_1}[\cdot]$ is also dominated by $\mathcal{E}^{g_2}[\cdot]$ in the following sense: for each $T > 0$ and Y, $Y' \in L^2(\mathbf{\Omega}, \mathcal{F}_T, P)$, we have

$$
\mathcal{E}_{t,T}^{g_1}[Y] - \mathcal{E}_{t,T}^{g_1}[Y'] \le \mathcal{E}_{t,T}^{g_2}[Y - Y']. \tag{3.27}
$$

If g is dominated by itself, then $\mathcal{E}_g[\cdot]$ is also dominated by itself.

Proof. We consider the following three BSDEs

$$
\begin{aligned}
-dy_s &= g_1(s, y_s, z_s) ds - z_s dB_s, \quad y_T = Y, \\
-dy'_s &= g_1(s, y'_s, z'_s) ds - z'_s dB_s, \quad y'_T = Y'
\end{aligned}
$$

and

$$
-dY_s = g_2(s, Y_s, Z_s) ds - Z_s dB_s, \quad Y_T = Y - Y'.
$$

We denote $(\hat{y}_s, \hat{z}_s) = (y_s - y'_s, z_s - z'_s)$ and $\hat{g}_s = g_1(s, y_s, z_s) - g_1(s, y'_s, z'_s)$

$$
-d\hat{y}_s = \hat{g}_s ds - \hat{z}_s dB_s, \quad \hat{y}_T = Y - Y'.
$$

Condition (3.26) implies $g_2(s, \hat{y}_s, \hat{z}_s) \ge \hat{g}_s$. It follows from Comparison Theorem that

$$
\hat{y}_t \le Y_t, \quad \forall t \in [0, T], \text{ a.s.}
$$

By the definition of $\mathcal{E}^g[\cdot]$ it follows that (3.27) holds. ∎

3.2.3   Example: Black–Scholes evaluations

Consider a financial market consisting of $d+1$ assets: a bond and d stocks. We denote by $P_0(t)$ the price of the bond and by $P_i(t)$ the price of ith stock at time t. We assume that $P_0(\cdot)$ is the solution of the ordinary differential equation

$$dP_0(t) = r(t)P_0(t)dt, \quad P_0(0) = 1,$$

$\{P_i(\cdot)\}_{i=1}^d$ is the solution of the following SDE

$$\begin{aligned} dP_i(t) &= P_i(t)[b_i(t)dt + \sum_{j=1}^d \sigma_{ij}(t)dB_t^j], \\ P_i(0) &= p_i, \quad i = 1, \cdots, d. \end{aligned}$$

Here r is the interest rate of the bond; $\{b_i\}_{i=1}^d$ is the rate of the expected return, $\{\sigma_{ij}\}_{i,j=1}^d$ the volatility of the stocks. We assume that r, b, σ and σ^{-1} are all $\mathcal{F}_t$– adapted and uniformly bounded processes on $[0, \infty)$. The problem is how a market evaluates an European type of derivative $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ with maturity T? To solve this problem we consider an investor who has, at a time $t \leq T$, $n_0(t)$ bonds and $n_i(t)$ i-stocks, $i = 1, \cdots, d$, i.e., he invests $n_0(t)P_0(t)$ in bond and $\pi_i(t) = n_i(t)P_i(t)$: in the ith stock. $\pi(t) = (\pi_1(t), \cdots, \pi_d(t))$, $0 \leq t \leq T$ is an R^d valued, square-integrable and adapted process. We define by $y(t)$ the investor's wealth invested in the market at time t:

$$y(t) = n_0(t)P_0(t) + \sum_{i=1}^d \pi_i(t).$$

We make the so called self–financing assumption: in the period $[0, T]$, the investor does not withdraw his money from, or put his money in his account y_t. Under this condition, his wealth y. evolves according to

$$dy(t) = n_0(t)dP_0(t) + \sum_{i=1}^d n_i(t)dP_i(t).$$

or

$$dy(t) = [r(t)y(t) + \sum_{i=1}^d (b_i(t) - r(t))\pi_i(t)]dt + \sum_{i,j=1}^d \sigma_{ij}(t)\pi_i(t)dB_t^j.$$

We denote

$$g(t, y, z) = -r(t)y - \sum_{i,j=1}^d (b_i(t) - r(t))\sigma_{ij}^{-1}(t)z_j.$$

Then, by variable change $z_j(t) = \sum_{i=1}^d \sigma_{ij}(t)\pi_i(t)$, the above equation is

$$-dy(t) = g(t, y(t), z(t))dt - z(t)dB_t.$$

We observe that the function g satisfies (3.5) and (3.6). It follows from the existence and uniqueness theorem of BSDE (Theorem 19) that for each derivative $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, there exists a unique solution $(y(\cdot), z(\cdot)) \in \mathcal{M}(0, T; R^{1+d})$ with the terminal condition $y_T = \xi$. This meaning is significant: in order to replicate the derivative ξ, the investor needs and only needs to invest $y(t)$ at the present time t

and then, during the time interval $[t, T]$, to perform the strategy $\pi_i(s) = \sigma_{ij}^{-1}(s)z_j(s)$. Furthermore, by Comparison Theorem of BSDE, if he wants to replicate a ξ' which is bigger than ξ, (i.e., $\xi' \geq \xi$, a.s., $P(\xi' \geq \xi) > 0$), then he must pay more, i.e., there this is an arbitrage–free strategy. This $y(t)$ is called the Black–Scholes price, or Black–Scholes evaluation, of ξ at the time t. We define, as in (3.24) $\mathcal{E}_{t,T}^g[\xi] = y_t$. We observe that the function g satisfies (i)–(iii) of condition (3.22). It follows from Theorem 28 that $\mathcal{E}_{t,T}^g[\cdot]$ satisfies of properties (i)–(iv) for $\mathcal{F}_t$–consistent evaluation.

3.2.4 g–Expectations

A particularly interesting situation of the above stochastic semigroups is when g satisfies $g(s, y, z)|_{z=0} \equiv 0$, i.e., it satisfy (i), (ii) and (iii') in (3.22). In this situation we have the following property

Proposition 30 *For each $T > 0$, and $Y \in L^2(\Omega, \mathcal{F}_T, P; \mathbf{R})$, we have*

$$\mathcal{E}_{t,T'}^g[Y] = \mathcal{E}_{t,T}^g[Y], \ \forall T' > T. \tag{3.28}$$

Proof.   We consider the solution (y', z') of (3.23) with the same terminal condition Y, but defined on in $[0, T']$:

$$y_t' = Y + \int_t^{T'} g(s, y_s', z_s')ds - \int_t^{T'} z_s'dB_s, \quad t \in [0, T']. \tag{3.29}$$

We have $y_t' = \mathcal{E}_{t,T'}^g[Y]$. But by Assumption (3.22)–(iii'), it is easy to check (y', z') has the following form

$$(y_t', z_t') = \begin{cases} (Y, 0), & t \in (T, T'], \\ (y_t, z_t), & t \in [0, T]. \end{cases}$$

We thus have (3.28). ∎

In this case, for each $Y \in L^2(\Omega, \mathcal{F}_\infty^0, P)$, $\mathcal{E}_{t,T}^g[Y]$ does not change value with large enough T.

Definition 31 *We define*

$$\mathcal{E}_g[Y] := \mathcal{E}_{0,T}^g[Y], \ \mathcal{E}_g[Y|\mathcal{F}_t] := \mathcal{E}_{t,T}^g[Y], \ \ Y \in L^2(\Omega, \mathcal{F}_T, P). \tag{3.30}$$

$\mathcal{E}_g[Y]$ *is called g–expectation of Y. In particular, if $g = \mu|z|$ then we denote $\mathcal{E}_g[Y] = \mathcal{E}^\mu[Y]$.*

g–expectations is nonlinear but it satisfies all other properties of a classical linear expectation.

Proposition 32   *We assume that g satsfies (i), (ii) and (iii') in (3.22). Then the g–expectation defined in (3.30) is a $\mathcal{F}$ nonlinear expectation defined on $L^2(\Omega, \mathcal{F}_\infty^0, P)$. That is, $\mathcal{E}_g[\cdot]$ satisfies monotonicity (i) and constant preserving (ii) in Definition 1. Moreover, $\mathcal{E}_g[\cdot]$ is dominated by $\mathcal{E}^\mu[\cdot]$ and $\mathcal{E}^{\mu,\nu}[\cdot]$ in the following sense:*

$$-\mathcal{E}^\mu[-Y] \leq \mathcal{E}_g[Y] \leq \mathcal{E}^\mu[Y], \ \ \forall Y \in L^2(\Omega, \mathcal{F}_\infty^0, P; \mathbf{R}).$$

and

$$\mathcal{E}_{t,T}^{-\mu,-\nu}[Y_1 - Y_2] \; \leq \; \mathcal{E}_g[Y_1] - \mathcal{E}_g[Y_2] \leq \mathcal{E}_{t,T}^{\mu,\nu}[Y_1 - Y_2],$$
$$\forall Y_1, Y_2 \; \in \; L^2(\Omega, \mathcal{F}_T, P; \mathbf{R}).$$

A very interesting property is that $\mathcal{E}_g$ is an $\mathcal{F}_t$–consistent nonlinear expectation

Proposition 33 *We assume that g satsfies (i), (ii) and (iii') in (3.22). Then the g–expectation defined in (3.30) is an $\mathcal{F}_t$–consistent nonlinear expectation, where for each t, the corresponding conditional g–expectation under $\mathcal{F}_t$ is*

$$\mathcal{E}_g[Y|\mathcal{F}_t] = \mathcal{E}_{t,T}^g[Y].$$

Proof.   From this definition and Theorem 28 (ii) and (iii), it is easy to check that

$$\mathcal{E}_g[1_A \mathcal{E}_g[Y|\mathcal{F}_t]] = \mathcal{E}_g[\mathcal{E}_g[1_A Y|\mathcal{F}_t]], \;\; \forall A \in \mathcal{F}_t.$$

∎

Remark 11 *If $\tau \leq T$ is a stopping time, we define similarly*

$$\mathcal{E}_g[Y|\mathcal{F}_\tau] = \mathcal{E}_{\tau,T}[Y].$$

Definition 34 *(g-**martingales**) A process $(Y_t)_{0 \leq t \leq T}$ such that $E[Y_t^2] < \infty$ for all t is a g-martingale (resp. g-supermartingale, g-submartingale) iff*

$$\mathcal{E}_g[Y_t|\mathcal{F}_s] = Y_s, \quad (resp. \leq Y_s, \geq Y_s), \quad \forall s \leq t \leq T.$$

We shall often have to assume that g does not depend on y.

$$g = g(\omega, s, z). \tag{3.31}$$

The importance of this special setting follows from the following economically meaningful property.

Lemma 35 *Let the function g satisfies (i), (ii) and (iii') of (3.22). Then*

$$\mathcal{E}_g[Y + \eta|\mathcal{F}_t] = \mathcal{E}_g[Y|\mathcal{F}_t] + \eta, \quad \forall \eta \in L^2(\Omega, \mathcal{F}_t, P) \tag{3.32}$$

if and only if g satisfies (3.31).

Proof.   We only prove the "if" part. Consider the BSDE

$$-dy_s = g(s, z_s)ds - z_s dB_s, \ t \le s \le T,$$
$$y_T = Y.$$

We have by the definition $\mathcal{E}_g[Y|\mathcal{F}_t] = y_t$. On the other hand, it is easy to check that $(y_s', z_s') := (y_s + \eta, z_s)$, $s \in [t, T]$ solve the above equation with the terminal condition $y_T' = Y + \eta$. It then follows that

$$\mathcal{E}_g[Y + \eta] = y_t' = y_t + \eta = \mathcal{E}_g[Y|\mathcal{F}_t] + \eta.$$

■

We will always write in the sequel $\mathcal{E}^\mu[Y] \equiv \mathcal{E}_g[Y]$ for $g = \mu|z|$ and $\mathcal{E}^{-\mu}[Y] = \mathcal{E}_g[Y]$ for $g \equiv -\mu|z|$. Note that

$$\forall c > 0, \quad \mathcal{E}^\mu[cY|\mathcal{F}_t] = c\mathcal{E}^\mu[Y|\mathcal{F}_t] \tag{3.33}$$

and

$$\forall C < 0, \quad \mathcal{E}^\mu[cY|\mathcal{F}_t] = -c\mathcal{E}^\mu[-Y|\mathcal{F}_t].$$

An important feature of $\mathcal{E}^\mu[\cdot]$ is

Proposition 36 *Let g satisfy (i), (ii) and (iii') of Assumption (3.22), then $\mathcal{E}_g[\cdot]$ is dominated by $\mathcal{E}^\mu[\cdot]$ in the following sense, for each $t \ge 0$,*

$$\mathcal{E}_g[Y|\mathcal{F}_t] - \mathcal{E}_g[Y'|\mathcal{F}_t] \le \mathcal{E}^\mu[Y - Y'|\mathcal{F}_t], \ \forall Y, Y' \in L^2(\Omega, \mathcal{F}_\infty^0, P). \tag{3.34}$$

In particular, $\mathcal{E}^\mu[\cdot]$ is dominated by itself:

$$\mathcal{E}^\mu[Y|\mathcal{F}_t] - \mathcal{E}^\mu[Y'|\mathcal{F}_t] \le \mathcal{E}^\mu[Y - Y'|\mathcal{F}_t], \ \forall Y, Y' \in L^2(\Omega, \mathcal{F}_\infty^0, P). \tag{3.35}$$

Proof.   We observe that $\mathcal{E}_{t,T}^{\mu,0}[Y] = \mathcal{E}^\mu[Y|\mathcal{F}_t]$. Thus (3.34) as well as (3.35) are directly derived by (iii) of Theorem 28.

The self–domination property (3.35) of $\mathcal{E}^\mu[\cdot]$ permit us to defined a norm

Definition 37 *We define*

$$\|Y\|_\mu := \mathcal{E}^\mu[|Y|], \ Y \in L^2(\Omega, \mathcal{F}_\infty^0, P).$$

Proposition 38 $\|\cdot\|_\mu$ *forms a norm in* $L^2(\Omega, \mathcal{F}_\infty^0, P)$.

Proof.   The triangle inequality $\|Y\|_\mu + \|Z\|_\mu \le \|Y + Z\|_\mu$ follows from (3.35) with $t = 0$. By (3.33) we also have $\|cY\|_\mu \le c\|Y\|_\mu$, $c \ge 0$. ■

Proposition 39 *Under* $\|\cdot\|_\mu$, $\mathcal{E}_g[\cdot|\mathcal{F}_t]$ *is a contract mapping:*

$$\|\mathcal{E}_g[Y|\mathcal{F}_t] - \mathcal{E}_g[Y'|\mathcal{F}_t]\|_\mu \le \|Y - Y'\|_\mu.$$

Proof.   It is an easy consequence of (3.34). ∎

Proposition 40 *For each $\mu > 0$, and $T > 0$, there exist a constant $c_{\mu,T}$ such that*

$$E[|Y|] \le \mathcal{E}^{\mu}[|Y|] \le c_{\mu,T}(E[|Y|^2])^{1/2}. \tag{3.36}$$

Proof.   By the definition of

$$
\begin{aligned}
\mathcal{E}^{\mu}[|Y||\mathcal{F}_t] &= |Y| + \int_t^T \mu|Z_s|ds - \int_t^T Z_s dB_s \\
&= |Y| + \int_t^T b_\mu(s)Z_s ds - \int_t^T Z_s dB_s,
\end{aligned}
\tag{3.37}
$$

where $b_\mu(s) = \mu \frac{Z_s}{|Z_s|}1_{|Z_s|>0}$. Let Q^{μ} be the solution of SDE

$$dQ_t^{\mu} = b_\mu(t)Q_t^{\mu}dB_t, \quad Q_0^{\mu} = 1.$$

Using Itô's formula to $Q^{\mu}\mathcal{E}^{\mu}[|Y||\mathcal{F}_t]$, we have

$$\mathcal{E}^{\mu}[|Y|] = \mathcal{E}^{\mu}[|Y||\mathcal{F}_0] = E[Q_T^{\mu}|Y|] \le E[(Q_T^{\mu})^2]^{1/2}E[|Y|]^{1/2}.$$

But since $|b_\mu| \le \mu$, there exists a constant $c_{\mu,T}$ depending only on μ and T, such that $E[(Q_T^{\mu})^2]^{1/2} \le c_{\mu,T}$. We thus have the second inequality of (3.36). The first inequality is derived by taking $t = 0$ on both sides of (3.37) and then taking expectation. ∎

We then have

Corollary 41 *Let T be fixed. Then the extension $L_\mu(\Omega, \mathcal{F}_T, P)$ of $L^2(\Omega, \mathcal{F}_T, P)$ under the norm $\|\cdot\|_\mu$ is a Banach space. $L_\mu(\Omega, \mathcal{F}_T, P)$ is a closed subspace of $L^1(\Omega, \mathcal{F}_T, P)$.*

Lemma 42 *We have for all $\mu > 0$ and $Y \in L^2(\Omega, \mathcal{F}_T, P)$,*

$$E[\mathcal{E}^{\mu}[Y|\mathcal{F}_t]^2] \le e^{\mu^2(T-t)}E[Y^2].$$

Proof.   By definition,

$$\mathcal{E}^{\mu}[Y|\mathcal{F}_t] = Y + \int_t^T \mu|Z_s|ds - \int_t^T Z_s dB_s.$$

Ito's formula gives

$$\mathcal{E}^{\mu}[Y|\mathcal{F}_t]^2 = Y^2 + \int_t^T 2\mu\mathcal{E}^{\mu}[Y|\mathcal{F}_s]|Z_s|ds - 2\int_t^T \mathcal{E}^{\mu}[Y|\mathcal{F}_s]Z_s dB_s - \int_t^T Z_s^2 ds.$$

Taking expectations, we deduce that

$$
\begin{aligned}
E[\mathcal{E}^{\mu}[Y|\mathcal{F}_t]^2] &= E[Y^2] + \int_t^T E[2\mu\mathcal{E}^{\mu}[Y|\mathcal{F}_s]|Z_s|]ds - \int_t^T E[Z_s^2]ds \\
&\le E[Y^2] + \mu^2 \int_t^T E[\mathcal{E}^{\mu}[Y|\mathcal{F}_s]^2]ds
\end{aligned}
$$

(because of $2ab \leq a^2 + b^2$). The claim follows then immediately from Gronwall's inequality. ∎

We end this Section by giving an appropriate version of a downcrossing inequality given in [**?**] as Theorem 6.

Let g satisfy (3.22) and (Y_t) be a g-supermartingale on $[0,T]$. Let $0 = t_0 < t_1 < \cdots < t_n = T$, and $a < b$ be two constants. Then there exists a constant $c > 0$ such that the number $D_a^b[Y,n]$ of downcrossings of $[a,b]$ by $\{Y_{t_j}\}_{0 \leq j \leq n}$ satisfies

$$\mathcal{E}^{-\mu}[D_a^b[Y,n]] \leq \frac{c}{b-a} \mathcal{E}^{\mu}[Y_0 \wedge b].$$

Remark 12 *Contrarily to Theorem 6 in [**?**], we need not assume that Y is positive: indeed, as $g(\cdot,y,0) = 0$, one checks easily that the proof given in [**?**] can be carried over for every g-supermartingale.*

Remark 13 *This proposition allows us to prove, by classical means, that a g-supemartingale (Y_t) admits a càdlàg modification if and only if the mapping $t \to \mathcal{E}_g(Y_t)$ is right-continuous. More details on this topic will be given in Lemma 60.*

3.3 A monotonic limit theorem of BSDE

For a given stopping time $\tau \leq T < \infty$, we consider a process (y_t) the solution of the following BSDE

$$y_t = \xi + \int_{t \wedge \tau}^{\tau} g(y_s, z_s, s) ds + (A_\tau - A_{t \wedge \tau}) - \int_{t \wedge \tau}^{\tau} z_s dB_s \qquad (3.38)$$

where $\xi \in L^2(\Omega, \mathcal{F}_\tau, P)$ and A is a given RCLL increasing process with $\mathbf{E}[(A_\tau)^2] < \infty$. The following terms will be frequently used in this paper.

Definition 44 *If (y_t, z_t) is a solution of BSDE of form (3.38) then we call (y_t) a g-supersolution on $[0,\tau]$. If $A_t \equiv 0$ on $[0,\tau]$, then we call (y_t) a g-solution on $[0,\tau]$.*

We recall that a g-solution (y_t) on $[0,\tau]$ is uniquely determined if its terminal condition $y_\tau = \xi$ is given, a g-supersolution (y_t) on $[0,\tau]$ is uniquely determined if y_τ and $(A_t)_{0 \leq t \leq \tau}$ are given. If (y_t) is a g-solution on $[0,\tau]$ and (y_t') is a g-supersolution on $[0,\tau]$ such that $y_\tau \leq y_\tau'$ a.s., then for all stopping time $\sigma \leq \tau$ we have also $y_\sigma \leq y_\sigma'$.

Proposition 45 *Let (y_t) be a g-supersolution defined on an interval $[0,\tau]$. Then there is a unique $(z_t) \in L^2(0,\tau; \mathbf{R}^d)$ and a unique increasing RCLL process (A_t) on $[0,\tau]$ with $\mathbf{E}[(A_\tau)^2] < \infty$ such that the triple (y_t, z_t, A_t) satisfies (3.38).*

Proof.   If both (y_t, z_t, A_t) and (y_t, z_t', A_t') satisfy (3.38), then we apply Itô's formula to $(y_t - y_t)^2 (\equiv 0)$ on $[0, \tau]$ and take expectation:

$$\mathbf{E} \int_0^\tau |z_t - z_t'|^2 ds + \mathbf{E}[\sum_{t \in (0, \tau]} (\Delta(A_t - A_t'))^2] = 0.$$

Thus $z_t \equiv z_t'$. From this it follows that $A_t \equiv A_t'$. ∎

Thus we can define

Definition 46 *Let (y_t) be a supersolution on $[0, \tau]$ and let (y_t, A_t, z_t) be the related unique triple in the sense of BSDE (1.7). Then we call (A_t, z_t) the (unique) decomposition of (y_t).*

Let us now consider the following sequence of g-supersolution (y_t^i) on $[0, T]$ i.e.,

$$y_t^i = y_T^i + \int_t^T g(y_s^i, z_s^i, s) ds + (A_T^i - A_t^i) - \int_t^T z_s^i dB_s, \qquad i = 1, 2, \cdots. \qquad (3.39)$$

Here the function g satisfies (3.22) and (A_t^i) are RCLL increasing processes with $A_0^i = 0$ and $E[(A_T^i)^2] < \infty$.

The following theorem prove that the limit of shows that the limit of $\{y^i\}_{i=1}^\infty$ is still a g–supersolution.

Theorem 47 *We assume that g satisfies (i) and (ii) of Assumptions (3.22). For each $i = 1, 2, \cdots$, A^i be an RCLL increasing processes with $A_0^i = 0$ and $E[(A_T^i)^2] < \infty$ and (y^i, z^i) be the solution of BSDE (3.39). If, as $i \to \infty$, $\{y^i\}_{i=1}^\infty$ converges monotonically up to a process y with $E[\mathrm{esssup}_{0 \le t \le T} |y_t|^2] < \infty$. Then this limit (y_t) is still a g-supersolution, i.e., there exists $z \in \mathcal{M}(0, T; R^d)$ and an RCLL increasing process with $E[(A_T)^2] < \infty$ such that*

$$y_t = y_T + \int_t^T g(y_s, z_s, s) ds + (A_T - A_t) - \int_t^T z_s dB_s, \quad t \in [0, T], \qquad (3.40)$$

To prove this theorem, we need following lemma. The lemma says that both $\{z^i\}$ and $\{(A_T^i)^2\}$ are uniformly bounded in L^2:

Under the assumptions of Theorem 47, there exists a constant C that is independent of i such that

$$\begin{array}{ll} (i) & \mathbf{E} \int_0^T |z_s^i|^2 ds \le C, \\ (ii) & \mathbf{E}[(A_T^i)^2] \le C. \end{array} \qquad (3.41)$$

Proof. From BSDE (3.39), we have

$$
\begin{aligned}
A_T^i &= y_0^i - y_T^i - \int_0^T g(y_s^i, z_s^i, s)ds + \int_0^T z_s^i dB_s \\
&\leq |y_0^i| + |y_T^i| + \int_0^T [\mu|y_s^i| + \mu|z_s^i| + |g(0,0,s)|]ds + |\int_0^T z_s^i dB_s|.
\end{aligned}
$$

We observe that $|y_t^i|$ is dominated by $|y_t^1| + |y_t|$. Thus there exists a constant, independent of i, such that

$$
\mathbf{E}[\sup_{0\leq t\leq T} |y_t^i|^2] \leq C. \tag{3.42}
$$

It follows that, there exists a constant C_1, independent of i, such that

$$
\mathbf{E}|A_T^i|^2 \leq C_1 + 2\mathbf{E}\int_0^T |z_s^i|^2 ds. \tag{3.43}
$$

On the other hand, we use Itô's formula applied to $|y_t^i|^2$:

$$
|y_0^i|^2 + \mathbf{E}\int_0^T |z_s^i|^2 ds = \mathbf{E}|y_T^i|^2 + 2\mathbf{E}\int_0^T y_s^i g(y_s^i, z_s^i, s)ds + 2\mathbf{E}\int_0^T y_s^i dA_s^i
$$

The last two terms are bounded by

$$
\begin{aligned}
2y_s^i g(y_s^i, z_s^i, s)ds &\leq 2|y_s^i|(\mu|y_s^i| + \mu|z_s^i| + |g(0,0,s)|) \\
&\leq 2(\mu + \mu^2)|y_s^i|^2 + \frac{1}{2}|z_s^i| + |g(0,0,s)|
\end{aligned}
$$

and $2\mathbf{E}\int_0^T |y_s^i|dA_s^i \leq 2[\mathbf{E}\sup_{0\leq s\leq T} |y_s^i|^2]^{1/2}[\mathbf{E}|A_T^i|^2]^{1/2}$. Thus

$$
\begin{aligned}
\mathbf{E}\int_0^T |z_s^i|^2 ds &\leq C + 4[\mathbf{E}\sup_{0\leq s\leq T} |y_s^i|^2]^{1/2}[\mathbf{E}|A_T^i|^2]^{1/2} \\
&\leq C + 16\mathbf{E}[\sup_{0\leq s\leq T} |y_s^i|^2] + \frac{1}{4}\mathbf{E}|A_T^i|^2 \\
&= C_1 + \frac{1}{4}\mathbf{E}|A_T^i|^2,
\end{aligned}
$$

where, from (3.42), the constants C and C_1 are independent of i. This with (3.43) it follows that (3.41)–(i) and then (3.41)–(ii) hold true. The proof is complete. ∎

Combining this Lemma with Theorem 80, we can easily prove Theorem 47.

Proof of Theorem 47. In (3.39), we set $g_t^i := -g(y_t^i, z_t^i, t)$; Since $\{z^i\}$ is bounded in $\mathcal{M}(0,T)$, thanks to the monotonic limit theorem of Itô processes (see Appendix: Theorem 80), there exists a $(z_t) \in \mathcal{M}(0,T;\mathbf{R}^d)$ such that, for each $p \in [0,2)$, (z^i) strongly converges to (z) in $L_{\mathcal{F}}^p(0,T)$.

As result, $\{g^i\} = \{-g(y^i, z^i, \cdot)\}$ strongly converges in $L_{\mathcal{F}}^p(0,T;\mathbf{R}^d)$ to g^0 and

$$
g^0(s) = -g(y_s, z_s, s), \quad \text{a.s., a.e.}
$$

From this it follows immediately that (y_t, z_t) is the solution of the BSDE (3.40). ∎

3.4 g–Martingale and decomposition theorem

An $\mathcal{F}_t$-progressively measurable real-valued process Y with

$$\mathbf{E}[\operatorname{ess\,sup}_{0 \leq t \leq T} |Y_t|^2] < \infty, \ \forall T < \infty.$$

is called a $\mathcal{E}$–supermartingale (resp. $\mathcal{E}$–submartingale) in the strong sense if, for each $T < \infty$ and for each stopping times$\sigma \leq \tau \leq T$,

$$\mathcal{E}_{\sigma,\tau}[Y_\tau] \leq Y_\sigma, \ (\text{resp. } \geq Y_\sigma) \text{ a.s.}$$

Y is called a $\mathcal{E}$–supermartingale (resp. $\mathcal{E}$–submartingale) in the weak sense if, for each $0 \leq t \leq T < \infty$

$$\mathcal{E}_{t,T}[Y_T] \leq Y_t, \ (\text{resp. } \geq Y_t) \text{ a.s.}$$

Certainly, An $\mathcal{E}$-supermartingale in strong sense is also a $\mathcal{E}$-supermartingale in weak sense. It is already shown that, under assumptions similar to the classical case, a $\mathcal{E}^g$-supermartingale in weak sense coincides with a $\mathcal{E}^g$-supermartingale in strong sense (see [CP]). This result corresponds the so-called Optional Stopping Theorem in theory of martingales.

In this section we will consider $\mathcal{E}^g$–supermartingales. By Comparison Theorem of BSDE, it is easy to prove the following result

Proposition 48 *We assume that g satisfies (i) and (ii) of (3.22). Let $(A_t)_{0 \leq t < \infty}$ be an RCLL increasing (resp. decreasing) process with $\mathbf{E}[(A_T)^2] < \infty$ for each $T > 0$. Let (y, z) be the solution of the following BSDE, for each $T > 0$,*

$$y_t = y_T + \int_t^T g(y_s, z_s, s)ds + (A_T - A_t) - \int_t^T z_s dB_s, \quad t \in [0, T], \tag{3.44}$$

Then $(y_t)_{0 \leq t \leq T}$ is a $\mathcal{E}^g$–supermartingale (resp. $\mathcal{E}^g$–supermartingale).

In this section we are concerned with the inverse problem: can we say that a right-continuous $\mathcal{E}^g$-supermartingale is also a $\mathcal{E}^g$-supersolution? This problem is more difficult since it is in fact a nonlinear version of Doob-Meyer Decomposition Theorem. We claim

Theorem 49 *We assume that g satisfies (i) and (ii) of (3.22). Let (Y_t) be a right-continuous $\mathcal{E}^g$–supermartingale on $[0, T]$ in the strong sense and let*

$$\mathbf{E}[\sup_{0 \leq t \leq T} |Y_t|^2] < \infty, \ \forall T > 0.$$

Then (Y_t) is an g-supersolution: there exists a unique RCLL increasing process (A_t) witt $\mathbf{E}[(A_T)^2] < \infty$, for each $T > 0$, such that (Y_t) coincides with the unique solution (y_t) of the BSDE. For each $T > 0$,

$$y_t = Y_T + \int_t^T g(y_s, z_s, s)ds + (A_T - A_t) - \int_t^T z_s dB_s, \quad t \in [0, T], \tag{3.45}$$

In order to prove this theorem, we consider the following family of BSDE parameterized by $i = 1, 2, \cdots$.

$$y_t^i = Y_T + \int_t^T g(y_s^i, z_s^i, s)ds + i \int_t^T (Y_s - y_s^i)ds - \int_t^T z_s^i dB_s. \qquad (3.46)$$

An important observation is that, for each i, (y_t^i) is bounded from above by (Y_t). Thus (y^i) is a $\mathcal{E}$-supersolution on $[0, T]$. Under this observation, (3.46) becomes a penalization problem introduced in [ELal].

Lemma 50 *We have, for each $i = 1, 2, \cdots$,*

$$Y_t \geq y_t^i.$$

Proof.

Proof. For a $\delta > 0$ and a given integer $i > 0$, we define

$$\sigma^{i,\delta} := \inf\{t; \ y_t^i \geq Y_t + \delta\} \wedge T.$$

If $P(\sigma^{i,\delta} < T) = 0$, for all i and δ, then the proof is done. If it is not the case, then there exist $\delta > 0$ and a positive integer i such that $P(\sigma^{i,\delta} < T) > 0$. We can then define the following stopping times

$$\tau := \inf\{t \geq \sigma^{i,\delta}; \ y_t^i \leq Y_t\}$$

It is clear that $\sigma^{i,\delta} \leq \tau \leq T$. Since $Y_\cdot - y_\cdot^i$ is RCLL, we have

$$y_\tau^i \leq Y_\tau.$$

But since $(Y(s) - y^i(s)) \leq 0$ on $[\sigma^{i,\delta}, \tau]$, by monotonicity of $\mathcal{E}^g[\cdot]$,

$$
\begin{aligned}
y_{\sigma^{i,\delta}}^i &\leq \mathcal{E}_{\sigma^{i,\delta},\tau}^g[y_\tau^i | \mathcal{F}_{\sigma^{i,\delta}}] \\
&\leq \mathcal{E}_{\sigma^{i,\delta},\tau}^g[Y_\tau | \mathcal{F}_{\sigma^{i,\delta}}] \\
&\leq Y_{\sigma^{i,\delta}} \qquad \text{(since } Y \text{ is an } \mathcal{E}^g\text{–supermartingale)}
\end{aligned}
$$

But on the other hand, we have $P(\sigma^{i,\delta} < T) > 0$ and, by the definition of $\sigma^{i,\delta}$, $y_{\sigma^{i,\delta}}^i \geq Y_{\sigma^{i,\delta}} + \delta$ on $\{\sigma^{i,\delta} < T\}$. This induces a contradiction. The proof is complete. ∎

Remark 14 *From the above result, the term $i(Y_s - y_s^i)$ in (3.46) equals to $i(Y_s - y_s^i)^+$. By Comparison Theorem y^i are pushed up to be above the supermartingale (Y_t), but in fact they can never surpass (Y_t). We will see that this effect will force y^i to converge to the supermartingale (Y_t) itself. Thus, by Limit Theorem 47 (Y) itself is also a form of (3.45). Specifically, we have:*

Proof of Theorem 49. The uniqueness is due to the uniqueness of $\mathcal{E}$-supersolution i.e. Prop. 1.6. We now prove the existence. We can rewrite BSDE (3.46) as

$$y_t^i = Y_T + \int_t^T g(y_s^i, z_s^i, s)ds + A_T^i - A_t^i - \int_t^T z_s^i dB_s,$$

where we denote

$$A_t^i := i \int_0^t (Y_s - y_s^i)ds.$$

From Lemma 50, $Y_t - y_t^i = |Y_t - y_t^i|$. It follows from the Comparison Theorem that $y_t^i \leq y_t^{i+1}$. Thus $\{y^i\}$ is a sequence of continuous $\mathcal{E}^g$-supermartingale that is monotonically converges up to a process (y_t). Moreover (y_t) is bounded from above by Y_t. It is then easy to check that all conditions in Theorem 47 are satisfied. (y_t) is a $\mathcal{E}^g$-supersolution on $[0, T]$ of the following form.

$$y_t = Y_T + \int_t^T g(y_s, z_s, s)ds + (A_T - A_t) - \int_t^T z_s dB_s, \quad t \in [0, T],$$

where (A_t) is a RCLL increasing process. It then remains to prove that $y = Y$. From Lemma 3.3–(ii) we have

$$\mathbf{E}|A_T^i|^2 = i^2 \mathbf{E}\left[\int_0^T |Y_t - y_t^i|dt\right]^2 \leq C.$$

It then follows that $Y_t \equiv y_t$. The proof is complete ∎.

Chapter 4

FINDING THE MECHANISM: IS AN $\mathcal{F}$–EXPECTATION A G–EXPECTATION?

4.1 $\mathcal{E}^\mu$-dominated $\mathcal{F}$-expectations

We will study now $\mathcal{F}$-expectations dominated by $\mathcal{E}^\mu$, for some large enough $\mu > 0$, according to the following

Definition 51 ($\mathcal{E}^\mu$-domination) *Given $\mu > 0$, we say that an $\mathcal{F}$-expectation $\mathcal{E}$ is dominated by $\mathcal{E}^\mu$ if*

$$\mathcal{E}[X + Y] - \mathcal{E}[X] \leq \mathcal{E}^\mu[Y], \ \forall X, Y \in L^2(\Omega, \mathcal{F}_T, P) \tag{4.1}$$

By Proposition 29, for any g satisfying (i), (ii) (iii") of (3.22), the associated g-expectation is dominated by $\mathcal{E}^\mu$, where μ is the Lipschitz constant in (3.22).

Lemma 52 *If $\mathcal{E}$ is dominated by $\mathcal{E}^\mu$ for some $\mu > 0$, then*

$$\mathcal{E}^{-\mu}[Y] \leq \mathcal{E}[X + Y] - \mathcal{E}[X] \leq \mathcal{E}^\mu[Y]. \tag{4.2}$$

Proof. It is a simple consequence of

$$\mathcal{E}^{-\mu}[Y|\mathcal{F}_t] = -\mathcal{E}^\mu[-Y|\mathcal{F}_t].$$

∎

Lemma 53 *If $\mathcal{E}$ is dominated by $\mathcal{E}^\mu$ for some $\mu > 0$, then $\mathcal{E}[\cdot]$ is a continuous operator on $L^2(\Omega, \mathcal{F}_T, P)$ in the following sense:*

$$\exists C > 0, \quad |\mathcal{E}[\xi_1] - \mathcal{E}[\xi_2]| \leq C \, \|\xi_1 - \xi_2\|_{L^2}, \quad \forall \xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}_T, P). \tag{4.3}$$

Proof. The claim follows easily from Lemma 52 above and Lemma 42. ∎

From now on we will deal with $\mathcal{F}$-expectations $\mathcal{E}[\cdot]$ also satisfying the following condition:

$$\mathcal{E}[X + Y|\mathcal{F}_t] = \mathcal{E}[X|\mathcal{F}_t] + Y, \quad \forall X \in L^2(\Omega, \mathcal{F}_T, P) \quad \text{and} \quad Y \in L^2(\Omega, \mathcal{F}_t, P) \tag{4.4}$$

Recall that, when $\mathcal{E}[\cdot]$ is a g-expectation, (4.4) means that g satisfies (3.31). We observe that an expectation $E_Q[\cdot]$ under a Girsanov transformation $\dfrac{dQ}{dP}$ satisfies this assumption.

Our first result connected to (4.4) will consist in deducing '$\mathcal{E}^\mu$-domination at time t' from (4.1). This will be correctly stated and proved in Lemma 55, but we need first to introduce some new notation.

For a given $\zeta \in L^2(\Omega, \mathcal{F}_T, P)$, we consider the mapping $\mathcal{E}_\zeta[\cdot]$ defined by

$$\mathcal{E}_\zeta[X] = \mathcal{E}[X + \zeta] - \mathcal{E}[\zeta] : L^2(\Omega, \mathcal{F}_T, P) \longmapsto R. \tag{4.5}$$

Lemma 54 *If $\mathcal{E}[\cdot]$ is an $\mathcal{F}$-expectation satisfying (4.1) and (4.4), then the mapping $\mathcal{E}_\zeta[\cdot]$ is also an $\mathcal{F}$-expectation satisfying (4.1) and (4.4). Its conditional expectation under $\mathcal{F}_t$ is*

$$\mathcal{E}_\zeta[X|\mathcal{F}_t] = \mathcal{E}[X + \zeta|\mathcal{F}_t] - \mathcal{E}[\zeta|\mathcal{F}_t]. \tag{4.6}$$

Proof.   It is easily seen that $\mathcal{E}_\zeta[\cdot]$ is a nonlinear expectation.

We now prove that the notion $\mathcal{E}_\zeta[X|\mathcal{F}_t]$ defined in (4.6) is actually the conditional $\mathcal{F}$-expectation induced by $\mathcal{E}_\zeta[\cdot]$ under $\mathcal{F}_t$.

Indeed, put $G(X, \zeta, \mathcal{F}_t) = \mathcal{E}[X + \zeta|\mathcal{F}_t] - \mathcal{E}[\zeta|\mathcal{F}_t]$. We want to show that, for all $A \in \mathcal{F}_t$, $\mathcal{E}_\zeta(G(X, \zeta, \mathcal{F}_t)1_A) = \mathcal{E}_\zeta(X1_A)$. Computations give:

$$
\begin{aligned}
\mathcal{E}_\zeta[G(X, \zeta, \mathcal{F}_t)] &= \mathcal{E}[\mathcal{E}[X + \zeta|\mathcal{F}_t] - \mathcal{E}[\zeta|\mathcal{F}_t] + \zeta] - \mathcal{E}[\zeta] \quad \text{(by (2.5))} \\
&= \mathcal{E}[\mathcal{E}[X + \zeta|\mathcal{F}_t] - \mathcal{E}[\zeta|\mathcal{F}_t] + \mathcal{E}[\zeta|\mathcal{F}_t]] - \mathcal{E}[\zeta] \quad \text{(by (4.4))} \\
&= \mathcal{E}[\mathcal{E}[X + \zeta|\mathcal{F}_t]] - \mathcal{E}[\zeta] \\
&= \mathcal{E}[X + \zeta] - \mathcal{E}[\zeta].
\end{aligned}
$$

Thus we have

$$\mathcal{E}_\zeta[G(X, \zeta, \mathcal{F}_t)] = \mathcal{E}_\zeta[X], \quad \forall X. \tag{4.7}$$

Now for each $A \in \mathcal{F}_t$, we have,

$$
\begin{aligned}
G(X1_A, \zeta, \mathcal{F}_t) &= \mathcal{E}[X1_A + \zeta1_A + \zeta1_{A^c}|\mathcal{F}_t] - \mathcal{E}[\zeta|\mathcal{F}_t] \\
&= \mathcal{E}[(X + \zeta)1_A + \zeta1_{A^c}|\mathcal{F}_t] - \mathcal{E}[\zeta|\mathcal{F}_t] \\
&= \mathcal{E}[X + \zeta|\mathcal{F}_t]1_A + \mathcal{E}[\zeta|\mathcal{F}_t]1_{A^c} - \mathcal{E}[\zeta|\mathcal{F}_t] \\
&= (\mathcal{E}[X + \zeta|\mathcal{F}_t] - \mathcal{E}[\zeta|\mathcal{F}_t])1_A \\
&= G(X, \zeta, \mathcal{F}_t)1_A.
\end{aligned}
$$

From this with (4.7) it follows that $\mathcal{E}_\zeta[X|\mathcal{F}_t]$ satisfies (2.3):

$$\mathcal{E}_\zeta[G(X, \zeta, \mathcal{F}_t)1_A] = \mathcal{E}_\zeta[G(X1_A, \zeta, \mathcal{F}_t)] = \mathcal{E}_\zeta[X1_A], \quad \forall A \in \mathcal{F}_t.$$

Thus $\mathcal{E}_\zeta[\cdot]$ is an $\mathcal{F}$-expectation with $\mathcal{E}_\zeta[\cdot|\mathcal{F}_t]$ given by (4.6).

We now check that (4.1) is satisfied. For each $X, Y \in L^2(\Omega, \mathcal{F}_T, P)$,

$$
\begin{aligned}
\mathcal{E}_\zeta[X + Y] - \mathcal{E}_\zeta[X] &= (\mathcal{E}[X + Y + \zeta] - \mathcal{E}[\zeta]) - (\mathcal{E}[X + \zeta] - \mathcal{E}[\zeta]) \\
&= \mathcal{E}[X + Y + \zeta] - \mathcal{E}[X + \zeta].
\end{aligned}
$$

Since $\mathcal{E}[\cdot]$ satisfies (4.1), $\mathcal{E}_\zeta[\cdot]$ satisfies

$$
\mathcal{E}_\zeta[X + Y] - \mathcal{E}_\zeta[X] \le \mathcal{E}^\mu[Y].
$$

Finally, let $Y \in L^2(\Omega, \mathcal{F}_t, P)$; since $\mathcal{E}[\cdot]$ satisfies property (4.4), thus

$$
\begin{aligned}
\mathcal{E}_\zeta[X + Y|\mathcal{F}_t] &= \mathcal{E}[X + \zeta|\mathcal{F}_t] - \mathcal{E}[\zeta|\mathcal{F}_t] + Y \\
&= \mathcal{E}_\zeta[X|\mathcal{F}_t] + Y.
\end{aligned}
$$

Thus $\mathcal{E}_\zeta[\cdot]$ also satisfies property (4.4). The proof is complete. ∎

Lemma 55 *Let $\mathcal{E}[\cdot]$ be an $\mathcal{F}$-expectation satisfying (4.1) and (4.4). Then, for each $t \le T$, we have a.s.*

$$
\mathcal{E}^{-\mu}[X|\mathcal{F}_t] \le \mathcal{E}_\zeta[X|\mathcal{F}_t] \le \mathcal{E}^\mu[X|\mathcal{F}_t], \ \forall X, \ \zeta \in L^2(\Omega, \mathcal{F}_T, P).
$$

This lemma is a simple consequence of the following one, whose proof is inspired by [1].

Lemma 56 *Let $\mathcal{E}_1[\cdot]$ and $\mathcal{E}_2[\cdot]$ be two $\mathcal{F}$-expectations satisfying (4.1) and (4.4). If*

$$
\mathcal{E}_1[X] \le \mathcal{E}_2[X], \quad \forall X \in L^2(\Omega, \mathcal{F}_T, P),
$$

then a.s. and for all t,

$$
\mathcal{E}_1[X|\mathcal{F}_t] \le \mathcal{E}_2[X|\mathcal{F}_t], \quad \forall X \in L^2(\Omega, \mathcal{F}_T, P).
$$

Proof. Indeed, for all $Y \in L^2(\mathcal{F}_T)$, we have by (4.4)

$$
\begin{aligned}
\mathcal{E}_1[Y - \mathcal{E}_1[Y|\mathcal{F}_t]] &= \mathcal{E}_1[\mathcal{E}_1[Y - \mathcal{E}_1[Y|\mathcal{F}_t]|\mathcal{F}_t]] \\
&= \mathcal{E}_1[\mathcal{E}_1[Y|\mathcal{F}_t] - \mathcal{E}_1[Y|\mathcal{F}_t]] \\
&= \mathcal{E}_1[0] = 0.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\mathcal{E}_1[Y - \mathcal{E}_1[Y|\mathcal{F}_t]] &\le \mathcal{E}_2[Y - \mathcal{E}_1[Y|\mathcal{F}_t]] \\
&= \mathcal{E}_2[\mathcal{E}_2[Y - \mathcal{E}_1[Y|\mathcal{F}_t]|\mathcal{F}_t]].
\end{aligned}
$$

Thus
$$\mathcal{E}_2[\mathcal{E}_2[Y|\mathcal{F}_t] - \mathcal{E}_1[Y|\mathcal{F}_t]] \geq 0, \quad \forall Y \in L^2(\mathcal{F}_T).$$

Now, for a fixed $X \in L^2(\mathcal{F}_T)$, we set $\eta = \mathcal{E}_2[X|\mathcal{F}_t] - \mathcal{E}_1[X|\mathcal{F}_t]$. Since

$$
\begin{aligned}
\eta 1_{\{\eta<0\}} &= 1_{\{\eta<0\}}\mathcal{E}_2[X|\mathcal{F}_t] - 1_{\{\eta<0\}}\mathcal{E}_1[X|\mathcal{F}_t] \\
&= \mathcal{E}_2[X1_{\{\eta<0\}}|\mathcal{F}_t] - \mathcal{E}_1[X1_{\{\eta<0\}}|\mathcal{F}_t],
\end{aligned}
$$

we have then
$$\mathcal{E}_2[\eta 1_{\{\eta<0\}}] = 0.$$

But since $\eta 1_{\{\eta<0\}} \leq 0$, it follows from the strict monotonicity of $\mathcal{E}_2[\cdot]$ that $\eta 1_{\{\eta<0\}} = 0$ a.s.. Thus
$$\mathcal{E}_2[X|\mathcal{F}_t] - \mathcal{E}_1[X|\mathcal{F}_t] \geq 0 \quad \text{a.s.}$$

The proof is complete. ∎

Lemma 57 *If $\mathcal{E}$ meets (4.1) and (4.4), there exists a positive constant C such that, for all X and Y in $L^2(\Omega, \mathcal{F}_T, P)$, and for all $t \geq 0$,*

$$\mathcal{E}[\mathcal{E}[X + Y|\mathcal{F}_t] - \mathcal{E}[X|\mathcal{F}_t]] \leq C\|Y\|_{L^2}.$$

Proof.   Indeed, Lemmas 54 and 55 above imply that

$$
\begin{aligned}
\mathcal{E}[\mathcal{E}[X + Y|\mathcal{F}_t] - \mathcal{E}[X|\mathcal{F}_t]] &= \mathcal{E}[\mathcal{E}_X[Y|\mathcal{F}_t]] \\
&\leq \mathcal{E}[\mathcal{E}^\mu[Y|\mathcal{F}_t]] \\
&\leq \mathcal{E}^\mu[\mathcal{E}^\mu[Y|\mathcal{F}_t]] = \mathcal{E}^\mu[Y] \leq C\|Y\|_{L^2}.
\end{aligned}
$$

(Last equality coming from Lemma 53)
∎

4.2 $\mathcal{F}_t$-consistent martingales

In this section we assume that $\mathcal{E}$ is an $\mathcal{F}$-expectation satisfying (4.1) for some $\mu > 0$, and (4.4) as well.

Definition 58 *A process $(X_t)_{t \in [0,T]} \in L^2_{\mathcal{F}}(0, T)$ is called an $\mathcal{E}$-martingale (resp. $\mathcal{E}$-supermartingale, -submartingale) if for each $0 \leq s \leq t \leq T$*

$$X_s = \mathcal{E}[X_t|\mathcal{F}_s], \ (\text{resp. } \geq \mathcal{E}[X_t|\mathcal{F}_s], \ \leq \mathcal{E}[X_t|\mathcal{F}_s]).$$

Lemma 59 *An $\mathcal{E}^\mu$-supermartingale (ξ_t) is both an $\mathcal{E}$–supermartingale and $\mathcal{E}^{-\mu}$-supermartingale. An $\mathcal{E}^{-\mu}$-submartingale (ξ_t) is both an $\mathcal{E}$- and $\mathcal{E}^\mu$-submartingale. An $\mathcal{E}$-martingale (ξ_t) is an $\mathcal{E}^{-\mu}$-supermartingale and an $\mathcal{E}^\mu$-submartingale.*

Proof.   It comes simply from the fact that, for each $0 \leq s \leq t \leq T$,

$$\mathcal{E}^{-\mu}[\xi_t|\mathcal{F}_s] \leq \mathcal{E}[\xi_t|\mathcal{F}_s] \leq \mathcal{E}^{\mu}[\xi_t|\mathcal{F}_s].$$

∎

Next result is the first step in a procedure that will eventually prove that every $\mathcal{E}$-martingale admits continuous paths.

Lemma 60 *For each $X \in L^2(\Omega, \mathcal{F}_T, P)$ the process $\mathcal{E}[X|\mathcal{F}_t]$, $t \in [0,T]$ admits a unique modification with a.s. càdlàg paths.*

Proof.   We can deduce from Lemma 59 that the process $\mathcal{E}[X|\mathcal{F}_t]$, $t \in [0,T]$, is an $\mathcal{E}^{-\mu}$-supermartingale. Hence we can apply the downcrossing inequality recalled in Proposition 43
This dowcrossing equality tells us that $\mathcal{E}[X|\mathcal{F}_t]$, $t \in [0,T]$ has P-a.s. finitely many downcrossings of every interval $[a,b]$ with rational $a < b$. By classical methods, this imply the almost sure existence of left and right limits for the paths of $\mathcal{E}[X|\mathcal{F}.]$.
Define now $Y_t = \lim\limits_{\substack{s \searrow t \\ s \in \mathbf{Q} \cap [0,T]}} \mathcal{E}[X|\mathcal{F}_s]$, whose existence a.s. has just been proved.
Take A in $\mathcal{F}_t$, we have that

$$Y_t 1_A = \lim_{\substack{s \searrow t \\ s \in \mathbf{Q} \cap [0,T]}} \mathcal{E}[X|\mathcal{F}_s]1_A,$$

the above limit being taken in L^2. From Lemma 53, it follows that

$$\mathcal{E}[Y_t 1_A] = \lim_{\substack{s \searrow t \\ s \in \mathbf{Q} \cap [0,T]}} \mathcal{E}[\mathcal{E}[X|\mathcal{F}_s 1_A]|.$$

But

$$\begin{aligned}
\mathcal{E}[\mathcal{E}[X|\mathcal{F}_s]1_A] &= \mathcal{E}[\mathcal{E}[\mathcal{E}[X|\mathcal{F}_s 1_A]|\mathcal{F}_t]] \\
&= \mathcal{E}[1_A \mathcal{E}[\mathcal{E}[X|\mathcal{F}_s]|\mathcal{F}_t g]] \\
&= \mathcal{E}[1_A \mathcal{E}[X|\mathcal{F}_t]].
\end{aligned}$$

It follows that a.s. $Y_t = \mathcal{E}[X|\mathcal{F}_t]$.
Now it's again classical to prove, using the existence of left and right limits, that the process Y defined above is a càdlàg modification of $\mathcal{E}[X|\mathcal{F}_t]$, $t \in [0,T]$, and the lemma is proved. ∎

Henceforth, and without needing to recall it, we will always consider the càdlàg modifications of the $\mathcal{E}$-martingales we have to deal with.

Lemma 60 has an immediate consequence as follows :

Lemma 61 *Let $\mathcal{E}[\cdot]$ be an $\mathcal{F}$-expectation satisfying (4.1) and (4.4). Then for each $X \in L^2(\Omega, \mathcal{F}_T, P)$ and $g \in L^2_{\mathcal{F}}(0,T)$ the process $\mathcal{E}[X + \int_t^T g_s ds|\mathcal{F}_t]$, $t \in [0,T]$ is càdlàg a.s.*

Proof.   Indeed, we can write

$$
\begin{aligned}
\mathcal{E}[X + \int_t^T g_s ds | \mathcal{F}_t] &= \mathcal{E}[X + \int_0^T g_s ds - \int_0^t g_s ds | \mathcal{F}_t] \\
&= \mathcal{E}[X + \int_0^T g_s ds | \mathcal{F}_t] - \int_0^t g_s ds
\end{aligned}
$$

because of (4.4). The claim follows then easily from Lemma 60. ■

Lemma 62 *For each $X \in L^2(\Omega, \mathcal{F}_T, P)$, let*

$$
y_t = \mathcal{E}[X | \mathcal{F}_t].
$$

Then there exists a pair $(g(\cdot), z(\cdot)) \in L^2_{\mathcal{F}}(0, T; R \times R^d)$ with

$$
|g_t| \leq \mu |z_t| \tag{4.8}
$$

such that

$$
y_t = X + \int_t^T g_s ds - \int_t^T z_s dB_s. \tag{4.9}
$$

Furthermore, take $X' \in L^2(\Omega, \mathcal{F}_T, P)$, put $y_t' = \mathcal{E}[X' | \mathcal{F}_t]$, and let $(g'(\cdot), z'(\cdot)) \in L^2_{\mathcal{F}}(0, T; R \times R^d)$ be the corresponding pair. Then we have

$$
|g_t - g_t'| \leq \mu |z_t - z_t'| \tag{4.10}
$$

Proof.   Since
$$
y_t = \mathcal{E}[X | \mathcal{F}_t], \quad 0 \leq t \leq T,
$$
is an $\mathcal{E}$-martingale, and since it is càdlàg, it is a right-continuous $\mathcal{E}^\mu$-submartingale (resp. $\mathcal{E}^{-\mu}$-supermartingale) and we know from the g-supermartingale decomposition theorem (Theorem 49) that there exist (z^μ, A^μ) and $(z^{-\mu}, A^{-\mu})$ in $L^2_{\mathcal{F}}([0, T]; R \times R^d)$ with A^μ and $A^{-\mu}$ càdlàg and increasing such that $A^\mu(0) = 0$, $A^{-\mu}(0) = 0$ and

$$
y_t = y_T + \int_t^T \mu |z_s^\mu| ds - A_T^\mu + A_t^\mu - \int_t^T z_s^\mu dB_s.
$$

$$
y_t = y_T - \int_t^T \mu |z_s^{-\mu}| ds + A_T^{-\mu} - A_t^{-\mu} - \int_t^T z_s^{-\mu} dB_s.
$$

Hence,

$$
\begin{aligned}
z_t^\mu &\equiv z_t^{-\mu}, \\
-\mu |z_t^\mu| dt + dA_t^\mu &\equiv \mu |z_t^\mu| dt - dA_t^{-\mu},
\end{aligned}
$$

whence

$$
2\mu |z_t^\mu| dt \equiv dA_t^\mu + dA_t^{-\mu}.
$$

It follows that A^μ and $A^{-\mu}$ are both absolutely continuous and we can write:

$$dA_t^\mu = a_t^\mu dt, \quad dA_t^{-\mu} = a_t^{-\mu} dt$$

with

$$0 \le a_t^\mu, \quad 0 \le a_t^{-\mu}.$$

We also have

$$a_t^\mu + a_t^{-\mu} \equiv 2\mu |z_t^\mu|,$$

so, if we define

$$
\begin{aligned}
z_t &= z_t^\mu \\
g_t &= \mu|z_t| - a_t^\mu,
\end{aligned}
$$

we get (4.9) and (4.8).

Now, we prove (4.10). We have

$$
\begin{aligned}
y_t - y_t' &= \mathcal{E}[X|\mathcal{F}_t] - \mathcal{E}[X'|\mathcal{F}_t] \\
&= \mathcal{E}[X - X' + X'|\mathcal{F}_t] - \mathcal{E}[X'|\mathcal{F}_t] \\
&= \mathcal{E}_{X'}[X - X'|\mathcal{F}_t]
\end{aligned}
$$

Recall (Lemma 54 in Section 5) that $\mathcal{E}_{X'}[\cdot]$ is another $\mathcal{F}$-expectation satisfying (4.1) and (4.4). Thus there also exists a pair $(\tilde{g}(\cdot), \tilde{z}(\cdot)) \in L_\mathcal{F}^2([0,T]; R \times R^d)$ with

$$|\tilde{g}_t| \le \mu |\tilde{z}_t| \tag{4.11}$$

such that the $\mathcal{E}_{X'}$-martingale $y_t - y_t'$ satisfies

$$y_t - y_t' = X - X' + \int_t^T \tilde{g}_s ds - \int_t^T \tilde{z}_s dB_s.$$

On the other hand, we have

$$y_t - y_t' = X - X' + \int_t^T [g_s - g_s'] ds - \int_t^T [z_s - z_s'] dB_s.$$

It follows then that

$$\tilde{g}_t \equiv g_t - g_t', \quad \text{and} \quad \tilde{z}_t \equiv z_t - z_t'.$$

This with (4.11) yields (4.10). The proof is complete. ∎

15

Remark 15 *From the above lemma, the result of Lemma 61 can be improved to: for each $X \in L^2(\Omega, \mathcal{F}_T, P)$ and $g \in L_\mathcal{F}^2(0,T)$, the process $\mathcal{E}[X + \int_t^T g_s ds|\mathcal{F}_t]$, $t \in [0,T]$ is continuous a.s..*

4.3   BSDE under $\mathcal{F}$–consistent nonlinear expectations

Here again, $\mathcal{E}$ denotes an $\mathcal{F}$-expectation satisfying (4.1) for some $\mu > 0$, and (4.4) as well. Let a function f be given

$$f(\omega, t, y) : \Omega \times [0, T] \times R \longmapsto R$$

satisfying, for some constant $C_1 > 0$,

$$\left\{ \begin{array}{ll} \text{(i)} & f(\cdot, y) \in L^2_{\mathcal{F}}(0, T), \quad \text{for each } y \in R; \\ \text{(ii)} & |f(t, y_1) - f(t, y_2)| \le C_1 |y_1 - y_2|, \quad \forall y_1, \ y_2 \in R. \end{array} \right. \tag{4.12}$$

For a given terminal data $X \in L^2(\Omega, \mathcal{F}_T, P)$, we consider the following type of equation:

$$Y_t = \mathcal{E}[X + \int_t^T f(s, Y_s)ds | \mathcal{F}_t] \tag{4.13}$$

Theorem 63 *We assume (4.12). Then there exists a unique process $Y(\cdot)$ solution of (4.13). Moreover, $Y(\cdot)$ admits continuous paths.*

Proof.   Define a mapping $\Lambda(y(\cdot)) : L^2_{\mathcal{F}}(0, T) \longmapsto L^2_{\mathcal{F}}(0, T)$ by

$$\Lambda_t(y(\cdot)) := \mathcal{E}[X + \int_t^T f(s, y_s)ds | \mathcal{F}_t].$$

Using Lemma 55,

$$\Lambda_t(y_1(\cdot)) - \Lambda_t(y_2(\cdot)) \le \mathcal{E}^{\mu}[\int_t^T (f(s, y_1(s)) - f(s, y_2(s)))ds | \mathcal{F}_t].$$

Thus

$$\begin{aligned} |\Lambda_t(y_1(\cdot)) - \Lambda_t(y_2(\cdot))| & \le & \mathcal{E}^{\mu}[\int_t^T (f(s, y_1(s)) - f(s, y_2(s)))ds | \mathcal{F}_t] \\ & \le & C_1 \mathcal{E}^{\mu}[\int_t^T |y_1(s) - y_2(s)|ds | \mathcal{F}_t], \text{ by (4.12).} \end{aligned}$$

Using Lemma 42, it follows that

$$\begin{aligned} E[|\Lambda_t(y_1(\cdot)) - \Lambda_t(y_2(\cdot))|^2] & \le & C_1^2 E[\mathcal{E}^{\mu}[\int_t^T |y_1(s) - y_2(s)|ds | \mathcal{F}_t]^2] \\ & \le & C_1^2 e^{\mu^2(T-t)} E[\int_t^T |y_1(s) - y_2(s)|ds]^2 \\ & \le & C_2 E[\int_t^T |y_1(s) - y_2(s)|^2 ds]. \end{aligned}$$

where $C_2 := T C_1^2 e^{\mu^2 T}$.

We observe that, for any finite number β, the following two norms are equivalent in $M_*(0, T; R^m)$

$$E \int_0^T |\phi_s|^2 dt \sim E \int_0^T |\phi_s|^2 e^{\beta t} dt.$$

Thus we multiply $e^{2C_2 t}$ on both sides of the above inequality and then integrate them on $[0, T]$. It follows that

$$
\begin{aligned}
E \int_0^T |\Lambda_t(y.) - \Lambda_t(y'.)|^2 e^{2C_1 t} dt &\leq C_2 E \int_0^T e^{2C_2 t} \int_t^T |y_s - y'_s|^2 ds dt \\
&= C_2 E \int_0^T \int_0^s e^{2C_2 t} dt |y_s - y'_s|^2 ds \\
&= (2C_2)^{-1} C_2 E \int_0^T (e^{2C_2 s} - 1)|y_s - y'_s|^2 ds.
\end{aligned}
$$

We then have

$$E \int_0^T |\Lambda_t(y.) - \Lambda_t(y'.)|^2 e^{2C_2 t} dt \leq \frac{1}{2} E \int_0^T |y_t - y'_t|^2 e^{2C_2 t} dt.$$

Namely, Λ is a contract mapping on $\mathcal{M}(0, T; R^m)$. It follows that this mapping has a unique fixed point Y:

$$Y_t = \mathcal{E}[X + \int_t^T f(s, Y_s) ds | \mathcal{F}_t].$$

Fanally, Lemma 61 and Remark 15 proves that the solution of (4.13) admits continuous paths, and the proof is complete. ∎

Theorem 64 (Comparison Theorem). *Let Y be the solution of (4.13) and let Y' be the solution of*

$$Y'_t = \mathcal{E}[X' + \int_t^T [f(s, Y'_s) + \phi_s] ds | \mathcal{F}_t]$$

where $X' \in L^2(\Omega, \mathcal{F}_T, P)$ and $\phi \in L^2_{\mathcal{F}}(0, T)$. If

$$X' \geq X, \quad \phi_t \geq 0, \quad dP \times dt\text{-}a.e., \tag{4.14}$$

then we have

$$Y'_t \geq Y_t, \quad dP \times dt\text{-}a.e. \tag{4.15}$$

(4.15) becomes equality if and only if (4.14) become equalities.

Proof.   We begin with the case $\phi_t \equiv 0$. For each $\delta > 0$, we define

$$\tau_1^\delta = \inf\{t \geq 0; Y_t' \leq Y_t - \delta\} \wedge T.$$

It is clear that if, for all $\delta > 0$, $\tau_1^\delta = T$ a.s., then (4.15) holds. Now if for some $\delta > 0$ we have

$$P(A) > 0, \quad \text{with } A = \left\{\tau_1^\delta < T\right\} \in \mathcal{F}_{\tau_1^\delta}$$

we then can define

$$\tau_2 = \inf\{t \geq \tau_1^\delta; Y_t' \geq Y_t\}.$$

Since $Y_T' = X' \geq X = Y_T$, thus $\tau_2 \leq T$ and $1_A Y'(\tau_2) = 1_A Y(\tau_2)$. It follows that, for $\tau \in [\tau_1^\delta, \tau_2]$,

$$1_A Y_\tau \;=\; \mathcal{E}[1_A Y_{\tau_2} + \int_\tau^{\tau_2} 1_A f(s, 1_A Y_s)ds|\mathcal{F}_\tau],$$

$$1_A Y_\tau' \;=\; \mathcal{E}[1_A Y_{\tau_2} + \int_\tau^{\tau_2} 1_A f(s, 1_A Y_s')ds|\mathcal{F}_\tau].$$

By the uniqueness result of Theorem 1, the solutions of the above two equations must coincide with each other. Thus $Y'_{\tau_1^\delta} 1_A = \bar{Y}_{\tau_1^\delta} 1_A$. This contradicts $P(A) > 0$.

In order to prove the general case when $\phi_s \geq 0$, we define for $n = 1, 2, 3, \cdots$, $Y^n(\cdot)$ to be the solution of

$$Y_t^n \;=\; \mathcal{E}\left[[X' + \int_{\frac{iT}{n}}^T \phi_s ds] + \int_t^T f(s, Y_s^n)ds|\mathcal{F}_t\right],$$

$$\text{for } t \;\in\; [t_i^n, t_{i+1}^n), \; t_i^n := \frac{iT}{n}, \;\; i = 0, 1, \cdots, n-1..$$

This equation can be written, piece by piece, as

$$Y_t^n \;=\; \mathcal{E}\left[[Y_{t_{i+1}^n}^n + \int_{t_t^n}^{t_{i+1}^n} \phi_s ds] + \int_t^{t_{i+1}^n} f(s, Y_s^n)ds|\mathcal{F}_t\right],$$

$$t \;\in\; [t_i^n, t_{i+1}^n), \; Y_T^n = Y_{t_n^n}^n = X'.$$

From the first part of the proof. We have, for $i = n-1$, $Y_t^n \geq Y_t$, $t \in [t_{n-1}^n, T)$. In particular, $Y_{t_{n-1}^n}^n \geq Y_{t_{n-1}^n}$. An obvious iteration of this algorithm gives

$$Y_t^n \geq Y_t, \; t \in [t_i^n, t_{i+1}^n), \;\; i = 0, \cdots, n-2.$$

Thus $Y_t^n \geq Y_t$, $t \in [0, T]$.

In order to prove that $Y_t' \geq Y_t$, It suffices to show the convergence of the sequence (Y^n) to Y'. A computation analogous to the proof of Lemma 53 shows that, for fixed $t \in t \in [t_i^n, t_{i+1}^n)$ and an appropriate constant C,

$$E[|Y_t^n - Y_t'|^2] \;\leq\; CE[\int_{\frac{iT}{n}}^t |\phi_s|ds + C_1 \int_t^T |Y_s^n - Y_s'|ds]^2$$

$$\leq\; 2C[E(\int_{\frac{iT}{n}}^t |\phi_s|ds)]^2 + C_1^2[E\int_t^T |Y_s^n - Y_s'|ds]^2$$

Using Schwards inequality, one has for all $t \in [0,T]$

$$E[|Y_t^n - Y_t'|^2] \leq 2C\frac{T}{n}E\int_0^T |\phi_s|^2 ds + 2CC_1^2 TE\int_t^T |Y_s^n - Y_s'|^2 ds. \qquad (4.16)$$

Gronwall's Lemma applied to the above inequality shows that

$$E[|Y_t^n - Y_t'|^2] \to 0,$$

and finally $Y_t' \geq Y_t$.

Finally, we investigate possible equality in (4.15). From $Y_t' \equiv Y_t$, one has

$$\mathcal{E}[X + \int_0^T f(s, Y_s)ds] = \mathcal{E}[X' + \int_0^T f(s, Y_s)ds + \int_0^T \Phi_s ds]$$

Since $X' \geq X$ and $\int_0^T \Phi_s ds \geq 0$, it follows from the strict monotonicity of $\mathcal{E}$ that $X' = X$ a.s., and $\int_0^T \Phi_s ds = 0$, whence $\Phi = 0$ $dt \times dP$ a.e. and the end of the proof. ∎

4.4 Decomposition theorem for $\mathcal{E}$-supermartingales

Our next result generalizes the decomposition theorem for g-supermartingales proved in [?] to continuous $\mathcal{E}$-supermartingales. The proof uses mainly arguments from [?].

Theorem 65 (Decomposition theorem for $\mathcal{E}$-supermartingales) *Let $\mathcal{E}[\cdot]$ be an $\mathcal{F}$-expectation satisfying (4.1) and (4.4), and let (Y_t) be a related continuous $\mathcal{E}$-supermartingale with*

$$E[\sup_{t\in[0,T]} |Y_t|^2] < \infty.$$

Then there exists an $A(\cdot) \in L_{\mathcal{F}}^2(0,T;R)$ such that $A(\cdot)$ is continuous and increasing with $A(0) = 0$, and such that $Y_t + A_t$ is an $\mathcal{E}$-martingale.

Proof. For $n \geq 1$, we define $y^n(\cdot)$, solution of the following BSDE:

$$y_t^n = \mathcal{E}[Y_T + \int_t^T n(Y_s - y_s^n)ds|\mathcal{F}_t]$$

We have then the following

Lemma 66 *We have, for each t and $n \geq 1$,*

$$Y_t \geq y_t^n.$$

Proof.   For a $\delta > 0$ and a given integer $n > 0$, we define

$$\sigma^{n,\delta} := \inf\{t; \ y_t^n \geq Y_t + \delta\} \wedge T.$$

If $P(\sigma^{n,\delta} < T) = 0$, for all n and δ, then the proof is done. If it is not the case, then there exist $\delta > 0$ and a positive integer n such that $P(\sigma^{n,\delta} < T) > 0$. We can then define the following stopping times

$$\tau := \inf\{t \geq \sigma^{n,\delta}; \ y_t^n \leq Y_t\}$$

It is clear that $\sigma^{n,\delta} \leq \tau \leq T$. Because of Theorem 6.1, $Y_t - y_t^n$ is **continuous**. Hence we have

$$y_\tau^n \leq Y_\tau \tag{4.17}$$

But since $(Y_s - y_s^n) \leq 0$ in $[\sigma^{n,\delta}, \tau]$, by monotonicity of $\mathcal{E}[\cdot]$,

$$
\begin{aligned}
y_{\sigma^{n,\delta}}^n &= \mathcal{E}[y_\tau^n + \int_{\sigma^{n,\delta}}^\tau n(Y_s - y_s^n)ds|\mathcal{F}_{\sigma^{n,\delta}}] \\
&\leq \mathcal{E}[y_\tau^n|\mathcal{F}_{\sigma^{n,\delta}}] \\
&\leq \mathcal{E}[Y_\tau|\mathcal{F}_{\sigma^{n,\delta}}]
\end{aligned}
$$

Finally, since Y is an $\mathcal{E}$-supermartingale,

$$Y_{\sigma^{n,\delta}} \geq y_{\sigma^{n,\delta}}^n.$$

But on the other hand, we have $P(\sigma^{n,\delta} < T) > 0$ and, by the definition of $\sigma^{n,\delta}$, $y_{\sigma^{n,\delta}}^n) \geq Y_{\sigma^{n,\delta}} + \delta$ on $\{\sigma^{n,\delta} < T\}$. This induces a contradiction. The proof is complete. ∎

Lemma 66 with Theorem 64 above imply that $y^n(\cdot)$ monotonically converges to some $Y^0(\cdot) \leq Y(\cdot)$. Indeed, writing $\phi_t = Y_t - y_t^{(n+1)} \geq 0$ shows that $(y^n(\cdot))$ is an increasing sequence of functions.

Observe then that $y_t^n + \int_0^t n(Y_s - y_s^n)ds$ is an $\mathcal{E}$-martingale. By Lemma 62, there exists $(g^n, z^n) \in L_{\mathcal{F}}^2(0, T; R \times R^d)$ with

$$|g_s^n| \leq \mu|z_s^n|, \quad n = 1, 2, \cdots, \tag{4.18}$$

such that

$$
\begin{aligned}
y_t^n + \int_0^t n(Y_s - y_s^n)ds &= \ y_T^n + \int_0^T n(Y_s - y_s^n)ds \\
&\quad + \int_t^T g_s^n ds - \int_t^T z_s^n dB_s,
\end{aligned}
$$

hence, as $y_T^n = Y_T$,

$$y_t^n = Y_T + \int_t^T [g_s^n + n(Y_s - y_s^n)]ds - \int_t^T z_s^n dB_s. \tag{4.19}$$

(4.10) also tells us that

$$|g_s^n - g_s^{(m)}| \leq \mu |z_s^n - z_s^{(m)}|, \quad n, m = 1, 2, \cdots \tag{4.20}$$

Let us denote, for each $n = 1, 2, \cdots$,

$$A_t^n = n \int_0^t (Y_s - y_s^n) ds$$

A^n is a continuous increasing process such that $A^n(0) = 0$.

We are now going to identify the limit of $y^n(\cdot)$. To this end, we shall use the following lemma :

Lemma 67 *There exists a constant C which is independent of n such that*

$$(i) \quad E \int_0^T |z_s^n|^2 ds \leq C; \quad (ii) \quad E[(A_T^n)^2] \leq C. \tag{4.21}$$

Proof. From (4.19) and (4.18), we take

$$
\begin{aligned}
A_T^n &= y^n(0) - y_T^n - \int_0^T g_s^n ds + \int_0^T z_s^n dB_s \\
&\leq |y^n(0)| + |y_T^n| + \int_0^T \mu |z_s^n| ds + |\int_0^T z_s^n dB_s|.
\end{aligned}
$$

Since $y_t^1 \leq y_t^n \leq Y_t$ for all t, we' have $|y_t^n| \leq |y_t^1| + |Y_t|$. Thus there exists a constant C, independent of n, such that

$$E[\sup_{0 \leq t \leq T} |y_t^n|^2] \leq C. \tag{4.22}$$

It follows readily that there exist two constants C_1 and C_2, independent of n, such that

$$E|A_T^n|^2 \leq C_1 + C_2 E \int_0^T |z_s^n|^2 ds. \tag{4.23}$$

On the other hand, Itô's formula applied to $|y^n(\cdot)|^2$ gives:

$$
\begin{aligned}
E[|y^n(0)|^2] &= E|Y_T|^2 + E \int_0^T [2y_s^n g_s^n - |z_s^n|^2] ds \\
&\quad + 2E \int_0^T y_s^n dA_s^n \\
&\leq E|Y_T|^2 + E \int_0^T [2\mu |y_s^n||z_s^n| - |z_s^n|^2] ds \\
&\quad + 2E[A_T^n \sup_{0 \leq s \leq T} |y_s^n|],
\end{aligned}
$$

whence, using that, for positive a, b and ε, $2ab \leq \varepsilon a^2 + b^2/\varepsilon$ (noting also that $E[|y^n(0)|^2] \geq 0$!), we get

$$
\begin{aligned}
E \int_0^T |z_s^n|^2 ds \ \leq \ & E|Y_T|^2 + E \int_0^T [2\mu^2|y_s^n|^2 + \frac{1}{2}|z_s^n|^2]ds \\
& + 2[E \sup_{0 \leq s \leq T} |y_s^n|^2]^{1/2} [E|A_T^n|^2]^{1/2},
\end{aligned}
$$

and using the same inequality with $\varepsilon = 4C_2$,

$$
\begin{aligned}
E \int_0^T |z_s^n|^2 ds \ \leq \ & 2E|Y_T|^2 + 4\mu^2 E \int_0^T |y_s^n|^2 ds \\
& + 8C_2[E \sup_{0 \leq s \leq T} |y_s^n|^2] + \frac{1}{2C_2}[E|A_T^n|^2] \\
\leq \ & 2E|Y_T|^2 + 4\mu^2 E \int_0^T |y_s^n|^2 ds \\
& + 8C_2[E \sup_{0 \leq s \leq T} |y_s^n|^2] + \frac{C_1}{2C_2} + \frac{1}{2} E \int_0^T |z_s^n|^2 ds.
\end{aligned}
$$

because of (4.23).

Finally, it comes

$$
\begin{aligned}
E \int_0^T |z_s^n|^2 ds \ \leq \ & 4E|Y_T|^2 + 8\mu^2 E \int_0^T |y_s^n|^2 ds \\
& + 16C_2[E \sup_{0 \leq s \leq T} |y_s^n|^2] + \frac{C_1}{C_2},
\end{aligned}
$$

and it is sufficient to note that, thanks to (4.22), the constant

$$
\sup_n \{4E|Y_T|^2 + 8\mu^2 E \int_0^T |y_s^n|^2 ds + 16C_2[E \sup_{0 \leq s \leq T} |y_s^n|^2] + \frac{C_1}{C_2}\} < \infty
$$

to conclude that (4.21)–(i) and then (using (4.23)), (4.21)–(ii) hold true. The lemma is proved. ∎

With the help of Lemma 67 above we can now end the proof of the Decomposition Theorem.

Note first that (4.21)–(i) with (4.18) also implies

$$
\mathbf{E} \int_0^T |g_s^n|^2 ds \leq \mu^2 C
$$

(4.21)–(ii) implies that
$$
y^n(\cdot) \nearrow Y(\cdot).
$$

From Theorem 2.1. in [**?**], it follows that we can write Y under the form

$$Y_t = Y_T + \int_t^T g_s ds + A_T - A_t - \int_t^T z_s dB_s \qquad (4.24)$$

for some $(g, z) \in L^2_{\mathcal{F}}(0, T; R \times R^d)$ and an increasing process A. Observe that $Y(\cdot)$ and then $A(\cdot)$ is continuous. Applying the result in the first part of the proof of Theorem 2.1 in [**?**], pp482–pp483 (see Appendix for details), we have

$$z^n(\cdot) \to z(\cdot), \quad \text{strongly in } L^2_{\mathcal{F}}(0, T).$$

It follows from (4.20) that

$$g^n(\cdot) \to g(\cdot), \quad \text{strongly in } L^2_{\mathcal{F}}(0, T).$$

And finally, (4.19) gives

$$A_t^n \longmapsto A_t, \quad \text{strongly in } L^2(\Omega, \mathcal{F}_T, P).$$

Thanks to Lemma 57, we can pass to the L^2-limit in both sides of

$$y_t^n = \mathcal{E}[Y_T + A_T^n - A_t^n | \mathcal{F}_t].$$

It follows that

$$Y_t = \mathcal{E}[Y_T + A_T - A_t | \mathcal{F}_t].$$

Thus $Y_t + A_t = \mathcal{E}[Y_T + A_T | \mathcal{F}_t]$ is an $\mathcal{E}$-martingale (because of 4.4)). Since A is increasing, the Theorem is proved. ∎

4.5   Finding a g–expectation to represent $\mathcal{E}[\cdot|\mathcal{F}_t]$

An $\mathcal{F}_t$–Consistent Expectation is a g-Expectation

     In this section, we will prove an important result: an $\mathcal{F}_t$–consistent nonlinear expectation can be identified as a g-expectation, provided that (4.1) and (4.4) hold.

Theorem 68 *We assume that an $\mathcal{F}$-expectation $\mathcal{E}[\cdot]$ satisfies (4.1) and (4.4) for some $\mu > 0$. Then there exists a function $g = g(t, z) : \Omega \times [0, T] \times R^d$ satisfying (i), (ii) and (iii") of (3.22) such that*

$$\mathcal{E}[X] = \mathcal{E}_g[X], \quad \forall X \in L^2(\Omega, \mathcal{F}_T, P).$$

In particular, every $\mathcal{E}$-martingale is continuous a.s.

     Moreover, we have $|g(t, z)| \le \mu|z|$ for all $t \in [0, T]$.

Proof.   For each given $z \in R^d$, we consider the following forward equation

$$\begin{cases} dY_t^z = -\mu|z|dt + zdB_t, \\ Y^z(0) = 0. \end{cases}$$

We have $E[\sup_{t\in[0,T]} |Y_t^z|^2] < \infty$. It is also clear that Y^z is an $\mathcal{E}^\mu$-martingale, thus an $\mathcal{E}[\cdot]$-supermartingale. Indeed, we can write $Y_t^z = \mathcal{E}^\mu[Y_T|\mathcal{F}_t]$.   From Theorem 65, there exists an increasing process $A^z(\cdot)$ with $A^z(0) = 0$ and $E[A_T^{z2}] < \infty$ such that

$$Y_t^z = \mathcal{E}[Y_T^z + A_T^z - A_t^z|\mathcal{F}_t].$$

Or

$$Y_t^z + A_t^z = \mathcal{E}[Y_T^z + A_T^z|\mathcal{F}_t], \quad t \in [0, T].$$

Then, from Lemma 62. there exists $(g(z, \cdot), Z^z(\cdot)) \in L_{\mathcal{F}}^2(0, T; R \times R^d)$ with $|g(z, t)| \leq \mu|Z_t^z|$ such that

$$Y_t^z + A_t^z = Y_T^z + A_T^z + \int_t^T g(z, s)ds - \int_t^T Z_s^z dB_s. \tag{4.25}$$

We also have

$$|g(z, t) - g(z', t)| \leq \mu|Z_t^z - Z_t^{z'}|. \tag{4.26}$$

But on the other hand, since

$$Y_t^z = Y_T^z + \int_t^T \mu|z|ds - \int_t^T zdB_s,$$

it follows that

$$\begin{aligned} A_t^z &\equiv \mu|z|t - \int_0^t g(z, s)ds \\ Z_t^z &\equiv z \end{aligned}$$

In particular, (4.26) becomes

$$|g(z, t) - g(z', t)| \leq \mu|z - z'|. \tag{4.27}$$

Moreover,

$$Y_t^z + A_t^z = Y^z(r) + A^z(r) - \int_r^t g(z, s)ds + \int_r^t zdB_s, \quad 0 \leq r \leq t \leq T,$$

and $Y_t^z + A_t^z$ is an $\mathcal{E}$-martingale. But with the assumption (4.4) one has, for each $z \in R^d$ and $r \leq t$

$$\mathcal{E}[-\int_r^t g(z, s)ds + \int_r^t zdB_s|\mathcal{F}_r] = \mathcal{E}[Y_t^z + A_t^z - (Y^z(r) + A^z(r))|\mathcal{F}_r],$$

i.e.
$$\mathcal{E}[-\int_r^t g(z,s)ds + \int_r^t zdB_s|\mathcal{F}_r] = 0 \quad 0 \le r \le t \le T \tag{4.28}$$

Now let $\{A_i\}_{i=1}^N$ be a $\mathcal{F}_r$-measurable partition of Ω (i.e., A_i are disjoint, $\mathcal{F}_r$-measurable and $\cup A_i = \Omega$) and let $z_i \in R^d$, $i = 1, 2, \cdots, N$. From Lemma 5. of Section 4., and the fact that $g(0,s) \equiv 0$, it follows that

$$\mathcal{E}[-\int_r^t g(\sum_{i=1}^N z_i 1_{A_i}, s)ds + \int_r^t \sum_{i=1}^N z_i 1_{A_i} dB_s|\mathcal{F}_r]$$

$$= \mathcal{E}[\sum_{i=1}^N 1_{A_i}\left(-\int_r^t g(z_i, s)ds + \int_r^t z_i dB_s\right)|\mathcal{F}_r]$$

$$= \sum_{i=1}^N 1_{A_i}\mathcal{E}[-\int_r^t g(z_i, s)ds + \int_r^t z_i dB_s|\mathcal{F}_r]$$

$$= 0$$

(because of (4.28)). In other words, for each simple function $\eta \in L^2(\Omega, \mathcal{F}_r, P)$,

$$\mathcal{E}[-\int_r^t g(\eta, s)ds + \int_r^t \eta dB_s|\mathcal{F}_r] = 0.$$

From this, the continuity of $\mathcal{E}[\cdot]$ in L^2 given by (4.3) and the fact that g is Lipschitz in z, it follows that the above equality holds for $\eta(\cdot) \in L^2_{\mathcal{F}}(0, T; R^d)$:

$$\mathcal{E}[-\int_r^t g(\eta_s, s)ds + \int_r^t \eta_s dB_s|\mathcal{F}_r] = 0. \tag{4.29}$$

We just have to prove now that

$$\mathcal{E}_g[X] = \mathcal{E}[X], \quad \forall X \in L^2(\Omega, \mathcal{F}_T, P).$$

To this end we first solve the following BSDE

$$-dy_s = g(t, z_s)ds - z_s dB_s,$$
$$y_T = X.$$

Since g is Lipschitz in z, there exists a unique solution $(y(\cdot), z(\cdot)) \in L^2_{\mathcal{F}}(0, T; R \times R^d)$. By the definition of g-expectation,

$$\mathcal{E}_g[X] = y(0).$$

On the other hand, using (4.29), one finds

$$\mathcal{E}[X] = \mathcal{E}[y(0) - \int_0^T g(z_s, s)ds + \int_0^T z_s dB_s]$$

$$= y(0) + \mathcal{E}[-\int_0^T g(z_s, s)ds + \int_0^T z_s dB_s]$$

$$= y(0) = \mathcal{E}_g[X].$$

It follows that this g-expectation $\mathcal{E}_g[\cdot]$ coincides with $\mathcal{E}[\cdot]$ and we are finished. ∎

4.6   How to test and find g?

Let $g(s,z)$ be the generator of the investigated agent. An very important problem is how to fin this function g. We will treat this problem for the case where g is a deterministic function: $g(t,z) : [0,\infty) \times \mathbf{R}^d \to \mathbf{R}$. We assume that

$$|g(t,z) - g(t,z')| \leq \mu|z - z'|, \ \forall t \geq 0, \ \ \forall z, z' \in \mathbf{R}^d,$$
$$g(t,0) \equiv 0, \ \ \forall t \geq 0, . \tag{4.30}$$

In this case we can find such g by the following testing method.

Proposition 69 *We assume (4.30). Let $\bar{z} \in \mathbf{R}^d$ be given, then*

$$\int_t^T g(s,\bar{z})ds = \mathcal{E}_g[\bar{z}B_T|\mathcal{F}_t] - \bar{z}B_t \tag{4.31}$$

In particular

$$\int_0^T g(s,\bar{z})ds = \mathcal{E}_g[\bar{z}B_T] \tag{4.32}$$

Proof.   We denote $Y_t := \mathcal{E}_g[\bar{z}B_T|\mathcal{F}_t]$, it is the solution of the following BSDE

$$Y_t = \bar{z}B_T + \int_t^T g(s,Z_s)ds - \int_t^T Z_s dB_s$$

Or

$$Y_t - \bar{z}B_t = \int_t^T g(s, Z_s - \bar{z} + \bar{z})ds - \int_t^T (Z_s - \bar{z})dB_s.$$

It follows that $(\bar{Y}_t, \bar{Z}_t) := (Y_t - \bar{z}B_t, Z_t - \bar{z})$ solves the BSDE

$$\bar{Y}_t = \int_t^T g(s, \bar{Z}_s + \bar{z})ds - \int_t^T \bar{Z}_s dB_s.$$

This BSDE has a unique solution $(\bar{Y}_t, \bar{Z}_t) \equiv (\int_t^T g(s,\bar{z})ds, 0)$. We thus have (4.31).

Remark 16 *It is meaningful to test the generator g of an agent: at a time $t \leq T$, we let the we let the agent evaluate $\bar{z}B_T$ and result $\mathcal{E}_g[\bar{z}B_T|\mathcal{F}_t]$. Then the deterministic data $\int_t^T g(s,\bar{z})ds$ is obtained by $= \mathcal{E}_g[\bar{z}B_T|\mathcal{F}_t] - \bar{z}B_t$, where B_t is a known value at the time t.*

Example 70 *If g is time–invariant: $g = g(z)$, then we have*

$$g(\bar{z})(T - t) = \mathcal{E}_g[\bar{z}B_T|\mathcal{F}_t] - \bar{z}B_t$$

and

$$g(\bar{z})T = \mathcal{E}_g[\bar{z}B_T], \ \ \bar{z} \in \mathbf{R}^d.$$

Example 71 *If we already know that $g = g_0(\theta, z)$, where $g_0 : [a, b] \times \mathbf{R}^d \to \mathbf{R}$ is a given function but we have to find the parameter $\theta \in [a, b]$, assume that for some $\bar{z} \in \mathbf{R}^d$, $g_0(\theta, z)$ is a strictly increasing function of θ in $[a, b]$. Then we can only test the agent once at the time, say $t = 0$. Using the formula*

$$g_0(\theta, \bar{z})T = \mathcal{E}_g[\bar{z}B_T],$$

we can uniquely determine θ.

Chapter 5
HOW TO SOLVE BSDE

5.1 BSDE, SDE and PDE: nonlinear Feynman—Kac formula

In this section we consider a classical (forward) stochastic differential equation (SDE), i.e., the well–known Itô's SDE and relate it to a BSDE. The related solution of BSDE is a type of nonlinear PDE of parabolic type.

Consider a SDE parameterized with the following initial condition $(t, Y) \in [0, T] \times L^2(\omega, \mathcal{F}_t, P; \mathbf{R}^n)$

$$
\begin{aligned}
dX_s^{t,\xi} &= b(\omega, s, X_s^{t,\xi})ds + \sigma(\omega, s, X_s^{t,\xi})dB_s, \quad s \in [t, T], \\
X_t^{t,\xi} &= \xi.
\end{aligned}
\tag{5.1}
$$

We are most interested in the case where ξ is a deterministic vector: $\xi = x \in \mathbf{R}^n$:

$$
\begin{aligned}
dX_s^{t,x} &= b(\omega, s, X_s^{t,x})ds + \sigma(\omega, s, X_s^{t,x})dB_s, \quad s \in [t, T], \\
X_t^{t,x} &= x,
\end{aligned}
\tag{5.2}
$$

Here each for each fixed $x \in \mathbf{R}^n$, $b(\cdot, x)$ $\sigma(\cdot, x)$ are respectively $\mathbf{R}^n$ valued $\mathbf{R}^{n \times d}$-valued bounded and $(\mathcal{F}_t)$-adapted processes. We also assume that b and σ satisfy Lipschitz condition in x: for each x, $x' \in \mathbf{R}^n$

$$
|b(\omega, t, x) - b(\omega, t, x')| + |\sigma(\omega, t, x) - \sigma(\omega, t, x')| \leq C|x - x'|.
\tag{5.3}
$$

and

$$
|b(\omega, t, 0)| + |\sigma(\omega, t, 0)| \leq C_0.
\tag{5.4}
$$

We know that under the above assumptions, there exists a unique strong solution of SDE (5.1). Moreover, for each $t \in [0, T)$, $\forall \zeta, \zeta' \in L^2(\omega, \mathcal{F}_t, P; \mathbf{R}^n)$ we have

$$
\mathbf{E}[\sup_{s \in [t,T]} |X_s^{t,\zeta} - X_s^{t,\zeta'}|^2 | \mathcal{F}_t] \leq C_0|\zeta - \zeta'|^2, \quad \text{a.s},
\tag{5.5}
$$

and for each $p \geq 2$, we have

$$
\mathbf{E}[\sup_{s \in [t,T]} |X_s^{t,\zeta}|^p] \leq C_p(1 + |\zeta|^p), \forall \zeta \in L^p(\omega, \mathcal{F}_t, P; \mathbf{R}^n), \quad \text{a.s.}
\tag{5.6}
$$

Here the constant C_0 depends only on T and the Lipschitz constant of b and σ. C_p depends only on p, T, the Lipschitz constant of b and σ and their bound C_0 in (5.4).

We now consider our BSDE. Let $f = f(\omega, t, x, y, z)$ and $\Phi(x)$ be $\mathbf{R}^m$-valued functions such that, for each $(x, y, z) \in \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^{m \times d}$, $f(\cdot, x, y, z)$ is a bounded and $(\mathcal{F}_t)$-adapted process, $\Phi(x)$ is a bounded $\mathcal{F}_T$-measurable random variable. They are Lipschitz continuous, in (x, y, z), i.e., for each (x, y, z), (x', y', z') in $\mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^{m \times d}$,

$$
\begin{aligned}
|\Phi(x) \;\; -\Phi(x')| + |f(t, x, y, z) - f(t, x', y', z')| \\
\leq C(|y - y'| + |z - z'| + |x - x'|),
\end{aligned}
\tag{5.7}
$$

We also assume that f and Φ satisfy the following linear growth condition in x:

$$
|f(t, x, 0, 0)| + |\Phi(x)| \leq C(1 + |x|), \quad \forall x \in \mathbf{R}^n.
\tag{5.8}
$$

Under the above assumptions it is easy to check that $f(t, X_t^{t,\zeta}, y, z)$ and $\xi = \Phi(X_T^{t,\zeta})$ satisfy all conditions required in Theorem 19. Thus the following BSDE has a unique solution:

$$
\begin{aligned}
-dY_s^{t,\zeta} &= \; f(s, X_s^{t,\zeta}, Y_s^{t,\zeta}, Z_s^{t,\zeta})ds - Z_s^{t,\zeta}dB_s, \quad s \in [t, T], \\
Y_T^{t,\zeta} &= \; \Phi(X_T^{t,\zeta}).
\end{aligned}
\tag{5.9}
$$

We first have the following estimate:

Proposition 72   *We assume (5.3)—(5.8). then for each $t < T$, and for each $\zeta, \zeta' \in L^2(\omega, \mathcal{F}_t, P; \mathbf{R}^n)$, we have*

$$
\begin{aligned}
&(i) \quad |Y_t^{t,\zeta} - Y_t^{t,\zeta'}| \leq C_0|\zeta - \zeta'|, \\
&(ii) \quad |Y_t^{t,\zeta}| \leq C_0(1 + |\zeta|).
\end{aligned}
\tag{5.10}
$$

Proof.   By Theorem 20,

$$
\begin{aligned}
|Y_t^{t,\zeta} - \;\; Y_t^{t,\zeta'}|^2 &\leq C\mathbf{E}^{\mathcal{F}_t}[e^{\beta T}|\Phi(X_T^{t,\zeta}) - \Phi(X_T^{t,\zeta'})|^2] \\
&\quad + C\mathbf{E}^{\mathcal{F}_t} \int_t^T |f(s, X_s^{t,\zeta}, Y_s^{t,\zeta}, Z_s^{t,\zeta}) - f(s, X_s^{t,\zeta'}, Y_s^{t,\zeta}, Z_s^{t,\zeta})|^2 ds \\
&\leq C\mathbf{E}^{\mathcal{F}_t}|X_T^{t,\zeta} - X_T^{t,\zeta'}|^2 + C\mathbf{E}^{\mathcal{F}_t} \int_t^T |X_s^{t,\zeta} - X_s^{t,\zeta'}|^2 ds
\end{aligned}
$$

This with (5.5)yields (5.10).
      We define

$$
u(t, x) := Y_s^{t,x}|_{s=t}, \;\; x \in \mathbf{R}^n. \tag{4.5}
$$

By (4.4), we have

$$
\begin{aligned}
&(i) \quad |u(t, x) - u(t, x')| \leq C_0|x - x'|^\alpha, \\
&(ii) \quad |u(t, x)| \leq C_0(1 + |x|).
\end{aligned}
\tag{5.11}
$$

Remark 17   *We note that, in general, u is a random function, i.e., for each $x \in \mathbf{R}^n$, $u(x, \cdot)$ is a ($\mathbf{R}^m$-valued) $(\mathcal{F}_t)$-adapted process. But if we assume furthermore that, for each (t, x, y, z),*

$$
b(t, x), \sigma(t, x), \Phi(x), f(t, x, y, z) \text{ are deterministic functions} \tag{5.12}
$$

Then u becomes a deterministic function of (t,x). In fact, it is easy to check that for each (t,x), the solution $X_s^{t,x}$ of SDE $(4.1-2)$ is $(\mathcal{F}_s^t)$-adapted. Thus $\Phi(X_T^{t,x})$ is $\mathcal{F}_T^t$-measurable, and for each $(y,z) \in \mathbf{R}^m \times \mathbf{R}^{m \times d}$, $f(s, X_s^{t,x}, y, z)$ is a $(\mathcal{F}_s^t)$-adapted process. By Proposition 2.4, the solution $(Y_s^{t,x}, Z_s^{t,x})$, $s \in [t,T]$, of BSDE (5.9) is also $(\mathcal{F}_s^t)$-adapted. In particular, $u(t,x) = Y_s^{t,x}|_{s=t}$ is deterministic.

Remark 18 *A more particular situation is when $m = 1$ and f is independent of (y,z), i.e., $f = f(t,x)$. In this case if we assume furthermore (5.12), then u has the following explicit expression:*

$$u(t,x) = \mathbf{E}[\int_t^T f(s, X_s^{t,x})ds + \Phi(X_T^{t,x})].$$

Exercise 5.1.1 *We assume (5.12), $m = 1$. We assume furthermore $f = c(x)y + f_0(t,x)$. Prove the well-known Feynman–Kac formula:*

$$u(t,x) = \mathbf{E}[\int_t^T e^{\int_t^s c(X_r^{t,x})dr} f_0(s, X_s^{t,x})ds + \Phi(X_T^{t,x})e^{\int_t^T c(X_r^{t,x})dr}].$$

Proof. What is the explicit expressions of u if, in Remark 18 and/or Exercise 5.1.1, we remove Assumption (5.12)?

In the following we study the situation without Assumption (5.12). u is a random function. We need to define

Definition 73 *For each fixed $t \in [0,T]$, a collection $\{A_i\}_{i=1}^N \subset \mathcal{F}_t$ is called an $\mathcal{F}_t$-partition if $\bigcup_{i=1}^N A_i = \omega$ and for each $i,j = 1, \cdots, N$, $A_i \cap A_j = \emptyset$.*

Theorem 74 *For each $\zeta \in L^2(\omega, \mathcal{F}_t, P, \mathbf{R}^n)$, we have*

$$u(t,\zeta) = Y_t^{t,\zeta}. \tag{5.13}$$

Proof. We first consider the case where ζ is a simple function:

$$\zeta = \sum_{i=1}^N I_{A_i} x_i, \tag{5.14}$$

where $\{A_i\}_{i=1}^N$ an $\mathcal{F}_t$–partition and $x_i \in \mathbf{R}^n$, $i = 1, 2, \cdots, N$. For each i, we denote

$$(X_s^i, Y_s^i, Z_s^i) \equiv (X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})|_{x=x_i}.$$

X^i is the solution of the following SDE

$$X_s^i = x_i + \int_t^s b(r, X_r^i)dr + \int_t^s \sigma(r, X_r^i)dB_r, \quad s \in [t,T]$$

(Y^i, Z^i) is the solution of BSDE

$$Y_s^i = \Phi(X_T^i) + \int_s^T f(r, X_r^i, Y_r^i, Z_r^i)dr - \int_s^T Z_r^i dB_r, \quad s \in [t, T].$$

We then multiple $I_{\mathbf{A_i}}$ on both sides of the above two equations. Then take summation over $i = 1, \cdot, N$. By the following simple rule $\sum_i \varphi(x_i)I_{A_i} = \varphi(\sum_i x_i I_{A_i})$ we derive

$$\sum_{i=1}^N I_{A_i} X_s^i = \sum_{i=1}^N I_{A_i} x_i + \int_t^s b(r, \sum_{i=1}^N I_{A_i} X_r^i)dr + \int_t^s \sigma(r, \sum_{i=1}^N I_{A_i} X_r^i)dB_r$$

and

$$\sum_{i=1}^N I_{A_i} Y_s^i = \Phi(\sum_{i=1}^N I_{A_i} X_T^i) +$$
$$\int_s^T f(r, \sum_{i=1}^N I_{A_i} X_r^i, \sum_{i=1}^N I_{A_i} Y_r^i, \sum_{i=1}^N I_{A_i} Z_r^i)dr - int_s^T \sum_{i=1}^N I_{A_i} Z_r^i dB_r,$$

It then follows from the existence of uniqueness Theorems of SDE and BSDE that

$$X_s^{t,\zeta} = \sum_{i=1}^N X_s^i I_{A_i},$$

and

$$(Y_s^{t,\zeta}, Z_s^{t,\zeta}) = (\sum_{i=1}^N I_{A_i} Y_s^i, \sum_{i=1}^N I_{A_i} Z_s^i).$$

By the definition of $u(t, x_i) = Y_t^i$,

$$Y_t^{t,\zeta} = \sum_{i=1}^N Y_t^i I_{A_i} = \sum_{i=1}^N u(t, x_i)I_{A_i}$$
$$= u(t, \sum_{i=1}^N x_i I_{A_i}) = u(t, \zeta).$$

Thus (5.13) holds for the case where ζ is a simple function.

But in general $\zeta \in L^2(\omega, \mathcal{F}_t, P; \mathbf{R}^n)$. In this case we choose a Cauchy sequence $\{\zeta_i\}$ that converges to ζ in $L^2(\omega, \mathcal{F}_t, P; \mathbf{R}^n)$. By estimates (5.10) and (5.11), we have

$$\mathbf{E}[|Y_t^{t,\zeta_i} - Y_t^{t,\zeta}|^2] \le C_0 \mathbf{E}|\zeta_i - \zeta|^2 \to 0$$

and

$$\mathbf{E}[|u(t, \zeta_i) - u(t, \zeta)|^2] \le C_0 \mathbf{E}|\zeta_i - \zeta|^2 \to 0.$$

By which and $u(t, \zeta_i) = Y_t^{t,\zeta_i}$ we finally derive (5.13). ∎

In (3.24)we have introduced the notion of backward semigroup $\mathcal{E}_{r,t}^g[\cdot]$. We now discuss use this backward semigroup property to discuss u. For each given $t_1 \in (t, T]$ and $\eta \in L^2(\omega, \mathcal{F}_{t_1}, P; \mathbf{R}^n)$, we have

$$-dY_s = f(s, X_s^{t,\zeta}, Y_s, Z_s)ds - Z_s dB_s, \quad s \in [0, t_1],$$
$$Y_{t_1} = \eta.$$

for each $r \in [t, t_1)$, we denote $\mathcal{E}_{r,t_1}^{g\zeta}[\eta] := Y_r$. By the definition of u (see (3.30)), for each $0 < \delta < T - t$,

$$u(t, x) = \mathcal{E}_{t,T}^x[\Phi(X_T^{t,x})] = G_{t,t+\delta}^x[Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t,x}}].$$

This with (5.13) we derive the following result.

Proposition 75 *We have*

$$u(t, x) = G_{t,t+\delta}^x[u(t + \delta, X_{t+\delta}^{t,x})], \quad 0 < \delta < T - t. \tag{5.15}$$

Remark 19 *We recall the example given in Remark 18. In this special case it is easy to check that, for a given smooth function $\varphi(t, x)$, we have*

$$G_{t,t+\delta}^x[\varphi(t + \delta, X_{t+\delta}^{t,x})] = \mathbf{E}[\int_t^{t+\delta} f(s, X_s^{t,x})ds + \varphi(t + \delta, X_{t+\delta}^{t,x})].$$

By (5.15) we have

$$u(t, x) = \mathbf{E}[\int_t^{t+\delta} f(s, X_s^{t,x})ds + u(t + \delta, X_{t+\delta}^{t,x})].$$

If u is a $C^{2,1}$-function, then we apply Itô's formula to $u(t + s, X_{t+s}^{t,x}) - u(t, x)$ in the interval $s \in [t, t + \delta]$:

$$\frac{1}{\delta}\mathbf{E}[\int_t^{t+\delta}[(\partial_t + \mathcal{L}u + f)(s, X_s^{t,x})ds] = 0.$$

where $\mathcal{L}$ is the following second order elliptic operator:

$$\mathcal{L}\varphi(t, x) = [\frac{1}{2}Tr(\sigma\sigma^*D^2\varphi) + < D\varphi, b >](t, x).$$

Let $\delta \to 0$. We derive that u is a solution of the following PDE:

$$\partial_t u(t, x) + \mathcal{L}u(t, x) + f(t, x) = 0.$$

The same argument applied to the case in Exercise5.1.1. can derive the well–known Feynman–Kac formula. Proposition 75 plays a key role to establish the relation between PDE and BSDE.

In the next section we will derive a generalized formulation of (5.15): which is in fact a generalized dynamic programming principle in stochastic optimal control theory. The related PDE will be a fully nonlinear second order PDE, a generalized version HJB equation and Feynman–Kac formula.

Exercise 5.1.2 *For the case $f = c(x)y + f_0(t,x)$ and with the assumption (5.12), give a similar formulation as in Remark 19.*

Exercise 5.1.3 *Let (H4.4) be hold and let $u(t,x) : \mathbf{R}^m \times [0,T] \to \mathbf{R}^m$ be a smooth solution of the following PDE*

$$\partial_t u(t,x) + \mathcal{L}u(t,x) + f(t,x,u,Du\sigma) = 0$$

with terminal condition $u(T,x) = \varphi(x)$. Prove that in this case the solution of BSDE (5.9) is $(u, Du\sigma)(s, X_s^{t,\zeta})$.

If we remove assumption (5.12), then $u(x, \cdot)$ is a $(\mathcal{F}_t)$-adapted process. In this case u the solution of the following backward stochastic PDE:

$$
\begin{aligned}
-du(t,x) &= [\mathcal{L}u(t,x) + f(t,x,u,Du\sigma + \phi) + D\phi\sigma(t,x)]dt \\
&\quad -[Du\sigma + \phi(t,x)]dB_t, \\
u(T,x) &= \Phi(T,x)
\end{aligned}
$$

(see [P4]). Prove that if (u, ψ) is smooth solution of the above SPDE, then the solution (Y^ζ, Z^ζ) of BSDE (5.9) is

$$Y_s^\zeta = u(s, X_s^{t,\zeta}), \quad Z_s^\zeta = (Du\sigma + \phi)(s, X_s^{t,\zeta}), \quad s \in [t,T].$$

5.2   Numerical solution of BSDEs

5.2.1   Euler's Approximation

Let $(\epsilon_i^n)_{i=1,2,\cdots,n}$ be a Bernouil sequence, i.e., an i.i.d. sequence such that with

$$P\{\epsilon_i^n = 1\} = P\{\epsilon_i^n = -1\} = \frac{1}{2}.$$

We set

$$B_k^n \quad := \quad \sqrt{n}\sum_{i=1}^k \epsilon_i^n, \quad \mathcal{F}_k^n := \sigma\{B_k^n; 1 \le k \le n\}$$

$$\Delta B_{k+1}^n \quad := \quad B_{k+1}^n - B_k^n = \sqrt{n}\epsilon_k^n,$$

Let ξ be $\mathcal{F}_k^n$–measurable. This implies that there exists a function: $\Phi : \{1,-1\}^k \to R$, such that

$$\xi^n = \Phi_n(\epsilon_1^n, \cdots, \epsilon_k^n).$$

All processes are assumed to be F_k^n–adapted. We make the following assumption

(H1) B^n converges to B in $\mathcal{S}^2$
(H2) ξ^n converges to ξ in $L^2(P)$.
f and $f^n : [0,1] \times \Omega \times R \times R \longrightarrow R$ such that for each $(y,z) \in R \times R$,

$\{f^n(t, y, z)\}_{0 \leq t \leq 1}$(resp. $\{f(t, y, z)\}_{0 \leq t \leq 1}$) are progressively measurable with respect to $\mathcal{F}_t^n$ (resp. to $\mathcal{F}_t$) such that
(H3)–(i):

$$|f^n(t, y, z) - f^n(t, y', z')| \leq C(|y - y'| + |z - z'|)$$
$$|f(t, y, z) - f(t, y', z')| \leq C(|y - y'| + |z - z'|)$$

(ii) For each (y, z) paths $\{f^n(t, y, z)\}_{0 \leq t \leq 1}$ have RCLL paths and converges to $\{f(t, y, z)\}_{0 \leq t \leq 1}$ in $\mathcal{S}^2(R)$ with

$$|Y|_{\mathcal{S}^2} := \{E[\sup_{0 \leq t \leq 1} |Y_t|^2]\}^{1/2}.$$

In this paper we set

$$f^n(t, y, z) \equiv g_k^n(y, z), \quad t \in [\frac{k}{n}, \frac{k+1}{n}), \ k = 0, 1, \cdots, n.$$

We set
$y_n^n = \xi^n$: a given $\mathcal{F}_n^n$–measurable random variable. Then we solve backwardly

$$y_k^n = y_{k+1}^n + g_k^n(y_k^n, z_k^n)\frac{1}{n} - z_k^n \Delta B_{k+1}^n, \ k = n - 1, \cdots, 3, 2, 1.$$

Or $y_t^n \equiv y_k^n$, $z_t^n \equiv z_k^n$, $t \in [\frac{k}{n}, \frac{k+1}{n})$. We call (y^n, z^n) the solution to (g, ξ).

$$dy_t^n = f^n(t, y_t^n, z_t^n)d\langle B^n \rangle_t - z_t^n dB_t^n,$$
$$y_T^n = \xi^n.$$

Theorem 76 *(Existence and Uniqueness and Comparison) Let*

$$g_k^n(\omega, y, z) : \Omega \times R \times R \to R, \ k = 1, \cdots, n - 1$$

be $\mathcal{F}_k^n$–measurable and C–Lipschitz with respect to y with $n > C$. Then there exists a unique $\mathcal{F}_k^n$–adapted pair (y^n, z^n), solution to (g, ξ). Moreover, if $(y^{n'}, z^{n'})$ is the solution corresponding to (g', ξ'), and if

$$g_k^{n'}(\omega, y, z) \geq g_k^n(\omega, y, z), \ \xi^{n'} \geq \xi^n,$$

then the corresponding solution $(y^{n'}, z^{n'})$ satisfies

$$y_k^{n'} \geq y_k^n.$$

COROLLARY. If $A_1(\cdot)$ and $A_2(\cdot)$ satisfies the above conditions with $A_1(y) \geq A_2(y)$, for all $y \in R$. Then $A_1^{-1}(x) \leq A_2^{-1}(x)$, for all $x \in R$.

PROOF of The Theorem. Assume that y_{k+1}^n are solved, we then solve (y_k^n, z_k^n).

$$y_k^n = y_{k+1}^n + g_k^n(y_k^n, z_k^n)\frac{1}{n} - z_k^n \Delta B_{k+1}^n \qquad (5.16)$$

Since y_{k+1}^n has the form: $y_{k+1}^n = \Phi_{k+1}(\epsilon_1, \cdots, \epsilon_{k+1})$. We set

$$
\begin{aligned}
y_{k+1}^{(+)} &:\ = \Phi_{k+1}(\epsilon_1, \cdots, 1), \\
y_{k+1}^{(-)} &:\ = \Phi_{k+1}(\epsilon_1, \cdots, -1).
\end{aligned}
$$

y_{k+1}^+ and y_{k+1}^- are $\mathcal{F}_k^n$–measurable.

We set $\epsilon_{k+1} = \pm 1$, in (5.16):

$$
\begin{aligned}
y_k^n &= y_{k+1}^+ + g_k^n(y_k^n, z_k^n)\frac{1}{n} + -z_k^n n^{-1/2} \\
y_k^n &= y_{k+1}^- + g_k^n(y_k^n, z_k^n)\frac{1}{n} + +z_k^n n^{-1/2}
\end{aligned}
$$

z_k^n can be uniquely solved by $z_k^n = \frac{y_{k+1}^{(+)} - y_{k+1}^{(-)}}{2}$. The equation for y_k^n is

$$y_k^n - g_k^n(y_k^n, z_k^n)\frac{1}{n} = \frac{y_{k+1}^{(+)} + y_{k+1}^{(-)}}{2} \qquad (5.17)$$

When $n > C$, the mapping $A(y) := y - g_k^n(y, z_k^n)\frac{1}{n}$ is strictly monotonic function of y with $A(y) \to +\infty$ (resp. $-\infty$) as $y \to +\infty$ (resp. $-\infty$). Thus the solution y_k^n of (3) exists and is unique. By Corollary, the comparison theorem also holds.

CONVERGENCE RESULT

We consider

(a) $y_t = \xi + \int_t^1 f(s, y_s, z_s)ds - \int_t^1 z_s dB_s$

(b)$_n$ $y_t^n = \xi^n + \int_t^1 f_n(s, y_s^n, z_s^n)d\langle B^n \rangle_t - \int_t^1 z_s^n dB_s^n$

Theorem 77 *(Briand, Delyon & Memin, 2001) We assume (H1), (H2) and (H3). Let (y^n, z^n) be the solution of (b)$_n$ and (y, z) be the solution of (a). Then, in $\mathcal{S}^2 \times \mathcal{S}^2$,*

$$\left(y^n, \int_0^{\cdot} z_s^n dB_s^n \right) \to \left(y, \int_0^{\cdot} z_s dB_s \right), \text{ as } n \to \infty$$

and in $\mathcal{S}^2 \times \mathcal{S}^2$

$$\left(\int_0^{\cdot} z_s^n d\langle B^n \rangle_s, \int_0^{\cdot} |z_s^n|^2 d\langle B^n \rangle_s \right) \to \left(\int_0^{\cdot} z_s^n d\langle B^n \rangle_s, \int_0^{\cdot} |z_s^n|^2 d\langle B^n \rangle_s \right) \text{ as } n \to \infty.$$

Chapter 6
APPENDIX

6.1 Martingale representation theorem

The existence theorem of BSDE requires the following result: an element $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ can be represent by

$$\xi = \mathbf{E}[\xi] + \int_0^T \phi_s dB_s.$$

Lemma 78 *Let $\zeta \in L^2(\Omega, \mathcal{F}_T, P)$ be given such that*

$$\mathbf{E}[\zeta(1 + \int_0^T \phi_s dB_s)] = 0, \quad \forall \phi \in L^2(\Omega, \mathcal{F}_T, P).$$

Then $\zeta = 0$, a.s..

Proof.   For each $\mu(\cdot) \in L^\infty([0,T]; \mathbf{R}^d)$, we denote X^μ, the solution of the following SDE

$$dX_t^\mu = \mu(t)X_t^\mu dB_t, \ X_0^\mu = 1.$$

It is sufficient to prove that

$$\mathbf{E}[\zeta X_T^\mu] = 0, \ \forall \mu \in L^\infty([0,T]; \mathbf{R}^d), \ \Rightarrow \zeta = 0, \text{ a.s.}$$

For each $N \in \mathbb{Z}$, $x = (x_1, \cdots, x_N) \in \mathbf{R}^\mathbb{N}$ and $0 < t_1 < \cdots < t_N \le T$, we have set $\mu(t) = i\sum_{j=1}^N x_j 1_{[0,t_j]}(t)$. It is easy to check that

$$X_t^\mu = \exp\{i\int_0^t \mu(s)dB_s - \frac{1}{2}\int_0^t |\mu(s)|^2 ds\} = e^{i\sum_{j=1}^N x_j B_{t_j \wedge t}} \exp\{-\frac{1}{2}\int_0^t |\mu(s)|^2 ds\}$$

Thus the condition $\mathbf{E}[\zeta X_T^\mu] = 0$ implies

$$\Phi_\mu(x) := E[\zeta e^{i\sum_{j=1}^N x_j B_{t_j}}] = 0,$$

Now for an arbitrary $g \in C_0^\infty(\mathbf{R}^\mathbb{N})$, let $\hat{g}$ be its Fourier transform. We then have

$$E[g(B_{t_1}, \cdots, B_{t_N})\zeta]$$
$$= E[(2\pi)^{-\frac{N}{2}} \int_{\mathbf{R}^\mathbb{N}} \hat{g}(x_0, x_1, \cdots, x_N)e^{i\sum_{j=1}^N x_j B_{t_j}} dx\zeta]$$
$$= (2\pi)^{-\frac{N}{2}} \int_{\mathbf{R}^\mathbb{N}} \hat{g}(x) \ \Phi_\mu(x) dx = 0.$$

Since

$$\{g(\cdot, W(t_1), \cdots, W(t_N)); 0 \le t_1, \cdots, t_N \le T, g \in C_0^\infty(\mathbf{R}^{\mathbb{N}}), \mathbb{N} \in \mathbb{Z}\}$$

is dense in $L^2(\Omega, \mathcal{F}_T, P)$, it follows that $\zeta = 0$.

Theorem 79 (*Representation theorem of an $L^2(\Omega, \mathcal{F}_T, P)$– random variable by Itô's integral*) *For each $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ there exists a unique $z \in \mathcal{M}^2([0, T]; \mathbf{R}^d)$ such that*

$$\xi = \mathbf{E}[\xi] + \int_0^T z_s dB_s. \tag{6.1}$$

Proof.   Let $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ be given. We define the following functional

$$f(\phi) := \mathbf{E}[\xi \int_0^T \phi_s dB_s], \ \phi \in \mathcal{M}^2([0, T]; \mathbf{R}^d).$$

By Schwards inequality $|f(\phi)| \le \mathbf{E}[|\xi|^2]^{1/2} \cdot \mathbf{E}[\int_0^T |\phi_s|^2 ds]^{1/2}$. Thus f is a bounded linear functional in $\mathcal{M}^2([0, T]; \mathbf{R}^d)$. It follows from Riez's representation theorem that, there exists a unique $z \in \mathcal{M}^2([0, T]; \mathbf{R}^d)$

$$f(\phi) = \mathbf{E}[\int_0^T \phi_s z_s ds], \ \forall \phi \in \mathcal{M}^2([0, T]; \mathbf{R}^d).$$

or

$$\mathbf{E}[\int_0^T \phi_s dB_s(\xi - \int_0^T z_s dB_s)] = 0, \ \forall \phi \in \mathcal{M}^2([0, T]; \mathbf{R}^d).$$

Thus we have

$$\mathbf{E}[(1 + \int_0^T \phi_s dB_s)(\xi - \mathbf{E}[\xi] - \int_0^T z_s dB_s)] = 0, \ \forall \phi \in \mathcal{M}^2([0, T]; \mathbf{R}^d).$$

By Lemma 78, we have (6.1). ∎

6.2   A monotonic limit theorem of Itô's processes

Then, using this convergence theorem, we study the limit theorem of g- super-solution. We first consider the following a family of semi-martingales:

$$y_t^i = y_0^i + \int_0^t g_s^i ds - A_t^i + \int_0^t z_s^i dW_s, \qquad i = 1, 2, \cdots. \tag{6.2}$$

Here, for each i, the adapted process $g^i \in L^2_{\mathcal{F}}(0, T, \mathbf{R})$ are given, we also assume that, for each i,

$$(A_t^i) \text{ is a continuous and increasing process with } \mathbf{E}(A_T^i)^2 < \infty, \tag{6.3}$$

We further assume that

(i) (g_t^i) and (z_t^i) are bounded in $L_{\mathcal{F}}^2(0,T)$: $\mathbf{E}\int_0^T[|g_s^i|^2 + |z_s^i|^2]ds \leq C$;

(ii) (y_t^i) increasingly converges to (y_t) with $\mathbf{E}\sup_{0\leq t\leq T}|y_t|^2 < \infty$;

$\hspace{11cm}$ (6.4)

It is clear that

$$(i)\quad \mathbf{E}[\sup_{0\leq t\leq T}|y_t^i|^2] \leq C;$$
$$(ii)\quad \mathbf{E}\int_0^T|y_t^i - y_t|^2ds \to 0,$$

$\hspace{11cm}$ (6.5)

where the constant C is independent of i.

Remark 20 *It is not hard to prove that the limit y_t has the following form*

$$y_t = y_0 + \int_0^t g_s^0 ds - A_t + \int_0^t z_s dW_s, \tag{6.6}$$

where (g_t^0), (z_t) and (A_t) are respectively the L^2-weak limit of (g_t^i), (z_t^i) and (A_t) is an increasing process. In general, we can not prove the strong convergence of $\left\{\int_0^T z_s^i dW_s\right\}_{i=1}^{\infty}$. Our new observation is: for each $p \in [1,2)$, $\{z^i\}$ converges strongly in L^p. This observation is crucially important in this paper, since we will treat nonlinear cases.

Theorem 80 *Assume (6.3) and (6.4) hold. Then the limit (y_t) of (y_t^i) has a form (6.6), where $(g_t^0) \in L_{\mathcal{F}}^2(0,T;\mathbf{R})$, (z_t) is the weak limit of (z_t^i), (A_t) is an RCLL square–integrable increasing process. Furthermore, for any $p \in [0,2)$, $(z_t^i)_{0\leq t\leq T}$ strongly converges to (z_t^i) in $L_{\mathcal{F}}^p(0,T,\mathbf{R}^d)$, i.e.,*

$$\lim_{i\to\infty}\mathbf{E}\int_0^T|z_s^i - z_s|^p ds = 0, \ \forall p \in [0,2). \tag{6.7}$$

If moreover $(y_t)_{t\in[0,T]}$ is continuous, then we have

$$\lim_{i\to\infty}\mathbf{E}\int_0^T|z_s^i - z_s|^2 ds = 0. \tag{6.8}$$

In order to prove this theorem, we need the several Lemmas. The following lemma will be applied to prove that the limit processes (y_t) is RCLL.

Lemma 81 *Let $\{x^i(\cdot)\}$ be a sequence of (deterministic) RCLL processes defined on $[0,T]$ that increasingly converges to $x(\cdot)$: for each $t \in [0,T]$, $x^i(t) \uparrow x(t)$, with $x(t) = b(t) - a(t)$, where $b(\cdot)$ is an RCLL process and $a(\cdot)$ is an increasing process with $a(0) = 0$ and $a(T) < \infty$. Then $x(\cdot)$ and $a(\cdot)$ are also RCLL processes.*

Proof. Since, for each t, $b(\cdot)$, $a(\cdot)$ and thus $x(\cdot)$ have left and right limits at t, thus we only need to check that $x(\cdot)$ is right-continuous.

Since, for each $t \in [0, T)$, $a(t+) \geq a(t)$, thus

$$x(t+) = b(t) - a(t+) \leq x(t). \tag{6.9}$$

On the other hand, for any $\delta > 0$, there exists a positive integer $j = j(\delta, t)$ such that $x(t) \leq x^j(t) + \delta$. Since $x^j(\cdot)$ is RCLL, thus there exists a positive number $\epsilon_0 = \epsilon_0(j, t, \delta)$ such that $x^j(t) \leq x^j(t + \epsilon) + \delta$, $\forall \epsilon \in (0, \epsilon_0]$. These imply that, for any $\epsilon \in (0, \epsilon_0]$,

$$x(t) \leq x^j(t + \epsilon) + 2\delta \leq x^{i+j}(t + \epsilon) + 2\delta \uparrow\uparrow x(t + \epsilon) + 2\delta.$$

Particularly $x(t) \leq x(t+) + 2\delta$ and thus $x(t) \leq x(t+)$. This with (6.9) implies the right continuity of $x(\cdot)$.

Lemma 82 . *Let (A_t) be an increasing RCLL process defined on $[0, T]$ with $A_0 = 0$ and $\mathbf{E}(A_T)^2 < \infty$. Then, for any $\epsilon > 0$, there exists a finite number of stopping times τ_k, $k = 0, 1, 2, \cdots N + 1$ with $\tau_0 = 0 < \tau_1 \leq \cdots \leq \tau_k \leq T = \tau_{N+1}$ and with disjoint graphs on $(0, T)$ such that*

$$\sum_{k=0}^{N} \mathbf{E} \sum_{t \in (\tau_k, \tau_{k+1})} (\Delta A_t)^2 \leq \epsilon. \tag{6.10}$$

Proof.   For each $\nu > 0$, we denote

$$A_t(\nu) = A_t - \sum_{s \leq t} \Delta A_s 1_{\{\Delta A_s > \nu\}}.$$

It has jumps smaller than ν. Thus there is a sufficiently small $\nu > 0$ such that

$$\mathbf{E}[\sum_{s \leq T} (\mathbf{\Delta A_s}(\nu))^2] \leq \frac{\epsilon}{2}.$$

Now let σ_k, $k = 1, 2, \cdots$ be the successive times of jumps of A with size bigger than ν; they are stopping times, and there is N such that

$$\mathbf{E}\left(\sum_{s \in (\sigma_N, T)} (\Delta A_s)^2 \right) \leq \frac{\epsilon}{2}.$$

Then $\tau_k = \sigma_k \wedge T$ for $k \leq N$, and $\tau_{N+1} = T$ satisfies (6.10). ∎
     For applying the formula of the integral by part to the process (y_t) (with jumps), the above open intervals (σ_k, σ_{k+1}) is not so convenient. Thus we will cut a sufficiently small part and only work on the remaining subintervals $(\sigma_k, \tau_k]$. This is possible since our filtration is continuous. In fact we have:

Lemma 83 *Let $0 < \sigma \leq T$ be a stopping time. Then there exists a sequence of $\mathcal{F}_t$–stopping times $\{\sigma_i\}$ with $0 < \sigma_i < \sigma$, a.s. for each $i = 1, 2, \cdots$, such that $\sigma_i \uparrow \sigma$.*

For the continuous filtration $\mathcal{F}_t$, this lemma is quite classical. The proof is omitted.

The following lemma tells that, for any given RCLL increasing process, the contribution of the jumps of (A_t) is mainly concentrated within a finite number of left–open right–closed intervals with "sufficiently small total length". Specifically, we have

Lemma 84 *Let (A_t) be an increasing RCLL process defined on $[0,T]$ with $A_0 = 0$ and $\mathbf{E} A_T^2 < \infty$. Then, for any $\delta, \epsilon > 0$, there exists a finite number of pairs of stopping times $\{\sigma_k, \tau_k\}$, $k = 0, 1, 2, \cdots N$ with $0 < \sigma_k \leq \tau_k \leq T$ such that*

$$
\begin{aligned}
&(i) && (\sigma_j, \tau_j] \cap (\sigma_k, \tau_k] = \emptyset \quad \text{for each } j \neq k; \\
&(ii) && \mathbf{E} \sum_{k=0}^{N} (\tau_k - \sigma_k) \geq T - \epsilon \\
&(iii) && \sum_{k=0}^{N} \mathbf{E} \sum_{\sigma_k < t \leq \tau_k} (\Delta A_t)^2 \leq \delta
\end{aligned}
$$

Proof. We first apply Lemma 82 to construct a sequence of non-decreasing stopping times $\{\sigma_k\}_{k=0}^{N+1}$ with $\sigma_0 = 0$ and $\sigma_{N+1} = T$ such that, $\sigma_k < \sigma_{k+1}$ whenever $\sigma_k < T$ and that

$$
\sum_{k=0}^{N} \mathbf{E} \sum_{t \in (\sigma_k, \sigma_{k+1})} (\Delta A_t)^2 \leq \delta.
$$

Then for each $0 \leq k \leq N$, we apply Lemma 83 to construct a stopping time $0 < \tau_k' < \sigma_{k+1}$, such that

$$
\mathbf{E} \sum_{k=0}^{N} (\sigma_{k+1} - \tau_k') \leq \epsilon.
$$

Finally we set

$$
\tau_0 = \tau_0', \quad \tau_1 = \sigma_1 \vee \tau_1', \quad \cdots, \quad \tau_N = \sigma_N \vee \tau_N'
$$

It is clear that $\tau_k \in [\sigma_k, \sigma_{k+1}) \cap [\tau_{k+1}', \sigma_{k+1}]$. We have also $\tau_k < \sigma_{k+1}$ whenever $\sigma_k < T$. Thus $(\sigma_k, \tau_k] \in (\sigma_k, \sigma_{k+1})$. It follows that

$$
\mathbf{E} \sum_{k=0}^{N} (\sigma_{k+1} - \tau_k) \leq \epsilon,
$$

or

$$
\mathbf{E} \sum_{k=0}^{N} (\tau_k - \sigma_k) \geq T - \epsilon,
$$

and

$$
\sum_{k=0}^{N} \mathbf{E} \sum_{t \in (\sigma_k, \tau_k]} (\Delta A_t)^2 \leq \sum_{k=0}^{N} \mathbf{E} \sum_{t \in (\sigma_k, \sigma_{k+1})} (\Delta A_t)^2 \leq \delta.
$$

Thus the above conditions (i)-(iii) are satisfied. ∎

We now give the

Proof of Theorem 80 Since (g^i) (resp. (z^i)) is weakly compact in $L^2_{\mathcal{F}}(0, T; \mathbf{R})$ (resp. $L^2_{\mathcal{F}}(0, T; \mathbf{R}^d)$), there is a subsequence, still denoted by (g^i) (resp. (z^i)) which converges weakly to (g^0_t) (resp. (z_t)).

Thus, for each stopping time $\tau \leq T$, the following weak convergence holds in $L^2(\Omega, \mathcal{F}_\tau, ; \mathbf{R})$.

$$\int_0^\tau z^i_s dW_s \rightharpoonup \int_0^\tau z_s dW_s, \qquad \int_0^\tau g^i_s ds \rightharpoonup \int_0^\tau g^0_s ds.$$

Since

$$A^i_\tau = -y^i_\tau + y^i_0 + \int_0^\tau g^i_s ds + \int_0^\tau z^i_s dW_s$$

thus we also have the weak convergence

$$A^i_\tau \rightharpoonup A_\tau := -y_\tau + y_0 + \int_0^\tau g^0_s ds + \int_0^\tau z_s dW_s..6$$

Obviously, $\mathbf{E} A^2_T < \infty$. For any two stopping times $\sigma \leq \tau \leq T$, we have $A_\sigma \leq A_\tau$ since $A^i_\sigma \leq A^i_\tau$. From this it follows that (A_t) is an increasing process. Moreover, from Lemma 2.2, both (A_t) and (y_t) are RCLL. Thus (y_t) has a form of (6.6). Since (y_t) is given, it is clear that (z_t) is uniquely determined. Thus not only a subsequence of (z^i) but also the sequence itself converges weakly to (z). Our key point is to show that $\{z^i\}$ converges to z in the strong sense of (6.7). In order to prove this we use Itô's formula applied to $(y^i_t - y_t)^2$ on a given subinterval $(\sigma, \tau]$. Here $0 \leq \sigma \leq \tau \leq T$ are two stopping times. Observe that $\Delta y_t \equiv \Delta A_t$ and the fact that y^i and then A^i are continuous. We have

$$\mathbf{E}|y^i_\sigma - y_\sigma|^2 + \mathbf{E} \int_\sigma^\tau |z^i_s - z_s|^2 ds$$

$$= \mathbf{E}|y^i_\tau - y_\tau|^2 - \mathbf{E} \sum_{t \in (\sigma, \tau]} (\Delta A_t)^2 - 2\mathbf{E} \int_\sigma^\tau (y^i_s - y_s)(g^i_s - g^0_s) ds$$

$$+ 2\mathbf{E} \int_{(\sigma, \tau]} (y^i_s - y_s) dA^i_s - 2\mathbf{E} \int_{(\sigma, \tau]} (y^i_s - y_{s-}) dA_s$$

$$= \mathbf{E}|y^i_\tau - y_\tau|^2 + \mathbf{E} \sum_{t \in (\sigma, \tau]} (\Delta A_t)^2 - 2\mathbf{E} \int_\sigma^\tau (y^i_s - y_s)(g^i_s - g^0_s) ds$$

$$+ 2\mathbf{E} \int_{(\sigma, \tau]} (y^i_s - y_s) dA^i_s - 2\mathbf{E} \int_{(\sigma, \tau]} (y^i_s - y_{s-}) dA_s$$

Since the $\int_{(\sigma, \tau]} (y^i_s - y_s) dA^i_s \leq 0$,

$$\mathbf{E} \int_\sigma^\tau |z^i_s - z_s|^2 ds \leq \mathbf{E}|y^i_\tau - y_\tau|^2 + \mathbf{E} \sum_{t \in (\sigma, \tau]} (\Delta A_t)^2 \qquad (6.11)$$

$$+ 2\mathbf{E} \int_\sigma^\tau |y^i_s - y_s||g^i_s - g^0_s| ds + 2\mathbf{E} \int_{(\sigma, \tau]} |y^i_s - y_s| dA_s.$$

The third term on the right side tends to zero since

$$\mathbf{E} \int_0^T |y_s^i - y_s||g_s^i - g_s^0|ds \leq C \left[\mathbf{E} \int_0^T |y_s^i - y_s|^2 ds \right]^{\frac{1}{2}} \to 0. \tag{6.12}$$

For the last term, we have, P–almost surely,

$$|y_s^1 - y_s| \geq |y_s^i - y_s| \to 0, \qquad \forall s \in [0, T].$$

Since

$$\mathbf{E} \int_0^T |y_s^1 - y_s|dA_s \leq (\mathbf{E}[\sup_s(|y_s^1 - y_s|^2)])^{\frac{1}{2}} (\mathbf{E}(A_T)^2)^{\frac{1}{2}} < \infty,$$

it then follows from Lebesgue's dominated convergence theorem that

$$\mathbf{E} \int_{(0,T]} |y_s^i - y_s|dA_s \to 0. \tag{6.13}$$

By convergence of (6.12) and (6.13), it is clear from the estimate (6.11) that, once A_t is continuous (thus $\Delta A_t \equiv 0$), then z^i tends to z strongly in $L_{\mathcal{F}}^2(0, T; \mathbf{R}^d)$. Thus the second assertion of the theorem follows.

But for the general case, the situation becomes complicated. Thanks to Lemma 84, for any $\delta, \epsilon > 0$, there exist a finite number of disjoint intervals $(\sigma_k, \tau_k]$, $k = 0, 1, \cdots, N$, such that $\sigma_k \leq \tau_k \leq T$ are all stopping times satisfying

$$\begin{array}{ll} \text{(i)} & \mathbf{E} \sum_{k=0}^N [\tau_k - \sigma_k](\omega) \geq T - \frac{\epsilon}{2}; \\ \text{(ii)} & \sum_{k=0}^N \sum_{\sigma_k < t \leq \tau_k} \mathbf{E}(\Delta A_t)^2 \leq \frac{\delta \epsilon}{3}. \end{array} \tag{6.14}$$

Now, for each $\sigma = \sigma_k$ and $\tau = \tau_k$, we apply estimate (6.11) and then take the sum. It follows that

$$\begin{aligned} \sum_{k=0}^N \mathbf{E} \int_{\sigma_k}^{\tau_k} |z_s^i - z_s|^2 ds \ \leq \ & \sum_{k=0}^N \mathbf{E} |y_{\tau_k}^i - y_{\tau_k}|^2 + \sum_{k=0}^N \mathbf{E} \sum_{t \in (\sigma_k, \tau_k]} (\Delta A_t)^2 \\ & + 2\mathbf{E} \int_0^T |y_s^i - y_s||g_s^i - g_s^0|ds + 2\mathbf{E} \int_{(0,T]} |y_s^i - y_s|dA_s. \end{aligned}$$

By using the convergence results (6.12) and (6.13) and taking in consideration of (6.14)-(ii), it follows that

$$\overline{\lim_{i \to \infty}} \sum_{k=0}^N \mathbf{E} \int_{\sigma_k}^{\tau_k} |z_s^i - z_s|^2 ds \leq \sum_{k=0}^N \mathbf{E} \sum_{t \in (\sigma_k, \tau_k]} (\Delta A_t)^2 \leq \frac{\epsilon \delta}{3}$$

Thus there exists an integer $l_{\epsilon\delta} > 0$ such that, whenever $i \geq l_{\epsilon\delta}$, we have

$$\sum_{k=0}^N \mathbf{E} \int_{\sigma_k}^{\tau_k} |z_s^i - z_s|^2 ds \leq \frac{\epsilon \delta}{2}$$

Thus, in the product space $([0,T] \times \Omega, \mathcal{B}([0,T]) \times \mathcal{F}, m \times P)$ (here m stands for the Lebesgue measure on $[0,T]$), we have

$$m \times P \left\{ (s,\omega) \in \bigcup_{k=0}^{N} (\sigma_k(\omega), \tau_k(\omega)] \times \Omega; \quad |z_s^i(\omega) - z_s(\omega)|^2 \geq \delta \right\} \leq \frac{\epsilon}{2}$$

This with (6.14)-(i) implies

$$m \times P \left\{ (s,\omega) \in [0,T] \times \Omega; \quad |z_s^i(\omega) - z_s(\omega)|^2 \geq \delta \right\} \leq \epsilon, \qquad \forall \ i \geq l_{\epsilon\delta}.$$

From this it follows that, for any $\delta > 0$,

$$\lim_{i \to \infty} m \times P \left\{ (s,\omega) \in [0,T] \times \Omega; \quad |z_s^i(\omega) - z_s(\omega)|^2 \geq \delta \right\} = 0.$$

Thus, on $[0,T] \times \Omega$, the sequence $\{(z^i)\}$ converges in measure to (z_t). Since (z_t^i) is also bounded in $L^2_{\mathcal{F}}(0,T;\mathbf{R}^d)$, then for each $p \in [1,2)$, it converges strongly in $L^p_{\mathcal{F}}(0,T;\mathbf{R}^d)$. ∎

6.3 Existence and basic estimates of SDE

We consider the following Itô process

$$dX_t = u_t dt + v_t dB_t, \ X_{t_0} = \xi_0, \tag{6.15}$$

where, for each $T > 0$

$$\begin{aligned} u &\in \mathcal{M}([0,T];\mathbf{R}), \ v \in \mathcal{M}([0,T];\mathbf{R}^d), \\ \xi_0 &\in L^2(\Omega, \mathcal{F}_{t_0}, P; \mathbf{R}). \end{aligned}$$

Lemma 85 *We have the following estimate, for each $\beta > 0$,*

$$\mathbf{E}^{\mathcal{F}_{t_0}}[|X_t|^2] + \frac{\beta}{2} \mathbf{E}^{\mathcal{F}_{t_0}} \left[\int_{t_0}^t e^{\beta(t-s)} |X_s|^2 ds \right] \tag{6.16}$$

$$\leq |X_{t_0}|^2 e^{\beta(t-t_0)} + \mathbf{E}^{\mathcal{F}_{t_0}} \int_{t_0}^t e^{\beta(t-s)} \left[\frac{2}{\beta}|u_s|^2 + |v_s|^2 \right] ds.$$

In particular

$$\mathbf{E}[|X_t|^2] + \frac{\beta}{2} \mathbf{E} \left[\int_{t_0}^t e^{\beta(t-s)} |X_s|^2 ds \right] \tag{6.17}$$

$$\leq \mathbf{E}[|X_{t_0}|^2] e^{\beta(t-t_0)} + \mathbf{E} \int_{t_0}^t e^{\beta(t-s)} \left[\frac{2}{\beta}|u_s|^2 + |v_s|^2 \right] ds.$$

Proof. We first prove the case where u and v are uniformly bounded. We apply Itô's formula to $|X_s|^2 e^{-\beta s}$ in the interval $[t, T]$:

$$|X_T|^2 e^{-\beta T} = |X_t|^2 e^{-\beta t} + \int_t^T e^{-\beta s}[-\beta|X_s|^2 + 2X_s u_s + |v_s|^2]ds + 2\int_t^T e^{-\beta s}X_s v_s dB_s$$

Then we take expectation $\mathbf{E}^{\mathcal{F}_{t_0}}$ and $\mathbf{E}$. By $2Xu \leq \frac{\beta}{2}|X|^2 + \frac{2}{\beta}|u|^2$, we have (6.16) and (6.17). To prove the general situation, i.e., $(u, v) \in \mathcal{M}^2([0, T]; \mathbf{R}^n \times \mathbf{R}^{n \times d})$, we let $\{(u^i, v^i)\}$ be a Cauchy sequences in $\mathcal{M}^2([0, T])$ which converge to $(u, v) \in \mathcal{M}^2([0, T]; \mathbf{R}^n \times \mathbf{R}^{n \times d})$ and X^i be the solution of (6.15) corresponding to (u^i, v^i). It follows that $X^i - X^j$ satisfies the same equation corresponding to $(u^i - u^j, v^i - v^j)$. It then follows from (6.17) that X^i is a Cauchy sequence in $\mathcal{M}^2([0, T]; \mathbf{R}^n)$. By passing to the limit, (6.16) and (6.17) also hold. ∎

We now consider to solve the following SDE

$$dX_t^{t_0, \xi_0} = b(t, X_t^{t_0, \xi_0})dt + \sigma(t, X_t^{t_0, \xi_0})dB_t, \ X_{t_0} = \xi_0. \tag{6.18}$$

We make the following assumptions, for each $x \in \mathbf{R}^n$,

$$b(\cdot, x) \in \mathcal{M}([0, T]; \mathbf{R}^n), \ \ \sigma(\cdot, x) \in \mathcal{M}([0, T]; \mathbf{R}^{n \times d}),$$

and for each $x, x' \in \mathbf{R}^n$,

$$|b(s, x) - b(s, x')| \leq C|x - x'|, \ |\sigma(s, x) - \sigma(s, x')| \leq C|x - x'|, \ \forall x, x' \in \mathbf{R}^n.$$

In the following we fix the constant $\beta > 0$ to be such that $C^2(\frac{2}{\beta} + 1) = \frac{\beta}{4}$.

We introduce the following mapping: $X_t = I_\xi(x)(t) : x \in \mathcal{M}([t_0, T]; \mathbf{R}^n) \rightarrow X \in \mathcal{M}([t_0, T]; \mathbf{R}^n)$

$$dX_t = b(t, x_t)dt + \sigma(t, x_t)dB_t, \ x_{t_0} = \xi.$$

Lemma 86 *We let $X^i := I_{\xi^i}(x^i)$, $i = 1, 2$, and let $\hat{X} = X^1 - X^2$, $\hat{x} = x^1 - x^2$, $\hat{\xi} = \xi^1 - \xi^2$. Then we have*

$$[|\hat{X}_t|^2] + \frac{\beta}{2}\mathbf{E}^{\mathcal{F}_{t_0}}\big[\int_{t_0}^t e^{\beta(t-s)}|\hat{X}_s|^2 ds\big]$$

$$\leq \mathbf{E}^{\mathcal{F}_{t_0}}[|\hat{\xi}|^2]e^{\beta(t-t_0)} + \frac{\beta}{4}\mathbf{E}^{\mathcal{F}_{t_0}}\big[\int_{t_0}^t e^{\beta(t-s)}|\hat{x}_s|^2\big]ds.$$

Proof. We let $\hat{b}_s = b(s, x_s^1) - b(s, x_s^2)$ and $\hat{\sigma}_s = \sigma(s, x_s^1) - \sigma(s, x_s^2)$. By (6.17) and Lipschitz conditions of b and σ,

$$\mathbf{E}^{\mathcal{F}_{t_0}}[|X_t|^2] + \frac{\beta}{2}\mathbf{E}^{\mathcal{F}_{t_0}}\big[\int_{t_0}^t e^{\beta(t-s)}|X_s|^2 ds\big]$$

$$\leq |X_{t_0}|^2 e^{\beta(t-t_0)} + \mathbf{E}^{\mathcal{F}_{t_0}}\int_{t_0}^t e^{\beta(t-s)}[\frac{2}{\beta}|\hat{b}_s|^2 + |\hat{\sigma}_s|^2]ds$$

$$\leq |X_{t_0}|^2 e^{\beta(t-t_0)} + C^2(\frac{2}{\beta} + 1)\mathbf{E}^{\mathcal{F}_{t_0}}\big[\int_{t_0}^t e^{\beta(t-s)}|x_s|^2\big]ds.$$

We then have (6.17). ∎

Theorem 87 *We assume (6.16) and (6.17). Then For each $0 \le t_0 < T < \infty$, and $\xi \in L^2(\Omega, \mathcal{F}_{t_0}, P; \mathbf{R}^n)$, Itô's equation (6.18) has a unique solution $X_s^{t_0,\xi}$, $s \in [t_0, T]$. Moreover we have*

$$[|X_t^{t_0,\xi} - X_t^{t_0,\xi'}|^2] + \frac{\beta}{2}\mathbf{E}^{\mathcal{F}_{t_0}}\big[\int_{t_0}^t e^{\beta(t-s)}|X_s^{t_0,\xi} - X_s^{t_0,\xi'}|^2 ds\big] \qquad (6.19)$$
$$\le \mathbf{E}^{\mathcal{F}_{t_0}}[|\xi - \xi'|^2]e^{\beta(t-t_0)} \ .$$

Proof. Let $\xi^1 = \xi^2$ in the above lemma. We have

$$\mathbf{E}\big[\int_{t_0}^T e^{\beta(T-s)}|X_s|^2 ds\big] \le \frac{1}{2}\mathbf{E}\big[\int_{t_0}^T e^{\beta(T-s)}|x_s|^2\big]ds$$

It follows that $I_{\xi_0}(\cdot)$ is a strict contract mapping in $\mathcal{M}([t_0, T]; \mathbf{R}^n)$. Thus there exists a unique fixed point $X_{\cdot}^{t_0,\xi}$. By the definition of $I_{\xi_0}(\cdot)$, this mapping $X_{\cdot}^{t_0,\xi}$ is the solution of (6.18). (6.19) is easy to derive by applying the above Lemma to $X_{\cdot}^{t_0,\xi} - X_{\cdot}^{t_0,\xi'}$. $\blacksquare$

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