Nonlinear Fluid Dynamics From Gravity

A Thesis

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for the degree of Doctor of Philosophy
in Physics

by

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DECLARATION

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgment of collaborative research and discussions.

The work was done under the guidance of Prof. Shiraz Minwalla at Tata Institute of Fundamental Research, Mumbai.

Sayantani Bhattacharyya

In my capacity as supervisor of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

Prof. Shiraz Minwalla

Date:

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Contents

0.1 SYNOPSIS

0.1.1 Introduction

This synopsis is based on the following three papers, [1], [2] and [3].

In general it is difficult to study the non-equilibrium strongly coupled dynamics . The AdS/CFT correspondence provides an important laboratory to explore such processes for at least a class of quamtum field theories. It relates string theories in AdS_{d+1} background to gauge theories in d dimensions. In particular examples the gauge theory at large N (where N is the rank of the gauge group) limit corresponds to the classical string theory. This theory further reduces to classical supergravity in the strong coupling limit. While classical supergravity is a well-studied system it is itself dynamically rather complicated. However for a class of questions we are able to exploit an additional simplification; supergravity admits a consistent truncation to much simpler dynamical system consisting of Einstein equations with negative cosmological constant. In this sector the metric is the only dynamical field. The existence of such truncation predicts that the corresponding large N strongly coupled gauge theory always has some sector of solutions where the stress-tensor (the field theory operator dual to the bulk metric under AdS/CFT correspondence) is the only dynamical operator.

Therefore just studying the Einstein equations with negative cosmological constant one might gather a lot of information about some strongly coupled field theory dynamics.

First in section 0.1.2 we use the gravitational dynamics to study field theory processes which are far from equilibrium. The field theory is perturbed by turning on some marginal operator for a very small duration. As a consequence of strong interaction the system then rapidly evolves to local thermal equilibrium. The dynamics of this equilibration is dual to the process of black hole formation via gravitational collapse. Gravitational collapse is fascinating in its own right but it gains additional interest in asymptotically AdS spaces because of its link to the field theory.

Once local equilibrium has been achieved (ie, a black hole has been formed) the system (if un-forced) slowly relaxes towards global equilibrium. This relaxation process happens on length and time scales that are both large compared to the inverse local temperature and so admits an effective description in terms of fluid dynamics. Therefore on the the dual gravitational picture, once the black hole is formed, Einstein's equations should reduce to the nonlinear equations of fluid dynamics in an appropriate regime of parameters. In section 0.1.3 we provide a systematic framework to construct this universal gravity dual to the nonlinear fluid dynamics, order by order in a boundary derivative expansion.

Then we proceed to study the causal structure of these gravity solutions. These solutions are regular everywhere away from a space-like surface, and moreover this singularity is shielded from the boundary of AdS space by an event horizon. Within derivative expansion the position of this event horizon can be determined. The area form on it translates into an expression for the entropy current in the boundary field theory. The positivity of its divergence follows from the classic area increase theorems in general relativity.

At the end of this section we quote the results that we found after implementing this framework.

Our work builds on earlier derivations of linearized fluid dynamics from linearized gravity by Policastro, Son and Starinets [4] and on earlier examples of the duality between nonlinear fluid dynamics and gravity [5–13] . There is a large literature in deriving linearized hydrodynamics from AdS/CFT, see [14] for a review and comprehensive set of references.

0.1.2 Weak Field Black Hole Formation

This section is based on [1].

An *AdS* collapse process that could result in black hole formation may be set up, following Yaffe and Chesler [15], as follows . Consider an asymptotically locally *AdS* spacetime, and let $\mathcal{R}$ denote a finite patch of the conformal boundary of this spacetime.

We choose our spacetime to be exactly AdS outside the causal future of $\mathcal{R}$. On $\mathcal{R}$ we turn on the non normalizable part of a massless bulk scalar field. This boundary condition sets up an ingoing shell of the corresponding field that collapses in AdS space. Under appropriate conditions the subsequent dynamics can result in black hole formation.

Translationally invariant asymptotically AdS_{d+1} collapse

First we analyze spacetimes that asymptote to Poincare patch AdS_{d+1} space and We choose our non normalizable data to be independent of boundary spatial coordinates and nonzero only in the time interval $v \in (0, \delta t)$. These boundary conditions create a translationally invariant wave of small amplitude ϵ near the boundary of AdS, which then propagates into the bulk of AdS space.

It turns out that in this case it will always results in black brane formation at small amplitude which, outside the event horizon, can be reliably described by a perturbation expansion. At leading order in perturbation theory the metric takes the following form.

$$ds^2 = 2drdv - \left(r^2 - \frac{M(v)}{r^{d-2}}\right) dv^2 + r^2 dx_i^2. \tag{0.1.1}$$

This form of the metric is exact for all r when $v < 0$, and is a good approximation to the metric for $r \gg \frac{\epsilon^{\frac{2}{d-1}}}{\delta t}$ when $v > 0$. The function $M(v)$ in (0.1.1) can be determined in terms of the non normalizable data at the boundary and turns out to be of order $\frac{\epsilon^2}{(\delta t)^d}$ [1]. $M(v)$ reduces to constant M for $v > \delta t$. The spacetime (0.1.1) describes the process of formation of a black brane of temperature $T \sim \frac{\epsilon^{\frac{2}{d}}}{\delta t}$ over the time scale of order δt. Using the fact that the time scale of formation of the brane is much smaller than its inverse temperature, one can explicitly compute the event horizon of the spacetime (0.1.1) in a power series in $\delta t T \sim \epsilon^{\frac{2}{d}}$ and can show that all of the spacetime outside the event horizon

[1] More precisely, let $\phi_0(v) = \epsilon\ \chi(\frac{v}{\delta t})$ where $\phi_0(v)$ is the non-normalizable part of the bulk scalar field and χ is a function that is defined on $(0, 1)$. Then the energy of the resultant black brane is $\frac{\epsilon^2}{(\delta t)^d} \times A[\chi]$ where $A[\chi]$ is a functional of $\chi(x)$.

(which is causally disconnected from the region inside) lies within the domain of validity of our perturbative procedure.

Spherically symmetric collapse in flat space

Next we consider a spherically symmetric shell, propagating inwards, focused onto the origin of an asymptotically flat space. Such a shell may qualitatively be characterized by its thickness and its Schwarzschild radius r_H associated with its mass. This collapse process may reliably be described in an amplitude expansion when $y \equiv \frac{r_H}{\delta t}$ is very small. The starting point for this expansion is the propagation of a free scalar shell. This free motion receives weak scattering corrections at small y, which may be computed perturbatively.

We demonstrate that this flat space collapse process may also be reliably described in an amplitude expansion at *large* y. The starting point for this expansion is a Vaidya metric similar to (0.1.1), whose event horizon we are able to reliably compute in a power series expansion in inverse powers of y. Outside this event horizon the dilaton is everywhere small and the Vaidya metric receives only weak scattering corrections that may systematically be computed in a power series in $\frac{1}{y}$ at large y. As in the previous subsection the breakdown of perturbation theory occurs entirely within the event horizon, and so does not impinge on our control of the solution outside the event horizon.

Spherically symmetric collapse in asymptotically global AdS

The process of spherically symmetric collapse in an asymptotically global AdS space constitutes a one parameter interpolation between the collapse processes described in subsections 0.1.2 and 0.1.2.

Here the collapse process is initiated by radially symmetric non normalizable boundary conditions that are turned on, uniformly over the boundary sphere of radius R and over a time interval δt. The amplitude ϵ of this source together with the dimensionless ratio

$x \equiv \frac{\delta t}{R}$, constitute the two qualitatively important parameters of the subsequent evolution. When $x \ll \epsilon^{\frac{2}{d}}$ it reduces to the Poincare patch collapse process described in subsection 0.1.2, and results in the formation of a black hole that is large compared to the AdS radius (and so locally well approximates a black brane). When $x \gg \epsilon^{\frac{2}{d}}$ the most interesting part of the collapse process takes place in a bubble of approximately flat space. In this case the solution closely resembles a wave propagating in AdS space at large r, glued onto a flat space collapse process described in subsection 0.1.2.

Following through the details of the gluing process, it turns out that the inverse of the effective flat space y parameter (see subsection 0.1.2) is given by $\frac{x^{\frac{2d-2}{d-2}}}{\epsilon^{\frac{2}{d-2}}}$. The parameter y is of order unity when $x \sim \epsilon^{\frac{1}{d-1}}$. So we conclude that the end point of the global AdS collapse process is a black hole for $x \ll \epsilon^{\frac{1}{d-1}}$ but a scattering dilaton wave for $x \gg \epsilon^{\frac{1}{d-1}}$.

Interpretation in dual field theory

We can interpret these results in dual field theoretic terms.

The gravity solution of subsection 0.1.2 describes a CFT in $\mathcal{R}^{(3,1)}$. The CFT is initially in its vacuum state and over the time period $(0, \delta t)$ it is perturbed by a translationally invariant time dependent source, of amplitude ϵ. The source couples to a marginal operator and pumps energy into the system which subsequently equilibrates.

Since the spacetime in (0.1.1) is identical to the spacetime outside a static uniform black brane (dual to field theory in thermal equilibrium) for $v > \delta t$, the response of the field theory to any boundary perturbation, localized at times $v > \delta t$, will be identical to that of a thermally equilibrated system. Also the expectation values of all local boundary operators (which are determined by the bulk solution in the neighborhood of the boundary) reduces instantaneously to their thermal values. So for this purposes the system seems to thermalize as soon as the external source is switched off. It is only the nonlocal gauge-invariant operators like Wilson loops (which will probe the space-time (0.1.1) away from the boundary depending on their non-locality) can distinguish the the system from being thermalized instantaneously.

Any CFT (with a two derivative gravity dual) when studied on sphere undergoes a first order finite temperature phase transition . The low temperature phase is a gas of 'glueballs' (dual to gravitons) while the high temperature phase is a strongly interacting, dissipative, 'plasma' (dual to the black hole).

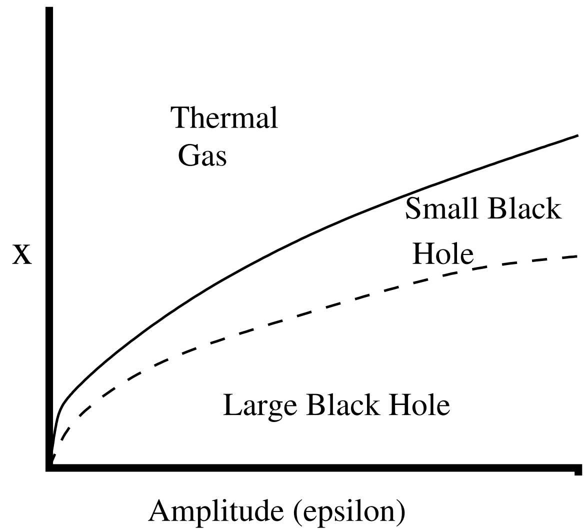

Figure 1: The 'Phase Diagram' for our dynamical stirring in global AdS. The final outcome is a large black hole for $x \ll \epsilon^{\frac{2}{d}}$ (below the dashed curve), a small black hole for $x \ll \epsilon^{\frac{1}{d-1}}$ (between the solid and dashed curve) and a thermal gas for $x \gg \epsilon^{\frac{1}{d-1}}$. The solid curve represents non analytic behavior (a phase transition) while the dashed curve is a crossover.

The gravitational solution of subsection 0.1.2 describes such a CFT on S^{d-1}, initially in its vacuum state. We then excite the CFT over a time δt by turning on a spherically symmetric source function that couples to a marginal operator.

Our solutions predict that the system settles in its free particle phase when $x \gg \epsilon^{\frac{1}{d-1}}$ but in the plasma phase when $x \ll \epsilon^{\frac{1}{d-1}}$. As in subsection 0.1.2 the equilibration in the high temperature phase is almost instantaneous. The transition between these two end points appears to be singular (this is the Choptuik singularity [16] in gravity) in the large N limit. This singularity is presumably smoothed out by fluctuations at finite N, a phenomenon that should be dual to the smoothing out of a naked gravitational singularity by quantum gravity fluctuations.

0.1.3 Fluid dynamics - Gravity correspondence

This section is based on [2] and [3].

Here we describe an unforced system which is already locally equilibrated and is evolving towards global equilibrium. From here onwards we will set $d = 4$ i.e. we will consider only asymptotically AdS_5 spaces.

Consider any two derivative theory of five dimensional gravity interacting with other fields, that has AdS_5 as a solution. The solution space of such systems has a universal sub-sector; the solutions of pure gravity with a negative cosmological constant. We will focus on this universal sub-sector in a particular long wavelength limit. Specifically, we study all solutions that tubewise approximate black branes in AdS_5. We will work in AdS spacetimes where the radial coordinate $r \in (0, \infty)$ and will refer to the remaining coordinates $x^\mu = (v, x_i) \in \mathcal{R}^{1,3}$ as field theory or boundary coordinates. The tubes referred to in the text cover a small patch in field theory directions, but include all values of r well separated from the black brane singularity at $r = 0$; typically $r \geq r_h$ where r_h is the scale set by the putative horizon. The temperature and boost velocity of each tube vary as a function of boundary coordinates x^μ on a length scale that is large compared to the inverse temperature of the brane. We investigate all such solutions order by order in a perturbative expansion; the perturbation parameter is the length scale of boundary variation divided by the thermal length scale. Within the domain of validity of our perturbative procedure we establish the existence of a one to one map between these gravitational solutions and the solutions of the equations of a distinguished system of boundary conformal fluid dynamics. Implementing our perturbative procedure to second order, we explicitly construct the fluid dynamical stress tensor of this distinguished fluid to second order in the derivative expansion. As an important physical input into our procedure, we follow [6, 17, 18] to demand that all the solutions we study are regular away from the $r = 0$ curvature singularity of black branes, and in particular at the the location of the horizon of the black brane tubes out of which our solution is constructed.

Causal structure

It is possible to foliate these gravity solutions into a collection of tubes, each of which is centered about a radial ingoing null geodesic emanating from the AdS boundary. This is sketched in figure 2 where we indicate the tubes on a local portion of the spacetime Penrose diagram.[2] The congruence of null geodesics (around which each of our tubes is centered) yields a natural map from the boundary of AdS space to the horizon of our solutions. When the width of these tubes in the boundary directions is small relative to the scale of variation of the dual hydrodynamic configurations, the restriction of the solution to any one tube is well-approximated by the metric of a uniform brane with the local value of temperature and velocity. This feature of the solutions – the fact that they are tube-wise indistinguishable from uniform black brane solutions – is dual to the fact that the Navier-Stokes equations describe the dynamics of locally equilibrated lumps of fluid.

Local entropy from gravity

In this subsection we restrict attention to fluid dynamical configurations that approach uniform homogeneous flow at some fixed velocity $u_\mu^{(0)}$ and temperature $T^{(0)}$ at spatial infinity. It seems intuitively clear from the dissipative nature of the Navier-Stokes equations that the late time behavior of all fluid flows with these boundary conditions will eventually become $u_\mu(x) = u_\mu^{(0)}$ and $T(x) = T^{(0)}$; The gravitational dual of this globally

[2] For a realistic collapse scenario, described by these nonuniform solutions, only the right asymptotic region and the future horizon and singularity are present.

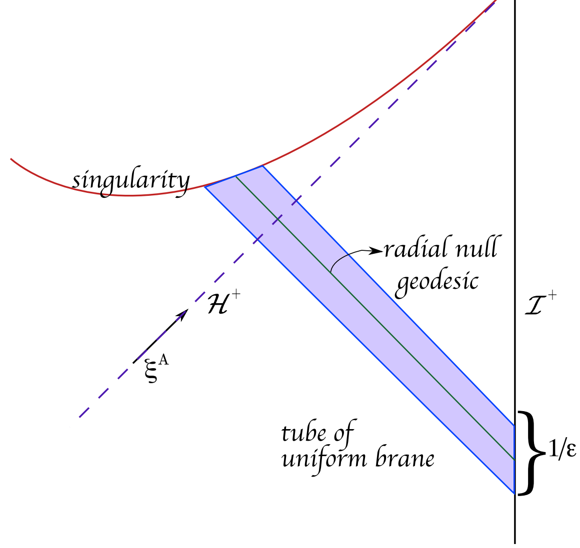

Figure 2: The causal structure of the spacetimes dual to fluid mechanics illustrating the tube structure. The dashed line denotes the future event horizon, while the shaded tube indicates the region of spacetime over which the solution is well approximated by a tube of the uniform black brane.

equilibrated fluid flow is just the uniform blackbrane metric with temperature $T(x) = T^{(0)}$ and boosted to the velocity $u_\mu(x) = u_\mu^{(0)}$. The equation for the event horizon of a uniform black brane is well known. The event horizon of the metric dual to the full non equilibrium fluid flow is the unique null hypersurface that joins with this late time event horizon in the asymptotic future. Within the derivative expansion it turns out that the radial location of the event horizon is determined locally by values and derivatives of fluid dynamical velocity and temperature at the corresponding boundary point. This is achieved using the boundary to horizon map generated by the congruence of ingoing null geodesics described above (see figure 2).

It is possible to define a natural area 3-form on any event horizon whose integral over any co-dimension one spatial slice of the horizon is simply the area of that submanifold. The positivity of the exterior derivative [3]of the area 3-form is a formal restatement of the

[3]The positivity of the top form on horizon submanifold is defined as follows. Choose coordinates $(\lambda, \alpha^1, \alpha^2, \alpha^3)$ on horizon such that α^i s are constant along the null geodesics (which are the generators

area increase theorem of general relativity that is local on the horizon. This statement can be linked to the positivity of the entropy production in the boundary theory by using a 'natural' map from the boundary to the horizon provided by the congruence of the null geodesics described above. The pullback of the area 3-form under this map now lives at the boundary, and also has a 'positive' exterior derivative. Consequently, the 'entropy current', defined as the boundary Hodge dual to the pull-back of the area 3-form on the boundary (with appropriate factors of Newton's constant), has non-negative divergence, and so satisfies a crucial physical requirement for an entropy current of fluid dynamics.

Results

In this subsection we present our explicit construction of the bulk metric, boundary stress tensor and the entropy current upto second order in derivatives computed following the procedure described above.

The metric upto second order is given by

$$\begin{aligned} ds^2 =& -2u_\mu(x^\mu)dx^\mu dr + r^2 f(b(x^\mu)r)u_\mu u_\nu dx^\mu dx^\nu + \mathcal{P}_{\mu\nu}dx^\mu dx^\nu \\ &+\left(2\ b\ r^2 F(br)\sigma_{\mu\nu} + \frac{2}{3}r\ \theta\ u_\mu u_\nu - r(a_\mu u_\nu + a_\nu u_\mu)\right)dx^\mu dx^\nu \\ &+3\ b^2 H\ u_\mu dx^\mu dr \\ &+\left(r^2 b^2 H\ \mathcal{P}_{\mu\nu} + \frac{1}{r^2 b^2}K\ u_\mu u_\nu + \frac{1}{r^2 b^2}\left(J_\mu u_\nu + J_\nu u_\mu\right) + r^2 b^2 \alpha_{\mu\nu}\right)dx^\mu dx^\nu \end{aligned} \tag{0.1.2}$$

In this equation the first line is simply the ansatz (which is the metric of a blackbrane written in a covariant way in Eddington-Finkelstein coordinate). $u_\mu(x^\mu)$ is the velocity of the dual fluid and the $b(x^\mu)$ is inversely related to the temperature $[T(x^\mu)]$ of the fluid.

$$b(x^\mu) = \frac{\pi}{T(x^\mu)}$$

These are two functional parameters of the whole solution. $\mathcal{P}_{\mu\nu}$ is the projection operator that projects in the direction perpendicular to u^μ. The function $f(s)$ and $\mathcal{P}_{\mu\nu}$ are defined

of the horizon) and λ is a future directed parameter along these geodesics. Then a 4-form defined on horizon will be called positive if it is a positive multiple of the 4-form $d\lambda \wedge d\alpha^1 \wedge d\alpha^2 \wedge d\alpha^3$

in the following way

$$\mathcal{P}_{\mu\nu} = u_\mu u_\nu + \eta_{\mu\nu}, \quad f(s) = 1 - \frac{1}{s^4} \tag{0.1.3}$$

The second line records corrections to this metric at first order in derivative, while the third and the fourth lines record the second order corrections to this ansatz.

In the rest of this section we will systematically define all the previously undefined functions that appear in (0.1.2). We will start by defining all scalar functions of the radial coordinate r that appear in (0.1.2), and then turn to the definition of the index valued forms that these functions multiply.

The only undefined function of r in the second line of (0.1.2) is $F(r)$ which is given by

$$F(r) = \frac{1}{4}\left[\ln\left(\frac{(1+r)^2(1+r^2)}{r^4}\right) - 2\,\arctan(r) + \pi\right] \tag{0.1.4}$$

The undefined functions on the third fourth and fifth line of the same equation are defined as

$$\begin{aligned} H &= h^{(1)}(br)\ \mathfrak{S}_4 + h^{(2)}(br)\ \mathfrak{S}_5 \\ K &= k^{(1)}(br)\ \mathfrak{S}_4 + k^{(2)}(br)\ \mathfrak{S}_5 + k^{(3)}(br)\ \mathcal{S} \\ J_\mu &= j^{(1)}(br)\ \mathbf{B}_\mu^\infty + j^{(2)}(br)\ \mathbf{B}_\mu^{\text{fin}} \\ \alpha_{\mu\nu} &= a_1(br)\mathcal{T}_{\mu\nu} + a_5(br)\ (T_5)_{\mu\nu} \\ &\quad + a_6(br)\ (T_6)_{\mu\nu} + a_7(br)\ (T_7)_{\mu\nu} \end{aligned} \tag{0.1.5}$$

where

$$\begin{aligned} h^{(1)}(r) &= -\frac{1}{12r^2} \\ h^{(2)}(r) &= -\frac{1}{6r^2} + \int_r^\infty \frac{dx}{x^5}\int_x^\infty dy\ y^4\left(\frac{1}{2}W_h(y) - \frac{2}{3y^3}\right) \end{aligned} \tag{0.1.6}$$

$$
\begin{aligned}
k^{(1)}(r) &= -\frac{r^2}{12} - \int_r^\infty \left(12x^3 h^{(1)}(x) + (3x^4-1)\frac{dh^{(1)}(x)}{dx} + \frac{1+2x^4}{6x^3} + \frac{x}{6}\right) \\
k^{(2)}(r) &= \frac{7r^2}{6} - \int_r^\infty \left(12x^3 h^{(2)}(x) + (3x^4-1)\frac{dh^{(2)}(x)}{dx} + \frac{1}{2}W_k(x) - \frac{7x}{3}\right) \\
k^{(3)}(r) &= r^2/2
\end{aligned}
\tag{0.1.7}
$$

$$
\begin{aligned}
j^{(1)}(r) &= \frac{r^2}{36} - \int_r^\infty dx\ x^3 \int_x^\infty dy \left(\frac{p(y)}{18y^3(y+1)(y^2+1)} - \frac{1}{9y^3}\right) \\
j^{(2)}(r) &= -\int_r^\infty dx\ x^3 \int_x^\infty dy \left(\frac{1}{18y^3(y+1)(y^2+1)}\right)
\end{aligned}
\tag{0.1.8}
$$

$$
\begin{aligned}
a_1(r) &= -\int_r^\infty \frac{dx}{x(x^4-1)} \int_1^x dy\ 2y \left(\left[\frac{3p(y)+11}{p(y)+5}\right] - 3yF(y)\right) \\
a_5(r) &= -\int_r^\infty \frac{dx}{x(x^4-1)} \int_1^x dy\ y \left(1 + \frac{1}{y^4}\right) \\
a_6(r) &= -\int_r^\infty \frac{dx}{x(x^4-1)} \int_1^x dy\ 2y \left(\frac{4}{y^2}\left[\frac{y^2 p(y) + 3y^2 - y - 1}{p(y)+5}\right] - 6yF(y)\right) \\
a_7(r) &= \frac{1}{4}\int_r^\infty \frac{dx}{x(x^4-1)} \int_1^x dy\ 2y \left(2\left[\frac{p(y)+1}{p(y)+5}\right] - 6yF(y)\right)
\end{aligned}
\tag{0.1.9}
$$

$$
\begin{aligned}
W_h(r) &= \frac{4}{3}\frac{(r^2+r+1)^2 - 2(3r^2+2r+1)F(r)}{r(r+1)^2(r^2+1)^2} \\
W_k(r) &= \frac{2}{3}\frac{4(r^2+r+1)(3r^4-1)F(r) - (2r^5+2r^4+2r^3-r-1)}{r(r+1)(r^2+1)} \\
p(r) &= 2r^3 + 2r^2 + 2r - 3
\end{aligned}
\tag{0.1.10}
$$

We now turn to defining all the terms that carry boundary index structure in (0.1.2). These terms are all expressed in terms of fixed numbers of boundary derivatives of the velocity.

Terms with a single boundary derivative

$$
\begin{aligned}
&\theta = \partial_\alpha u^\alpha, \quad a_\mu = (u.\partial)u_\mu, \quad l^\mu = \epsilon^{\alpha\beta\gamma\mu} u_\alpha \partial_\beta u_\gamma \\
&\sigma_{\mu\nu} = \frac{1}{2}\mathcal{P}^{\mu\alpha}\mathcal{P}^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha\right) - \frac{1}{3}\mathcal{P}_{\mu\nu}\theta
\end{aligned}
\tag{0.1.11}
$$

The quantity l_μ defined here does not appear in the first order correction to the ansatz metric, but does appear, multiplied by other first order terms in the second order metric. We now describe all terms with two boundary derivatives. We sub-classify these terms as scalar like, vector like or tensor like, depending on their transformation properties under the $SO(3)$ rotation group that is left unbroken by the velocity u_μ

Scalar terms with two derivatives

$$\begin{aligned} \mathcal{S} &= \left(-\frac{4}{3}\mathbf{s3} + 2\ \mathfrak{S}1 - \frac{2}{9}\mathfrak{S}3\right) \\ \mathfrak{S}2 &= l_\mu a^\mu, \quad \mathfrak{S}4 = l_\mu l^\mu, \quad \mathfrak{S}5 = \sigma_{\mu\nu}\ \sigma^{\mu\nu} \end{aligned} \tag{0.1.12}$$

where

$$\mathfrak{S}1 = a_\mu a^\mu, \quad \mathfrak{S}3 = \theta^2, \quad \mathbf{s3} = \frac{1}{b}\mathcal{P}^{\alpha\beta}\partial_\alpha\partial_\beta\ b \tag{0.1.13}$$

Vector terms with two derivatives

$$\begin{aligned} \mathbf{B}^\infty &= 4\ (10\,\mathbf{v}4 + \mathbf{v}5 + 3\,\mathfrak{V}1 - 3\,\mathfrak{V}2 - 6\,\mathfrak{V}3) \\ \mathbf{B}^{\text{fin}} &= 9\ (20\,\mathbf{v}4 - 5\,\mathfrak{V}2 - 6\,\mathfrak{V}3) \end{aligned} \tag{0.1.14}$$

where

$$\begin{aligned} (\mathbf{v}4)_\nu &= \frac{9}{5}\left[\frac{1}{2}\mathcal{P}^\alpha_\nu\mathcal{P}^{\beta\gamma}\left(\partial_\beta u_\gamma + \partial_\gamma u_\beta\right) - \frac{1}{3}\mathcal{P}^{\alpha\beta}\mathcal{P}^\gamma_\nu\partial_\gamma\partial_\alpha\ u_\beta\right] - \mathcal{P}^{\alpha\beta}\mathcal{P}^\gamma_\nu\partial_\alpha\partial_\beta\ u_\gamma \\ (\mathbf{v}5)_\nu &= \mathcal{P}^{\alpha\beta}\mathcal{P}^\gamma_\nu\partial_\alpha\partial_\beta\ u_\gamma \end{aligned} \tag{0.1.15}$$

$$\mathfrak{V}1_\nu = \theta a_\nu, \quad \mathfrak{V}2_\nu = \epsilon_{\alpha\beta\gamma\nu}u^\alpha\ a^\beta\ l^\gamma, \quad \mathfrak{V}3_\nu = a^\alpha\ \sigma_{\alpha\nu}$$

Tensor terms with two derivatives

$$\begin{aligned} \mathfrak{T}_{\mu\nu} &= (\mathfrak{T}_1)_{\mu\nu} + \frac{1}{3}(\mathfrak{T}_4)_{\mu\nu} + (\mathfrak{T}_3)_{\mu\nu} \\ (\mathfrak{T}_5)_{\mu\nu} &= l_\mu l_\nu - \frac{1}{3}\mathcal{P}_{\mu\nu}\ \mathfrak{S}4 \\ (\mathfrak{T}_6)_{\mu\nu} &= \sigma_{\mu\alpha}\ \sigma^\alpha_\nu - \frac{1}{3}\mathcal{P}_{\mu\nu}\ \mathfrak{S}5 \\ (\mathfrak{T}_7)_{\mu\nu} &= \left(\epsilon^{\alpha\beta\gamma\mu}\ \sigma^\nu_\gamma + \epsilon^{\alpha\beta\gamma\nu}\ \sigma^\mu_\gamma\right) u_\alpha\ l_\beta \end{aligned} \tag{0.1.16}$$

where

$$
\begin{aligned}
(\mathfrak{T}_1)_{\mu\nu} &= a_\mu a_\nu - \frac{1}{3}\mathcal{P}_{\mu\nu}\ \mathfrak{S}_1 \\
(\mathfrak{T}_3)_{\mu\nu} &= \frac{1}{2}\mathcal{P}_\mu^\alpha \mathcal{P}_\nu^\beta\ (u.\partial)\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha\right) - \frac{1}{3}\mathcal{P}_{\mu\nu}\mathcal{P}^{\alpha\beta}(u.\partial)(\partial_\alpha u_\beta) \\
(\mathfrak{T}_4)_{\mu\nu} &= \sigma_{\mu\nu}\theta
\end{aligned}
\tag{0.1.17}
$$

The stress tensor may be determined for the field theory configurations dual to this solution using the following formula

$$
16\pi G_5 T^\mu_\nu = \lim_{r\to\infty}\left[2r^4(K_{\alpha\beta}h^{\alpha\beta}\delta^\mu_\nu - K^\mu_\nu)\right] - 6\delta^\mu_\nu \tag{0.1.18}
$$

The extrinsic curvature of the regulated boundary is defined via the normal lie-derivative of the induced metric - $K_{\mu\nu} \equiv \frac{1}{2}\mathfrak{L}_n h_{\mu\nu}$. All the indices in the above formulas are raised using the induced metric on the regulated boundary. We find

$$
\begin{aligned}
16\pi G_5 T^{\mu\nu} &= (\pi T)^4\left(g^{\mu\nu} + 4\,u^\mu u^\nu\right) - 2\,(\pi T)^3\,\sigma^{\mu\nu} \\
&\quad + (\pi T)^2\left[\left(\frac{\ln 2}{2}\right)(\mathfrak{T}_7)^{\mu\nu} + 2\,(\mathfrak{T}_6)^{\mu\nu} + (2-\ln 2)\,\mathfrak{T}^{\mu\nu}\right]
\end{aligned}
\tag{0.1.19}
$$

Further, the spacetime configuration presented in (0.1.2) is a solution to the Einstein equation with negative cosmological constant if and only if the velocity and temperature fields obey the constraint

$$
\partial_\mu T^{\mu\nu} = 0 \tag{0.1.20}
$$

The position of the event horizon for the space-time described above is given by the following expression. It is correct upto second order in derivatives.

$$
r_H = \frac{1}{b} + \frac{b}{4}\left(s_b^{(2)} + \frac{1}{3}\,\sigma_{\mu\nu}\,\sigma^{\mu\nu}\right) + \cdots \tag{0.1.21}
$$

Where

$$
s_b^{(2)} = -\frac{2}{3}\,\mathbf{s}_3 + \mathfrak{S}_1 - \frac{1}{9}\,\mathfrak{S}_3 - \frac{1}{12}\,\mathfrak{S}_4 + \mathfrak{S}_5\,\left(\frac{1}{6} + \mathcal{C} + \frac{\pi}{6} + \frac{5\,\pi^2}{48} + \frac{2}{3}\ln 2\right) \tag{0.1.22}
$$

and $\quad \mathcal{C} =$ Catalan number

The expression of local entropy current is given by

$$\begin{aligned}4\,G_N^{(5)}\,b^3\,J_S^\mu = u^\mu\,&\left(1-\frac{b^4}{4}\,F^2\,\sigma_{\alpha\beta}\,\sigma^{\alpha\beta}+\frac{b^2}{4}\sigma_{\alpha\beta}\,\sigma^{\alpha\beta}+\frac{3\,b^2}{4}\,s_b^{(2)}+s_a^{(2)}\right)\\ &+b^2\,P^{\mu\nu}\left[-\frac{1}{2}\left(\partial^\alpha\sigma_{\alpha\nu}-3\,\sigma_{\nu\alpha}\,u^\beta\,\partial_\beta u^\alpha\right)+j_\nu^{(2)}\right]\,.\end{aligned} \tag{0.1.23}$$

Where

$$\begin{aligned}s_a^{(2)} &= \frac{b^2}{16}\left(2\,\mathfrak{S}_4-\mathfrak{S}_5\,\left(2+12\,\mathcal{C}+\pi+\pi^2-9\,(\ln 2)^2-3\pi\,\ln 2+4\,\ln 2\right)\right)\\ j_\mu^{(2)} &= \frac{1}{16}\mathbf{B}^\infty-\frac{1}{144}\mathbf{B}^{\rm fin}\end{aligned} \tag{0.1.24}$$

One can explicitly check that upto third order in derivative the divergence of this entropy current is always positive.

0.1.4 Discussion

This synopsis is about the non-equilibrium dynamics of field theory via AdS/CFT correspondence.

We first studied the approach to equilibrium by rapid forcing and it turns out that for many purposes the system locally equilibrates almost instantaneously and sets the initial conditions for the subsequent slowly varying fluid dynamical evolution.

One can make direct contact with the construction in section 0.1.3 by introducing a forcing that is pulse-like in time but has a slow (compared to the inverse temperature of the black brane that is set up in our solutions) variation in space. Here we expect the resultant thermalization process to be described by a dual metric which can be approximated tubewise by the solutions described in section 0.1.2. The metric will then be corrected in a power series expansion in two variables; The amplitude of the forcing function (as described in section 0.1.2) and a spatial derivative expansion weighted by inverse temperature. The last expansion should reduce exactly to the fluid dynamical expansion described in section 0.1.3.

Roughly speaking, the construction in section 0.1.3 may be regarded as the 'Chiral Lagrangian' for brane horizons. The isometry group of AdS_5 is $SO(4,2)$. The Poincare algebra plus dilatations form a distinguished subalgebra of this group; one that acts mildly on the boundary. The rotations $SO(3)$ and translations $\mathcal{R}^{3,1}$ that belong to this subalgebra annihilate the static black brane solution in AdS$_5$. However the remaining symmetry generators – dilatations and boosts – act nontrivially on this brane, generating a 4 parameter set of brane solutions. These four parameters are simply the temperature and the velocity of the brane. Our construction effectively promotes these parameters to collective coordinate fields and determines the effective dynamics of these collective coordinate fields, order by order in the derivative expansion, but making no assumption about amplitudes.

Next we studied the causal structure of the spacetime constructed in section 0.1.3. We computed the position of the event horizon and derived an expression for the entropy current of the boundary field theory.

While field theoretic conserved currents are most naturally evaluated at the boundary of AdS, this entropy current most naturally lives on the horizon. This is probably related to the fact that while field theoretic conserved currents are microscopically defined, the notion of a local entropy is an emergent long distance concept, and so naturally lives in the deep IR region of geometry, which, by the UV/IR map, is precisely the event horizon. In the limits studied in this synopsis, the shape of the event horizon is a local reflection of fluid variables,. which is reminiscent of the membrane paradigm of black hole physics.

Publications

[A1] P. Basu, J. Bhattacharya, S. Bhattacharyya, R. Loganayagam, S. Minwalla and V. Umesh
"Small hairy blackholes in global AdS spacetime"
arXiv:1003.3232[hep-th].

[A2] S. Bhattacharyya, S. Minwalla and K. Papadodimas
"Small hairy blackholes in $\mathrm{AdS}_5 \times S^5$"
arXiv:1005.1287[hep-th].

[A3] S. Bhattacharyya and S. Minwalla,
"Weak Field Black Hole Formation in Asymptotically AdS Spacetimes,"
JHEP **0909**, 034 (2009), [arXiv:0904.0464 [hep-th]].

[A4] S. Bhattacharyya, S. Minwalla and S. R. Wadia,
"The Incompressible Non-Relativistic Navier-Stokes Equation from Gravity,"
JHEP **0908** (2009) 059, [arXiv:0810.1545 [hep-th]].

[A5] S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla and A. Sharma,
"Conformal Nonlinear Fluid Dynamics from Gravity in Arbitrary Dimensions,"
JHEP **0812**, 116 (2008), [arXiv:0809.4272 [hep-th]].

[A6] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta,
R. Loganayagam and P. Surowka, "Hydrodynamics from charged black branes,"
arXiv:0809.2596 [hep-th].

[A7] S. Bhattacharyya, R. Loganayagam, S. Minwalla, S. Nampuri,
S. P. Trivedi and S. R. Wadia, "Forced Fluid Dynamics from Gravity,"
JHEP **0902**, 018 (2009), [arXiv:0806.0006 [hep-th]].

[A8] S. Bhattacharyya, V. E. Hubeny, R. Loganayagam, G. Mandal, S. Minwalla, T. Morita, M. Rangamani and H. S. Reall,
"Local Fluid Dynamical Entropy from Gravity,"
JHEP **0806**, 055 (2008), [arXiv:0803.2526 [hep-th]].

[A9] J. Bhattacharya, S. Bhattacharyya, S. Minwalla and S. Raju,
"Indices for Superconformal Field Theories in 3,5 and 6 Dimensions,"
JHEP **0802**, 064 (2008), [arXiv:0801.1435 [hep-th]].

[A10] S. Bhattacharyya, V. E. Hubeny, S. Minwalla and M. Rangamani,
"Nonlinear Fluid Dynamics from Gravity,"
JHEP **0802**, 045 (2008), [arXiv:0712.2456 [hep-th]].

[A11] S. Bhattacharyya, S. Lahiri, R. Loganayagam and S. Minwalla,
"Large rotating AdS black holes from fluid mechanics,"
JHEP **0809**, 054 (2008), [arXiv:0708.1770 [hep-th]].

[A12] S. Bhattacharyya and S. Minwalla,
"Supersymmetric states in M5/M2 CFTs,"
JHEP **0712** (2007) 004, [arXiv:hep-th/0702069].

Chapter 1

Introduction

1.1 The AdS/CFT correspondence

Field theories are most usually studied in a series expansion in the coupling constant. But this perturbation theory is not applicable if the coupling constant is large and in a generic case it is difficult to analyze field theory dynamics which are strongly coupled. However field theories with non-abelian gauge symmetries become simplify in the limit of large number of colors (denoted by N) (see [19] and references therein). This is because in a gauge theory the effective coupling constant between two external physical particles is always multiplied by explicit factors of N coming from the number of color degrees of freedom that can run in the internal loop. Therefore in the limit of large N, one can use another alternative expansion where N is the expansion parameter and the theory is expanded around $N = \infty$.

The large N perturbation theory can be organized as follows. Within propagators each of the gauge group indices propagates along a directed line. The lines corresponding to fundamental and antifundamental indices are distinguished by their directions. In this convention all the fields that transform in adjoint or bifundamental representation are represented by double lines whereas fields transforming in the fundamental or antifundamental representation of the gauge group run along single lines.

It turns out that at a given order in $\frac{1}{N}$ such diagrams look like some two dimensional oriented surfaces with some holes and handles. The presence of fundamental or antifundamental matter fields provides single lines which make the boundaries of these surfaces. In the absence of any fundamental fields these surfaces are closed. The genus of these surfaces is related to the order of the expansion in $\frac{1}{N}$ such that the higher order diagrams have higher genus.

One does a similar sort of genus expansion in the case of the perturbative string theory, which is an expansion in the string coupling constant g_s around the value $g_s = 0$. Here also as the order of expansion increases one has to compute the world-sheet areas of two dimensional surfaces with higher and higher genus. In closed string theories one has closed surfaces and in open string theories one has surfaces with boundaries.

This analogy with the string theory indicates that string theory might be a dual description for gauge theory. It also suggests that the parameter $\frac{1}{N}$ in gauge theory is roughly equal to the string coupling constant g_s and the relation is more visible when N is large or g_s is small so that the dual string description is weakly coupled and can be described in perturbation [20], [21].

The AdS/CFT correspondence is a conjecture [22] for what this dual string theory should be in the case of a particular large N gauge theory. According to this conjecture Type IIB string theory in $\mathrm{AdS}_5 \times S^5$ is dual to the 4 dimensional $\mathcal{N} = 4$ supersymmetric Yang -Mills theory living on the boundary of AdS spaces. It is a strong-weak duality in the sense that when the string theory propagates on a highly curved space the dual field theory is weakly coupled. In the strongest form of the conjecture the duality holds at all values of coupling at the both sides.

This conjecture is motivated from the open and close string duality within the string theory itself [21].

The low energy dynamics (the dynamics which includes only the massless fields) of 10 dimensional type IIB string theory is given by 10 dimensional supergravity action which contains dilaton, graviton, R-R field strength and their fermionic superpartners. The supergravity approximation is valid provided the characteristic length scale of the solution is much much greater than the string scale so that all the massive modes of the string theory (whose masses are of the order of the string scale or higher) can be safely ignored. An additional approximation reduces the quantum supergravity to classical, where quantum loop corrections are also ignored. This is valid if the value of the dilaton (which sets the effective value for the string coupling) in the classical solution is small everywhere.

There exists classical $p+1$ dimensional black-brane solutions for this SUGRA action which are charged under the R-R $p+1$-form gauge fields [23, 24]. Such solutions have spherical symmetry in the rest of the $9-p$ directions with a source of R-R field sitting at the origin. For a generic p these solutions contain a spacetime singularity shielded by some inner and outer horizons. These solutions can be parametrized by their outer horizon radius (denoted by r_+), the value of the quantized R-R charge (proportional to N) and the value of the dilaton which characterizes the effective string coupling (the effective value of g_s). Whenever the solution becomes singular the supergravity approximation fails and one has to consider the full string theory to get any consistent answer. Hence for a generic p, the classical supergravity solution can be trusted only in a limited region of the space-time (outside the horizon), which is away from the singularity.

However it turns out that for $p=3$ and in the extremal limit (in the limit where the inner and the outer horizon of the solution meet at say, $r=r_+$) the classical supergravity solution becomes non-singular everywhere in the spacetime. For these solutions space-time curvature never diverges and r_+ can be used to characterize the effective length scale of the solution everywhere. One can choose r_+ arbitrarily large compared to length scale set by the string tension. On the other hand for these solutions the dilaton becomes constant, which again can be chosen arbitrarily small.

Hence in this case the classical supergravity can be trusted uniformly everywhere in space-time. It also has been shown that the mass and the charge of this extremal D3 brane solution saturate the supersymmetric bound .

Therefore from the closed string point of view this D3-brane is a massive and charged solitonic object which curves the space-time around it. The mass of this solution turns out to be proportional to $\frac{1}{g_s}$ and so they are non-perturbative in string coupling constant. One can build up the full string theory in an perturbative expansion around this solitonic solution. Here the higher derivative corrections to the SUGRA action as well as various massive modes of the string theory will contribute, which can be treated quantum mechanically.

On the other hand in perturbative string theory one also has D-brane like objects. These are the hypersurfaces where an open string can end. By the world-sheet duality these hypersurfaces can also act like a source for the closed strings and therefore they can carry R-R charges. It has been shown that a stack of N Dp-brane carries exactly the same amount of R-R charge and preserves the same amount of super symmetry as that of the extremal p-brane SUGRA solution, described above [25]. So it is believed that these two are the same object having two different descriptions at the two opposite range of the parameters [26]. The D-brane description is effectively computable when

$$g_s N \leq 1$$

so that perturbative string theory is valid, whereas SUGRA is applicable when

$$g_s N \geq 1$$

(This bound arises from the fact that the characteristic length of the SUGRA solution has to be much greater than the string scale.)

The AdS/CFT conjecture arises from a further low energy limit taken on both sides.

In the SUGRA description there can be two types of low energy excitations as observed from asymptotic infinity. There are excitations localized far from the brane and have low energies. But there are excitations which are localized near the brane surface. Because of the large redshift factor (caused by the large gravitational potential of the brane) these excitations will also have low energies from the perspective of the observer sitting at the asymptotic infinity though they might have large proper energy. It turns out that these two types of excitation decouple in the strict low energy limit compared to the string scale (or equivalently the strict limit of $\alpha' \to 0$). The intuitive reason is the following. The low energy excitations away from the horizons have very large wavelengths compared to the typical size of the brane. Therefore in the strict limit, the perturbation to these huge waves due to the fluctuations on the brane (ie. the near horizon excitations) goes to zero. On the other hand as $\alpha' \to 0$ the near horizon excitations find it more and more difficult to climb up the gravitational potential and to escape to the asymptotic infinity. This intuition has been verified by calculating the low energy absorption cross section of the brane geometry [27, 28].

The near horizon geometry of the D3-brane solution is that of $\text{AdS}_5 \times S^5$. Therefore one can say that the system of low energy excitations around a D3-brane geometry gets decoupled into two subsystems. One subsystem (excitations localized in the near horizon region) consists of full string theory in $\text{AdS}_5 \times S^5$ geometry and the other subsystem is that of a low energy closed string excitations in the flat 10 dimensional space which is described by the flat 10 dimensional supergravity.

A similar decoupling happens in the dual picture of D-branes where the perturbative string theory can be applied. First, one has to consider the effective action for the massless excitations by integrating out the massive modes of the string theory. Schematically this effective action will be a sum of three parts. One is the action describing the effective dynamics of the open string or the brane. It is described by the appropriate $U(N)$ supersymmetric Yang-Mills theory and some higher derivative corrections to it. The second part of the action gives the low energy closed string dynamics in the flat 10 dimensional space. This is in general described by the 10 dimensional flat space supergravity and its higher derivative corrections. Then the last part should describe the interaction between these two types of excitations. It turns out that here also if one takes the strict $\alpha' \to 0$ limit the interaction action goes to zero so that the dynamics on the brane decouples from that of the bulk. All the higher derivative corrections to both of these decoupled systems vanish in this limit. So in the end one has two decoupled subsystems one describing a 3+1 dimensional $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang-Mills theory and the other describing a flat 10 dimensional supergravity.

Since in both the pictures one of the two decoupled subsystems is flat 10 dimensional supergravity, it is natural to identify the other system. This leads to the conjecture that 3+1 dimensional $\mathcal{N} = 4$ supersymmetric $U(N)$ Yang-Mills theory is dual to the the string theory in $\mathrm{AdS}_5 \times S^5$ [22].

1.1.1 Mapping between the parameters of string theory and gauge theory

Perturbative string theory in $\mathrm{AdS}_5 \times S^5$ has two dimensionless parameters, string coupling constant g_s and the radius of curvature of S^5 (denoted as R) in the units of string scale α'. Conventionally R is set to one and α' is treated as the parameter. In this unit $\alpha' \sim \frac{1}{\sqrt{g_s N}}$, where N equals to the number of the quanta of R-R field strength flux passing through S^5.

On the other hand $\mathcal{N} = 4$ supersymmetric Yang-Mills theory has two dimensionless parameters, the rank of the gauge group N and the coupling constant g. Here the perturbative expansion is controlled by $\lambda = g^2 N$

Under the duality the parameters are mapped in a simple way [21]. The number of the quanta of field strength in string theory side is mapped to the rank of the gauge group in the field theory. The gauge coupling g^2 is mapped to the string coupling g_s. Hence α' (in units where $R = 1$) maps to $\frac{1}{\sqrt{\lambda}}$.

The low energy classical supergravity limit is valid when both α' and g_s corrections are ignored beyond leading order. In the field theory side this corresponds to $\lambda \to \infty$ and $g^2 \to 0$ limit. This is a limit where N necessarily goes to infinity. Thus the weakly coupled classical SUGRA in $\mathrm{AdS}_5 \times S^5$ is the dual description for the strongly coupled 4 dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory .

The the dual gravity description being weakly coupled and classical is computable and one can use it to study the strongly coupled dynamics of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory where usual field theory perturbation technique does not work.

1.1.2 Expectation values of CFT operators using duality

It has been argued [29] that each field propagating in AdS space is in one to one correspondence with some operator in the conformal field theory. There is a prescription for how to compute the expectation values for the field theory operators using this dual theory [29, 30].

For example, all the massless bulk fields are dual to some field theory operators of dimension 4, like bulk metric corresponds to stress tensor operator or bulk dilaton field to lagrangian operator. The partition function of the field theory in the presence of an external source coupled to one such dimension 4 operator is given by the following.

$$\langle \mathrm{e}^{\int d^4x \psi_0(x) O(x)} \rangle_{\mathrm{CFT}} = Z_{\mathrm{string}} \left[\lim_{r\to\infty} \psi(r, x) = \psi_0(x) \right] \tag{1.1.1}$$

where the left hand side is the generating functional for the correlation functions of the dimension 4 field theory operator $O(x)$. Here x is the field theory coordinate and therefore the boundary coordinate for the AdS space. $\psi(r, x)$ is the propagating massless field in AdS which corresponds to the operator $O(x)$. Here r is the fifth coordinate in AdS space other than the four boundary coordinates denoted collectively by x. The boundary of the AdS space is at $r \to \infty$. The right hand side of the equation (1.1.1) is the full string theory partition function with a fixed boundary condition for the $\psi(r,x)$ at infinity which is given by $\psi_0(x)$. Therefore both sides of the equation will be a functional of $\psi_0(x)$. According to this prescription the boundary value of the field in AdS space acts as an external source for the corresponding operator in the field theory side. For the massive bulk fields $\psi_0(x)$ generalizes to the leading coefficient arising in the expansion of the corresponding classical bulk solution around $r = \infty$.

In general it is not possible to compute the full non perturbative string partition function. However in the limit of $\alpha' \to 0$ and $g_s \to 0$ (ie. in the limit of infinite N and infinite λ in gauge theory side) the string theory reduces to weakly-coupled classical supergravity in AdS space. The evaluation of the string partition function amounts to the evaluation of the supergravity action on classical solutions with appropriate boundary conditions for the relevant fields. In the large N and strong coupling limit the field theory correlation functions can be computed from the functional derivative of the classical supergravity action with respect to these boundary conditions [31].

For the case of massless bulk fields the expectation value of the dual operator is determined by the following prescription.

$$\begin{aligned}
\lim_{N,\lambda\to\infty} \langle O(x) \rangle &= \lim_{N,\lambda\to\infty} \frac{\delta}{\delta\psi_0(x)} \left(Z_{\text{string}} \left[\lim_{r\to\infty} \psi(r,x) = \psi_0(x) \right] \right)_{\psi_0(x)=0} \\
&\propto \frac{\delta}{\delta\psi_0(x)} \left(I_{\text{SUGRA}} \left[\lim_{r\to\infty} \psi(r,x) = \psi_0(x) \right] \right)_{\psi_0(x)=0} \\
&= \lim_{r\to\infty} \Pi_{\psi(r,x)}
\end{aligned} \tag{1.1.2}$$

where $I_{\text{SUGRA}} =$ Supergravity action and

$\Pi_{\psi(r,x)} =$ conjugate momentum corresponding to the r evolution of the field $\psi(r,x)$

Using this prescription one can compute the field theory partition functions and some multipoint correlation functions of the field theory operators at large N and strong coupling limit. Some of these quantities are protected due to some symmetries of the theory (eg. the spectrum of chiral operators are non-renormalizable because of supersymmetry) and therefore should match the perturbative field theory computation at small λ. The agreement of those quantities that can reliably be computed on both sides is a non-trivial check for this conjecture. On the other hand one can also use this duality to predict about the strongly coupled dynamics of the field theory.

1.2 Consistent truncation to pure gravity

To evaluate the supergravity action at classical level one has to extremize the action which gives a set of coupled differential equations involving all the supergravity fields. This problem is still quite hard to solve. However, the equations of classical supergravity admit a consistent truncation to a much simpler dynamical system, consisting only of Einstein equations with negative cosmological constant. In this case the metric is the only non-zero bulk field where all other supergravity fields have been set to zero consistently.

This is possible because supergravity is a two derivative theory of gravity containing no fields of spin two or higher other than the graviton itself. Therefore the only fields that can couple linearly with gravity are spin zero fields. The most general two-derivative action involving only linear coupling with gravity has to be of the following form

$$\mathcal{S} = \int \sqrt{g}\,[(A + B\chi_1 + R(C + D\chi_2)]$$

where χ_1 and χ_2 are two scalars, R is the Ricci Scalar and A, B, C and D are some constants. Now one can perform a Weyl transformation of the metric to get rid of χ_2. This transforms the action to 'Einstein Frame'. The fact that the pure AdS is a solution now tells that B is zero and hence the action is consistently truncated only to Einstein gravity with negative cosmological constant.

The equations can be viewed as r evolution of the induced metric on the constant r slices of the space-time. The induced metric in the $r \to \infty$ limit is the boundary metric for the field theory which can act as a source for the stress tensor operator.

Therefore the existence of such a truncation in Supergravity first of all predicts the similar existence of a sector in the dual large N field theory, where the stress-tensor is the only operator with non-zero dynamical expectation value. Secondly just by studying the classical gravity one can extract a lot of information about the strongly coupled dynamics of the stress tensor in the dual large N gauge theory. For example, the conjugate momentum to the induced metric in any gravity theory is given by the extrinsic curvature, which, in the end after appropriate renormalization and removal of divergences, will give the expectation value for the stress tensor in the field theory.

In this thesis we shall primarily study the system of Einstein equations with negative cosmological constant and look for solutions which are locally asymptotically AdS. We shall particularly be interested in the approach towards equilibrium in this truncated sector through the evolution of the stress tensor. In general we shall see that there are two stages in this equilibration process, one, in which the system, driven far from equilibrium by some external source, rapidly approaches to some near equilibrium phase and then a slow hydrodynamic evolution towards the final global equilibrium.

1.2.1 Different equilibrium solutions in gravity

Even with a fixed boundary condition it is possible that there exist more than one solution to a set of differential equations. In this case it implies the existence of more than one saddle points for the partition function. The solution which evaluates to the global minima for the supergravity action is the one that dominates the path integral. There exist phase transitions where one saddle point wins over the other at some particular values of the parameters of the solution.

There are some well-known asymptotically AdS solutions .

- Pure AdS solution.

The metric for this solution is given by

$$ds^2 = \frac{dr^2}{r^2} + r^2\left(-dt^2 + d\vec{x}^2\right) \tag{1.2.3}$$

or

$$ds^2 = \frac{dr^2}{r^2+1} - (r^2+1)dt^2 + r^2 d\Omega_3^2 \tag{1.2.4}$$

Both of these metrics satisfy the Einstein equations with negative cosmological constant (where units are chosen such that the radius of the AdS space is 1). The first metric in equation (1.2.3) has a coordinate singularity at $r = 0$ and its boundary has the topology of $R^{(3,1)}$. The second metric in (1.2.4) is regular everywhere and its boundary has the topology of $S^3 \times R$. The metric in equation (1.2.3) does not cover the full AdS space as it has a horizon. It actually covers a patch (called Poincare patch) of the space (global AdS space) described by the metric in equation (1.2.4).

One can weakly perturb the supergravity equations around both of these metric. But only in the case of the second metric one gets a regular solution which is like a gas of weakly coupled graviton in AdS space.

- Black-brane solution:

The metric for this solution is given by

$$ds^2 = \frac{dr^2}{1-\frac{r_+^4}{r^4}} - r^2\left(1-\frac{r_+^4}{r^4}\right)dt^2 + r^2 d\vec{x}^2 \tag{1.2.5}$$

This solution has a real curvature singularity at $r = 0$. The singularity is shielded from the boundary by a translationally invariant horizon. The horizon is at $r = r_+$. The whole solution has translational invariance. In the limit of $r \to \infty$ it approaches the metric of equation (1.2.3) and therefore the boundary of this solution also has the topology of $R^{(3,1)}$.

In the Euclidean continuation ($t \to i\tau$) this space-time has a conical singularity at $r = r_+$ and one has to compactify the τ coordinate with a specific radius in order to have a non-singular metric. The periodicity of the time circle is given by the surface-gravity of the black-brane solution and is related to the temperature of the black-brane thermodynamics.

- Black hole solution:

  Here the metric is given by

$$ds^2 = \frac{dr^2}{r^2 + 1 - \frac{m}{r^2}} - \left(r^2 + 1 - \frac{m}{r^2}\right) dt^2 + r^2 d\Omega_3^2 \tag{1.2.6}$$

  This metric also has a curvature singularity at $r = 0$. The singularity is shielded by a horizon at $r = r_+$ where r_+ is the largest root of the function

$$f(r) = r^2 + 1 - \frac{m}{r^2}$$

  In the limit of $r \to \infty$ this metric approaches the metric in equation (1.2.4). This is a spherically symmetric solution where both the horizon and the boundary have the topology of $S^3 \times R$.

  In the Euclidean continuation here also one has to compactify the imaginary time circle in order to remove the conical singularity at $r = r_+$ and this gives the temperature for the black hole thermodynamics.

The pure AdS solution is dual to the vacuum of the field theory. In case of Poincare patch solution (ie. equation (1.2.3)) the dual field theory lives on $R^{(3,1)}$ and for the global AdS metric (equation (1.2.4)) it lives on $S^3 \times R$.

The black-branes and black holes in gravity correspond to the deconfined thermal phase of the gauge theory [32]. The temperature of the field theory is determined from the temperature of the black solution considered. In the same way, the area of the horizon gives the entropy of the gravity solution as well as the leading order entropy of the dual field theory.

When studied on $S^3 \times R$, it is possible to have a phase transition between the confined and the deconfined phase of the gauge theory in the limit of infinite N.
At low temperatures the system should be in a confining phase where the Hilbert space of the theory consists of colour singlet particles. The vacuum energy of the theory is of order N^2 as there are N^2 gluons running in each vacuum diagram. The mass and the multiplicities of the colour singlet excitations in this phase are of order 1 and therefore their contribution to the free energy vanishes in the limit of infinite N. In this phase the free energy of the system is equal to the vacuum free energy and is inversely proportional to the temperature (coming from the integration over the Euclideanized time circle of length $\frac{1}{T}$ whereas the integrand, being equal to the vacuum energy, is independent of T) at leading order in large N.
At high temperature the system is in another phase which consists of deconfined plasma of quarks and gluons with a free energy which is also of order N^2 as there are N^2 species in the theory. But in this phase the temperature dependence of the free energy is determined by conformal invariance and goes as T^4.
Because of the conformal symmetry of the theory the transition temperature between these two phases should be proportional to the inverse radius of the S^3 as this is the only length scale available.

One can take an infinite volume limit on the system by taking the radius of S^3 to infinity. In this limit it is not possible to have a phase transition for a conformal field theory. The theory will always be in the high temperature phase as the temperature scale set by the inverse radius goes to zero.

A phase transition in the infinite N limit of the gauge theory should map to a phase transition between two classical solutions of gravity in presence of negative cosmological constant. One can see such phase transition in asymptotically global AdS solution where the pure AdS solution (as described in (1.2.4)) and the black hole solution (as described in (1.2.6)) compete [33]. The free energies of these two solutions are computed by evaluating the action on them. It turns out that that their difference can have either sign depending on the horizon radius or the temperature of the black hole solution (1.2.6). At high temperature it is the black hole solution which has lower value for the free energy. At the transition temperature, which is determined by the inverse radius of the boundary S^3, both the solution evaluates to the same value of the action. Then at low temperature the pure AdS solution wins.

By taking infinite radius limit on the boundary S^3 of the asymptotically global AdS solutions one reaches the solutions in Poincare patch as described in (1.2.3) and in (1.2.5). In this limit there is no phase transition between the two solutions and it is always the black-brane solution (1.2.5) that contributes to the evaluation of the action or free energy.

The gas of gravitons in pure global AdS corresponds to the weakly interacting gas of colorless particles in the confined phase. The black hole or black brane solution is dual to the deconfined plasma as mentioned above [32]. Therefore the phase transition in the gravity solution matches exactly with the phase transition picture in the gauge theory.

1.2.2 Gravitational collapse and thermalization

There exists an extensive literature on the formation of black hole due to gravitational collapse both in asymptotically flat and AdS spaces. In AdS space it becomes particularly interesting because of its connection to the field theory. Since a black hole (or black brane) solution is dual to a field theory in thermal equilibrium, the gravitational collapse should be dual to the process of thermalization.

The collapse process can be initiated by adding a small perturbation to the pure AdS solution. One can choose the perturbation to be a departure from asymptotic AdS condition to an Asymptotically locally AdS metric for a finite duration. This means that in the limit of $r \to \infty$ the induced metric on constant r slices is no longer flat (or that of $S^3 \times R$ in case of global AdS) but some time dependent metric which reduces to the usual one after the perturbation is switched of. It has been shown below in chapter 2 that such a perturbation might result in black brane or black hole formation at some temperature depending on the nature and the strength of the perturbation. In fact if the perturbation is added to the pure AdS solution in Poincare patch it can be shown that it will always collapse to form a black brane. However in the case of global AdS there are two phases available for the perturbation to equilibrate in. One is the solution with a gas of gravitons and the other is the black hole. The strength and the duration of the perturbation will determine the final phase of the equilibrium solution.

It turns out that in certain weak field limit we have some analytic control over the collapse process, thus initiated.

The choice of such perturbation can directly be interpreted within field theory. Here the boundary field theory is also not on the usual flat space (or $S^3 \times R$) and the fluctuation of the boundary metric acts as a time-dependent external source. Since the boundary metric couples to the stress tensor operator, this source pumps energy into the field theory system through the dynamics of the stress tensor. Thermalization takes place once the perturbation is switched off.

If the field theory is being studied on a flat space, the disturbance, thus created, eventually equilibrates to form a soup of deconfined plasma at a particular temperature whose magnitude will depend on the strength and the nature of the perturbation.

While studied on $S^3 \times R$ the system can settle either in a confined phase or in a deconfined phase once the perturbation is switched off. Using the gravity analysis one can predict the nature of this transition and also the final equilibrium temperature.

This is the first stage of the equilibration process as mentioned in the beginning of this section. It turns out that the formation of the black-brane is almost instantaneous. However if there are slow (compared to the average temperature of the black-brane just formed) spatial inhomogeneities in the external source function, they do not smooth out within this rapid initial stage. Then it forms a black-brane, whose temperature and other parameters has slow spatial variations determined by the profile of the external source. At each point locally the solution is exactly that of a black-brane but globally it is not, since the parameters are space-time dependent though in a slow fashion. One expects similar behavior in the dual field theory solution ie. a near equilibrium state with a slowly varying temperature or velocity. This is exactly the situation for the onset of fluid dynamics.

1.2.3 Gravity in the regime of hydrodynamics

It is expected that any strongly interacting field theory system at high enough densities admits an effective description in terms of fluid dynamics. As mentioned in the previous subsection, fluid dynamics describes the approach from local equilibrium towards global equilibrium. Naively the system enters the hydrodynamic regime once, it has already equilibrated over a distance scale of the mean free path but over a distance scale, which is much larger than the mean free path the parameters of the system vary. Therefore fluid dynamics is valid only above a certain length scale set by the mean free path. It is a description where modes with higher energy compared to this scale are integrated out and therefore one has to deal with fewer degrees of freedom than that of the original theory.

An uncharged relativistic conformal fluid in equilibrium is described by following two parameters

- Constant unit normalized four velocity u^μ
- Constant temperature T

In the near equilibrium hydrodynamic situation both u^μ and T become slowly varying function of the space time (compared to the local value of the temperature). The evolution of these fields are determined from the equations given by the conservation of stress tensor. There are four functions that are to be determined, three arising form the three independent components of the unit normalized vector field $u^\mu(x)$ and one from the scalar field temperature, $T(x)$. In 3+1 dimensions conservation of stress tensor is also equivalent to exactly four differential equations. Thus, given a set of initial or boundary conditions for the four functions, the system is completely determined.

The constitutive relation of fluid dynamics expresses stress tensor as a function of these four-velocity and temperature. It is generally a relation which depends on the microscopic details of the theory and in usual situations where such microscopic description is not available or not computable, this relation is determined phenomenologically in order to match the experimental results.

Because fluid dynamics applies in a long wavelength limit, one can expand this constitutive relation in terms of the derivatives of velocity and temperature. Successive terms have more derivatives and therefore more suppressed. The symmetries of the theory largely determines this expansion. For example in case of a conformal theory like the theory of $\mathcal{N} = 4$ SYM at each order in derivatives all terms are determined upto some numerical constants. The determination of these constants requires a field theory calculation for the strongly coupled system which can be performed using the duality. Weakly coupled classical gravity calculation can provide these numbers.

There is a large literature for such computations [4–13], see [14] for a review and comprehensive set of references.

As described before in the previous subsection a black-brane solution is dual to a fluid in equilibrium. The velocity of the fluid u^μ is dual to the unique Killing vector of the black-brane solution which goes null on the Killing horizon and the temperature of the fluid is given by the black-brane temperature .

At the first derivative order only one new term can occur in the stress tensor of a fluid with conformal symmetry. The coefficient of this new term is related to the shear viscosity of the fluid. The shear viscosity measures the rate at which first order transverse fluctuations (small space-time dependent fluctuations in the direction perpendicular to the global motion of the fluid) dissipate into equilibrium. Such coefficients can be related to the two point correlators of the stress tensor . Such two-point functions of stress tensor are computable in the dual picture of gravity using the usual AdS/CFT prescription. In the gravity side one has to compute the on-shell bulk action when the asymptotic AdS condition is perturbed by some fluctuation in the boundary metric in the transverse direction (metric components in the directions transverse to r and time) . Since the fluctuation appears in the action quadratically, for two point correlator it turns out that it is enough to solve the bulk equations only at linear order . However as one goes higher and higher order in derivative expansion, the number of possible terms in the stress tensor, allowed by the symmetries of the theory, increases. For example at two derivative order for a conformal uncharged fluid there are a total of five new terms. The transport coefficients for these terms occurring at higher order in derivative expansion, are in principle, related to some higher point correlators of the stress-tensor. But it becomes increasingly difficult to determine these relations and then compute the correlators using the gravity dual, where one needs to solve the bulk equation upto some non-linear orders.

However one might expect a more direct relation between gravity and hydrodynamic states of the dual field theory.

The Stress-tensor operator has a unique expression when evaluated on each hydrodynamic state of the field theory. General space-time translational invariance of the field theory implies the conservation of this stress-tensor. According to the AdS/CFT prescription, in gravity side this specifies the boundary value of the momentum corresponding to the r evolution of the induced metric. Similarly the metric of the space-time on which the field theory lives determines the boundary value of the induced metric itself. It has been shown in [34] how, given any boundary metric and conserved stress tensor at $r \to \infty$ one can integrate the Einstein equations inside the bulk and get the metric as an expansion around $r = \infty$. It has also been proved that these two sets of boundary data (metric and stress tensor) is sufficient to have a unique solution. Intuitively the Einstein equations are second order differential equations in r for the induced metric on constant r slices and therefore once the value of this induced metric and its corresponding momentum is defined on any r slice one can expect a unique solution. Thus, if Einstein equations are viewed as a Hamiltonian system of r evolution, it is a completely determined system once one has completely specified the dual field theory hydrodynamic state by providing the expression for stress tensor and the boundary metric. This, on the other hand, implies the existence of a unique gravity solution corresponding to each hydrodynamic state of the field theory.

According to this argument any boundary data will give a unique gravity solution as long as the boundary data satisfies the conservation equation. If we fix the boundary metric to be flat, the boundary data will contain nine functions of the space-time given as nine independent components of the four-dimensional traceless symmetric stress-tensor (it is traceless because the theory is conformally invariant). But it is known that the fluid dynamical data is always expressible in terms of four functions and their derivatives whose evolution is completely determined by the four components of conservation equation. Therefore within gravity itself there should be some principle which will cut down the admissible boundary data from nine to four functions, constrained by the conservation law. This principle is provided by the condition of regularity.
As explained in the previous subsection the gravity solution has entered the near equilibrium hydrodynamic regime due to the perturbation created at the boundary and for a generic perturbation one does not expect a singularity to develop within those region of the space time which is causally connected to the asymptotic infinity (ie. the boundary of the AdS space). It turns out that the boundary data (ie. the stress tensor) that can produce a regular bulk solution can be expressed in terms of four functions like a hydrodynamic stress tensor. This makes a one-to-one correspondence between each four dimensional uncharged hydrodynamic state of the field theory to a regular asymptotically AdS metric in five dimension.

This condition of regularity imposes some sort of mixed boundary condition on the solutions to the Einstein equations. For some part of the bulk metric, the boundary condition is determined only at $r = \infty$ in the usual way ie. by specifying the induced metric and the relevant part of the stress tensor. In these sectors the solution is automatically regular, whereas for the rest of the metric the two sets of required boundary conditions are provided by specifying the induced metric at $r = \infty$ and the condition of regularity at the other causal end of the bulk space-time that is the horizon. As mentioned before, once these two conditions are satisfied at both ends in this sector the stress tensor is completely determined in terms of the data provided in the other sectors and thus removing the unphysical non-hydrodynamic modes from the solution.

It turns out that one can formulate an algorithm to solve the gravity system of Einstein equations with negative cosmological constant which can be implemented order by order in a derivative expansion.The zeroth order solution is same as the black-brane solution but with a slowly varying velocity and temperature. At each order the metric receives new corrections to take care of the higher order space-time derivatives of $u^\mu(x)$ and $T(x)$, the parameters of the zeroth order solution. At each order one can evaluate the stress tensor at infinity. This provides a way to compute all the transport coefficients for the fluid which is dual to the particular gravity system being solved. In our case it is the fluid of $\mathcal{N} = 4$ supersymmmetric Yang-Mill's theory.

1.2.4 Entropy current for the near equilibrium solution

The entropy of the black-brane solution is proportional to the horizon area which is also equal to the entropy density of the dual Yang-Mill's theory on $R^{(3,1)}$. For a solution in equilibrium the entropy density (denoted by s) is constant on both sides and one can define an entropy current, given by entropy density times velocity, which is divergenceless if the zeroth order stress tensor is conserved.

For a near equilibrium system one can also try to define an entropy current which need not be divergenceless. But the divergence of this current should always be positive according to the second law of thermodynamics. The value of this divergence will give a measure for the entropy production as the system approaches towards equilibrium.

As explained in the previous subsection, a near equilibrium hydrodynamic system behaves as if in equilibrium over length scale of mean free path. Therefore over this length scale the entropy current also should reduce to its local equilibrium value and in a fluid system one could expect that the expression for the entropy current is also expressible in derivative expansion at each point in space-time. Further, for a conformally invariant fluid like the one that we are considering here, this expression should have a fixed transformation property under conformal transformation ie. it should transform covariantly at all orders in derivative expansion. But even after imposing all these constraints together the expression for entropy current for a conformally invariant fluid is not uniquely determined. The gravity solution provides a very natural construction of an entropy current for the dual fluid living on the boundary. In a non-static solution for Einstein equations it is the horizon area that is always increasing as proved in the area-increase theorem in gravity. Therefore this can be pulled back to the boundary and can be identified with a choice of entropy current for the dual fluid. It is a two step process. The first step involves finding the location of the horizon for the non-static solution and determining the ever-increasing area form on a spacelike slice of the horizon. The second step involves the pull-back of this area form to the boundary. Both of these two steps can be performed order by order in derivative expansion provided the fluid at infinite future approaches global equilibrium. This last assumption is true for a generic situation because of the dissipative nature of the equations governing the fluid motion. Because of this assumption it is possible to determine the horizon (the unique null hypersurface that asymptotes to the horizon of a static black-brane solution at infinite future) in a local fashion, which otherwise requires the explicit global solution at all future time.

In this thesis chapter 2 describes the collapse situation which is mentioned in subsection 1.2.2. In chapter 3 we have described the detail algorithm of how to construct the fluid -gravity duality in a near equilibrium situation. This has been briefly described in subsection 1.2.3. The topic of the last subsection 1.2.4 is described in detail in chapter 4.

Chapter 2

Weak Field Black Hole Formation

This chapter is based on [1].

In this chapter we use the gravitational dynamics to study field theory processes which are far from equilibrium. The field theory is perturbed by turning on some marginal operator for a very small duration. As a consequence of strong interaction the system then rapidly evolves to local thermal equilibrium. The dynamics of this equilibration is dual to the process of black hole formation via gravitational collapse. Gravitational collapse is fascinating in its own right but it gains additional interest in asymptotically AdS spaces because of its link to the field theory.

An AdS collapse process that could result in black hole formation may be set up, following Yaffe and Chesler [15], as follows . Consider an asymptotically locally AdS spacetime, and let $\mathcal{R}$ denote a finite patch of the conformal boundary of this spacetime. We choose our spacetime to be exactly AdS outside the causal future of $\mathcal{R}$. On $\mathcal{R}$ we turn on the non normalizable part of a massless bulk scalar field. This boundary condition sets up an ingoing shell of the corresponding field that collapses in AdS space. Under appropriate conditions the subsequent dynamics can result in black hole formation.

2.1 Translationally invariant collapse in AdS

In this section we study asymptotically planar (Poincare patch) AdS_{d+1} solutions to negative cosmological constant Einstein gravity interacting with a minimally coupled massless scalar field (note that this system obeys the null energy condition). We focus on solutions in which the boundary value of the scalar field takes a given functional form $\phi_0(v)$ in the interval $(0, \delta t)$ but vanishes otherwise. The amplitude of $\phi_0(v)$ (which we schematically refer to as ϵ below) will be taken to be small in most of this section. The boundary dual to our setup is a d dimensional conformal field theory on $R^{d-1,1}$, perturbed by a spatially homogeneous and isotropic source function, $\phi_0(v)$, multiplying a marginal scalar operator.

Note that our boundary conditions preserve an $R^{d-1} \times SO(d-1)$ symmetry (the R^{d-1} factor is boundary spatial translations while the $SO(d-1)$ is boundary spatial rotations). In this section we study solutions on which $R^{d-1} \rtimes SO(d-1)$ lifts to an isometry of the full bulk spacetime. In other words the spacetimes studied in this section preserve the maximal symmetry allowed by our boundary conditions. As a consequence all bulk fields in our problem are functions of only two variables; a radial coordinate r and an Eddington Finkelstein ingoing time coordinate v. The chief results of this section are as follows:

- The boundary conditions described above result in black brane formation for an arbitrary (small amplitude) source functions $\phi_0(v)$.

- Outside the event horizon of our spacetime, we find an explicit analytic form for the metric as a function of $\phi_0(v)$. Our metric is accurate at leading order in the ϵ expansion, and takes the Vaidya form (0.1.1) with a mass function that we determine explicitly as a function of time.

- In particular, we find that the energy density of the resultant black brane is given, to leading order, by

$$C_2 = \frac{2^{d-1}}{(d-1)} \left(\frac{(\frac{d-1}{2})!}{(d-1)!} \right)^2 \int_{-\infty}^{\infty} \left(\left(\partial_t^{\frac{d+1}{2}} \phi_0(t) \right)^2 \right) \tag{2.1.1}$$

in odd d and by

$$C_2 = -\frac{d^2}{(d-1)2^d}\frac{1}{\left(\frac{d}{2}!\right)^2}\int dt_1 dt_2 \partial_{t_1}^{\frac{d+2}{2}}\phi_0(t_1)\ln(t_1-t_2)\theta(t_1-t_2)\partial_{t_2}^{\frac{d+2}{2}}\phi_0(t_2) \quad (2.1.2)$$

in even d. Note that, in each case, $C_2 \sim \frac{\epsilon^2}{(\delta t)^d}$.

- We find an explicit expression for the event horizon of the resultant solutions, at leading order, and thereby demonstrate that singularities formed in the process of black brane formation are always shielded by a regular event horizon at small ϵ.
- Perturbation theory in the amplitude ϵ yields systematic corrections to this leading order metric. We unravel the structure of this perturbation expansion in detail and compute the first corrections to the leading order result.

While every two derivative theory of gravity that admits an AdS solutions admits a consistent truncation to Einstein gravity with a negative cosmological constant, the same statement is clearly not true of gravity coupled to a minimally coupled massless scalar field. It is consequently of considerable interest to note that results closely analogous to those described above also apply to the study of Einstein gravity with a negative cosmological constant. In § 2.4 we analyze the process of black brane formation by gravitational wave collapse in the theory of pure gravity (similar to the set up of [15]), and find results that are qualitatively very similar to those reported in this section. The solutions of § 2.4 yield the dual description of a class of thermalization processes in every 3 dimensional conformal field theory that admits a dual description as a two derivative theory of gravity. In fact, the close similarity of the results of § 2.4 with those of this section, lead us to believe that the results reported in this section are qualitatively robust. In particular we think it is very likely that results of this section will qualitatively apply to the most general small amplitude translationally invariant collapse process in the systems we study.

2.1.1 The set up

Consider a minimally coupled massless scalar (the 'dilaton') interacting with negative cosmological constant Einstein gravity in $d+1$ spacetime dimensions

$$S = \int d^{d+1}x\sqrt{g}\left(R - \frac{d(d-1)}{2} - \frac{1}{2}(\partial\phi)^2\right) \tag{2.1.3}$$

The equations of motion that follow from the Lagrangian (2.1.3) are

$$\begin{aligned} &E_{\mu\nu} \equiv G_{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\left(-\frac{d(d-1)}{2} + \frac{1}{4}(\partial\phi)^2\right) = 0 \\ &\nabla^2\phi = 0 \end{aligned} \tag{2.1.4}$$

where the indices μ, ν range over all $d+1$ spacetime coordinates. As mentioned above, in this section we are interested in locally asymptotically AdS_{d+1} solutions to these equations that preserve an $R^{d-1} \times SO(d-1)$ symmetry group. This symmetry requirement forces the boundary metric to be Weyl flat (i.e. Weyl equivalent to flat $R^{d-1,1}$); however it allows the boundary value of the scalar field to be an arbitrary function of boundary time v. We choose this function as

$$\begin{aligned} &\phi_0(v) = 0 \quad (v < 0) \\ &\phi_0(v) < \epsilon \quad (0 < v < \delta t) \\ &\phi_0(v) = 0 \quad (v > \delta t) \end{aligned} \tag{2.1.5}$$

(we also require that $\phi_0(v)$ and its first few derivatives are everywhere continuous.[1]).

Everywhere in this chapter we adopt the 'Eddington Finkelstein' gauge $g_{rr} = g_{ri} = 0$ and $g_{rv} = 1$. In this gauge, and subject to our symmetry requirement, our spacetime takes the form

$$\begin{aligned} &ds^2 = 2drdv - g(r,v)dv^2 + f^2(r,v)dx_i^2 \\ &\phi = \phi(r,v). \end{aligned} \tag{2.1.6}$$

[1] We expect that all our main physical conclusions will continue to apply if we replace our ϕ_0 - which is chosen to strictly vanish outside $(0, \delta t)$ - by any function that decays sufficiently rapidly outside this range.

The mathematical problem we address in this subsection is to solve the equations of motion (2.1.4) for the functions ϕ, f and g, subject to the pure AdS initial conditions

$$\begin{aligned} g(r,v) &= r^2 \quad (v<0) \\ f(r,v) &= r \quad (v<0) \\ \phi(r,v) &= 0 \quad (v<0) \end{aligned} \tag{2.1.7}$$

and the large r boundary conditions

$$\begin{aligned} &\lim_{r\to\infty} \frac{g(r,v)}{r^2} = 1 \\ &\lim_{r\to\infty} \frac{f(r,v)}{r} = 1 \\ &\lim_{r\to\infty} \phi(r,v) = \phi_0(v) \end{aligned} \tag{2.1.8}$$

The Eddington Finkelstein gauge we adopt in this chapter does not completely fix gauge redundancy (see [15] for a related observation). The coordinate redefinition $r = \tilde{r} + h(v)$ respects both our gauge choice as well as our boundary conditions. In order to completely define the mathematical problem of this section, we must fix this ambiguity. We have assumed above that $f(r,v) = r + \mathcal{O}(1)$ at large r. It follows that under the unfixed diffeomorphism, $f(r,v) \to f(r,v) + h(v) + \mathcal{O}(1/r)$. Consequently we can fix this gauge redundancy by demanding that $f(r,v) \approx r + \mathcal{O}(1/r)$ at large r. We make this choice in what follows. As we will see below, it then follows from the equations of motion that $g(r) = r^2 + \mathcal{O}(1)$. Consequently, the boundary conditions (2.1.8) on the fields g, f and ϕ, may be restated in more detail as

$$\begin{aligned} g(r,v) &= r^2\left(1 + \mathcal{O}(\frac{1}{r^2})\right) \\ f(r,v) &= r\left(1 + \mathcal{O}(\frac{1}{r^2})\right) \\ \phi(r,v) &= \phi_0(v) + \mathcal{O}(\frac{1}{r}) \end{aligned} \tag{2.1.9}$$

Equations (2.1.4), (2.1.6), (2.1.7) and (2.1.9) together constitute a completely well defined dynamical system. Given a particular forcing function $\phi_0(v)$, these equations and boundary conditions uniquely determine the functions $\phi(r,v)$, $g(r,v)$ and $f(r,v)$.

2.1.2 Structure of the equations of motion

The nonzero equations of motion (2.1.4) consist of four nontrivial Einstein equations E_{rr}, E_{rv}, E_{vv} and $\sum_i E_{ii}$ (where the index i runs over the $d-1$ spatial directions) together with the dilaton equation of motion. For the considerations that follow below, we will find it convenient to study the following linear combinations of equations

$$
\begin{aligned}
E_c^1 &= g^{v\mu} E_{\mu r} \\
E_c^2 &= g^{v\mu} E_{\mu v} \\
E_{ec} &= g^{r\mu} E_{\mu r} \\
E_d &= \sum_{i=1}^{d-1} E_{ii} \\
E_\phi &= \nabla^2 \phi
\end{aligned}
\tag{2.1.10}
$$

Note that the equations E_c^1 and E_c^2 are constraint equations from the point of view of v evolution.

It is possible to show that E_d and $\frac{d(rE_{ec})}{dr}$ both automatically vanish whenever $E_c^1 = E_c^2 = E_\phi = 0$. This implies that this last set of three independent equations - supplemented by the condition that $rE_{ec} = 0$ at any one value of r - completely exhaust the dynamical content of (2.1.4). As a consequence, in the rest of this chapter we will only bother to solve the two constraint equations and the dilaton equation, but take care to simultaneously ensure that $rE_{ec} = 0$ at some value of r. It will often prove useful to impose the last equation at arbitrarily large r. This choice makes the physical content of $rE_{ec} = 0$ manifest; this is simply the equation of energy conservation in our system. [2]

[2] It turns out that both E_d and the dilaton equation of motion are automatically satisfied whenever E_{ec} together with the two Einstein constraint equations are satisfied. Consequently E_{ec} plus the two Einstein constraint equations form another set of independent equations. This choice of equations has the advantage that it does not require the addition of any additional condition analogous to energy conservation. However it turns out to be an inconvenient choice for implementing the ϵ expansion of this chapter, and we do not adopt it in this chapter.

Explicit form of the constraints and the dilaton equation

With our choice of gauge and notation the dilaton equation takes the minimally coupled form

$$\partial_r \left(f^{d-1} g \partial_r \phi\right) + \partial_v \left(f^{d-1} \partial_r \phi\right) + \partial_r \left(f^{d-1} \partial_v \phi\right) = 0 \tag{2.1.11}$$

Appropriate linear combinations of the two constraint equations take the form

$$\begin{aligned} (\partial_r \phi)^2 &= -\frac{2(d-1)\partial_r^2 f}{f} \\ \partial_r \left(f^{d-2} g \partial_r f + 2 f^{d-2} \partial_v f\right) &= f^{d-1} d \end{aligned} \tag{2.1.12}$$

Note that the equations (2.1.12) (together with boundary conditions and the energy conservation equation) permit the unique determination of $f(r, v_0)$ and $g(r, v_0)$ in terms of $\phi(r, v_0)$ and $\dot{\phi}(r, v_0)$ (where v_0 is any particular time). It follows that f and g are not independent fields. A solution to the differential equation set (2.1.11) and (2.1.12) is completely specified by the value of ϕ on a constant v slice (note that the equations are all first order in time derivatives, so $\dot{\phi}$ on the slice is not part of the data of the problem) together with the boundary condition $\phi_0(v)$.

2.1.3 Explicit form of the energy conservation equation

In this section we give an explicit form for the equation $E_{ec} = 0$ at large r. We specialize here to $d = 3$ but see § 2.5.1 for arbitrary d. Using the Graham Fefferman expansion to solve the equations of motion in a power series in $\frac{1}{r}$ we find

$$\begin{aligned} f(r,v) &= r\left(1 - \frac{\dot{\phi}_0{}^2}{8r^2} + \frac{1}{r^4}\left(\frac{1}{384}(\dot{\phi}_0)^4 - \frac{1}{8}L(v)\dot{\phi}_0\right) + \mathcal{O}(\frac{1}{r^5})\right) \\ g(r,v) &= r^2\left(1 - \frac{3(\dot{\phi}_0)^2}{4r^2} - \frac{M(v)}{r^3} + \mathcal{O}(\frac{1}{r^4})\right) \\ \phi(r,v) &= \phi_0(v) + \frac{\dot{\phi}_0}{r} + \frac{L(v)}{r^3} + \mathcal{O}(\frac{1}{r^4}) \end{aligned} \tag{2.1.13}$$

where the functions $M(v)$ and $L(v)$ are undetermined functions of time that are, however, constrained by the energy conservation equation E_{ec}, which takes the explicit form

$$\dot{M} = \dot{\phi}_0 \left(\frac{3}{8}(\dot{\phi}_0)^3 - \frac{3L}{2} - \frac{1}{2}\dddot{\phi}_0. \right) \tag{2.1.14}$$

In all the equations in this subsection and in the rest of the chapter, the symbol $\dot{P}$ denotes the derivative of P with respect to our time coordinate v. Solving for $M(v)$ we have

$$M(v) = \frac{1}{2} \int_0^v dt \left(\left(\ddot{\phi}_0 \right)^2 + \frac{3}{4} \left(\dot{\phi}_0 \right)^4 - 3\dot{\phi}_0 L(t) \right) \tag{2.1.15}$$

[3]

2.1.4 The metric and event horizon at leading order

Later in this section we will solve the equations of motion (2.1.11), (2.1.12) and (2.1.14) in an expansion in powers of ϵ, the amplitude of the forcing function $\phi_0(v)$. In this subsection we simply state our result for the spacetime metric at leading order in ϵ. We

[3] We note parenthetically that (2.1.14) may be rewritten as

$$\dot{T}_0^0 = \frac{1}{2}\dot{\phi}_0 \mathcal{L} \tag{2.1.16}$$

where the value $\mathcal{L}$ of the operator dual to the scalar field ϕ and the stress tensor $T_{\alpha\beta}$ are given by

$$\begin{aligned} \mathcal{L} &\equiv \lim_{r\to\infty} r^3 \left(\partial_n \phi + \partial^2 \phi \right) \\ T^\mu_\nu &= \lim_{r\to\infty} r^3 \left(K^\mu_\nu - (K-2)\delta^\mu_\nu - \mathcal{G}^\mu_\nu + \frac{\partial^\mu \phi \partial_\nu \phi}{2} - \frac{(\partial\phi)^2 \delta^\mu_\nu}{4} \right). \end{aligned} \tag{2.1.17}$$

Where

$$\begin{aligned} K^\mu_\nu &= \text{Extrinsic curvature of the constant } r \text{ surfaces}, \quad K = K^\mu_\mu \\ \mathcal{G}^\mu_\nu &= \text{Einstein tensor evaluated on the induced metric of the constant } r \text{ surfaces} \end{aligned} \tag{2.1.18}$$

yielding

$$\begin{aligned} T_0^0 &= -2T_x^x = -2T_y^y = M(v) \\ \mathcal{L} &= \frac{3}{4}\dot{\phi}_0^3 - 3L(v) - \partial_v^3 \phi_0 \end{aligned} \tag{2.1.19}$$

then proceed to compute the event horizon of our spacetime to leading order in ϵ. We present the computation of the event horizon of our spacetime before actually justifying the computation of the spacetime itself for the following reason. In the subsections below we will aim to construct the spacetime that describes black hole formation only outside the event horizon. For this reason we will find it useful below to have a prior understanding of the location of the event horizon in the spacetimes that emerge out of perturbation theory.

We will show below that to leading order in ϵ, our spacetime metric takes the Vaidya form (0.1.1). The mass function $M(v)$ that enters this Vaidya metric is also determined very simply. As we will show below, it turns out that $L(v) \sim \mathcal{O}(\epsilon^3)$ on our perturbative solution. It follows immediately from (2.1.15) that the mass function $M(v)$ that enters the Vaidya metric, is given to leading order by

$$\begin{aligned} M(v) &= C_2(v) + \mathcal{O}(\epsilon^4) \\ C_2(v) &= -\frac{1}{2}\int_{-\infty}^{v} dt \dot{\phi}_0(t) \dddot{\phi}_0(t) \end{aligned} \tag{2.1.20}$$

(Here C_2 is the approximation to the mass density, valid to second order in the amplitude expansion, see below).

Note that, for $v > \delta t$, $C_2(v)$ reduces to a constant $M = C_2$ given by

$$C_2 = \frac{1}{2}\int_{-\infty}^{\infty} dt \left(\ddot{\phi}_0(t)\right)^2 \sim \frac{\epsilon^2}{(\delta t)^3} \tag{2.1.21}$$

In the rest this subsection we proceed to compute the event horizon of the leading order spacetime (0.1.1) in an expansion in $\epsilon^{\frac{2}{3}}$ expansion. Let the event horizon manifold of our spacetime be given by the surface $S \equiv r - r_H(v) = 0$. As the event horizon is a null manifold, it follows that $\partial_\mu S \partial_\nu S g^{\mu\nu} = 0$, and we find

$$\frac{dr_H(v)}{dv} = \frac{r_H^2(v)}{2}\left(1 - \frac{M(v)}{r_H^3(v)}\right) \tag{2.1.22}$$

As $M(v)$ reduces to the constant $M = C_2$ for $v > \delta t$, it follows that the event horizon must reduce to the surface $r_H = M^{\frac{1}{3}}$ at late times. It is then easy to solve (2.1.22) for

$v < 0$ and $v > \delta t$; we find

$$
\begin{aligned}
r_H(v) &= M^{\frac{1}{3}}, \quad v \geq \delta t \\
r_H(v) &= M^{\frac{1}{3}} x(\frac{v}{\delta t}), \quad 0 < v < \delta t \\
\frac{1}{r_H(v)} &= -v + \frac{1}{M^{\frac{1}{3}} x(0)}, \quad v \leq 0
\end{aligned}
\tag{2.1.23}
$$

where $x(y)$ obeys the differential equation

$$
\begin{aligned}
\frac{dx}{dy} &= \alpha \frac{x^2}{2} \left(1 - \frac{M(y\delta t)}{M x^3}\right) \\
\alpha &= M^{\frac{1}{3}} \delta t \sim \epsilon^{\frac{2}{3}}
\end{aligned}
\tag{2.1.24}
$$

and must be solved subject to the final state conditions $x = 1$ for $y = 1$. (2.1.24) is easily solved in a perturbation series in α. We set

$$
x(y) = 1 + \sum_n \alpha^n x_n(y) \tag{2.1.25}
$$

and solve recursively for $x_n(t)$. To second order we find[4]

$$
\begin{aligned}
x_1(y) &= -\int_y^1 dz \, \left(\frac{1 - \frac{M(z\delta t)}{M}}{2}\right) \\
x_2(y) &= -\int_y^1 dz \; x_1(z) \left(1 + \frac{M(z\delta t)}{2M}\right)
\end{aligned}
\tag{2.1.26}
$$

In terms of which

$$
r_H(v) = M^{\frac{1}{d}} \left(1 + \alpha \; x_1(\frac{v}{\delta t}) + \alpha^2 x_2(\frac{v}{\delta t}) + \mathcal{O}(\alpha^3)\right) \quad (0 < v < \delta t) \tag{2.1.27}
$$

Note in particular that, to leading order, $r_H(v)$ is simply given by the constant $M^{\frac{1}{3}}$ for all $v > 0$.

[4] In this section we only construct the event horizon for the Vaidya metric. The actual metrics of interest to this chapter receive corrections away from the Vaidya form, in powers of $M\delta t$. Consequently, the event horizons for the actual metrics determined in this chapter will agree with those of this subsection only at leading order in $M\delta t$. The determination of the event horizon of the Vaidya metric at higher orders in $M\delta t$, is an academic exercise that we solve in this subsection largely because it illustrates the procedure one could adopt on the full metric.

2.1.5 Formal structure of the expansion in amplitudes

In this subsection we will solve the equations (2.1.11), (2.1.12) and (2.1.14) in a perturbative expansion in the amplitude of the source function $\phi_0(v)$. In order to achieve this we formally replace $\phi_o(v)$ with $\epsilon\phi_0(v)$ and solve all equations in a power series expansion in ϵ. At the end of this procedure we can set the formal parameter ϵ to unity. In other words ϵ is a formal parameter that keeps track of the homogeneity of ϕ_0. Our perturbative expansion is really justified by the fact that the amplitude of ϕ_0 is small.

In order to proceed with our perturbative procedure, we set

$$\begin{aligned} f(r,v) &= \sum_{n=0}^{\infty} \epsilon^n f_n(r,v) \\ g(r,v) &= \sum_{n=0}^{\infty} \epsilon^n g_n(r,v) \\ \phi(r,v) &= \sum_{n=0}^{\infty} \epsilon^n \phi_n(r,v) \end{aligned} \tag{2.1.28}$$

with

$$f_0(r,v) = r, \quad g_0(r,v) = r^2, \quad \phi_0(r,v) = 0. \tag{2.1.29}$$

We then plug these expansions into the equations of motion, expand these equations in a power series in ϵ, and proceed to solve these equations recursively, order by order in ϵ.

The formal structure of this procedure is familiar. The coefficient of ϵ^n in the equations of motion take the schematic form

$$H^i_j \chi^j_n(r,v)) = s^i_n \tag{2.1.30}$$

Here χ^i_N stands for the three dimensional 'vector' of n^{th} order unknowns, i.e. $\chi^1_n = f_n$, $\chi^2_n = g_n$ and $\chi^3_n = \phi_n$. The differential operator H^i_j is universal (in the sense that it is the same at all n) and has a simple interpretation; it is simply the operator that describes linearized fluctuations about AdS space. The source functions s^i_n are linear combinations of products of χ^i_m $(m < n)$; the sum over m over fields that appear in any particular term adds up to n.

The equations (2.1.30) are to be solved subject to the large r boundary conditions

$$\begin{aligned}
\lim_{r\to\infty}\phi_1(r,v) &= \phi_0(r) \\
\phi_n(r,v) &\leq \mathcal{O}(1/r), \quad n\geq 2 \\
f_n(r,v) &\leq \mathcal{O}(1/r), \quad n\geq 1 \\
g_n(r,v) &\leq \mathcal{O}(r), \quad n\geq 1
\end{aligned} \tag{2.1.31}$$

together with the initial conditions

$$\phi_n(r,v)=g_n(r,v)=f_n(r,v)=0 \quad \text{for} \;\; v<0 \;\; (n\geq 1) \tag{2.1.32}$$

These boundary and initial conditions uniquely determine ϕ_n, g_n and f_n in terms of the source functions.

All sources vanish at first order in perturbation theory (i.e the functions s_1^i are zero). Consequently, the functions f_1 and g_1 vanish but ϕ_1 is forced by its boundary condition to be nonzero. As we will see below, it is easy to explicitly solve for the function ϕ_1. This solution, in turn, completely determines the source functions at $\mathcal{O}(\epsilon^2)$ and so the equations (2.1.30) unambiguously determine g_2, ϕ_2 and f_2. This story repeats recursively. The solution to perturbation theory at order $n-1$ determine the source functions at order n and so permits the determination of the unknown functions at order n. The final answer, at every order, is uniquely determined in terms of $\phi_0(v)$.

To end this subsection, we note a simplifying aspect of our perturbation theory. It follows from the structure of the equations that ϕ_n is nonzero only when n is odd while f_m and g_m are nonzero only when m is even. We will use this fact extensively below.

2.1.6 Explicit results for naive perturbation theory to fifth order

We have implemented the naive perturbative procedure described above to $\mathcal{O}(\epsilon^5)$. Before proceeding to a more structural discussion of the nature of the perturbative expansion, we pause here to record our explicit results.

At leading (first and second) order we find

$$\begin{aligned}
\phi_1(r,v) &= \phi_0(v) + \frac{\dot\phi_0}{r} \\
f_2(r,v) &= -\frac{\dot\phi_0^2}{8r} \\
g_2(r,v) &= -\frac{C_2(v)}{r} - \frac{3}{4}\dot\phi_0^2
\end{aligned} \tag{2.1.33}$$

At the next order

$$\begin{aligned}
\phi_3(r,v) &= \frac{1}{4r^3}\int_{-\infty}^{v} B(x)\ dx \\
f_4(r,v) &= \frac{\dot\phi_0}{384r^3}\left\{\dot\phi_0^3 - 12\int_{-\infty}^{v} B(x)\ dx\right\} \\
g_4(r,v) &= \frac{C_4(v)}{r} + \frac{\dot\phi_0}{24r^2}\left\{-\dot\phi_0^3 + 3\int_{-\infty}^{v} B(x)\ dx\right\} \\
&\quad + \frac{1}{48r^3}\left(3B(v)\dot\phi_0 - 4\dot\phi_0^3\ddot\phi_0 + 3\ddot\phi_0\int_{v}^{\infty} B(t)dt\right)
\end{aligned} \tag{2.1.34}$$

while ϕ_5 is given by

$$\begin{aligned}
\phi_5(r,v) &= \frac{1}{8r^5}\int_{-\infty}^{v} B_1(x)\ dx \\
&\quad + \frac{1}{6r^4}\int_{-\infty}^{v} B_3(x)\ dx + \frac{5}{24r^4}\int_{-\infty}^{v} dy \int_{-\infty}^{y} B_1(x)\ dx \\
&\quad + \frac{1}{4r^3}\int_{-\infty}^{v} B_2(x)\ dx + \frac{1}{6r^3}\int_{-\infty}^{v} dy \int_{-\infty}^{y} B_3(x)\ dx \\
&\quad + \frac{5}{24r^3}\int_{-\infty}^{v} dz \int_{-\infty}^{z} dy \int_{-\infty}^{y} B_1(x)\ dx
\end{aligned} \tag{2.1.35}$$

In the equations above

$$\begin{aligned}
B(v) &= \dot\phi_0\left[-C_2(v) + \dot\phi_0\ddot\phi_0\right] \\
B_1(v) &= \left(-\frac{9}{4}C_2(v) + \frac{7}{8}\dot\phi_0\ddot\phi_0\right)\int_{-\infty}^{v} B(x)\ dx \\
&\quad + \frac{1}{2}C_2(v)\dot\phi_0^3 + \frac{3}{8}\dot\phi_0^2 B(v) - \frac{1}{6}\dot\phi_0^4\ddot\phi_0 \\
B_2(v) &= C_4(v)\dot\phi_0 \\
B_3(v) &= \frac{1}{24}\left(-30\dot\phi_0^2\int_{-\infty}^{v} B(x)\ dx + 7\dot\phi_0^5\right)
\end{aligned} \tag{2.1.36}$$

and the energy functions $C_2(v)$ and $C_4(v)$ (obtained by integrating the energy conservation equation) are given by

$$
\begin{aligned}
C_2(v) &= -\int_{-\infty}^{v} dt \frac{1}{2} \dot{\phi}_0 \dddot{\phi}_0 \\
C_4(v) &= \int_{-\infty}^{v} dt \frac{3}{8} \dot{\phi}_0 \left(-\dot{\phi}_0^3 + \int_{-\infty}^{t} B(x)\ dx \right)
\end{aligned}
\tag{2.1.37}
$$

For use below, we note in particular that at $v = \delta t$ the mass of the black brane is given by $C_2(\delta t) - C_4(\delta t) + \mathcal{O}(\epsilon^6)$ while the value of the dilaton field is given by

$$
\begin{aligned}
\phi(r, \delta t) = &\frac{1}{4r^3} \int_{-\infty}^{\delta t} B(x)\ dx \\
&+ \frac{1}{4r^3} \int_{-\infty}^{\delta t} B_2(x)\ dx + \frac{1}{6r^3} \int_{-\infty}^{\delta t} dy \int_{-\infty}^{y} B_3(x)\ dx \\
&+ \frac{5}{24r^3} \int_{-\infty}^{\delta t} dz \int_{-\infty}^{z} dy \int_{-\infty}^{y} B_1(x)\ dx \\
&+ \frac{5}{24r^4} \int_{-\infty}^{\delta t} dy \int_{-\infty}^{y} B_1(x)\ dx + \frac{1}{6r^4} \int_{-\infty}^{\delta t} B_3(x)\ dx \\
&+ \frac{1}{8r^5} \int_{-\infty}^{\delta t} B_1(x)\ dx \quad + \mathcal{O}(\epsilon^7)
\end{aligned}
\tag{2.1.38}
$$

2.1.7 The analytic structure of the naive perturbative expansion

In this subsection we will explore the analytic structure of the naive perturbation expansion in the variables v (for $v > \delta t$) and r. It is possible to inductively demonstrate that

- 1. The functions ϕ_{2n+1}, g_{2n+2} and f_{2n+2} have the following analytic structure in the variable r

$$
\begin{aligned}
\phi_{2n+1}(r, v) &= \sum_{k=0}^{2n-2} \frac{\phi_{2n+1}^k(v)}{r^{2n+1-k}}, \quad (n \geq 2) \\
f_{2n}(r, v) &= r \sum_{k=0}^{2n-6} \frac{f_{2n}^k(v)}{r^{2n-k}}, \quad (n \geq 3) \\
g_{2n}(r, v) &= \frac{C_{2n}(\delta t)}{r} + r \sum_{k=0}^{2n-5} \frac{g_{2n-3}^k(v)}{r^{2n-k}}, \quad (n \geq 3)
\end{aligned}
\tag{2.1.39}
$$

Moreover, when $v > \delta t$ $\phi_1(r,v) = f_2(r,v) = f_4(r,v) = 0$ while $g_2(r,v) = -\frac{C_2(\delta t)}{r}$ and $g_4(r,v) = \frac{C_4(\delta t)}{r}$.

- 2. The functions $\phi^k_{2n+1}(v)$, $f^k_{2n}(v)$ and $g^k_{2n}(v)$ are each functionals of $\phi_0(v)$ that scale like $\lambda^{-2n-1+k}$, λ^{-2n+k} and $\lambda^{-2n+k-1}$ respectively under the scaling $v \to \lambda v$.

- 3. For $v > \delta t$ the functions $\phi^k_{2n+1}(v)$ are all polynomials in v of a degree that grows with n. In particular the degree of ϕ^k_{2n+1} at most $n-1+k$; the degree of f^k_{2n} is at most $n-3+k$ and the degree of g^k_{2n} is at most $n-4+k$.

The reader may easily verify that all these properties hold for the explicit low order solutions of the previous subsection.

2.1.8 Infrared divergences and their cure

The fact that $\phi_{2n+1}(v)$ are polynomials in time whose degree grows with n immediately implies that the naive perturbation theory of the previous subsection fails at late positive times. We pause to characterize this failure in more detail. As we have explained above, the field $\phi(r,v)$ schematically takes the form

$$\sum_{n,k} \frac{\epsilon^{2n+1}\phi^k_{2n+1}}{r^{2n+1-k}}$$

where $\phi^k_{2n+1} \sim \frac{v^{n-1+k}}{(\delta t)^{3n}}$ at large times. Let us examine this sum in the vicinity $r \sim \frac{\epsilon^{\frac{2}{3}}}{\delta t}$, a surface that will turn out to be the event horizon of our solution. The term with labels n, k scales like $\epsilon \times (\epsilon^{\frac{2}{3}} \frac{v}{\delta t})^{n-1+k}$. Now $\frac{\epsilon^{\frac{2}{3}}}{\delta t} = T$ is approximately the temperature of a black brane of event horizon r_H. We conclude that the term with labels n, k scales like $(vT)^{n-1+k}$. It follows that, at least in the vicinity of the horizon, the naive expansion for ϕ is dominated by the smallest values of n and k when $\delta t T \ll 1$. On the other hand, at times large compared to the inverse temperature, this sum is dominated by the largest values of k and n. As the sum over n runs to infinity, it follows that naive perturbation theory breaks down at time scales of order T^{-1}.

A long time or IR divergence in perturbation theory usually signals the fact that the perturbation expansion has been carried out about the wrong expansion point; i.e. the zero order 'guess' with which we started perturbation (empty AdS space) does not everywhere approximate the true solution even at arbitrarily small ϵ. Recall that naive perturbation theory is perfectly satisfactory for times of order δv so long as $r \gg \frac{\epsilon}{\delta t}$. Consequently this perturbation theory may be used to check if our spacetime metric deviates significantly from the pure AdS in this range of r and at these early times. The answer is that it does, even in the limit $\epsilon \to 0$. In order to see precisely how this comes about, note that the most singular term in g_{2n} is of order $r \times \frac{1}{r^{2n}}$ for $n \geq 1$, the exact value of $g_0 = r^2 = (r \times \frac{1}{r^0} \times r)$. In other words g_0 happens to be less singular, near $r = 0$, than one would expect from an extrapolation of the singularity structure of g_n at finite n down to $n = 0$. As a consequence, even though g_0 is of lowest order in ϵ, at small enough r it is dominated by the most singular term in $g_2(r, v)$. Moreover this crossover in dominance occurs at $r \sim \frac{\epsilon^{\frac{2}{3}}}{\delta t} \gg \frac{\epsilon}{\delta t}$ and so occurs well within the domain of applicability of perturbation theory. In other words, in the variable range $r \gg \frac{\epsilon}{\delta t}$, $g(r, v)$ is not uniformly well approximated by $g_0 = r^2$ at small ϵ but instead by

$$g(r, v) \approx r^2 - \frac{C_2(v)}{r}.$$

This implies that, in the appropriate parameter range, the true metric of the spacetime is everywhere well approximated by the Vaidya metric (0.1.1), with $M(v)$ given by (2.1.20) in the limit $\epsilon \to 0$.

Of course even this corrected estimate for $g(r, v)$ breaks down at $r \sim \frac{\epsilon}{\delta t}$. However, as we have indicated above, this will turn out to be irrelevant for our purposes as our spacetime develops an event horizon at $r \sim \frac{\epsilon^{\frac{2}{3}}}{\delta t}$.

We will now proceed to argue that the metric is well approximated by the Vaidya form at all times (not just at early times) outside its event horizon, so that the Vaidya metric (0.1.1) rather than empty AdS space, constitutes the correct starting point for the perturbative expansion of our solution.

2.1.9 The metric to leading order at all times

The dilaton field and spacetime metric begin a new stage in their evolution at $v = \delta t$. At later times the solution is a normalizable, asymptotically AdS solution to the equations of motion. This late time motion is unforced and so is completely determined by two pieces of initial data; the mass density $M(\delta t)$ and the dilaton function $\phi(r, \delta t)$. As the naive perturbation expansion described in subsection 2.1.7 is valid at times of order δt, it determines both these quantities perturbatively in ϵ. The explicit results for these quantities, to first two nontrivial orders in ϵ, are listed in (2.1.38).

The leading order expression for the mass density is simply given by C_2 in (2.1.20). Now if one could ignore $\phi(r, \delta t)$ (i.e. if this function were zero) this initial condition would define a unique, simple subsequent solution to Einstein's equations; the uniform black brane with mass density C_2. While $\phi(r, \delta t)$ is not zero, we will now show it induces only a small perturbation about the black brane background.

In order to see this it is useful to move to a rescaled variable $\tilde{r} = \frac{r}{C_2^{\frac{1}{3}}}$. In terms of this rescaled variable, our solution at $v = \delta t$ is a black brane of unit energy density, perturbed by $\phi(r, \delta t)$. With this choice of variable the background metric is independent of ϵ, so that all ϵ dependence in our problem lies in the perturbation. It follows that, to leading order in ϵ (recall $\phi_1(r, \delta t) = 0$)

$$\phi(r, \delta t) = \frac{\phi_3^0(\delta t)}{r^3}\left(1 + \mathcal{O}(\epsilon^{\frac{2}{3}})\right) = \frac{1}{\tilde{r}^3} \times \frac{\phi_3^0(\delta t)}{M}\left(1 + \mathcal{O}(\epsilon^{\frac{2}{3}})\right) \sim \frac{\epsilon}{\tilde{r}^3} \tag{2.1.40}$$

where, from subsection 2.1.6

$$\phi_3^0(\delta t) = \frac{1}{4}\int_{-\infty}^{\delta t} B(x)\, dx \tag{2.1.41}$$

The important point here is that the perturbation is proportional to ϵ and so represents a small deformation of the dilaton field about the unit energy density black brane initial condition. Moreover, any regular linearized perturbation about the black brane may be re expressed as a linear sum of quasinormal modes about the black brane and so decays exponentially over a time scale of order the inverse temperature. It follows that the initialy small dilaton perturbation remains small at all future times and in fact decays exponentially to zero over a finite time. The fact that perturbations about the Vaidya metric (0.1.1) are bounded both in amplitude as well as in temporal duration allows us to conclude that the event horizon of the true spacetime is well approximated by the event horizon of the Vaidya metric at small ϵ, as described in subsection 2.1.4.

2.1.10 Resummed versus naive perturbation theory

Let us define a resummed perturbation theory which uses the corrected metric (0.1.1) (rather than the unperturbed AdS metric) as the starting point of an amplitude expansion. This amounts to correcting the naive perturbative expansion by working to all orders in $M \sim \epsilon^2$, while working perturbatively in all other sources of ϵ dependence. [5] As we have argued above, resummed perturbation theory (unlike its naive counterpart) is valid at all times.

We have seen above that the naive perturbation theory gives reliable results when $vT \ll 1$. This fact has a simple 'explanation'; we will now argue that the resummed perturbation theory (which is always reliable at small ϵ) agrees qualitatively with naive perturbation theory $vT \ll 1$.

At each order, resummed perturbation theory involves solving the equation

$$\partial_r \left[r^4 \left(1 - \frac{M(v)}{r^3} \right) \partial_r \phi \right] + 2r\partial_v \partial_r (r\phi) = \text{source} \tag{2.1.42}$$

The naive perturbation procedure requires us to solve an equation of the same form but with M set to zero. In the vicinity of the horizon, the two terms in the expression

[5] This is conceptually similar to the coupling constant expansion in finite temperature weak coupling QED. There, as in our situation, naive perturbation theory leads to IR divergences, which are cured upon exactly accounting for the photon mass (which is of order g_{YM}^2). Resummed perturbation theory in that context corresponds to working with a modified propagator which effectively includes all order effects in the photon mass, while working perturbatively in all other sources of the fine structure constant α.

$(1 - \frac{M(v)}{r^3})$ are comparable, so that the resummed and naive perturbative expansions can agree only when the entire first term on the LHS of (2.1.42) is negligible compared to the second term on the LHS of the same equation. The ratio of the first term to the second may be approximated by rv where v is the time scale for the process in question. Now the term multiplying the mass in (2.1.42) is only important in the neighborhood of the horizon, where $r \sim M^{\frac{1}{3}} \sim T$ where T is the temperature of the black brane. It follows that resummed and naive perturbation expansions will differ substantially from each other only at time scales of order and larger than the inverse temperature.

Let us restate the point in a less technical manner. The evolution of a field ϕ, outside the horizon of a black brane of temperature T, is not very different from the evolution of the same field in Poincare patch AdS space, over time scales v where $vT \ll 1$. However the two motions differ significantly over time scales of order or greater than the inverse temperature. In particular, in the background of the black brane, the field ϕ outside the horizon decays exponentially with time over a time scale set by the inverse temperature; i.e. the solution involves factors like e^{-vT}. As the temperature is itself of order $\epsilon^{\frac{2}{3}}$, naive perturbation theory deals with these exponentials by power expanding them. Truncating to any finite order then gives apparently divergent behavior at large times. Resummed perturbation theory makes it apparent that these divergences actually resum into completely convergent, decaying, exponentials.

2.1.11 Resummed perturbation theory at third order

In the previous subsection we have presented explicit results for the behavior of the dilaton and metric fields, at small ϵ and for early times $vM^{\frac{1}{3}} \ll 1$. The resummed perturbation theory outlined in this section may be used to systematically correct the leading order spacetime (0.1.1) at all times, in a power series in $\epsilon^{\frac{2}{3}}$. In this section we explicitly evaluate the leading order correction in terms of a universal (i.e. ϕ_0 independent) function $\psi(x, y)$, whose explicit form we are able to determine only numerically.

Let us define the function $\psi(x,y)$ as the unique solution of the differential equation

$$\partial_x\left(x^4\left(1-\frac{1}{x^3}\right)\partial_x\psi\right)+2x\partial_y\partial_x\left(x\psi\right)=0 \tag{2.1.43}$$

subject to the boundary condition $\psi \sim \mathcal{O}(\frac{1}{x^3})$ at large x and the initial condition $\psi(x,0) = \frac{1}{x^3}$. The leading order solution to the resummed perturbation theory for ϕ, for $v > \delta t$, is given by

$$\phi = \frac{\phi_3^0(\delta t)}{M}\psi(\frac{r}{M^{\frac{1}{3}}},(v-\delta t)M^{\frac{1}{3}}) \tag{2.1.44}$$

Unfortunately, the linear differential equation (2.1.43) - appears to be difficult to solve analytically. In this section we present a numerical solution of (2.1.43). Although we are forced to resort to numerics to determine $\psi(x,y)$, we emphasize that a single numerical evaluation suffices to determine the leading order solution at all values of the forcing function $\phi_0(v)$. This may be contrasted with an ab initio numerical approach to the full nonlinear differential equations, which require the re running of the full numerical code for every initial function ϕ_0. In particular the ab initio numerical method cannot be used to prove general statements about a wide class of forcing functions ϕ_0.

In Figure 2.1 we present a plot of $\psi(\frac{1}{u},y)$ against the variables u and y. The exterior of the event horizon lives in the compact interval $\frac{1}{x} = u \in (0,1)$, and in our figure y runs from zero to three.

In order to obtain this plot we rewrote the differential equation (2.1.43) in terms of the variable $u = \frac{1}{x}$ (as explained above) and worked with the field variable $\chi(u,y) = (1-u)\psi(\frac{1}{u},y)$. Recall that our original field ψ is expected to be regular at the horizon $u = 1$ at all times. This expectation imposes the boundary condition $\chi(.999999,y) = 0$. We further imposed the condition of normalizability $\chi(0,y) = 0$ and the initial condition $\chi(u,0) = (0.999999-u)u^3$. Of course 0.999999 above is simply a good approximation to 1 that avoids numerical difficulties at unity. The partial differential equation solving routine of Mathematica-6 was able to solve our equation subject to these boundary and initial conditions, with a step size of 0.0005 and an accuracy goal of 0.001; we have displayed this Mathematica output in figure 2.1. In order to give a better feeling for the function $\psi(x,y)$

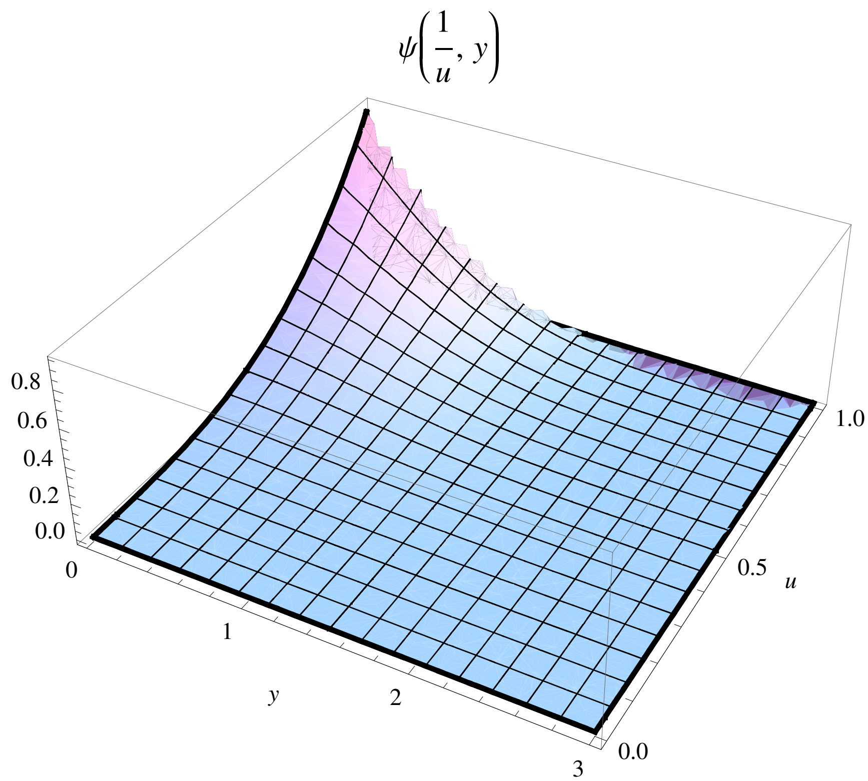

Figure 2.1: Numerical solution for dilaton to the leading order in amplitude at late time

in figure 2.2 we present a graph of $\psi(\frac{1}{0.7}, y)$ (i.e. as a function of time at a fixed radial location). Notice that this graph decays, roughly exponentially for $v > 0.5$ and that this

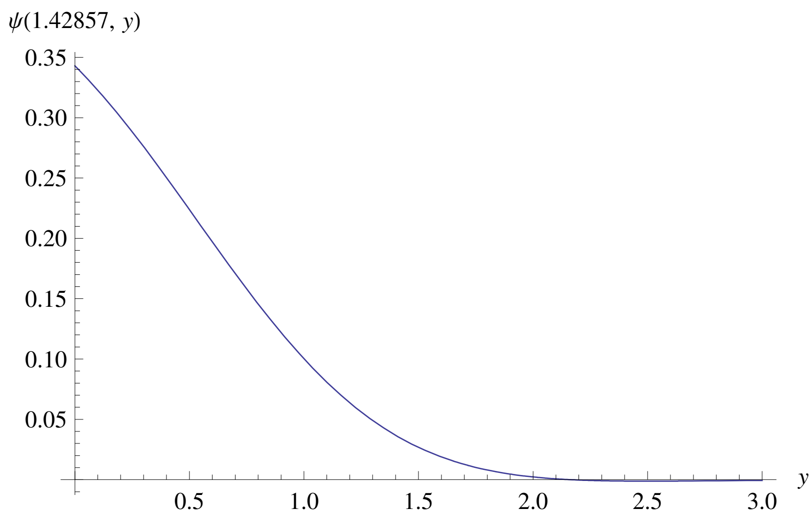

Figure 2.2: A plot of $\psi(\frac{1}{0.7}, y)$ as a function of y

exponential decay is dressed with a sinusodial osciallation, as expected for quasinormal type behavior. A very very rough estimate of this decay constant ω_I may be obtained from equation $\frac{\psi(\frac{1}{0.7},1.5)}{\psi(\frac{1}{0.7},.5)} = e^{-\omega_I}$ which gives $\omega_I \approx 8.9T$ (here T is the temperature of our black brane given by $T = \frac{4\pi}{3}$). This number is the same ballpark as the decay constant for the first quasi normal mode of the uniform black brane, $\omega_I = 11.16T$, quoted in [35].

2.2 Spherically symmetric asymptotically flat collapse

2.2.1 The Set Up

In this section[6] we study spherically symmetric asymptotically flat solutions to Einstein gravity (with no cosmological constant) interacting with a minimally coupled massless

[6] We thank B. Kol and O. Aharony for discussions that led us to separately study collapse in flat space.

scalar field, in 4 bulk dimensions. The Lagrangian for our system is

$$S = \int d^4x\sqrt{g}\left(R - \frac{1}{2}(\partial\phi)^2\right) \tag{2.2.45}$$

We choose a gauge so that our metric and dilaton take the form

$$\begin{aligned} ds^2 &= 2drdv - g(r,v)dv^2 + f^2(r,v)d\Omega_2^2 \\ \phi &= \phi(r,v). \end{aligned} \tag{2.2.46}$$

where $d\Omega_2^2$ is the line element on a unit two sphere. We will explore solutions to the equations of motion of this system subject to the pure flat space initial conditions

$$\begin{aligned} g(r,v) &= 1, \quad (v<0) \\ f(r,v) &= r, \quad (v<0) \\ \phi(r,v) &= 0, \quad (v<0) \end{aligned} \tag{2.2.47}$$

and the large r boundary conditions

$$\begin{aligned} g(r,v) &= 1 + \mathcal{O}(\frac{1}{r}) \\ f(r,v) &= r\left(1 + \mathcal{O}(\frac{1}{r^2})\right) \\ \phi(r,v) &= \frac{\psi(v)}{r} + \mathcal{O}(\frac{1}{r^2}) \end{aligned} \tag{2.2.48}$$

where $\psi(v)$ takes the form

$$\begin{aligned} &\psi(v) = 0, \quad (v<0) \\ &\psi(v) < \epsilon_f \delta t, \quad (0<v<\delta t) \\ &\psi(v) = 0 \quad (v>\delta t), \end{aligned} \tag{2.2.49}$$

In other words our spacetime starts out in its vacuum, but has a massless pulse of limited duration focused to converge at the origin at $v = 0$. This pulse could lead to interesting behavior - like black hole formation, as we explore in this section.

The structure of the equations of motion of our system was described in subsection 2.1.2. As in that subsection, the independent dynamical equations for our system may be chosen to be the dilaton equation of motion plus the two constraint equations, supplemented by an energy conservation equation. The explicit form of the dilaton and constraint equations is given by

$$\begin{aligned} &\partial_r\left(f^2 g \partial_r \phi\right) + \partial_v\left(f^2 \partial_r \phi\right) + \partial_r\left(f^2 \partial_v \phi\right) = 0 \\ (\partial_r \phi)^2 = -&\frac{4\partial_r^2 f}{f} \\ &\partial_r\left(fg\partial_r f + 2f\partial_v f\right) = 1 \end{aligned} \tag{2.2.50}$$

As in the previous section, we may choose to evaluate the energy conservation equation at large r. As we have explained, the large r behavior of the function g is given by

$$g(r,v) = 1 - \frac{M(v)}{r} + \mathcal{O}(\frac{1}{r^2}) \tag{2.2.51}$$

The energy conservation equation, evaluated at large r, yields

$$\dot{M} = -\frac{\psi\dddot{\psi}}{2} \tag{2.2.52}$$

The equations(2.2.50) together with (2.2.52) constitute the full set of dynamical equations for our problem.

By integrating (2.2.52) we find an exact expression for $M(v)$

$$M(v) = \frac{-\psi\ddot{\psi} + \int_{-\infty}^{v} \ddot{\psi}^2}{2} \tag{2.2.53}$$

Note in particular that $M(v)$ reduces to a constant M for $v > \delta t$ where

$$M = \frac{\int_{-\infty}^{\delta t} \ddot{\psi}^2}{2} \sim \epsilon_f^2 \delta t \tag{2.2.54}$$

2.2.2 Regular Amplitude Expansion

Our equations may be solved in the amplitude expansion formally described in (2.1.5), i.e. in an expansion in powers of the function $\psi(v)$. As we will argue in this chapter, there are two inequivalent valid amplitude expansions of these equations. In the first, the spacetime is everywhere regular and the dilaton is everywhere small. In the second, the spacetime is singular at small r but this singularity is shielded from asymptotic infinity by a regular event horizon. The second amplitude expansion reliably describes the spacetime only outside the event horizon; this expansion works because the dilaton is uniformly small outside the event horizon. As we will see two amplitude expansions described above have non overlapping regimes of validity, and so describe dynamics in different regimes of parameter space.

In this subsection we briefly comment on the more straightforward fully regular expansion. At every order in perturbation theory, the requirement or regularity uniquely determines the solution. Explicitly at first order we have

$$\phi_1(r,v) = \frac{\psi(v) - \psi(v-2r)}{r} \tag{2.2.55}$$

The perturbation expansion that starts with this solution is valid only when $\phi(r)$ is everywhere small. $\phi(r)$ reaches its maximum value near the origin, and $\phi_1(0,v) \sim 2\dot{\psi}(v) \sim \epsilon_f$. Consequently the regular perturbation expansion, sketched in this section, is valid only when $\epsilon_f \ll 1$ i.e. when $\frac{\delta t}{M} \gg 1$.

At next order in the amplitude expansion we find

$$\begin{aligned} f_2(r,v) &= \frac{1}{4}\left(r\int_r^\infty \rho\left[\partial_\rho\phi_1(\rho,v)\right]^2\, d\rho - \int_r^\infty \rho^2\left[\partial_\rho\phi_1(\rho,v)\right]^2\, d\rho\right) \\ g_2(r,v) &= -2\partial_v f_2(r,v) - \frac{f_2(r,v)-f_2(0,v)}{r} - \partial_r f_2(r,v) \end{aligned} \tag{2.2.56}$$

The integration limits in the expression for $f_2(r,v)$ in 2.2.56 are fixed such that at large r $f(r,v)$ decays like $\frac{1}{r}$. The integration constant in $g_2(r,v)$ is fixed by the requirement that the solution be regular at $r=0$.

Regularity implies energy conservation

In this subsection we pause to explain an interesting technical subtlety that arises in carrying out the regular amplitude expansion. The discussion of this subsection will play no role in the analysis of spacetimes that describe black hole formation, so the reader who happens to be uninterested in the regular expansion could skip to the next section.

Note that in order to obtain (2.2.56) we did not make any use of the energy conservation equation. We will now verify (first in terms of the answer, and then more abstractly) that (2.2.56) automatically obeys the energy conservation equation. At large r, these functions have the following expansion

$$\begin{aligned} \phi_1(r,v) &= \frac{\psi(v)}{r} \\ f_2(r,v) &= -\frac{\psi(v)^2}{8r} \\ g_2(r,v) &= -\frac{C_2(v)}{r}, \quad \text{where} \\ C_2(v) &= -\frac{\psi(v)\dot{\psi}(v)}{2} - f_2(0,v) \end{aligned} \tag{2.2.57}$$

If our solution does indeed obey the energy conservation relation, we should find that $C_2(v)$ is equal to $M(v)$ in (2.2.54). We will now proceed to directly verify that this is the case.

The first term in $C_2(v)$ comes from the coefficient of $\frac{1}{r}$ in $\partial_v f_2(r,v)$. For the second term in the expression for $C_2(v)$, $f_2(0)$, is given by

$$f_2(0,v) = -\frac{1}{4}\int_0^\infty \rho^2 \left[\partial_\rho \phi_1(\rho,v)\right]^2 \, d\rho$$

The integrand in this expression may be split into four terms in the following way.

$$\begin{aligned} r^2\left[\partial_r\phi_1(r,v)\right]^2 &= 2\psi(v)\partial_r\left[\frac{\psi(v-2r)}{r}\right] + \frac{\psi(v)^2}{r^2} + r^2\left[\partial_r\left(\frac{\psi(v-2r)}{r}\right)\right]^2 \\ &= 2\psi(v)\partial_r\left[\frac{\psi(v-2r)}{r}\right] + \frac{\psi(v)^2}{r^2} + 4\left[\dot{\psi}(v-2r)\right]^2 - \partial_r\left[\frac{\psi^2(v-2r)}{r}\right] \end{aligned} \tag{2.2.58}$$

Now each of the terms can be integrated.

$$
\begin{aligned}
&\int_0^r 2\psi(v)\partial_\rho\left[\frac{\psi(v-2\rho)}{\rho}\right]d\rho = -2\lim_{r\to 0}\frac{\psi(v)\psi(v-2r)}{r} = -2\lim_{r\to 0}\frac{\psi(v)^2}{r}\\
&\int_0^r \frac{\psi(v)^2}{\rho^2}d\rho = \lim_{r\to 0}\frac{\psi(v)^2}{r}\\
&\int_0^r 4\left[\dot{\psi}(v-2\rho)\right]^2 d\rho = 2\int_{-\infty}^{v}\dot{\psi}(t)^2\,dt\\
&-\int_0^r \partial_\rho\left[\frac{\psi^2(v-2\rho)}{\rho}\right]d\rho = \lim_{r\to 0}\frac{\psi(v-2r)^2}{r} = \lim_{r\to 0}\frac{\psi(v)^2}{r}
\end{aligned}
\tag{2.2.59}
$$

Adding all the terms one finally finds

$$
-f_2(0,v) = \frac{1}{2}\int_{-\infty}^{v}\dot{\psi}(t)^2\,dt \tag{2.2.60}
$$

This implies

$$
C_2(v) = -\frac{\psi(v)\dot{\psi}(v)}{2} + \frac{1}{2}\int_{-\infty}^{v}\dot{\psi}(t)^2\,dt = M(v) \tag{2.2.61}
$$

as expected from energy conservation.

Let us summarize In order to obtain our result for g_2 above, we were required to fix the value of an integration constant. The value of this constant may determined in two equally valid ways

- By imposing the energy conservation equation E_{ec}
- By demanding regularity of the solution at $r = 0$

In fact these two conditions are secretly the same, as we now argue. As we have explained in subsection 2.1.2, $\partial_r(rE_{ec})$ automatically vanishes whenever the three equations (2.2.50) are obeyed. Consequently, if rE_{ec} vanishes at any one value of r it automatically vanishes at every r. Now the equation E_{ec} evaluates to a finite value at $r = 0$ provided our solution is regular at $r = 0$. It follows that the regular solution automatically has $rE_{ec} = 0$ everywhere.

Configurations in the amplitude expansion of the previous section (or the singular amplitude expansion we will describe shortly below), on the other hand, are all singular at $r = 0$. rE_{ec} does not automatically vanish on these solutions, and the energy conservation equation E_{ec} is not automatic but must be imposed as an additional constraint on solutions.

It would be a straightforward - if cumbersome - exercise to explicitly implement the perturbation theory, described in this subsection, to higher orders in ϵ_f. As our main interest is black hole formation, we do not pause to do that.

2.2.3 Leading order metric and event horizon for black hole formation

In the rest of this section we will describe the formation of black holes in flat space in an amplitude expansion. In contrast with the previous subsection, our amplitude expansion will be justified by the small parameter $\frac{1}{\epsilon_f}$. Our analysis will reveal that our spacetime takes the Vaidya form to leading order in $\frac{1}{\epsilon_f^2}$,

$$ds^2 = 2drdv - \left(1 - \frac{M(v)}{r}\right) dv^2 + r^2 d\Omega_2^2 \tag{2.2.62}$$

where $M(v)$ is given by (2.2.53).

In this subsection we will compute the event horizon of the spacetime (2.2.62) at large ϵ_f. We present the computation of this event horizon even before we have justified the form (2.2.62), as our aim in subsequent subsections is to have a good perturbative expansion of the true solution only outside the event horizon; consequently the results of this subsection will guide the construction of the amplitude expansion in subsequent subsections.

As in the previous section the event horizon takes the form

$$\begin{aligned} r_H(v) &= M, \quad (v > \delta t) \\ r_H(v) &= Mx(\frac{v}{\delta t}), \quad (0 < v < \delta t) \\ r_H(v) &= Mx(0) + v, \quad (-x(0) < v < 0) \end{aligned} \tag{2.2.63}$$

where the function $x(t)$ may easily be evaluated in a power series in $\frac{\delta t}{M} \sim \frac{1}{\epsilon_f^2}$. We find

$$\begin{aligned} x(t) &= 1 + \left(\frac{\delta t}{M}\right) x_1(t) + \left(\frac{\delta t}{M}\right)^2 x_2(t) + \dots \\ x_1(t) &= -\int_t^1 dy \left(\frac{1 - \frac{M(y\delta t)}{M}}{2}\right) \\ x_2(t) &= -\int_t^1 dy x_1(y) \frac{M(y\delta t)}{M}. \end{aligned} \tag{2.2.64}$$

In particular $r_H = M$ for all $v > 0$ at leading order.

2.2.4 Amplitude expansion for black hole formation

Let us now construct an amplitude expansion (i.e. expansion in powers of $\psi(v)$) of our solution in the opposite limit to that of the previous subsection, namely $\frac{M}{\delta t} \sim \epsilon_f^2 \gg 1$. It is intuitively clear that such a dilaton shell will propagate into its own Schwarzschild radius and then cannot expand back out to infinity. In other words the second term in (2.2.55) cannot form a good approximation to the leading order solution for the collapse of such a shell. Now (2.2.55) deviates from

$$\phi_1(r, v) = \frac{\psi(v)}{r}; \tag{2.2.65}$$

only at spacetime points that feel the back scattered expanding wave in (2.2.55). This observation suggests that (2.2.65) itself is the appropriate starting point for the amplitude expansion at large ϵ_f, and this is indeed the case.

The incident dilaton pulse (2.2.65) will back react on the metric; above we have derived an exact expression for one term - roughly the Newtonian potential - (see (2.2.54)) of this back reacted metric. Including this backreaction (all others turn out to be negligible at large ϵ_f) the spacetime metric takes the form

$$ds^2 = 2dvdr - dv^2(1 - \frac{M(v)}{r}) + r^2 d\Omega_2^2 \tag{2.2.66}$$

As we have explained in the previous subsection, this solution has an event horizon located at $r_H \sim M \sim \epsilon_f^2 \delta t$ for $v > 0$ (see below). Consequently, $\phi_1(r, v)$ outside the event horizon

$\leq \frac{\psi}{r_H} \sim \frac{1}{\epsilon_f} \sim \sqrt{\frac{\delta t}{r_H}}$, i.e. is parametrically small at large ϵ_f. This fact allows us to construct a large ϵ_f amplitude expansion for the solution outside its event horizon.

The perturbation expansion of our solutions in $\frac{\delta t}{M}$ is similar in many ways to the perturbation theory described in detail in § 2.1. As in that section, the true (resummed) expansion (built around the starting metric (2.2.66)) is well approximated at early times by a naive expansion built around unperturbed flat space. Naive and resummed expansions agree whenever the first term in the first equation of (2.2.50) is negligible compared to the other terms in that equation, i.e. for $v \ll M \sim \epsilon_f^2 \delta t$. As ϵ_f is large in this subsection, naive and resummed perturbation theory are simultaneously valid for times that are of order δt. However we expect the naive expansion to break down at $v \gg M$. We will now study the naive expansion in more detail and confirm these expectations.

2.2.5 Analytic structure of the naive perturbation expansion

In this subsection we describe the structure of a perturbative expansion built starting from the flat space metric. We expand the full solution as

$$\begin{aligned} \phi(r,v) &= \sum_{n=0}^{\infty} \Phi_{2n+1} \\ f(r,v) &= r + \sum_{n=1}^{\infty} F_{2n}(r,v) \\ g(r,v) &= 1 + \sum_{n=1}^{\infty} G_{2n}(r,v) \end{aligned} \tag{2.2.67}$$

where, by definition, the functions Φ_m F_m and G_m are each of homogeneity m in the source function $\psi(v)$. As explained above we take

$$\Phi_1(r,v) = \frac{\psi(v)}{r} \tag{2.2.68}$$

By studying the formal structure of the perturbation expansion, it is not difficult to inductively establish that

- 1. The functions Φ_{2n+1}, F_{2n} and G_{2n} have the following analytic structure in the variable r

$$\begin{aligned}
\Phi_{2n+1}(r,v) &= \sum_{m=0}^{\infty} \frac{\Phi^m_{2n+1}(v)}{r^{2n+m+1}} \\
F_{2n}(r,v) &= r\sum_{m=0}^{\infty} \frac{F^m_{2n}(v)}{r^{2n+m}} \\
G_{2n}(r,v) &= -\delta_{n,1}\frac{M(v)}{r} + r\sum_{m=0}^{\infty} \frac{G^m_{2n}(v)}{r^{2n+m}}
\end{aligned} \tag{2.2.69}$$

- 2. The functions $\Phi^m_{2n+1}(v)$, $F^m_{2n}(v)$ and $G^m_{2n}(v)$ are each functionals of $\psi(v)$ that scale like λ^m λ^m and λ^{m-1} under the the scaling $v \to \lambda v$.

- 3. For $v > \delta t$ the $\Phi^m_{2n+1}(v)$ are polynomials in v of degree $\leq n+m-1$; $F^m_{2n}(v)$ and G^m_{2n} are polynomials in v of degree $\leq n+m-3$ and $n+m-4$ respectively.

It follows that, say, $\phi(r,v)$, is given by a double sum

$$\phi(r,v) = \sum_n \Phi_{2n+1}(r,v) = \sum_{n,m=0}^{\infty} \frac{\Phi^m_{2n+1}(v)}{r^{2n+m+1}}.$$

Now sums over m and n are controlled by the effective expansion parameters $\sim \frac{v}{r}$ (for m) and $\frac{\psi^2 v}{\delta t r^2} \sim \frac{v}{\delta t \epsilon_f^2} \sim \frac{v}{M}$ (for n; recall that in the neighborhood of the horizon $r_H \sim \delta t \epsilon_f^2$).

It follows that the sum over m is well approximated by its first few terms if only $v \ll M$ (recall we are interested in the solution only for $r > M$). The sum over n may also be truncated to leading order only for $v \ll M$. As anticipated above, therefore, our naive perturbation expansion breaks over time scales v of order and larger than M.

Let us now focus on times v of order δt. Over these time scales naive perturbation theory is valid for $r \gg \epsilon_f \delta t$ (recall that this domain of validity includes the event horizon surface $r_H \sim \epsilon_f^2 \delta t$). Focusing on the region of interest, $r \geq r_H$, $\frac{\Phi^m_{2n+1}}{r_H^{2n+m+1}}$ scales like $\frac{1}{\epsilon_f^{2n+2m+1}}$. It follows that Φ^m_{2n+1}, with equal values of $n+m$ are comparable at times of order δt. For this reason we find it useful to define the resummed fields

$$\begin{aligned}
\phi_{2n+1}(r,v) &= \sum_{k=0}^{n-1} \frac{\Phi^k_{2n+1-2k}(r,v)}{r^{2n+1-k}} \\
f_{2n}(r,v) &= r\delta_{n,2}F^0_2 + r\sum_{k=0}^{n-2} \frac{F^k_{2n-2k}(r,v)}{r^{2n-k}} \\
g_{2n}(r,v) &= r\delta_{n,2}G^0_2 + r\sum_{k=0}^{n-2} \frac{G^k_{2n-2k}(r,v)}{r^{2n-k}}
\end{aligned} \tag{2.2.70}$$

ϕ_{2n-1}, unlike Φ_{2n-1}, receives contributions from only a finite number of terms at any fixed n, and so is effectively computable at low orders. According to our definitions, ϕ_m, f_m and g_m capture all contributions to our solutions of order $\frac{1}{\epsilon_f^m}$, at time scales of order δt.

We now present explicit computations of the fields ϕ_m, f_m and g_m up to 5th order. We find

$$\begin{aligned}
f_2(r,v) &= -\frac{\psi(v)^2}{8r} \\
g_2(r,v) &= -\frac{M(v)}{r} \\
f_4(r,v) &= \frac{\psi(v)^4}{384r^3} - \frac{\psi(v)B(v)}{32r^3} \\
g_4(r,v) &= -\frac{\dot\psi(v)\psi(v)^3}{48r^3} - \frac{M(v)\psi(v)^2}{16r^3} + \frac{\dot\psi(v)B(v)}{16r^3} \\
\phi_3(r,v) &= \frac{B(v)}{4r^3} \\
\phi_5(r,v) &= \frac{\int_{-\infty}^v \left(48B(t) - 16\psi(t)^3\right)\, dt}{192r^4} \\
&\quad + \frac{\int_{-\infty}^v \left[\psi(t)\dot\psi(t)\left\{5\psi(t)^3 + 21B(t)\right\} + 3M(t)\left\{\psi(t)^3 - 18B(t)\right\}\right]\, dt}{192r^5}
\end{aligned} \tag{2.2.71}$$

Where

$$B(v) = \int_{-\infty}^{v} \psi(t)\left(-M(t) + \psi(t)\dot\psi(t)\right)\, dt$$

2.2.6 Resummed perturbation theory at third order

As in the previous subsection, even at times of order δt (where naive perturbation theory is valid) naive perturbation theory yields a spacetime metric that is not uniformly well approximated by empty flat space over its region of validity $r \gg \delta t \epsilon_f$. The technical reason for this fact is very similar to that outlined in the previous section; g_0 is a constant, so is smaller at $r \sim r_H$ than one would have guessed from the naive extrapolation of (2.2.69) to $n = 0$. It follows that, in the previous section that, even at arbitrarily small ϵ, the resultant solution is well approximated by

$$g(r, v) \approx 1 - \frac{M(v)}{r}$$

rather than the flat space result $g(r, v) = 1$, over the full domain of the amplitude expansion. It follows that the correct (resummed) amplitude expansion should start with the Vaidya solution (2.2.66) rather than the empty flat space. The IR divergences of the naive expansion are a consequence of the incorrect choice of starting point for the perturbative expansion.

At $v = \delta t$ our metric, to leading order, is the Schwarzschild metric of a black hole Schwarzschild radius M with a superposed dilaton (and consequently metric) perturbation. We will now demonstrate that these pertubrations are small. As in the previous section, it is useful to define rescaled radial and time variables $x = \frac{r}{M}$ and $y = \frac{v}{M}$. In terms of the rescaled variables, the leading order metric takes the form

$$ds^2 = M^2 \left(2dxdy - dy^2 \left(1 - \frac{1}{x} \right) + x^2 d\Omega_2^2 \right) \tag{2.2.72}$$

while the ϕ perturbation is given to leading order by

$$\frac{\phi_3^0(\delta t)}{r^3} = \frac{\phi_3^0(\delta t)}{M^3 x^3} \sim \frac{1}{\epsilon_f^3 x^3} \tag{2.2.73}$$

(recall from (2.2.71) that

$$\phi_3^0(\delta t) = \frac{1}{4} \int_0^{\delta t} \psi(v) \left[-M(v) + \psi(v)\dot{\psi}(v) \right] dv \tag{2.2.74}$$

and $M(v)$ is given in (2.2.53)).

As a constant rescaling of the metric is an invariance of the equations of motion of the Einstein dilaton system, the factor of M^2 in (2.2.72) is irrelevant for dynamics. As the dilaton perturbation above is parametrically small ($\mathcal{O}(1/\epsilon_f^3)$) the subsequent evolution of the dilaton field is linear to leading order in the $\frac{1}{\epsilon_f}$ expansion.

Let $\chi(x,y)$ denote the unique solution to

$$\partial_x \left(x^2 \left(1 - \frac{1}{x} \right) \partial_x \chi \right) + 2x \partial_y \partial_x \left(x\chi \right) = 0 \tag{2.2.75}$$

subject to the boundary condition $\chi \sim \mathcal{O}(\frac{1}{x^3})$ at large x and the initial condition $\chi(x,0) = \frac{1}{x^3}$. The leading order solution to the resummed perturbation theory for ϕ, for $v > \delta t$, is given by

$$\phi = \frac{\phi_3^0(\delta t)}{M^3} \chi \left(\frac{r}{M}, \frac{(v-\delta t)}{M} \right) \tag{2.2.76}$$

Unfortunately, the function $\chi(x,y)$ appears to be difficult to determine analytically. As in § 2.1 this solution may presumably be determined numerically with a little effort. We will not attempt the requisite numerical calculation here. In the rest of this subsection we will explain in an example how the general analysis of this subsection yields useful precise information about the subleading solution even in the absence of detailed knowledge of the function $\chi(x,y)$.

Consider a spherically symmetric shell, of the form discussed in this section, imploding inwards to form a black hole. On general grounds we expect some of the energy of the incident shell to make up the mass of the black hole, while the remaining energy is reflected back out in the form of an outgoing wave that reaches $\mathcal{I}^+$. Let the fraction of the mass that is reflected out to $\mathcal{I}^+$ be denoted by f. [7]. f is one of the most interesting and easily measured observables that characterize black hole formation.

At leading order in the expansion in $\frac{1}{\epsilon_f}$ our spacetime metric takes the Vaidya form with no outgoing wave, and so $f = 0$. This prediction is corrected at first subleading order, as we now explain. It follows on general grounds that, at late times

$$\chi(x,y) \approx \frac{\zeta(y-2x)}{x}$$

[7] In fancy parlance $f = \frac{A-B}{A}$ where A is the ADM mass of the spacetime and B is the late time Bondi mass.

for some function $\zeta(v)$. Note that ζ, like the function χ, is universal (i.e. independent of the initial condition $\psi(v)$). It follows that at late times (and to leading order)

$$\phi = M\frac{\phi_3^0(\delta t)}{M^3}\frac{\zeta\left(\frac{v-2r}{M}\right)}{r}. \tag{2.2.77}$$

It then follows from (2.2.54) (but now applied to an outgoing rather than an ingoing wave) that the energy[8] carried by this pulse is

$$\left(M\frac{\phi_3^0(\delta t)}{M^3}\right)^2 \times \frac{1}{2}\int dt\left(\partial_t\zeta(\frac{t}{M})\right)^2 = M\times\left(\frac{\phi_3^0(\delta t)}{M^3}\right)^2\times\frac{1}{2}\int_{-\infty}^{\infty}dy\left(\dot{\zeta}(y)\right)^2 \tag{2.2.78}$$

It follows that

$$\begin{aligned} f &= A\left(\frac{\phi_3^0(\delta t)}{M^3}\right)^2 \\ A &= \frac{1}{2}\int_{-\infty}^{\infty}\dot{\zeta}^2 \end{aligned} \tag{2.2.79}$$

(2.2.79) analytically determines the dependence of f on the shape of the incident wave packet, $\psi(v)$ (recall that $\phi_3^0(\delta t)$ and M are determined in terms of $\psi(v)$ by (2.2.74) and (2.2.54)). Detailed knowledge of function $\chi(x, y)$ is required only to determine the precise value of universal dimensionless number A.

2.3 Spherically symmetric collapse in global AdS

We now turn to the study of black hole formation induced by an ingoing spherically symmetric dilaton pulse in an asymptotically AdS_{d+1} space in global coordinates. As in § 2.1 our bulk dynamics is described by the Einstein Lagrangian with a negative cosmological constant and a minimally coupled dilaton. However as in § 2.2 we study solutions that preserve an $SO(d)$ invariance; this $SO(d)$ may be thought of as the group of rotations of the boundary S^{d-1}. As in both sections 2.1 and 2.2 our symmetry requirement determines

[8] We have chosen our units of energy so that a black hole with horizon radius r_H has energy M.

our metric up to three unknown functions of the two variables; the time coordinate v and the radial coordinate r. Our solutions are completely determined by the boundary value, $\phi_0(v)$ of the dilaton field. As in § 2.1 we assume that $\phi_0(v)$ is everywhere bounded by ϵ and vanishes outside the interval $(0, \delta t)$. Through out this section we will focus on the regime $\delta t \ll R$ (where R is the radius of the boundary sphere) and $\epsilon \ll 1$. The complementary regime $\delta t \gg R$ and arbitrary ϵ is under independent current investigation [36].

The collapse process studied in this section depends crucially on two independent dynamical parameters; $x = \frac{\delta t}{R}$ together with ϵ of previous subsections. We study the evolution of our systems in a limit in which x and ϵ are both small. The problem of asymptotically AdS spherically symmetric collapse is dynamically richer than the collapse scenarios studied in sections 2.1 and 2.2, and indeed reduces to those two special cases in appropriate limits.

2.3.1 Set up and equations

The equations of motion for our system are given by (2.1.4). The form of our metric and dilaton is a slight modification of (2.1.6)

$$\begin{aligned} ds^2 &= 2drdv - g(r,v)dv^2 + f^2(r,v)d\Omega_{d-1}^2 \\ \phi &= \phi(r,v). \end{aligned} \tag{2.3.80}$$

where $d\Omega_{d-1}^2$ represents the metric of a unit $d-1$ sphere. Our fields are subject to the pure global AdS initial conditions

$$\begin{aligned} g(r,v) &= r^2 + \frac{1}{R^2}, \quad (v<0) \\ f(r,v) &= rR, \quad (v<0) \\ \phi(r,v) &= 0, \quad (v<0) \end{aligned} \tag{2.3.81}$$

and the large r boundary conditions

$$
\begin{aligned}
g(r,v) &= r^2\left(1+\mathcal{O}(\frac{1}{r^2})\right)\\
f(r,v) &= r\left(R+\mathcal{O}(\frac{1}{r^2})\right)\\
\phi(r,v) &= \phi_0(v)+\mathcal{O}(\frac{1}{r})
\end{aligned}
\tag{2.3.82}
$$

Equations (2.1.4), (2.3.80), (2.3.81) and (2.3.82) together constitute a completely well defined dynamical system. Given a particular forcing function $\phi_0(v)$, these equations and boundary conditions uniquely determine the functions $\phi(r,v)$, $g(r,v)$ and $f(r,v)$.

The structure of the equations of motion of our system was described in subsection 2.1.2. In particular, we may choose the dilaton equation of motion, together with the two constraint equations, as our independent equations of motion; this set is supplemented by the energy conservation relation. With our choice of gauge and notation, the dilaton equation of motion and constraint equations take the explicit form

$$
\begin{aligned}
&\partial_r\left(f^{d-1}g\partial_r\phi\right)+\partial_v\left(f^{d-1}\partial_r\phi\right)+\partial_r\left(f^{d-1}g\partial_v\phi\right)=0\\
&\left(\partial_r\phi\right)^2+\frac{2(d-1)\partial_r^2 f}{f}=0\\
&\partial_r\left(f^{d-2}g\partial_r f+2f^{d-2}\partial_v f\right)-d\ f^{d-1}-(d-2)f^{d-3}=0
\end{aligned}
\tag{2.3.83}
$$

As in section 2, the initial data needed to specify a solution to these equations is given by the value of $\phi(r)$ on a given time slice, supplemented by the initial value of the mass, and boundary conditions at infinity. In order to obtain an explicit form for the energy conservation equation we specialize to $d=3$ and explicitly 'solve' our system at large r a la Graham and Fefferman. We find

$$
\begin{aligned}
f(r,v) &= Rr\left(1-\frac{\dot\phi_0^2}{8r^2}+\mathcal{O}(\frac{1}{r^4})\right)\\
g(r,v) &= r^2\left(\frac{1}{R^2}+\frac{1-\frac{3\dot\phi_0^2}{4}}{r^2}-\frac{M(v)}{r^3}+\mathcal{O}(\frac{1}{r^4})\right)\\
\phi(r,v) &= \phi_0(v)+\frac{\dot\phi_0}{r}+\frac{L(v)}{r^3}+\mathcal{O}(\frac{1}{r^4})
\end{aligned}
\tag{2.3.84}
$$

The energy conservation equation constrains the (otherwise arbitrary) functions $M(v)$ and $L(v)$ to obey

$$\dot{M} = -\frac{\dot{\phi}_0}{8}\left(12L(v) + 4\frac{\dot{\phi}_0}{R^2} - 3\left(\dot{\phi}_0\right)^3 + 4\dddot{\phi}_0(v)\right) \tag{2.3.85}$$

9

2.3.2 Regular small amplitude expansion

As in § 2.2 there are two legitimate amplitude expansions of spacetime we wish to determine. In this subsection we discuss the expansion analogous to the expansion of subsection 2.2.2. That is we expand all our fields as in (2.1.28) (where the functions f_n, g_n and ϕ_n are all defined to be of homogeneity n in the boundary field ϕ_0) and demand that all functions are everywhere regular. The requirement of regularity, together with our boundary and initial conditions, uniquely specifies all functions in (2.1.28). Explicitly, to second order

[9] Note that the stress tensor and Lagrangian $\mathcal{L}$ of our system are given by

$$\begin{aligned} T_v^v &= M(v) \\ T_\theta^\theta = T_\phi^\phi &= -\frac{M(v)}{2} \\ \mathcal{L} &= -3L(v) - \frac{\dot{\phi}_0}{R^2} + \frac{3}{4}\left(\dot{\phi}_0\right)^3 - \dddot{\phi}_0(v) \end{aligned} \tag{2.3.86}$$

It follows that (2.3.85) may be rewritten as $\dot{M} = \frac{\dot{\phi}_0 \mathcal{L}}{2}$.

we find

$$
\begin{aligned}
\phi_1(r,v) &= \sum_{m=0}^{\infty}(-1)^m\Big[\phi_0(v-m\pi R)+\frac{\dot\phi_0(v-m\pi R)}{r}+\phi_0(v-Rm\pi-2R\tan^{-1}(rR))\\
&\qquad-\frac{\dot\phi_0(v-m\pi R-2R\tan^{-1}(rR))}{r}\Big]\\
f_2(r,v) &= \frac{R}{4}\left(r\int_r^\infty \rho K(\rho,v)\,d\rho-\int_r^\infty \rho^2 K(\rho,v)\,d\rho\right)\\
g_2(r,v) &= -\frac{1}{4r}\Big[\frac{2r}{R^2}\int_r^\infty \rho K(\rho,v)\,d\rho+2r^2\int_r^\infty \rho^2 K(\rho,v)\,d\rho\\
&\qquad+\int_0^r \rho^2(\frac{1}{R^2}+\rho^2)K(\rho,v)\,d\rho\Big]-\frac{2}{R}\partial_v f_2(r,v)\\
&\text{where}\\
K(\rho,v) &= (\partial_r\phi_1(r,v))^2
\end{aligned}
\tag{2.3.87}
$$

The perturbation expansion in this section is valid only if $\phi(r,v)$ is everywhere small on the solution. $\phi_1(r,v)$ reaches its maximum value in the neighborhood of the origin where it is given approximately by $\phi_0+\ddot\phi_0\sim\epsilon+\frac{\epsilon}{x^2}$. Consequently the validity of the amplitude expansion sketched in this section requires both that $\epsilon\ll 1$ and that $x^2\gg\epsilon$.

We have chosen integration constants to ensure that the solution in (2.3.87) is regular at $r=0$. In particular

$$
g_2(0,v)=\frac{1}{2}\left(\int_0^\infty \rho^2\partial_v K(\rho,v)\,d\rho-\frac{1}{R^2}\int_0^\infty \rho K(\rho,v)\,d\rho\right).
$$

As in subsection 2.2.2, this choice automatically implies the energy conservation equation. In particular, expanding $g_2(r,v)$ at large r we find

$$
-M(v)=-\frac{1}{4}\left(\int_0^\infty \rho^2(\frac{1}{R^2}+\rho^2)K(\rho,v)\,d\rho-\dot\phi_0(v)^2-2\dot\phi_0(v)\ddot\phi_0(v)\right)
\tag{2.3.88}
$$

(this equation is valid only for $v<\pi R$; it turns out that $M(v)$ is constant for $v>\delta t$) in agreement with the energy conservation equation.

Finally, let us focus on the coordinate range rR, $\frac{v}{R} \ll 1$ and also require that x is small so that the time scale in ϕ_0 is also smaller than the AdS radius. In this parameter and coordinate range (2.3.87) should reduce to a solution of the flat space propagation equation (2.2.55); this is easily verified to be the case. In the given variable and parameter regime, all terms with (2.3.87) with $m \neq 0$ vanish; $\tan^{-1}(Rr) \approx rR$ and the first and the third terms in (2.3.87) are negligible compared to the second and fourth as x is small. Putting all this together, (2.3.87) reduces to (2.2.55) under the identification $\psi(v) = R^2 \dot{\phi}_0(v)$, once we also identify the coordinate r of § 2.2 with $R^2 r$ in this section. Notice that this replacement implies that $\epsilon_f = \frac{\epsilon}{x^2}$ (where ϵ_f was the perturbative expansion of § 2.2). This identification of parameters is consistent with the fact that the expansion of this section breaks down when $\frac{\epsilon}{x^2}$ becomes large, while the expansion of subsection 2.2.2 breaks down at large ϵ_f.

2.3.3 Spacetime and event horizon for black hole formation

In the rest of this section we will describe the process of black hole formation via collapse in an amplitude expansion. As in earlier sections, the spacetime that describes this collapse process will turn out to be given, to leading order, by the Vaidya form

$$
\begin{aligned}
ds^2 &= 2drdv - \left(\frac{1}{R^2} + r^2 - \frac{M(v)}{r}\right) dv^2 + R^2 r^2 d\Omega_2^2 \\
\phi(r,v) &= \phi_0(v) + \frac{\dot{\phi}_0}{r}
\end{aligned}
\tag{2.3.89}
$$

where $M(v)$ is approximated by $C_2(v)$, the order ϵ^2 piece of (2.3.85)

$$
C_2(v) = -\frac{1}{2} \int_{-\infty}^{v} dt \dot{\phi}_0(t) \left(\dddot{\phi}_0(t) + \frac{\dot{\phi}_0(t)}{R^2} \right) \tag{2.3.90}
$$

In this subsection we will compute the event horizon of the spacetime (2.3.89) in a perturbation expansion in a small parameter, whose nature we describe below. The horizon is determined by the differential equation

$$
2\frac{dr_H}{dv} = \frac{1}{R^2} + r_H^2 - \frac{M(v)}{r_H} \tag{2.3.91}
$$

where $M(v)$ reduces to a constant M for $t > \delta t$. At late times the event horizon surface must reduce to the largest real solution of the equation

$$\frac{1}{R^2} + (r_H^0)^2 - \frac{M}{r_H^0} = 0.$$

It then follows from (2.3.91) that

$$\begin{aligned} r_H(v) &= r_H^0, \quad (v > \delta t) \\ r_H(v) &= r_H^0 x(\frac{v}{\delta t}), \quad (0 < v < \delta t) \\ \tan^{-1}(r_H(v)) &= \tan^{-1}\left(r_H^0 x(0)\right) + v \quad (v < 0), \quad \tan^{-1}(r_H(v)) > 0 \end{aligned} \tag{2.3.92}$$

As in previous subsections, the function $x(t)$ is easily generated in a perturbation expansion

$$x(t) = 1 + \left(\frac{M\delta t}{(r_H^0)^2}\right) x_1(t) + \left(\frac{M\delta t}{(r_H^0)^2}\right)^2 x_2(t) + \ldots \tag{2.3.93}$$

The small parameter for this expansion is $\frac{M\delta t}{(r_H^0)^2}$. This parameter varies from approximately $\epsilon^{\frac{2}{3}}$ when $x \ll \epsilon^{\frac{2}{3}}$ to $\frac{x^4}{\epsilon^2}$ when $x \gg \epsilon^{\frac{2}{3}}$ and is always small provided $x \ll \sqrt{\epsilon}$ and $\epsilon \ll 1$. These conditions will always be met in our amplitude constructions below. Note that the event horizon of our solution is created (at $r = 0$) at the time $v = -\tan^{-1}(r_H^0)+$ subleading.

Explicitly working out the perturbation series we find

$$\begin{aligned} x_1(t) &= -\int_t^1 dt \frac{1 - \frac{M(y\delta t)}{M}}{2} \\ x_2(t) &= -\int_t^1 dy \left(\frac{2(r_H^0)^3}{M} + \frac{M(y\delta t)}{M}\right) \end{aligned} \tag{2.3.94}$$

2.3.4 Amplitude expansion for black hole formation

The amplitude expansion of the previous subsection breaks down for $x^2 \ll \epsilon$. As in § 2.2, we have a new amplitude expansion in this regime. As in § 2.2, the starting point for this expansion is the Vaidya metric and dilaton field (2.3.89). As in sections 2.1 and 2.2, the perturbation expansion based on (2.3.89) is technically difficult to implement at late times. However as in earlier sections, at early times - i.e. times of order δt - the perturbative expansion is well approximated by the naive expansion based on the solution (2.3.89) with $M(v)$ set equal to zero. Following the terminology of previous sections we refer to this simplified expansion as the naive expansion. In the rest of this subsection we will elaborate on the analytic structure of the naive perturbative expansion.

In order to build the naive expansion, we expand the fields $f(r,v)$, $g(r,v)$ and $\phi(r,v)$ in the form (2.1.28). It is not too difficult to inductively demonstrate that

- 1. The functions ϕ_{2n+1}, g_{2n} and f_{2n} have the following analytic structure in the variable r

$$
\begin{aligned}
\phi_{2n+1}(r,v) &= \sum_{m=0}^{\infty} \frac{1}{R^{2m}} \sum_{k=0}^{2n+m-2} \frac{\phi_{2n+1}^{k,m}(v)}{r^{2n+1-k+m}} \quad (n \geq 1) \\
f_{2n}(r,v) &= rR \sum_{m=0}^{\infty} \frac{1}{R^{2m}} \sum_{k=0}^{2n-4} \frac{f_{2n}^{k,m}(v)}{r^{2n-k+m}} \quad (n \geq 2) \\
g_{2n}(r,v) &= -\frac{C_{2n}(v)}{r} + r \sum_{m=0}^{\infty} \frac{1}{R^{2m}} \sum_{k=0}^{2n-3} \frac{g_{2n}^{k,m}(v)}{r^{2n-k+m}} \quad (n \geq 2)
\end{aligned}
\tag{2.3.95}
$$

- 2. The functions $\phi_{2n+1}^{k,m}(v)$, $f_{2n}^{k,m}(v)$ and $g_{2n}^{k,m}(v)$ are functionals of $\phi_0(v)$ that scale like $\lambda^{-2n-1+m+k}$, $\lambda^{-2n+m+k}$ and $\lambda^{-2n+m+k-1}$ respectively under the scaling $v \to \lambda v$.

- 3. For $v > \delta t$ we have some additional simplifications in structure. At these times $f_4(r,v) = 0$ and $g_4(r,v) = -\frac{C_4(v)}{r}$. Further, the sums over k in the second and third of the equations above run from 0 to $2n-6+m$ and $2n-5+m$ respectively. Finally, functions $\phi_{2n+1}^{k,m}(v)$ are all polynomials in v of a degree that grows with n. In particular the degree of $\phi_{2n+1}^{k,m}$ is at most $n-1+k+m$; the degree of $f_{2n}^{k,m}$ is at most $n-3+k+m$ and the degree of $g_{2n}^{k,m}$ is at most $n-4+k+m$.

As we have explained above,

$$
\phi(r,v) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{1}{R^{2m}} \sum_{k=0}^{2n-2+m} \frac{\phi_{2n+1}^{k,m}(v)}{r^{2n+1-k+m}}.
$$

We will now discuss the relative orders of magnitude of different terms in this summation. Abstractly, at times that are larger than or of order δt, the effective weighting factor for the sum over n, m, k respectively are approximately given by $\frac{\epsilon^2 v}{r^2(\delta t)^3}$, $\frac{v}{R^2 r}$ and vr respectively. We will try to understand the implications of these estimates in more detail.

Let us first suppose that $x \ll \epsilon^{\frac{2}{3}}$. In this case the black hole that is formed has a horizon radius of order $\frac{\epsilon^{\frac{2}{3}}}{\delta t} \gg \frac{1}{R}$ (this estimate is corrected in a power series in $\frac{x^2}{\epsilon^{\frac{4}{3}}}$). Consequently, the resultant black hole is large compared to the AdS radius. At $m = 0$, this regime, the summation over k and n simply reproduce the solution of § 2.1. As in § 2.1 these summations are dominated by the smallest values of k and n for $vr_H \sim vT \ll 1$, in the neighborhood of the horizon. As in § 2.1 the sum over k is dominated by the largest value of k at large enough r. The new element here is the sum over m; this summation is dominated by small m when $vT \ll \frac{\epsilon^{\frac{4}{3}}}{x^2}$. When $x \ll \epsilon^{\frac{2}{3}}$, this condition automatically follows whenever $vT \ll 1$. Consequently, naive perturbation theory is always good for times small compared to the inverse black hole temperature, in this regime.

We emphasize that naive perturbation theory is always good at times of order δt. Over such time scales (and for $r \sim r_H$) we note that the sum over n and k are weighted by $\epsilon^{\frac{2}{3}}$ (this is as in § 2.1) while the sum over m is weighted by $\epsilon^{\frac{2}{3}}(\frac{x}{\epsilon^{\frac{2}{3}}})^2$. Note that the weighting factor for the sum over m is smaller than the weighting factor for the sum over, for instance n, provided $x \ll \epsilon^{\frac{2}{3}}$. It follows that our naive perturbation theory represents a weak departure from the black brane formation solution of § 2.1 when $x \ll \epsilon^{\frac{2}{3}}$.

Now let us turn to the the parameter regime $x \gg \epsilon^{\frac{2}{3}}$. In this regime $r_H R \sim \frac{\epsilon^2}{x^3}$, so that black holes that are formed in the collapse process are always small in units of the AdS radius. At times that are larger or of order δt, the sum over m and n are dominated by their smallest values provided $\frac{v}{R} \ll \frac{\epsilon^2}{x^3}$. Making the replacement $\epsilon = x^2 \epsilon^f$, this condition reduces to $v \ll \epsilon_f^2 \delta t$ which was exactly the condition for applicability of naive perturbation theory in flat space in section 2.2. The new element here is the sum over k. k is zero in § 2.2, and the sum over k here is dominated by $k = 0$ near $r = r_H$

for $\frac{v}{R} \ll \frac{x^3}{\epsilon^2}$, a condition that is automatically implied by $\frac{v}{R} \ll \frac{\epsilon^2}{x^3}$. Note, however, that, as in the previous paragraph, the sum over k is always dominated by the largest value of k at sufficiently large r. This reflects the fact that AdS space is never well approximated by a flat bubble at large r. Finally, specializing to v of order δt and $r \sim r_H$, the sum over n and m are each weighted by $\frac{x^4}{\epsilon^2} \sim \frac{1}{\epsilon_f^2}$ while the sum over k is weighted by $\epsilon \frac{\epsilon}{x^2}$. In particular naive perturbation theory is good at times of order δt provided $x \ll \sqrt{\epsilon}$.

Let us summarize in broad qualitative terms. Naive perturbation theory is a good expansion to the true solution when $vT \ll 1$ for $\frac{v}{R} \ll \frac{\epsilon^2}{x^3}$. In particular, this condition is always obeyed for times of order δt when $x \ll \sqrt{\epsilon}$.

2.3.5 Explicit results for naive perturbation theory

As we have explained above, the functions ϕ, f and g may be expanded in an expansion in ϵ as

$$\begin{aligned} \phi(r,v) &= \epsilon\phi_1(r,v) + \epsilon^3\phi_3(r,v) + \mathcal{O}(\epsilon)^5 \\ f(r,v) &= rR\left(1 + \epsilon^2 f_2(r,v) + \epsilon^4 f_4(r,v) + \mathcal{O}(\epsilon)^6\right) \\ g(r,v) &= r^2 + \frac{1}{R^2} + \epsilon^2 g_2(r,v) + \epsilon^4 g_4(r,v) + \mathcal{O}(\epsilon)^6 \end{aligned} \tag{2.3.96}$$

Moreover the functions ϕ_{2n+1}, f_n and g_n may themselves each be expanded as a sum over two integer series (see (2.3.95)). The sum over k runs over a finite number of values in (2.3.95) and we will deal with this summation exactly below. However the sum over m runs over all integers, and is computatble only after truncation to some finite order. This truncation is justified as the sum over m is effectively weighted by a small parameter as explained in the section above. In this section we present exact expressions for the functions ϕ_1, g_2 and f_2, and expressions for ϕ_3, f_4 and g_4 to the first two orders in the expansion over the integer m (this summation is formally weighted by $\frac{1}{R}$);

The solutions are given as

$$\begin{aligned}
\phi_1(r,v) &= \phi_0(v) + \frac{\dot\phi_0(v)}{r} \\
f_2(r,v) &= -\frac{\dot\phi_0^2}{8r^2} \\
g_2(r,v) &= -\frac{3\dot\phi_0^2}{4} - \frac{C_2(v)}{r} \\
\phi_3(r,v) &= \frac{K(v)}{r^3} \\
&+ \frac{1}{R^2}\left[\frac{\int_{-\infty}^{v}\left(3K(t)-\dot\phi_0(t)^3\right)dt}{12r^4} + \frac{\int_{-\infty}^{v}dt_1\int_{-\infty}^{t_1}dt_2\left(3K(t_2)-\dot\phi_0(t_2)^3\right)}{12r^3}\right] \qquad (2.3.97)\\
&+ \mathcal{O}\left(\frac{1}{R}\right)^4 \\
f_4(r,v) &= \left(\frac{\dot\phi_0^4}{384r^4} - \frac{A_3(v)}{32r^4}\right) + \frac{1}{R^2}\left(\frac{A_1(v)}{96r^4} + \frac{A_2(v)}{120r^5}\right) + \mathcal{O}\left(\frac{1}{R}\right)^4 \\
g_4(r,v) &= -\frac{C_4(v)}{r} + \frac{3A_3(v)-\dot\phi_0^4}{24r^2} + \frac{1}{48r^3}\left(3\dot A_3(v) - 4\dot\phi_0^3\ddot\phi_0\right) \\
&- \frac{1}{R^2}\left[\frac{A_1(v)}{24r^2} + \frac{\dot A_1(v)}{48r^3} + \frac{15A_3(v)+4A_2(v)-\dot\phi_0^4}{240r^4}\right] + \mathcal{O}\left(\frac{1}{R}\right)^4
\end{aligned}$$

Where

$$\begin{aligned}
K(v) &= \int_{-\infty}^{v} dt\ \dot\phi_0\left(-C_2(t) + \dot\phi_0\ddot\phi_0\right) \\
A_1(v) &= \dot\phi_0(v)\int_{-\infty}^{v}dt_1\int_{-\infty}^{t_1}dt_2\left(-3K(t_2)+\dot\phi_0^3(t_2)\right) \\
A_2(v) &= \dot\phi_0(v)\int_{-\infty}^{v}dt\left(-3K(t)+\dot\phi_0^3(t)\right) \\
A_3(v) &= \dot\phi_0 K(v)
\end{aligned} \qquad (2.3.98)$$

$$\begin{aligned}
C_2(v) &= -\frac{1}{2}\int_{-\infty}^{v}dt\ \dot\phi_0(t)\left(\frac{\dot\phi_0(t)}{R^2} + \dddot\phi_0(t)\right) \\
C_4(v) &= -\frac{3}{8}\int_{-\infty}^{v}dt\ \dot\phi_0(t)\left(K(t)-\dot\phi_0(t)^3\right) \\
&- \frac{1}{8R^2}\int_{-\infty}^{v}dt_1\ \dot\phi_0(t_1)\int_{-\infty}^{t_1}dt_2\int_{-\infty}^{t_2}dt_3\left(3K(t_3)-\dot\phi_0(t_3)^3\right) + \mathcal{O}\left(\frac{1}{R}\right)^4
\end{aligned} \qquad (2.3.99)$$

2.3.6 The solution at late times

As in previous sections, our solution is normalizable (unforced) for $v > \delta t$. Naive perturbation theory reliably establishes the initial conditions for this unforced evolution at $v = \delta t$. To leading order, this evolution is given by global AdS black hole metric with $M = C_2(\delta t)$ (see (2.3.90)), perturbed by $\phi(\delta t) = \frac{K(\delta t)}{r^3}$ see (2.3.97). As in the previous two subsections, the qualitatively important point is that this represents a small perturbation about the black hole background. Moreover, it follows on general grounds that perturbations in a black hole background in AdS space never grow unboundedly (in fact they decay) with time. Consequently, we may reliably conclude that our spacetime takes the Vaidya form (2.3.89) at all times to leading order in the amplitude expansion.

In order to determine an explicit expression for the subsequent dilaton evolution, one needs to solve for the linear, minimally coupled, evolution of a $\frac{1}{r^3}$ initial condition in the background of global AdS with a Schwarzschild black hole of arbitrary mass. As in the previous two sections, the linear differential equation one needs to solve appears to be analytically intractable, but could easily be solved numerically. We will not, however, attempt this evaluation in this chapter.

2.4 Translationally invariant graviton collapse

In sections 2.1 and 2.3 above we have studied the collapse triggered by a minimally coupled scalar wave in an asymptotically AdS background. Our study was, in large part, motivated by potential applications to CFT dynamics via the AdS/CFT correspondence. From this point of view the starting point of our analyses in e.g. § 2.1 has a drawback as not every bulk system that arises in the study of the AdS/CFT correspondence, admits a consistent truncation to the theory of gravity coupled to a minimally coupled massless scalar field.

On the other hand, every two derivative theory of gravity that admits AdS space as a solution admits a consistent truncation to Einstein gravity with a negative cosmological constant. Consequently, any results that may be derived using the graviton instead of dilaton waves, applies universally to all examples of the AdS/CFT correspondence with two derivative gravity duals. In this section we study a situation very analogous to the set up of § 2.1, with, however, a transverse graviton playing the place of the dilaton field of § 2.1. All the calculations of this section apply universally to any CFT that admits a two derivative gravitational dual.

While the equations that describe the propagation of gravity waves are more complicated in detail than those that describe the propagation of a massless minimally coupled scalar field, it turns out that the final results of the calculations presented in this subsection are extremely similar to those of § 2.1. We take this to suggest that all the qualitative results of sections 2.1 and 2.3 would continue to qualitatively apply to the most general approximately translationally invariant gravitational perturbations of Poincare Patch AdS space or approximately spherically symmetric gravitational perturbation of global AdS space. If this guess is correct, it suggests that the qualitative lessons have a wide degree of applicability.

In this section we restrict our attention to the simplest dimension $d = 3$. It should we possible, with some additional effort, to extend our results at least to all odd d, and also to work out the corresponding results for even d. We leave this extension to future work.

The set up of this section is very closely analogous to that employed by Yaffe and Chesler in [15]. The main differences are as follows. Yaffe and Chesler worked in $d = 4$; they numerically studied the effect of a specific large amplitude non normalizable deformation on the gravitational bulk. We work in $d = 3$, and analytically study the the effect of the arbitrary small amplitude deformation on the gravitational bulk.

2.4.1 The set up and summary of results

In this section we study solutions to pure Einstein gravity with a negative cosmological constant. We study solutions that preserve an $R^2 \times Z_2 \times Z_2$ symmetry. Here R^2 denotes the symmetry of translations in spatial field theory directions, while the two Z_2s respectively denote the spatial parity flip and the discrete exchange symmetry between the two Cartesian spatial boundary coordinates x and y.

As in § 2.1, our symmetry requirements determine our metric up to three unknown functions of v and r. With the same choice of gauge as in § 2.1, our metric takes the form

$$ds^2 = -2\ dv\ dr + g(r,v)\ dv^2 + f^2(r,v)(dx^2 + dy^2) + 2r^2 h(r,v) dx\ dy \tag{2.4.100}$$

The boundary conditions on all fields are given by (2.1.9) under the replacement $\phi(r,v) \to h(r,v)$ and $\phi_0(v) \to h_0(v)$. Here $h_0(v)$ gives the boundary conditions on the off diagonal mode, g_{xy}, of the boundary metric. $h_0(v)$ is taken to be of order ϵ. Physically, our boundary conditions set up a graviton wave, with polarization parallel to the spatial directions of the brane.

As in § 2.1, in order to solve Einstein's equations with the symmetries above, it turns out to be sufficient to solve the three equations E_C^2, E_C^1 and E_{xy} (see (2.1.10)) (plus the energy conservation condition rE_{ec} at one r).

As in § 2.1 it is possible to solve these equations order by order in ϵ. We present our solution later in this section. To end this subsection, we list the principal qualitative results of this section. We are able to show that

- The boundary conditions described above result in black brane formation for an arbitrary (small amplitude) source function $h_0(v)$.

- Outside the event horizon of our spacetime, we find an explicit analytic form for the metric as a function of $h_0(v)$. Our metric is accurate at leading order in the ϵ expansion, and takes the Vaidya form (0.1.1) with a mass function

$$M(v) = -\frac{1}{2}\int_{-\infty}^{v} dt \dot{h}_0 \dddot{h}_0 \tag{2.4.101}$$

- In particular, we find that the energy density of resultant black brane is given by

$$M \approx -E_2 = \frac{1}{2}\int_{-\infty}^{\infty} dt \ddot{h}_0^2 \tag{2.4.102}$$

  Note that $E_2 \sim \frac{\epsilon^2}{(\delta t)^3}$.

- As this leading order metric is of the same form as that in the previous subsection, the analysis of the event horizons presented above continues to apply. In particular it follows that singularities formed in the process of black brane formation are always shielded by a regular event horizon at small ϵ.

- Going beyond leading order, perturbation theory in the amplitude ϵ yields systematic corrections to this metric at higher orders in ϵ. We unravel the structure of this perturbation expansion in detail and work out this perturbation theory explicitly to fifth order at small times.

2.4.2 The energy conservation equation

As we have explained above, the equations of motion for our system include the energy conservation relation, in addition to the one dynamical and two constraint equations. The form of the dynamical and constraint equations is easily determined using Mathematica-6;

these equations turn out to be rather lengthy and we do not present them here. In this section we content ourselves with presenting an explicit form for the energy conservation equation. As in § 2.1, it is possible to solve for the functions $\frac{f}{r}$, $\frac{g}{r^2}$ and h in a power series in $\frac{1}{r}$. This solution is simply the Graham Fefferman expansion. To order $\frac{1}{r^3}$ (relative to the leading result) we find

$$
\begin{aligned}
f(r,v) &= r\left(1+\frac{\frac{\left[\dot{h}_0\right]^2}{8\left(1-h_0^2\right)}}{r^2}+\frac{\frac{1}{2}h_0\sigma(v)}{r^3}+\cdots\right)\\
g(r,v) &= r^2\left(1+\frac{\frac{1}{4\left(-1+h_0^2\right)^2}\left[\left(1+3h_0^2\right)\left[\dot{h}_0\right]^2-4h_0\left(-1+h_0^2\right)\partial_v^2 h_0\right]}{r^2}-\frac{M(v)}{r^3}+\cdots\right)\\
h(r,v) &= \left(h_0+\frac{\dot{h}_0}{r}+\frac{\frac{h_0\dot{h}_0^2}{4\left(-1+h_0^2\right)}}{r^2}+\frac{\sigma(v)}{r^3}+\cdots\right)
\end{aligned}
\tag{2.4.103}
$$

where the parameters M and σ are constrained by the energy conservation equation

$$
\begin{aligned}
\dot{M} = &-\frac{\dot{h}_0}{2\left(-1+h_0^2\right)^4}\Big[+3M(v)h_0\left(-1+h_0^2\right)^3-3\left(-1+h_0^2\right)^3\sigma\\
&-4\left(-1+h_0^2\right)h_0\dot{h}_0\partial_v^2h_0+\left(-1+h_0^2\right)^2\partial_v^3h_0+\left(1+3h_0^2\right)\left[\dot{h}_0\right]^3\Big]
\end{aligned}
\tag{2.4.104}
$$

[10]

[10] The stress tensor is given by

$$
\begin{aligned}
T_{tt} &= M\\
T_{xx} &= T_{yy} = -\frac{M}{2}\\
T_{xy} &= -\frac{1}{2\left(-1+h_0^2\right)^3}\Big[-3\left(-1+h_0^2\right)^3\sigma(v)-4\left(-1+h_0^2\right)h_0\dot{h}_0\partial_v^2h_0\\
&\quad+\left(-1+h_0^2\right)^2h_0^3+\left(1+3h_0^2\right)\left[\dot{h}_0\right]^3\Big]
\end{aligned}
\tag{2.4.105}
$$

Using these relations, it may be verified that (2.4.104) is simply a statement of the conservation of the stress tensor.

In the perturbative solution we list below, we will find that $\sigma \sim \mathcal{O}(\epsilon^3)$. It follows that, to order $\mathcal{O}(\epsilon^2)$, the function $M(v)$ is given by (2.4.101).

2.4.3 Structure of the amplitude expansion

As in subsection 2.1 we set up a naive amplitude expansion by formally replacing h_0 with ϵh_0 and then solving our equations in a power series in ϵ. We expand

$$\begin{aligned} f(r,v) &= \sum_{n=0}^{\infty} \epsilon^n f_n(r,v) \\ g(r,v) &= \sum_{n=0}^{\infty} \epsilon^n g_n(r,v) \\ h(r,v) &= \sum_{n=0}^{\infty} \epsilon^n h_n(r,v) \end{aligned} \tag{2.4.106}$$

with

$$f_0(r,v) = r, \qquad g_0(r,v) = r^2, \quad h_0(r,v) = 0. \tag{2.4.107}$$

The formal structure of this expansion is identical to that described in § 2.1.5; in particular f_n and g_n are nonzero only for even n while h_n is nonzero only for odd n. At first order we find

$$h_1(r,v) = h_0(r,v) + \frac{\dot{h}_0(r,v)}{r} \tag{2.4.108}$$

which then leads to simple expressions (see below) for f_2 and g_2. In particular h_1 and f_2 vanish for $v \geq \delta t$ while $g_2 = M/r$ for $v \geq \delta t$.

Turning to higher orders in the perturbative expansion, it is possible to inductively demonstrate that for $n \geq 1$

- 1. The functions h_n, g_n and f_n have the following analytic structure in the variable

r

$$
\begin{aligned}
h_{2n+1}(r,v) &= \sum_{k=2}^{2n+1} \frac{\phi_n^k(v)}{r^k} \\
f_{2n+2}(r,v) &= r\sum_{k=2}^{2n+2} \frac{f_n^k(v)}{r^k} \\
g_{2n+2}(r,v) &= r\sum_{k=1}^{n} \frac{g_n^k(v)}{r^k}
\end{aligned}
\tag{2.4.109}
$$

- 2. The functions $h^k_{2n+1}(v)$, $f^k_{2n+2}(v)$ and $g^k_{2n+2}(v)$ are each functionals of $h_0(v)$ that scale like λ^{-k} under the scaling $v \to \lambda v$.

- 3. For $v > \delta t$ these functions are all polynomials in v of a degree that grows with n. For example, the degree of h^k_{2n+1} is of at most $3n - k$.

As in the § 2.1, this structure ensures that naive perturbation theory is good for times $v \ll M^{\frac{1}{3}}$, but fails for later times. As in section (2.1), the correct perturbative expansion uses the Vaidya metric (0.1.1) as the zero order solution.

2.4.4 Explicit results up to 5th order

At leading order we have

$$
\begin{aligned}
h_1(r,v) &= h_0(v) + \frac{\dot{h}_0}{r} \\
f_2(r,v) &= \frac{\left[\dot{h}_0\right]^2}{8r} \\
g_2(r,v) &= \frac{E_2(v)}{r} + \frac{1}{4}\left[\dot{h}_0\right]^2 + \dot{h}_0 \partial_v^2 h_0
\end{aligned}
\tag{2.4.110}
$$

At next order

$$h_3(r,v) = \frac{1}{4r^3}\left\{\int_{-\infty}^{v} E_2(x)\partial_x h_0 \; dx - r \; h_0 \left[\dot{h}_0\right]^2\right\}$$

$$\begin{aligned}
f_4(r,v) &= \frac{h_0^2(v)\left[\dot{h}_0\right]^2}{8r} + \frac{D(v)h_0(v)}{8r^2} - \frac{\dot{h}_0}{128r^3}\left(-12D(v) + \left[\dot{h}_0\right]^3\right) \\
g_4(r,v) &= \frac{E4(v)}{r} + \frac{5}{4}h_0(v)^2\left[\dot{h}_0\right]^2 + h_0(v)^3\partial_v^2 h0 \\
&\quad + \frac{\dot{h}_0}{8r^2}\left[D(v) + 4E_2(v)h_0(v)\right] + \frac{1}{16r^3}\left(E_2(v)\left[\dot{h}_0\right]^2 + D(v)\partial_v^2 h_0\right) \\
h_4(r,v) &= 0 \\
\text{where } \; D(v) &= \int_{-\infty}^{v} E_2(x)\partial_x h_0 \; dx
\end{aligned} \tag{2.4.111}$$

Finally at the next order

$$\begin{aligned}
h_5(r,v) &= \frac{D_1(v)}{2r^2} \\
&\quad + \frac{1}{24r^3}\Bigg[6\int_{-\infty}^{v} D_2(x)\; dx + 5\left\{\int_{-\infty}^{v} dz \int_{-\infty}^{z} dy \int_{-\infty}^{y} D_4(x)\; dx\right\} \\
&\qquad + 4\left\{\int_{-\infty}^{v} dy \int_{-\infty}^{y} D_3(x)\; dx\right\}\Bigg] \\
&\quad + \frac{1}{r^4}\left[\frac{5}{24}\left\{\int_{-\infty}^{v} dy \int_{-\infty}^{y} D_4(x)\; dx\right\} + \frac{1}{6}\left\{\int_{-\infty}^{v} D_3(x)\; dx\right\}\right] \\
&\quad + \frac{1}{8r^5}\left[\int_{-\infty}^{v} D_4(x)\; dx\right]
\end{aligned} \tag{2.4.112}$$

where

$$\begin{aligned}
D_1(x) &= -h_0(x)^3\left[\partial_x h_0\right]^2 \\
D_2(x) &= E_4(x)\partial_x h_0 + \frac{1}{4}D(x)h_0(x)\partial_x h_0 + E_2(x)h_0(x)^2\partial_x h_0 \\
D_3(x) &= \frac{1}{8}\left[5D(x)\left[\partial_x h_0\right]^2 + 15E_2(x)h_0(x)\left[\partial_x h_0\right]^2 + 15D(x)h_0(x)\partial_x^2 h_0\right] \\
D_4(x) &= \frac{1}{8}\left[18D(x)E_2(x) + 5E_2(x)\left[\partial_x h_0\right]^3 + 7D(x)h_0(x)\partial_x^2 h_0\right]
\end{aligned} \tag{2.4.113}$$

and (this follows from energy conservation)

$$\begin{aligned}\dot{E}_2 &= \frac{1}{2}\dot{h}_0\partial_v^3 h_0 \\ \dot{E}_4 &= \frac{3}{8}D(v)\dot{h}_0 + \frac{\dot{h}_0}{2}\left[3E_2(v)h_0(v) + \left[\dot{h}_0\right]^3 + 4h_0(v)\dot{h}_0\partial_v^2 h_0 + 2h_0^2\partial_v^3 h_0\right]\end{aligned} \tag{2.4.114}$$

It follows in particular that the the 'initial' condition for normalizable evolution at $v = \delta t$ is given, to leading order, by

$$h(r,\delta t) = \frac{1}{8r^3}\int_{-\infty}^{v}\left(\int_{-\infty}^{x} dy\left(\partial_y h_0 \partial_y^3 h_0\right)\partial_x h_0(x)dx\right) \tag{2.4.115}$$

This initial condition is of order $\frac{\epsilon^3}{(\delta t)^3 r^3}$ i.e. of order $\frac{\epsilon}{\tilde{r}^3}$ where $\tilde{r} = \frac{r}{E_2}$. This demonstrates that, for $v > \delta t$, our solution is a small perturbation about the black brane of energy density E_2.

2.4.5 Late Times Resummed perturbation theory

To leading order, the initial condition for the normalizable evolution of resummed perturbation theory for the field $h(r,v)$ is given by

$$h(\delta t) = \frac{1}{4r^3}\left(\int_{-\infty}^{\delta t} E_2(x)\partial_x h_0 \; dx\right) \equiv \frac{h_3^0(\delta t)}{r^3}$$

Now, at the linearized level the equation of motion for the function h is simply the minimally coupled scalar equation. It follows that the subsequent evolution of the field h is simply given by

$$h = \frac{h_3^0(\delta t)}{M}\psi\left(\frac{r}{M^{\frac{1}{3}}}, (v-\delta t)M^{\frac{1}{3}}\right) \tag{2.4.116}$$

where the universal function ψ was defined in § 2.1. As in § 2.1, this perturbation is small initially, and at all subsequent times, justifying the resummed perturbation procedure.

2.5 Generalization to Arbitrary Dimension

2.5.1 Translationally Invariant Scalar Collapse in Arbitrary Dimension

In this subsection we will investigate how the results of section 2.1, which were worked out for the special case $d = 3$, generalize to $d \geq 3$. The mathematical problem we will investigate in this section was already set up in general d in subsection 2.1.1. It turns out that the dynamical details of collapse processes in odd and even dynamics are substantially different, so we will deal with those two cases separately.

Odd d

The general structure of the solutions that describe collapse in odd $d \geq 5$ is similar in many ways to the solution reported in § 2.1. The energy conservation equations may be studied via a large r Graham Fefferman expansion closely analogous to that described in § 2.1. The functions ϕ f and g may be expanded at large r as

$$\begin{aligned}
\phi(r,v) &= \sum_{n=0}^{\infty} \frac{A_\phi^n(v)}{r^n} \\
f(r,v) &= r\left(\sum_{n=0}^{\infty} \frac{A_f^n(v)}{r^n}\right) \\
g(r,v) &= r^2\left(\sum_{n=0}^{\infty} \frac{A_g^n(v)}{r^n}\right)
\end{aligned} \tag{2.5.117}$$

For $n \leq d-1$ the equations of motion locally determine $A_\phi^n(v)$, $A_f^n(v)$ and $A_g^n(v)$ in terms of $\phi_0(v)$. Each of these functions is a local expression (of n^{th} order in v derivatives) of $\phi_0(v)$. However local analysis does not determine $A_g^d(v) \equiv M(v)$ and $A_\phi^d(v) \equiv L(v)$ in terms of $\phi_0(v)$. $M(v)$ and $L(v)$ are however constrained to obey an energy conservation equation that takes the form

$$\dot{M} = k\dot{\phi}L(v) + \text{local} \tag{2.5.118}$$

where k is a constant and 'local' represents the a set of terms built out of products of derivatives of $\phi_0(v)$ that we will return to below. As in $d = 3$, $L(v) = \mathcal{O}(\epsilon^3)$, so the first term in (2.5.118) does not contribute at lowest order of the amplitude expansion of interest to this chapter. The local terms in this equation (2.5.118) are easily worked out at lowest order, $\mathcal{O}(\epsilon^2)$, in the amplitude expansion, and we find $M(v) = C_2(v) + \mathcal{O}(\epsilon^4)$ with

$$C_2(v) = -\frac{2^{d-2}}{(d-2)}\left(\frac{(\frac{d-1}{2})!}{(d-1)!}\right)^2 \int_{-\infty}^{v} dt\left[\left(\partial_t^{\frac{d+3}{2}}\phi_0\right)\left(\partial_t^{\frac{d-1}{2}}\phi_0\right) - \frac{d-3}{d-1}\left(\partial_t^{\frac{d+1}{2}}\phi_0\right)^2\right] \tag{2.5.119}$$

$$C_2 = \frac{2^{d-1}}{(d-1)}\left(\frac{(\frac{d-1}{2})!}{(d-1)!}\right)^2 \int_{-\infty}^{\infty} dt\left(\partial_t^{\frac{d+1}{2}}\phi_0(t)\right)^2 \sim \frac{\epsilon^2}{(\delta t)^d}, \tag{2.5.120}$$

the generalization of (2.1.20) and (2.1.21) to arbitrary odd d. (2.5.120) gives the leading order expression for the mass of the black brane that is eventually formed at the end of the thermalization process.

Let us now turn to the naive amplitude expansion in arbitrary odd d. The first term in this expansion, ϕ_1 is easily determined and we find

$$\phi_1(r,v) = \sum_{k=0}^{\frac{d-1}{2}} \frac{2^k}{k!}\frac{\left(\frac{d-1}{2}\right)!}{(d-1)!}\frac{(d-1-k)!}{(\frac{d-1-2k}{2})!}\frac{\partial_v^k\phi_0}{r^k} \tag{2.5.121}$$

Equations (2.1.12) then immediately determine f_2 and g_2. Turning to higher orders, it is possible to demonstrate that

- 1. The functions ϕ_{2n+1}, g_{2n} and f_{2n} have the following analytic structure in the variable r

$$\begin{aligned}
\phi_{2n+1}(r,v) &= \sum_{k=0}^{\frac{(2n+1)(d-1)}{2}-p(n)} \frac{\phi^k_{2n+1}(v)}{r^{\frac{(2n+1)(d-1)}{2}-k}} \\
f_{2n}(r,v) &= r\left(\sum_{k=0}^{n(d-1)-f(n)} \frac{f^k_{2n}(v)}{r^{n(d-1)-k}}\right) \\
g_{2n}(r,v) &= -\frac{C_{2n}(v)}{r^{d-2}} + r\left(\sum_{k=0}^{n(d-1)-g(n)} \frac{g^k_{2n}(v)}{r^{n(d-1)-k}}\right)
\end{aligned} \tag{2.5.122}$$

where

$$p(n) = d, \quad (2n+1 \geq d), \quad p(n) = 2n+1 \quad (2n+1 \leq d),$$

$$f(n) = d, \quad (2n \geq d), \quad f(n) = 2n \quad (2n \leq d),$$

$$g(n) = d-1, \quad (2n \geq d-1), \quad g(n) = 2n-1 \quad (2n \leq d).$$

- 2. The functions $\phi^k_{2n+1}(v)$, $f^k_{2n}(v)$ and $g^k_{2n}(v)$ are each functionals of $\phi_0(v)$ that scale like λ^{-k} under the scaling $v \to \lambda v$.

- 3. For $v > \delta t$ $f_2 = f_4 = 0$, $g_2 = -\frac{C_2}{r^{d-2}}$ and $g_4 = \frac{-C_4}{r^{d-2}}$. Further, effectively, $p(n) = d$, $f(n) = 2d$ and $g(n) = 2d-1$ for $v > \delta t$ (all additional terms present in (2.5.122) vanish at these late times). Moreover the functions $\phi^k_{2n+1}(v)$, $f^k_{2n}(v)$ and $g^k_{2n}(v)$ are all polynomials in v whose degrees are bounded from above by $n+k-1$, $n+k-3$ and $n+k-4$ respectively.

As in $d = 3$, the polynomial growth in v of the coefficients of the naive perturbative expansion invalidates this expansion for large enough v. More specifically, the sums over k and n in the expressions above are weighted by rv and $\frac{\epsilon^2 v}{r^{d-1}}$ respectively. In the neighborhood of the horizon, $r \sim r_H \sim T \sim \frac{\epsilon^{\frac{2}{d}}}{\delta t}$ each of these sums is effectively weighted by the factor vT. Consequently, naive perturbation theory fails at times large compared to the inverse temperature of the brane. At times of order δt and for $r \sim r_H$ the sum over k and n are each weighted effectively by $\epsilon^{\frac{2}{d}}$. More generally, naive perturbation theory is good at times of order δt provided $r\delta t \gg \epsilon^{\frac{2}{d-1}}$, a condition that is satisfied at the event horizon.

As in $d = 3$ the IR divergence of the naive perturbation expansion has a simple explanation. Even within the validity of the naive perturbation expansion, the spacetime is not well approximated by empty AdS space, but rather by the Vaidya metric (0.1.1). The naive expansion, which may be carried out with comparative ease up to $v = \delta t$, may be used to supply initial conditions for the subsequent unforced normalizable evolution for resummed perturbation theory. For $v \geq \delta t$, the spacetime metric is given, to leading order, by the Vaidya form (0.1.1), with $C_2(v)$ given by the constant C_2 listed in (2.5.120)

Consequently, the spacetime metric for $v \geq \delta t$ is the black brane metric with temperature of order $\frac{\epsilon^{\frac{2}{d}}}{\delta t}$, perturbed by a propagating ϕ field and consequent spacetime ripples. The initial conditions at $v = \delta t$, that determine these perturbations at later times, are given to leading order in ϵ (read off from the most small r singular term in ϕ_3) as

$$\phi(r, v) = \frac{A}{r^{\frac{3(d-1)}{2}}}$$

where

$$A = \frac{(d-1)^2}{2(d-2)} \int_{-\infty}^{\infty} dt \Bigg[(d-2) \left(2^{\frac{d-1}{2}} \frac{\left(\frac{d-1}{2}\right)!}{(d-1)!} \right) C_2(t) \left(\partial_t^{\frac{d-1}{2}} \phi_0 \right) - \left(2^{\frac{d-1}{2}} \frac{\left(\frac{d-1}{2}\right)!}{(d-1)!} \right)^3 \left(\partial_t^{\frac{d-1}{2}} \phi_0 \right)^2 \left(\partial_t^{\frac{d+1}{2}} \phi_0 \right) \Bigg] \tag{2.5.123}$$

In terms of the normalized variable $x = \frac{r}{M^{\frac{1}{d}}}$ and $y = vM^{\frac{1}{d}}$ this initial condition takes the form

$$\phi(x) \sim \frac{\epsilon^{\frac{3}{d}}}{x^{\frac{3(d-1)}{2}}} \tag{2.5.124}$$

It follows that the solution at $v \geq \delta t$ is (in the appropriate x, y coordinates) an order $\epsilon^{\frac{3}{d}}$ perturbation about the uniform black brane. The coefficient of this perturbation is bounded for all y, and decays exponentially for large y over a time scale of order unity in that variable. The explicit form of the solution for ϕ, for $v > \delta t$, may be obtained in terms of a universal function, $\psi_d(x,y)$ as in § 2.1. The equation that we need to solve is

$$\partial_x \left(x^{d+1} \left(1 - \frac{1}{x^d}\right) \partial_x \psi_d \right) + 2x^{\frac{d-1}{2}} \partial_x \partial_y \left(x^{\frac{d-1}{2}} \psi_d \right) = 0 \qquad (2.5.125)$$

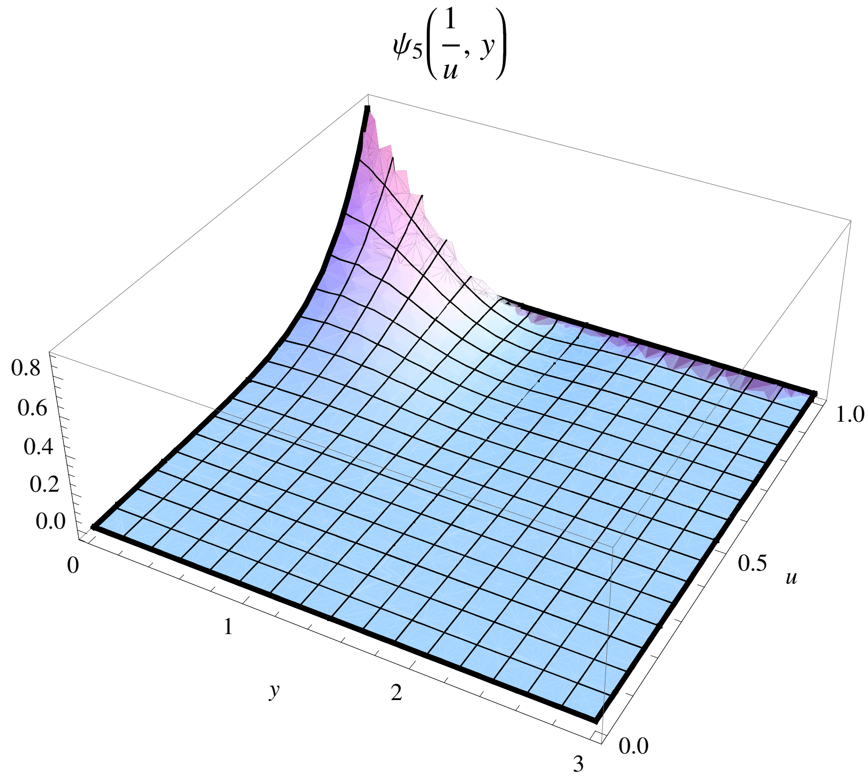

Figure 2.3: Numerical solution for the dilaton at late time in $d = 5$

As in § 2.1, this universal function appears to be difficult to obtain analytically, but is easily evaluated numerically. As an example in Figure 2.3 we present a numerical plot of this function in $d = 5$. As in § 2.1 we find it convenient to display the numerical output for the function $\psi_5(\frac{1}{x}, y)$ over the full exterior of the event horizon, $u \in (0, 1)$.

[11] In figure 2.4 we present a graph of $\psi_5(\frac{1}{0.7}, y)$ (i.e. as a function of time at a fixed radial location) Notice that this graph decays, roughly exponentially for $v > 0.5$ and that this exponential decay is dressed with a sinusodial osciallation, as expected for quasinormal

[11] In order to obtain this plot, as in 2.1, we worked with the redefined field $\chi_5(u, y) = (1 - u)\psi_5(\frac{1}{u}, y)$ and imposed Dirichlet boundary conditions on this field at $u = 0$ and $u = 0.999999$. We also imposed

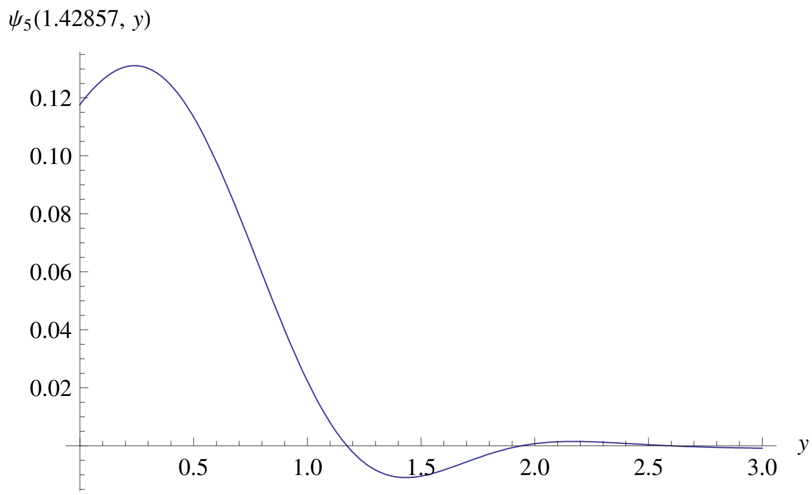

Figure 2.4: A plot of $\psi_5(\frac{1}{0.7}, y)$ as a function of y

type behavior. A very very rough estimate of this decay constant is provided by the equation ω_I using the equation $\frac{\psi_5(\frac{1}{0.7},1)}{\psi_5(\frac{1}{0.7},.5)} = e^{-0.5\omega_I}$ which gives $\omega_I \approx 8.2T$ (here T is the temperature of our black brane given by $T = \frac{4\pi}{5}$). This number is the same ballpark as the decay constants for the first quasi normal mode of the uniform black brane reported in [35] (unfortunately those authors have not reported the precise numerical value for $d = 5$) .

Even d

In our analyses above we have so far focused attention on odd d (recall that d is the spacetime dimension of the dual field theory). In this subsection we will study how our results generalize to even d. While all the broad qualitative conclusions of the odd d analysis plausibly continue to apply, several intermediate details are quite different.The analysis of all equations is more difficult in even than in odd dimensions. In this appendix we aim only to initiate a serious analysis of these equations, and to carry this analysis far

the initial conditions $\chi_5 = (0.999999 - u)u^6$. The Figure 2.3 was outputted by Mathematica-6's partial differential equation solver, with a step size of 0.0005 and an accuracy goal of 0.001.

enough to have a plausible guess for the behavior of our system. We leave a systematic analysis of these equations to future work.

The qualitative differences between even and odd d show themselves already in the Graham Fefferman expansion. We illustrate this by working out this expansion in $d = 4$. In this dimension the expansion of f, g, ϕ at large r take the form

$$\begin{aligned}
f(r,v) &= r - \frac{(\dot{\phi}_0)^2}{12r} - \frac{\ddot{\phi}_0\dot{\phi}_0}{36r^2} + \frac{-3(\dot{\phi}_0)^4 + 2\dddot{\phi}_0\dot{\phi}_0 - (\partial_v^2\phi_0)^2}{288r^3} \\
&\quad + \frac{-19\ddot{\phi}_0(\dot{\phi}_0)^3 - 1440L(v)\dot{\phi}_0 - 18\partial_v^4\phi_0\partial_v\phi_0 + 45\ddot{\phi}_0\dddot{\phi}_0}{21600r^4} \\
&\quad - \frac{\log(r)\dot{\phi}_0\left(\partial_v^4\phi_0 - 2(\dot{\phi}_0)^2\partial_v^2\phi_0\right)}{240r^4} + \dots \\
g(r,v) &= r^2 - \frac{5}{12}(\dot{\phi}_0)^2 - \frac{M(v)}{r^2} + \frac{\log(r)\left(-(\dot{\phi}_0)^4 + 2\dddot{\phi}_0\dot{\phi}_0 - (\partial_v^2\phi_0)^2\right)}{24r^2} + \dots \\
\phi(r,v) &= \phi_0 + \frac{\dot{\phi}_0}{r} + \frac{\partial_v^2\phi_0}{4r^2} + \frac{\frac{5}{36}(\dot{\phi}_0)^3 - \frac{1}{12}\dddot{\phi}_0}{r^3} + \frac{L}{r^4} + \dots \\
&\quad + \frac{\log(r)\left(\partial_v^4\phi_0 - 2(\dot{\phi}_0)^2\partial_v^2\phi_0\right)}{16r^4}
\end{aligned} \tag{2.5.126}$$

The energy conservation equation is

$$\dot{M} = \frac{1}{144}\left(40\ddot{\phi}_0(\dot{\phi}_0)^3 - 192L(v)\dot{\phi}_0 - 17\partial_v^4\phi_0\dot{\phi}_0 + 6\ddot{\phi}_0\dddot{\phi}_0\right) \tag{2.5.127}$$

and at quadratic order in ϵ we have

$$\begin{aligned}
M(v) &= C_2(v) + \mathcal{O}(\epsilon^4) \\
C_2(v) &= \frac{1}{144}\int_{-\infty}^{v} dt\left(-192L(t)\dot{\phi}_0 - 17\ddddot{\phi}_0\dot{\phi}_0 + 6\partial_t^2\phi_0\dddot{\phi}_0\right)
\end{aligned} \tag{2.5.128}$$

Unlike in even dimensions, it turns out that in odd dimensions $L(v)$ is nonzero at order ϵ. This is fortunate, as all the local terms in (2.5.128) are total derivatives, and so vanish when v is taken to be larger than δt. The full contribution to the mass of the black brane that is eventually formed from our collapse process arises from the term in (2.5.128) that is proportional to $L(v)$. As a consequence, the mass of the eventual black brane is not determined simply by Graham Fefferman analysis, but requires the details of the full dynamical process. These details may be worked out at lowest order in the ϵ expansion, (see below) and we will find

$$L(v) = \left(\frac{-7+12\log 2}{192}\right)\partial_v^4\phi_0 + \frac{1}{16}\int_{-\infty}^{v} dt\log(v-t)\ \partial_t^5\phi_0(t) + \mathcal{O}(\epsilon^3) \tag{2.5.129}$$

Plugging into (2.5.128) we find that $C_2(v)$ reduces to the constant C_2 for $v > \delta t$, and we have

$$C_2 = -\frac{1}{12}\int_{-\infty}^{\infty} dt_1 dt_2 \left(\partial_{t_1}^3\phi_0(t_1)\log(t_1-t_2)\Theta(t_1-t_2)\partial_{t_2}^3\phi_0(t_2)\right) \tag{2.5.130}$$

Let us now turn to the amplitude expansion of our solutions. We will work this expansion out only at leading order; already the leading order solution turns out to have qualitative differences (and to be much harder to determine and manipulate) than the corresponding solution in odd d.

Recall that ϕ_1 (2.5.121) is extremely simple when d was odd. To start with, the solution is local in time, i.e. $\phi_1(r, v_0)$ is completely determined by the value, and a finite number of derivatives, of $\phi_0(v_0)$. Relatedly $\phi(r, v)$ has a very simple analytic expression in r; it is a polynomial in $\frac{1}{r}$ of degree $\frac{d-1}{2}$. In even d, on the other hand the dependence of $\phi_1(r, v)$ on $\phi_0(v)$ is not local in time. Relatedly, the expansion of $\phi_1(r, v)$ in a power series in $\frac{1}{r}$ has terms of every order in $\frac{1}{r}$. Explicitly we find

$$\begin{aligned}
\phi_1(r,v) &= \int_0^{\infty} \partial_v^{d+1}\phi_0(v-t)\left(\frac{h(rt)}{r^d}\right)dt \\
h(x) &= \int_0^x dy \frac{(y(y+2))^{\frac{d-1}{2}}}{(d-1)!} \\
&= (-1)^{\frac{d}{2}}\binom{d}{\frac{d}{2}}\frac{\theta}{2^d} + \frac{1}{2^{d-1}}\sum_{k=0}^{\frac{d}{2}-1}\frac{(-1)^k}{d-2k}\binom{d}{k}\sinh\left((d-2k)\theta\right) \\
&\quad \text{where}\quad \cosh\theta = 1+x
\end{aligned} \tag{2.5.131}$$

Note that the function $h(x)$ admits the following large x expansion

$$
\begin{aligned}
h(x) = {} & \frac{x^d}{(d-1)!} + \sum_{k=1}^{d-1} \frac{x^{d-k}}{(d-k)k!(d-1)!}\left(\prod_{m=1}^{k}(d-2m-1)\right) \\
& + \frac{(-1)^{\frac{d}{2}+1}(d)!}{(d-1)!2^d((\frac{d}{2})!)^2}\left(\sum_{p=0}^{\frac{d}{2}-1}\frac{1}{(d-2p)(d-2p-1)}\right) + \frac{(-1)^{\frac{d}{2}}(d)}{2^d\left(\frac{d}{2}!\right)^2}\ln(2x) + \mathcal{O}(\frac{\ln x}{x})
\end{aligned}
\tag{2.5.132}
$$

The fact that $h(x)$ grows (rather than decays) with x may cause the reader to worry that $\phi_(r,v)$ blows up at large v. That this is not the case may be seen by noting that $v^k\partial_v^{d+1}\phi_0$ may be rewritten as a sum of total derivatives when $k \leq d+1$ and so integrates to zero when $v > \delta t$ (in general it integrates to a simple local expression even for $v < \delta t$). Explicitly, plugging (2.5.132) into (2.5.131) and integrating by parts we find that $\phi_1(r,v)$ has the following large rt behavior

$$
\begin{aligned}
\phi_1(r,v) &= \sum_{i=0}^{d}\frac{A_i(v)}{r^i} + \frac{B(v)\ln(r)}{r^d} + \mathcal{O}(\frac{\ln r}{r^{d+1}}) \\
&= \phi_0(v) \\
&+ \sum_{k=1}^{d-1}\frac{\partial_v^k\phi_0(v)}{r^k}\left[\frac{(d-k-1)!}{k!(d-1)!}\left(\prod_{m=1}^{k}(d-2m-1)\right)\right] \\
&+ \frac{\partial_v^d\phi_0(v)}{r^d}\left[\frac{(-1)^{\frac{d}{2}+1}(d)!}{(d-1)!2^d((\frac{d}{2})!)^2}\left(\sum_{p=0}^{\frac{d}{2}-1}\frac{1}{(d-2p)(d-2p-1)}\right)\right] \\
&+ \int_0^\infty dt\frac{\partial_v^{d+1}\phi_0(v-t)}{r^d}\ln(2rt)\left[\frac{(-1)^{\frac{d}{2}}(d)}{2^d\left(\frac{d}{2}!\right)^2}\right] \\
&+ \mathcal{O}(\frac{\ln(r)}{r}^{d+1})
\end{aligned}
\tag{2.5.133}
$$

(where the functions $A_i(v)$ and $B(v)$ are defined by this equation). On the other hand at small x we have

$$
h(x) = \frac{(2x)^{\frac{d+1}{2}}}{(d+1)(d-1)!}\left(1+\mathcal{O}(x)\right) \tag{2.5.134}
$$

from which it follows that

$$\phi_1(r,v) = \frac{1}{r^{\frac{d-1}{2}}}\frac{1}{(d+1)(d-1)!}\int_{-\infty}^{v} dt(2(v-t))^{\frac{d+1}{2}}\partial_t^{d+1}\phi_0(t) + \mathcal{O}(\frac{1}{r^{\frac{d-3}{2}}}), \qquad (2.5.135)$$

an expression that is valid at small rv. Note, in particular, that for $\delta t \ll v$, (2.5.135) reduces to

$$\phi_1(r,v) = \frac{2^{\frac{d+1}{2}}\int_0^{\delta t}\phi_0(t)dt}{r^{\frac{d-1}{2}}v^{\frac{d+1}{2}}}\frac{1}{(d+1)(d-1)!} + \mathcal{O}(\frac{1}{r^{\frac{d-3}{2}}}) + \mathcal{O}(\frac{1}{t^{\frac{d+3}{2}}}) \qquad (2.5.136)$$

In particular this formula determines the behavior of the field ϕ_1 in the neighborhood of the event horizon $r_H \sim T$ for times that are large compared to δt but small compared to T^{-1}.

The functions f_2 and g_2 are easily expressed in terms of the function ϕ_0. We find

$$\begin{aligned} f_2(r,v) &= -\frac{1}{2(d-1)}\left[r\int_r^{\infty}(\partial_\rho\phi_1)^2 d\rho - \int_r^{\infty}\rho^2(\partial_\rho\phi_1)^2 d\rho\right] \\ g_2(r,v) &= -\big(2\partial_v f_2(r,v) + (d-2)r f_2(r,v) + r^2\partial_r f_2(r,v)\big) \\ &\quad + \frac{d(d-1)}{r^{d-2}}\int_0^r \rho^{d-2} f_2(\rho,v)d\rho - \frac{D_2(v)}{r^{d-2}} \end{aligned} \qquad (2.5.137)$$

The function $D_2(v)$ is determined by the requirement that the coefficient of $\frac{1}{r^{d-2}}$, in the large r expansion of $g_2(r,v)$ is $-C_2(v)$ (see (2.5.128)); in particular, for $v > \delta t$, $D_2(v) = C_2(v)$. At small r and for $v > \delta t$

$$\begin{aligned} f_2(r,v) &= -\frac{K^2(v)}{2(d-1)(d-2)(d-3)r^{d-2}} + \mathcal{O}(\frac{1}{r^{d-3}}) \\ g_2(r,v) &= -\frac{C_2}{r^{d-2}} + \frac{\partial_v K^2(v)}{(d-1)(d-2)(d-3)r^{d-2}} + \mathcal{O}(\frac{1}{r^{d-3}}) \\ K(v) &= \frac{1}{(d+1)(d-1)!}\int_{-\infty}^{v} dt(2(v-t))^{\frac{d+1}{2}}\partial_t^{d+1}\phi_0(t) \\ &\approx \frac{2^{\frac{d+1}{2}}\int_0^{\delta t}\phi_0(t)dt}{v^{\frac{d+1}{2}}}\frac{1}{(d+1)(d-1)!} \quad (v \gg \delta t) \end{aligned} \qquad (2.5.138)$$

We would like to draw attention to several aspects of these results. First note that $\phi_1(r,v)$ is small provided $(r\delta t)^{\frac{d-1}{2}} \gg \epsilon$. Consequently, we expect a perturbative analysis to correctly capture the dynamics of our situation over this range of coordinates; note that this is exactly the same estimate as for odd d. Next note that the maximal singularity, at small r, in the functions f_2 and g_2, are both of order $\frac{1}{r^{d-2}}$; this is the same as the maximal singularity in the analogous functions in odd d (see the previous subsection). As the function $g_0(r,v) = r^2$, it follows, as in the previous function, that our spacetime metric is not uniformly well approximated by the empty AdS space over the full range of validity of perturbation theory. Over this entire range, however, it is well approximated by a Vaidya type metric, where the mass function for this metric is given at leading order by the coefficient of $-\frac{1}{r^{d-2}}$ in $g_2(r,v)$ above.

Unlike the situation in odd dimensions, the leading order mass function $M(v)$, in the effective Vaidya metric, is not given simply by $C_2(v)$. In particular, when $v \gg \delta t$ we have from (2.5.138) that

$$\frac{C_2 - M(v)}{C_2} \sim \left(\frac{\delta t}{v}\right)^{d+2}.$$

In other words, the leading order metric for the thermalization process, in even d, is not given precisely by the metric of the uniform black brane for $v > \delta t$. However it decays, in a power law fashion, to the black brane metric at times larger than δt. As a consequence at times $\delta t \ll v \ll T^{-1}$ the leading order metric that captures the thermalization process is arbitrarily well approximated by the metric of a uniform black brane. It follows that, while the spacetime described in this subsection does not capture the dual of instantaneous field theory thermalization (as was the case in odd d), it yields the dual of a thermalization process that occurs over the time scale of the forcing function rather than the much longer linear response time scale of the inverse temperature.

We will not, in this chapter, continue the perturbative expansion to higher orders in ϵ. We suspect, however, that the computation of ϕ_3 when carried through will yield a term proportional to $\frac{\epsilon^3}{r^{\frac{3(d-1)}{2}}}$ that is constant in time. This term will dominate the decaying tail of $\phi_1(r,v)$ at a time intermediate between δt and T^{-1} and will set the initial condition for the late time decay of the ϕ field (over time scale T^{-1}) as was the case in odd dimensions.

It would be very interesting to verify or correct this guess.

2.5.2 Spherically Symmetric flat space collapse in arbitrary dimension

Odd d

The discussion of § 2.2 also extends to the study of spherically symmetric collapse in a space that is asymptotically flat $R^{d,1}$ for arbitrary odd d. In this section we will very briefly explain how this works, focussing on the limit $y = \frac{r_H}{\delta t} \gg 1$.

To lowest order in the amplitude expansion we find

$$\phi_1(r,v) = \sum_m^{\frac{d-3}{2}} 2^{\frac{d-3}{2}-m} \frac{(-1)^m}{m!} \frac{\left(\frac{d-3}{2}+m\right)!}{\left(\frac{d-3}{2}-m\right)!} \frac{\partial_v^{\frac{d-3}{2}-m}\psi(v)}{r^{\frac{d-1}{2}+m}} \tag{2.5.139}$$

Here $\psi(v)$ is a function of time that we take, as usual, to vanish outside $v \in (0, \delta t)$, and be of order $\epsilon_f (\delta t)^{\frac{d-1}{2}}$, where ϵ_f is a dimensionless number such that $\epsilon_f \gg 1$. As in § 2.2 the parameter that will justify the amplitude expansion will be $\frac{1}{\epsilon_f}$.

(2.5.139) together with constraint equations immediately yields an expression for the functions f_2 and g_2. In particular, the leading large r approximation to g_2 is given by

$$\begin{aligned} g_2(r,v) &= -\frac{M(v)}{r^{d-2}} \\ M(v) &= -\frac{2^{(d-4)}}{d-1} \int_{-\infty}^{v} dt \left[\left(\partial_t^{\frac{(d-3)}{2}} \psi(t)\right) \left(\partial_t^{\frac{(d+1)}{2}} \psi(t)\right) - \frac{d-3}{d-2} \left(\partial_t^{\frac{(d-1)}{2}} \psi(t)\right)^2 \right] \end{aligned} \tag{2.5.140}$$

Note that $\phi_1 \ll 1$ whenever $r^{\frac{d-1}{2}} \ll (\delta t)^{\frac{d-1}{2}} \epsilon_f$ so we expect the amplitude expansion to reliably describe dynamics over this range of parameters. As in § 2.2, however, g_2 cannot be ignored in comparison to $g_0 = 1$ throughout this parameter regime. As in § 2.2, this implies that our spacetime is well approximated by a Vaidya type metric rather than empty flat space even at arbitrarily small $\frac{1}{\epsilon_f}$. The mass function of this Vaidya metric is given by $M(v)$ in (2.5.140).

As in § 2.2 one may ignore this complication at early times $v \ll r_H$ over which the solution is well approximated by a naive perturbation expansion that uses empty flat space as its starting point. It is possible to demonstrate that this naive expansion has the following analytic structure in the variables r and v

- 1. The functions Φ_{2n+1}, F_{2n} and G_{2n} have the following analytic structure in the variable r

$$\begin{aligned}
\Phi_{2n+1}(r,v) &= \sum_{m=0}^{\infty} \frac{\Phi_{2n+1}^m(v)}{r^{(2n+1)\frac{d-1}{2}+m}} \\
F_{2n}(r,v) &= r\sum_{m=0}^{\infty} \frac{F_{2n}^m(v)}{r^{n(d-1)+m}} \\
G_{2n}(r,v) &= -\delta_{n,1}\frac{M(v)}{r^{d-2}} + r\sum_{m=0}^{\infty} \frac{G_{2n}^m(v)}{r^{n(d-1)+m}}
\end{aligned} \tag{2.5.141}$$

- 2. The functions $\Phi_{2n+1}^m(v)$, $F_{2n}^m(v)$ and $G_{2n}^m(v)$ are each functionals of $\psi(v)$ that scale like $\lambda^{m-(2n+1)\frac{d-3}{2}}$ $\lambda^{m-n(d-3)}$ and $\lambda^{m-n(d-3)-1}$ under the the scaling $v \to \lambda v$. $M(v)$ scales like λ^{2-d} under the same scaling.

- 3. For $v > \delta t$ the $\Phi_{2n+1}^m(v)$ is polynomials in v of degree $\leq n+m-1$; $F_{2n}^m(v)$ and G_{2n}^m are polynomials in v of degree $\leq n+m-3$ and $n+m-4$ respectively.

It follows that, say, $\phi(r,v)$, is given by a double sum

$$\phi(r,v) = \sum_n \Phi_{2n+1}(r,v) = \sum_{n,m=0}^{\infty} \frac{\Phi_{2n+1}^m(v)}{r^{(2n+1)\frac{d-1}{2}+m}}.$$

Now sums over m and n are controlled by the effective expansion parameters $\sim \frac{v}{r}$ (for m) and $\frac{\psi^2 v}{(\delta t)^{d-2} r^{d-1}} \sim \frac{v}{\delta t \epsilon_f^{\frac{2}{d-2}}} \sim \frac{v}{r_H}$ (for n; recall that in the neighborhood of the horizon $r_H^{d-2} \sim (\delta t)^{d-2}\epsilon_f^2$).

As in § 2.2, it follows that the naive perturbation expansion breaks down for times $v \gg r_H$. However this expansion is valid everwhere outside the event horizon at times of order δt, and so may be used to set the initial conditions for a resummed perturbation expansion that uses the Vaidya metric as its starting point. For $v > \delta t$ the mass function of the Vaidya metric reduces to a constant. At long times our solution is given by a small perturbation around a black hole of mass M. This perturbation is best analyzed in the coordinates $x = \frac{r}{M^{\frac{1}{d-2}}}$ and $y = \frac{v}{M^{\frac{1}{d-2}}}$. In these coordinates the leading order tail of ϕ, at long times, is given by motion about a black hole of unit Schwarzschild radius perturbed by the ϕ field with initial condition

$$\phi(x,0) = \frac{\phi_3^0(\delta t)}{M^{\frac{3(d-1)}{2(d-2)}} x^{\frac{3(d-1)}{2}}} \sim \frac{1}{\epsilon_f^{\frac{3}{d-2}}}$$

The smallness of this perturbation justifies linearized treatment of the subsequent dynamics.

Even d

We will not, in this chapter, attempt an analysis of the spherically symmetric collapse to form a black hole asymptotically $R^{d,1}$ for even d. Here we simply note that the leading order large ϵ_f solution for $\phi_1(v)$ may formally be expressed as

$$\phi_1(r,v) = \int d\omega \left(q(\omega) e^{i\omega(v-r)} \frac{H^{(1)}_{\frac{d-2}{2}}(r\omega)}{r^{\frac{d-2}{2}}} \right) \tag{2.5.142}$$

for any function $q(\omega)$ where $H_n(x)$ is the n^{th} Hankel function of the first kind, i.e.

$$H_n^{(1)}(x) \approx \sqrt{\frac{2}{\pi x}} \left(e^{i(x - \frac{\pi}{4} - \frac{n\pi}{2})} + \mathcal{O}(\frac{1}{x}) \right)$$

Using this expansion, it is easily verified that $\phi_1(r,v)$ reduces, at large r, to an incoming wave that takes the form $\frac{\psi(v)}{r^{\frac{d-1}{2}}}$. The evolution of this wave to small r is implicitly given by (2.5.142). It should be possible to mimic the analysis of subsubsection 2.5.1 to explicitly express $\phi_1(r,v)$ as a spacetime dependent Kernel function convoluted against $\psi(v)$. In analogy with subsection 2.5.1 it should also be possible to expand $g_2(r,v)$ about small r. It is tempting to guess that such an analysis would reveal that the leading singularity in $g_2(r,v)$ scales like $\frac{1}{r^{d-2}}$, so that the metric is well approximated by a spacetime of the Vaidya form. We leave the verification of these guesses to future work.

2.5.3 Spherically symmetric asymptotically AdS collapse in arbitrary dimension

It should be straightforward to generalize the analysis of § 2.3 to arbitrary odd d, and perhaps also to arbitrary even d. We do not explicitly carry out this generalization in this chapter. However it is a simple matter to infer the various scales that will appear in this generalization using the intuition and results of subsections 2.5.1 and 2.5.2, and the fact that the results of global spherically symmetric AdS collapse must reduce to Poincare patch collapse in one limit and flat space collapse in another. We have reported these scales in the introductionto § 2.3.

Chapter 3

Fluid dynamics - Gravity correspondence

This chapter is based on [2].

In this chapter we describe an unforced system which is already locally equilibrated and is evolving towards global equilibrium. As described in the introduction this relaxation process happens on length and time scales that are both large compared to the inverse local temperature and so admits an effective description in terms of fluid dynamics. Therefore on the the dual gravitational picture, once the black hole is formed, Einstein's equations should reduce to the nonlinear equations of fluid dynamics in an appropriate regime of parameters.

In this chapter we provide a systematic framework to construct this universal gravity dual to the nonlinear fluid dynamics, order by order in a boundary derivative expansion.

From here onwards we will set $d = 4$ i.e. we will consider only asymptotically AdS_5 spaces.

3.1 Fluid dynamics from gravity

We begin with a description of the procedure we use to construct a map from solutions of fluid dynamics to solutions of gravity. We then summarize the results obtained by implementing this procedure to second order in the derivative expansion.

Consider a theory of pure gravity with a negative cosmological constant. With a particular choice of units ($R_{AdS} = 1$) Einstein's equations are given by[1]

$$\begin{aligned} E_{MN} &= R_{MN} - \frac{1}{2} g_{MN} R - 6\, g_{MN} = 0 \\ \Longrightarrow R_{MN} &+ 4\, g_{MN} = 0, \qquad R = -20. \end{aligned} \tag{3.1.1}$$

Of course the equations (3.1.1) admit AdS_5 solutions. Another class of solutions to these equations is given by the 'boosted black branes' [2]

$$ds^2 = -2\, u_\mu\, dx^\mu dr - r^2\, f(b\, r)\, u_\mu\, u_\nu\, dx^\mu dx^\nu + r^2\, P_{\mu\nu}\, dx^\mu dx^\nu\ , \tag{3.1.2}$$

with

$$\begin{aligned} f(r) &= 1 - \frac{1}{r^4} \\ u^v &= \frac{1}{\sqrt{1-\beta^2}} \\ u^i &= \frac{\beta_i}{\sqrt{1-\beta^2}}\ , \end{aligned} \tag{3.1.3}$$

where the temperature $T = \frac{1}{\pi\, b}$ and velocities β_i are all constants with $\beta^2 = \beta_j\, \beta^j$, and

$$P^{\mu\nu} = u^\mu u^\nu + \eta^{\mu\nu} \tag{3.1.4}$$

is the projector onto spatial directions. The metrics (3.1.2) describe the uniform black brane written in ingoing Eddington-Finkelstein coordinates, at temperature T, moving at

[1] We use upper case Latin indices $\{M, N, \cdots\}$ to denote bulk directions, while lower case Greek indices $\{\mu, \nu, \cdots\}$ refer to field theory or boundary directions. Finally, we use lower case Latin indices $\{i, j, \cdots\}$ to denote the spatial directions in the boundary.

[2] The indices in the boundary are raised and lowered with the Minkowski metric ie. $u_\mu = \eta_{\mu\nu}\, u^\nu$.

velocity β^i.[3]

Now consider the metric (3.1.2) with the constant parameter b and the velocities β_i replaced by slowly varying functions $b(x^\mu), \beta_i(x^\mu)$ of the boundary coordinates.

$$ds^2 = -2\, u_\mu(x^\alpha)\, dx^\mu\, dr - r^2\, f\left(b(x^\alpha)\, r\right)\, u_\mu(x^\alpha)\, u_\nu(x^\alpha)\, dx^\mu\, dx^\nu + r^2\, P_{\mu\nu}(x^\alpha)\, dx^\mu\, dx^\nu\ . \tag{3.1.6}$$

Generically, such a metric (we will denote it by $g^{(0)}(b(x^\mu), \beta_i(x^\mu))$ is not a solution to Einstein's equations. Nevertheless it has two attractive features. Firstly, away from $r = 0$, this deformed metric is everywhere non-singular. This pleasant feature is tied to our use of Eddington-Finkelstein[4] coordinates. [5] Secondly, if all derivatives of the parameters $b(x^\mu)$ and $\beta_i(x^\mu)$ are small, $g^{(0)}$ is tubewise[6] well approximated by a boosted black brane. Consequently, for slowly varying functions $b(x^\mu)$, $\beta_i(x^\mu)$, it might seem intuitively plausible that (3.1.6) is a good approximation to a true solution of Einstein's equations with a regular event horizon. The main result of our chapter is that this intuition is correct, provided the functions $b(x^\mu)$ and $\beta_i(x^\mu)$ obey a set of equations of motion, which turn out simply to be the equations of boundary fluid dynamics.

Einstein's equations, when evaluated on the metric $g^{(0)}$, yield terms of first and second order in field theory (ie. $(x_i, v) \equiv x^\mu$) derivatives of the temperature and velocity fields.[7]

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[3] As we have explained above, the 4 parameter set of metrics (3.1.2) may all be obtained from

$$ds^2 = 2\, dv\, dr - r^2\, f(r)\, dv^2 + r^2\, d\mathbf{x}^2\ , \tag{3.1.5}$$

with $f = 1 - \frac{1}{r^4}$ via a coordinate transform. The coordinate transformations in question are generated by a subalgebra of the isometry group of AdS_5.

[4] It is perhaps better to call these generalized Gaussian null coordinates as they are constructed with the aim of having the putative horizon located at the hypersurface $r(x^\mu) = r_h$.

[5] A similar ansatz for a black branes in (for instance) Fefferman-Graham coordinates ie. Schwarzschild like coordinates respecting Poincare symmetry, is singular at $r\, b = 1$.

[6] As explained above, any given tube consists of all values of r well separated from $r = 0$, but only a small region of the boundary coordinates x^μ.

[7] As $g^{(0)}$ is an exact solution to Einstein's equations when these fields are constants, terms with no derivatives are absent from this expansion.

By performing a scaling of coordinates to set b to unity (in a local patch), it is possible to show that field theory derivatives of either $\ln b(x^\mu)$ or $\beta_i(x^\mu)$ always appear together with a factor of b. As a result, the contribution of n derivative terms to the Einstein's equations is suppressed (relative to terms with no derivatives) by a factor of $(b/L)^n \sim 1/(T\,L)^n$. Here L is the length scale of variations of the temperature and velocity fields in the neighbourhood of a particular point, and T is the temperature at that point. Therefore, provided $L\,T \gg 1$, it is sensible to solve Einstein's equations perturbatively in the number of field theory derivatives.[8]

In 3.2 we formulate the perturbation theory described in the previous paragraph, and explicitly implement this expansion to second order in $1/(L\,T)$. As we have mentioned above it turns out to be possible to find a gravity solution dual to a boundary velocity and temperature profile only when these fields obey the equation of motion

$$\partial_\mu T^{\mu\nu} = 0 \tag{3.1.7}$$

where the rescaled[9] stress tensor $T^{\mu\nu}$ (to second order in derivatives) is given by

$$\begin{aligned} T^{\mu\nu} =& (\pi\,T)^4\,(\eta^{\mu\nu} + 4\,u^\mu u^\nu) - 2\,(\pi\,T)^3\,\sigma^{\mu\nu} \\ &+ (\pi T)^2\left((\ln 2)\;T^{\mu\nu}_{2a} + 2\,T^{\mu\nu}_{2b} + (2-\ln 2)\left[\frac{1}{3}\,T^{\mu\nu}_{2c} + T^{\mu\nu}_{2d} + T^{\mu\nu}_{2e}\right]\right) \end{aligned} \tag{3.1.8}$$

[8]Note that the variation in the radial direction, r, is never slow. Although we work order by order in the field theory derivatives, we will always solve all differential equations in the r direction exactly.

[9]Throughout this chapter $T^{\mu\nu} = 16\pi\,G_5\,t^{\mu\nu}$ where G_5 is the five dimensional Newton and $t^{\mu\nu}$ is the conventionally defined stress tensor, ie. the charge conjugate to translations of the coordinate v.

where

$$
\begin{aligned}
\sigma^{\mu\nu} &= P^{\mu\alpha}P^{\nu\beta}\;\partial_{(\alpha}u_{\beta)} - \frac{1}{3}\,P^{\mu\nu}\,\partial_\alpha u^\alpha \\
T^{\mu\nu}_{2a} &= \epsilon^{\alpha\beta\gamma(\mu}\,\sigma^{\nu)}{}_{\gamma}\,u_\alpha\,\ell_\beta \\
T^{\mu\nu}_{2b} &= \sigma^{\mu\alpha}\sigma^{\nu}{}_{\alpha} - \frac{1}{3}\,P^{\mu\nu}\,\sigma^{\alpha\beta}\sigma_{\alpha\beta} \\
T^{\mu\nu}_{2c} &= \partial_\alpha u^\alpha\,\sigma^{\mu\nu} \\
T^{\mu\nu}_{2d} &= \mathcal{D}u^\mu\,\mathcal{D}u^\nu - \frac{1}{3}\,P^{\mu\nu}\,\mathcal{D}u^\alpha\,\mathcal{D}u_\alpha \\
T^{\mu\nu}_{2e} &= P^{\mu\alpha}\,P^{\nu\beta}\,\mathcal{D}\left(\partial_{(\alpha}u_{\beta)}\right) - \frac{1}{3}\,P^{\mu\nu}\,P^{\alpha\beta}\,\mathcal{D}\left(\partial_\alpha u_\beta\right) \\
\ell_\mu &= \epsilon_{\alpha\beta\gamma\mu}\,u^\alpha\partial^\beta u^\gamma.
\end{aligned}
\tag{3.1.9}
$$

Our conventions are $\epsilon_{0123} = -\epsilon^{0123} = 1$ and $\mathcal{D} \equiv u^\alpha\partial_\alpha$ and the brackets () around the indices to denote symmetrization, ie. $a^{(\alpha}b^{\beta)} = (a^\alpha b^\beta + a^\beta b^\alpha)/2$.

These constraints are simply the equations of fluid dynamics expanded to second order in the derivative expansion. The first few terms in the expansion (3.1.8) are familiar. The derivative free terms describe a perfect fluid with pressure (ie. negative free energy density) $\pi^4\,T^4$, and so (via thermodynamics) entropy density $s = 4\pi^4\,T^3$. The viscosity η of this fluid may be read off from the coefficient of $\sigma^{\mu\nu}$ and is given by $\pi^3\,T^3$. Notice that $\eta/s = 1/(4\pi)$, in agreement with the famous result of Policastro, Son and Starinets [4].

Our computation of the two derivative terms in (3.1.8) is new; the coefficients of these terms are presumably related to the various 'relaxation times' discussed in the literature (see for instance [37]). As promised earlier, the fact that we are dealing with a particular conformal fluid, one that is dual to gravitational dynamics in asymptotically AdS spacetimes, leads to the coefficients being determined as fixed numbers.

In 3.5.1 we have checked that the minimal covariantization of the stress tensor (3.1.8) transforms as $T^{\mu\nu} \to e^{-6\phi}\,T^{\mu\nu}$ under the Weyl transformation $\eta_{\mu\nu} \to e^{2\phi}\,\eta_{\mu\nu}$, $T \to e^{-\phi}\,T$, $u^\alpha \to e^{-\phi}\,u^\alpha$, for an arbitrary function $\phi(x^\mu)$. Note that we have computed the fluid dynamical stress tensor only in flat space. The generalization of our expression above to an arbitrary curved space could well include contributions proportional to the spacetime curvature tensor. The fact that (3.1.8) is Weyl invariant by itself is a bit of a (pleasant) surprise. It implies that that the sum of all curvature dependent contributions to the stress tensor must be independently Weyl invariant.

3.2 The perturbative expansion

As we have described in 3.1, our goal is to set up a perturbative procedure to solve Einstein's equations in asymptotically AdS spacetimes order by order in a boundary derivative expansion. In this section we will explain the structure of this perturbative expansion, and outline our implementation of this expansion to second order, leaving the details of computation to future sections.

3.2.1 The basic set up

In order to mathematically implement our perturbation theory, it is useful to regard b and β_i described in 3.1 as functions of the rescaled field theory coordinates $\epsilon\, x^\mu$ where ϵ is a formal parameter that will eventually be set to unity. Notice that every derivative of β_i or b produces a power of ϵ, consequently powers of ϵ count the number of derivatives. We now describe a procedure to solve Einstein's equations in a power series in ϵ. Consider the metric[10]

$$g = g^{(0)}(\beta_i, b) + \epsilon\, g^{(1)}(\beta_i, b) + \epsilon^2\, g^{(2)}(\beta_i, b) + \mathcal{O}\left(\epsilon^3\right), \tag{3.2.10}$$

where $g^{(0)}$ is the metric (3.1.6) and $g^{(1)}, g^{(2)}$ etc are correction metrics that are yet to be determined. As we will explain below, perturbative solutions to the gravitational equations exist only when the velocity and temperature fields obey certain equations of motion. These equations are corrected order by order in the ϵ expansion; this forces us to correct the velocity and temperature fields themselves, order by order in this expansion.

[10]For convenience of notation we are dropping the spacetime indices in $g^{(n)}$. We also suppress the dependence of b and β_i on x^μ.

Consequently we set

$$\beta_i = \beta_i^{(0)} + \epsilon\,\beta_i^{(1)} + \mathcal{O}\left(\epsilon^2\right), \qquad b = b^{(0)} + \epsilon\, b^{(1)} + \mathcal{O}\left(\epsilon^2\right), \tag{3.2.11}$$

where $\beta_i^{(m)}$ and $b^{(n)}$ are all functions of $\epsilon\, x^\mu$.

In order to proceed with the calculation, it will be useful to fix a gauge. We work with the 'background field' gauge

$$g_{rr} = 0\ , \qquad g_{r\mu} \propto u_\mu\ , \qquad \mathrm{Tr}\left((g^{(0)})^{-1} g^{(n)}\right) = 0 \quad \forall\, n > 0. \tag{3.2.12}$$

Notice that the gauge condition at the point x^μ is given only once we know $u_\mu(v, x^i)$. In other words, the choice above amounts to choosing different gauges for different solutions, and is conceptually similar to the background field gauge routinely used in effective action computations for non abelian gauge theories.

3.2.2 General structure of perturbation theory

Let us imagine that we have solved the perturbation theory to the $(n-1)^{\text{th}}$ order, ie. we have determined $g^{(m)}$ for $m \leq n-1$, and have determined the functions $\beta_i^{(m)}$ and $b^{(m)}$ for $m \leq n-2$. Plugging the expansion (3.2.10) into Einstein's equations, and extracting the coefficient of ϵ^n, we obtain an equation of the form

$$H\left[g^{(0)}(\beta_i^{(0)}, b^{(0)})\right] g^{(n)}(x^\mu) = s_n. \tag{3.2.13}$$

Here H is a linear differential operator of second order in the variable r alone. As $g^{(n)}$ is already of order ϵ^n, and since every boundary derivative appears with an additional power of ϵ, H is an ultralocal operator in the field theory directions. It is important to note that H is a differential operator only in the variable r and does not depend on the variables x^μ. Moreover, the precise form of this operator at the point x^μ depends only on the values of $\beta_i^{(0)}$ and $b^{(0)}$ at x^μ but not on the derivatives of these functions at that point. Furthermore, the operator H is independent of n; we have the same homogeneous operator at every order in perturbation theory.

The source term s_n however is different at different orders in perturbation theory. It is a local expression of $n^{\rm th}$ order in boundary derivatives of $\beta_i^{(0)}$ and $b^{(0)}$, as well as of $(n-k)^{\rm th}$ order in $\beta_i^{(k)}$, $b^{(k)}$ for all $k \leq n-1$. Note that $\beta_i^{(n)}$ and $b^{(n)}$ do not enter the $n^{\rm th}$ order equations as constant (derivative free) shifts of velocities and temperatures solve the Einstein's equations.

The expressions (3.2.13) form a set of $5 \times 6/2 = 15$ equations. It turns out that four of these equations do not involve the unknown function $g^{(n)}$ at all; they simply constrain the velocity functions b and β_i. There is one redundancy among the remaining 11 equations which leaves 10 independent 'dynamical' equations. These may be used to solve for the 10 unknown functions in our gauge fixed metric correction $g^{(n)}$, as we describe in more detail below.

Constraint equations

By abuse of nomenclature, we will refer to those of the Einstein's equations that are of first order in r derivatives as constraint equations. Constraint equations are obtained by dotting the tensor E_{MN} with the vector dual to the one-form dr. Four of the five constraint equations (ie. those whose free index is a μ index) have an especially simple boundary interpretation; they are simply the equations of boundary energy momentum conservation. In the context of our perturbative analysis, these equations simply reduce to

$$\partial_\mu T^{\mu\nu}_{(n-1)} = 0 \tag{3.2.14}$$

where $T^{\mu\nu}_{(n-1)}$ is the boundary stress tensor dual the solution expanded up to $\mathcal{O}\left(\epsilon^{n-1}\right)$. Recall that each of $g^{(0)}, g^{(1)}...$ are local functions of b, β_i. It follows that the stress tensor $T^{\mu\nu}_{(n-1)}$ is also a local function (with at most $n-1$ derivatives) of these temperature and velocity fields. Of course the stress tensor $T^{\mu\nu}_{(n-1)}$ also respects 4 dimensional conformal invariance. Consequently it is a 'fluid dynamical' stress tensor with $n-1$ derivatives, the term simply being used for the most general stress tensor (with $n-1$ derivatives), written as a function of u^μ and T, that respects all boundary symmetries.

Consequently, in order to solve the constraint equations at n^{th} order one must solve the equations of fluid dynamics to $(n-1)^{\mathrm{th}}$ order. As we have already been handed a solution to fluid dynamics at order $n-2$, all we need to do is to correct this solution to one higher order. Though the question of how one goes about improving this solution is not the topic of our chapter (we wish only to establish a map between the solutions of fluid mechanics and gravity, not to investigate how to find the set of all such solutions) a few words in this connection may be in order. The only quantity in (3.2.14) that is not already known from the results of perturbation theory at lower orders are $\beta_i^{(n-1)}$ and $b^{(n-1)}$. The four equations (3.2.14) are linear differential equations in these unknowns that presumably always have a solution. There is a non-uniqueness in these solutions given by the zero modes obtained by linearizing the equations of stress energy conservation at zeroth order. These zero modes may always be absorbed into a redefinition of $\beta_i^{(0)}, b^{(0)}$, and so do not correspond to a physical non-uniqueness (ie. this ambiguity goes away once you specify more clearly what your zeroth order solution really is).

Our discussion so far may be summarized as follows: the first step in solving Einstein's equations at n^{th} order is to solve the constraint equations – this amounts to solving the equations of fluid dynamics at $(n-1)^{\mathrm{th}}$ order (3.2.14). As we explain below, while it is of course difficult in general to solve these differential equations throughout $\mathbf{R}^{3,1}$, it is easy to solve them locally in a derivative expansion about any point; this is in fact sufficient to implement our ultralocal perturbative procedure.

Dynamical equations

The remaining constraint E_{rr} and the 'dynamical' Einstein's equations $E_{\mu\nu}$ may be used to solve for the unknown function $g^{(n)}$. Roughly speaking, it turns out to be possible to make a judicious choice of variables such that the operator H is converted into a decoupled system of first order differential operators. It is then simple to solve the equation (3.2.13) for an arbitrary source s_n by direct integration. This procedure actually yields a whole linear space of solutions. The undetermined constants of integration in this procedure are arbitrary functions of x^μ and multiply zero modes of the operator (3.2.13). As we will see below, for an arbitrary non-singular and appropriately normalizable source s_n (of the sort that one expects to be generated in perturbation theory[11]), it is always possible to choose these constants to ensure that $g^{(n)}$ is appropriately normalizable at $r = \infty$ and non-singular at all nonzero r. These requirements do not yet completely specify the solution for $g^{(n)}$, as H possesses a set of zero modes that satisfy both these requirements. A basis for the linear space of zero modes, denoted g_b and g_i, is obtained by differentiating the 4 parameter class of solutions (3.1.2) with respect to the parameters b and β_i. In other words these zero modes correspond exactly to infinitesimal shifts of $\beta_i^{(0)}$ and $b^{(0)}$ and so may be absorbed into a redefinition of these quantities. They reflect only an ambiguity of convention, and may be fixed by a 'renormalization' prescription, as we will do below.

Summary of the perturbation analysis

In summary, it is always possible to find a physically unique solution for the metric $g^{(n)}$, which, in turn, yields the form of the n^{th} order fluid dynamical stress tensor (using the usual AdS/CFT dictionary). This process, being iterative, can be used to recover the fluid dynamics stress tensor to any desired order in the derivative expansion.

In 3.2.3 and 3.2.4 we will provide a few more details of our perturbative procedure, in the context of implementing this procedure to first and second order in the derivative expansion.

[11]Provided the solution at order $n-1$ is non-singular at all nonzero r, it is guaranteed to produce a non-singular source at all nonzero r. Consequently, the non-singularlity of s_n follows inductively. We think is possible to make a similar inductive argument for the large r behaviour of the source, but have not yet formulated this argument precisely enough to call it a proof.

3.2.3 Outline of the first order computation

We now present the strategy to implement the general procedure discussed above to first order in the derivative expansion.

Solving the constraint equations

The Einstein constraint equations at first order require that the zero order velocity and temperature fields obey the equations of perfect fluid dynamics

$$\partial_\mu T^{\mu\nu}_{(0)} = 0 \ , \tag{3.2.15}$$

where up to an overall constant

$$T^{\mu\nu}_{(0)} = \frac{1}{(b^{(0)})^4}\left(\eta^{\mu\nu} + 4\, u^\mu_{(0)}\, u^\nu_{(0)}\right) . \tag{3.2.16}$$

While it is difficult to find the general solution to these equations at all x^μ, in order to carry out our ultralocal perturbative procedure at a given point y^μ, we only need to solve these constraints to first order in a Taylor expansion of the fields b and β_i about the point y^μ. This is, of course, easily achieved. The four equations (3.2.15) may be used to solve for the 4 derivatives of the temperature field at y^μ in terms of first derivatives of the velocity fields at the same point. This determines the Taylor expansion of b to first order about y^μ in terms of the expansion, to first order, of the field β_i about the same point. We will only require the first order terms in the Taylor expansion of velocity and temperature fields in order to compute $g^{(1)}(y^\mu)$.

Solving the dynamical equations

As described in the previous section, we expand Einstein's equations to first order and find the equations (3.2.13). Using the 'solution' of 3.2.3, all source terms may be regarded as functions of first derivatives of velocity fields only. The equations (3.2.13) are then easily integrated subject to boundary conditions and we find (3.2.13) is given by

$$g^{(1)} = g^{(1)}_P + f_b(x_i, v)\, g_b + f_i(x_j, v)\, g_i, \tag{3.2.17}$$

where $g_P^{(1)}$ is a particular solution to (3.2.13), and f_b and f_i are a basis for the zero modes of H that were described in the 3.2.2. Plugging in this solution, the full metric $g^{(0)} + g^{(1)}$, when expanded to order first order in ϵ, is (3.2.10)

$$g = g^{(0)} + \epsilon \left(g_P^{(1)} + (f_b + b^{(1)}) g_b + (f_i + \beta^{(1)}) g_i \right), \tag{3.2.18}$$

where the four functions of x^μ, $f_b + b^{(1)}$, $f_i + \beta_i^{(1)}$ are all completely unconstrained by the equations at order ϵ.

The 'Landau' Frame

Our solution (3.2.17) for the first order metric has a four function non-uniqueness in it. As f_b and f_i may be absorbed into $b^{(1)}$ and $\beta_i^{(1)}$ this non-uniqueness simply represents an ambiguity of convention, and may be fixed by a 'renormalization' choice. We describe our choice below.

Given $g^{(1)}$, it is straightforward to use the AdS/CFT correspondence to recover the stress tensor. To first order in ϵ the boundary stress tensor dual to the metric (3.2.18) evaluates to

$$T^{\mu\nu} = \frac{1}{b^4} \left(\eta^{\mu\nu} + 4\, u^\mu u^\nu \right) - \frac{2}{b^3} T^{\mu\nu}_{(1)}, \tag{3.2.19}$$

where

$$\begin{aligned} b &= b^{(0)} + \epsilon (b^{(1)} + f_b) \\ \beta_i &= \epsilon (\beta_i^{(1)} + f_i) \end{aligned} \tag{3.2.20}$$

where $T^{\mu\nu}_{(1)}$, defined by (3.2.19), is an expression linear in x^μ derivatives of the velocity fields and temperature fields. Notice that our definition of $T^{\mu\nu}_{(1)}$, via (3.2.19), depends explicitly on the value of the coefficients f_i, f_b of the homogeneous modes of the differential equation (3.2.13). These coefficients depend on the specific choice of the particular solution $g_P^{(1)}$, which is of course ambiguous up to addition of homogenous solutions. Any given solution (3.2.17) may be broken up in many different ways into particular and homogeneous solutions, resulting in an ambiguity of shifts of the coefficients of f_b, f_i and thereby an ambiguity in $T^{\mu\nu}_{(1)}$. It is always possible to use the freedom provided by this ambiguity to set $u_{(0)\mu} T^{\mu\nu}_{(1)} = 0$. This choice completely fixes the particular solution $g^{(1)}_P$. We adopt this convention for the particular solution and then simply simply set $g^{(1)} = g^{(1)}_P$ ie. choose $f_b = f_i = 0$. $T^{\mu\nu}_{(1)}$ is now unambiguously defined and may be evaluated by explicit computation; it turns out that

$$T^{\mu\nu}_{(1)} = \sigma^{\mu\nu}.$$

The discussion of the previous paragraph has a natural generalization to perturbation theory at any order. As the operator H is the same at every order in perturbation theory, the ambiguity for the solution of $g^{(n)}$ in perturbation theory is always of the form described in (3.2.18). We will always fix the ambiguity in this solution by choosing $u_\mu T^{\mu\nu}_{(k)} = 0$. The convention dependence of this procedure has a well known counterpart in fluid dynamics; it is simply the ambiguity of the stress tensor under field redefinitions of the temperature and u^μ. Indeed this field redefinition ambiguity is standardly fixed by precisely the 'gauge' choice $u_\mu T^{\mu\nu}_{(1)} = 0$. This is the so called 'Landau frame' widely used in studies of fluid dynamics.[12]

We present the details of the first order computation in 3.3 below.

3.2.4 Outline of the second order computation

Assuming that we have implemented the first order calculation described in 3.2.3, it is then possible to find a solution to Einstein's equations at the next order. In this case care should be taken in implementing the constraints as we discuss below.

[12] Conventionally, one writes in fluid mechanics the stress tensor as the perfect fluid part and a dissipative part ie. $T^{\mu\nu} = T^{\mu\nu}_{perfect} + T^{\mu\nu}_{dissipative}$. The Landau gauge condition we choose at every order simply amounts to $u_\mu T^{\mu\nu}_{dissipative} = 0$.

The constraints at second order

The general discussion of 3.2.2 allows us to obtain the second order solution to Einstein's equations once we have solved the first order system as outlined in 3.2.3. However, we need to confront an important issue before proceeding, owing to the way we have set up the perturbation expansion. Of course perturbation theory at second order is well defined only once the first order equations have been solved. While in principle we should solve these equations everywhere in $\mathbf{R}^{3,1}$, in the previous subsection we did not quite achieve that; we were content to solve the constraint equation (3.2.15) only to first order in the Taylor expansion about our special point y^μ. While that was good enough to obtain $g^{(1)}$, in order to carry out the second order calculation we first need to do better; we must ensure that the first order constraint is obeyed to second order in the Taylor expansion of the fields $b^{(0)}$ and $\beta_i^{(0)}$ about y^μ. That is, we require

$$\partial_\lambda \partial_\mu T^{\mu\nu}_{(0)}\left(y^\alpha\right) = 0. \tag{3.2.21}$$

Essentially, we require that $T^{\mu\nu}_{(0)}$ satisfy the conservation equation (3.2.15) to order ϵ^2 before we attempt to find the second order stress tensor. In general, we would have need (3.2.15) to be satisfied globally before proceeding; however, the ultralocality manifest in our set-up implies that it suffices that the conservation holds only to the order we are working. If we were interested in say the n^{th} order stress tensor $T^{\mu\nu}_{(n)}$ we would need to ensure that the stress tensor up to order $n-1$ satisfies the conservation equation to order $\mathcal{O}\left(\epsilon^{n-1}\right)$.

The equations (3.2.21) may be thought of as a set of 16 linear constraints on the coefficients of the (40+78) two derivative terms involving $b^{(0)}$ and $\beta_i^{(0)}$. We use these equations to solve for 16 coefficients, and treat the remaining coefficients as independent. This process is the conceptual analogue of our zeroth order 'solution' of fluid dynamics at the point y^μ (described in the previous subsection), obtained by solving for the first derivatives of temperature in terms of the first derivatives of velocities. Indeed it is an extension of that procedure to the next order in derivatives. See 3.4 for the details of the implementation of this procedure. In summary, before we even start trying to solve for $g^{(2)}$, we need to plug a solution of (3.2.21) into $g^{(0)} + g^{(1)}$ expanded in a Taylor series expansion about y^μ. Otherwise we would be expanding the second order equations about a background that does not solve the first order fluid dynamics.

Nature of source terms

As we have explained above, the Einstein's equations, to second order, take the schematic form described in (3.2.13)

$$H\left[g^{(0)}(\beta_i^{(0)}, b^{(0)})\right]\, g^{(2)} = s_a + s_b \tag{3.2.22}$$

We have broken up the source term above into two pieces, s_a and s_b, for conceptual convenience. s_a is a local functional of $\beta_i^{(0)}$ and $b^{(0)}$ of up to second order in field theory derivatives. Terms contributing to s_a have their origin both in two field theory derivatives acting on the metric $g^{(0)}$ and exactly one field theory derivative acting on $g^{(1)}$ (recall that $g^{(1)}$ itself is a local function of $\beta_i^{(0)}$ and $b^{(0)}$ of first order in derivatives). The source term s_b is new: it arises from first order derivatives of the velocity and temperature corrections $\beta_i^{(1)}$ and $b^{(1)}$. This has no analogue in the first order computation.

As we have explained above, $\beta_i^{(0)}, b^{(0)}$ are absolutely any functions that obey the equations (3.2.15). In particular, if it turns of that the functions $\beta_i^{(0)} + \epsilon\, \beta_i^{(1)}$ and $b^{(0)} + \epsilon\, b^{(1)}$ obey that equation (to first order in ϵ) then $\beta_i^{(1)}$ and $b^{(1)}$ may each simply be set to zero by an appropriate redefinition of $\beta_i^{(0)}$ and $b_i^{(0)}$. This results in a 'gauge' ambiguity of the functions $\beta_i^{(1)}, b^{(1)}$. In our ultralocal perturbative procedure, we choose to fix this ambiguity by setting $\beta_i^{(1)}$ to zero (at our distinguished point y^μ) while leaving $b^{(1)}$ arbitrary.[13]

[13] The functions $b_i^{(1)}, \beta_i^{(1)}$ have sixteen independent first derivatives, all but four of which may be fixed by the gauge freedom. We choose use this freedom to set all velocity derivatives to zero.

Solution of the constraint equations

With the source terms in place, the procedure to solve for $g^{(2)}$ proceeds in direct imitation of the first order calculation. The constraint equations reduce to the expansion to order ϵ of the equation of conservation of the stress tensor

$$T^{\mu\nu} = \frac{1}{b^4}\,(4\,u^\mu u^\nu + \eta^{\mu\nu}) - 2\,\epsilon\,\frac{1}{b^3}\,\sigma^{\mu\nu} \tag{3.2.23}$$

with $\beta^i = \beta^{(0)}$, $b = b^{(0)} + \epsilon\, b^{(1)}$. These four equations may be used to solve for the four derivatives $\partial_\mu b^{(1)}$ at x^μ. Consequently the constraint equations plus our choice of gauge, uniquely determined the first order correction of the temperature field $b^{(1)}$ and velocity field $\beta_i^{(1)}$ as a function of the zeroth order solution.

Note that the gauge $\beta_i^{(1)}(y^\mu) = 0$ may be consistently chosen at any one point y^μ, but not at all x^μ. Nonetheless the results for $g^{(2)}$ that we obtain using this gauge will, when appropriately covariantized be simultaneously applicable to every spacetime point x^μ. The reason for this is that all source terms depend on $b^{(1)}$ and $\beta_i^{(1)}$ only through the expansion to order ϵ of $\partial_\mu T^{\mu\nu} = 0$ with $T^{\mu\nu}$ given by (3.2.23). Note that this source term is 'gauge invariant' (recall that 'gauge' transformations are simply shifts of $b^{(1)}$ and $\beta_i^{(1)}$ by zero modes of this equation). It follows that $g^{(2)}$ determined via this procedure does not depend on our choice of gauge, which was made purely for convenience.

Solving for $g^{(2)}$ and the second order stress tensor

Now plugging this solution for $b^{(1)}$ into the source terms it is straightforward to integrate (3.2.22) to obtain $g^{(2)}$. We fix the ambiguity in the choice of homogeneous mode in this solution as before, by requiring $T^{\mu\nu}_{(2)}\,u_{(0)\nu} = 0$. This condition yields a unique solution for $g^{(2)}$ as well as for the second order correction to the fluid dynamical stress tensor $T^{\mu\nu}_{(2)}$, giving rise to the result (3.1.9). We present the details of the second order computation in 3.4.

In the rest of this chapter we will present our implementation of our perturbative procedure described above, to first and second order in the derivative expansion.

3.3 The metric and stress tensor at first order

In this section we will determine the solution, to first order in the derivative expansion. As we have described in 3.2, the equations that determine $g^{(1)}$ at x^μ are ultralocal; consequently we are able to solve the problem point by point. It is always possible to choose coordinates to set $u^\mu = (1,0,0,0)$ and $b^{(0)} = 1$ at any given point x^μ. Making that choice, the metric (3.1.6) expanded to first order in derivatives in the neighbourhood of x^μ (chosen to be the origin of $\mathbf{R}^{3,1}$ for notational simplicity) is given by

$$\begin{aligned} ds^2_{(0)} &= 2\,dv\,dr - r^2\,f(r)\,dv^2 + r^2\,dx_i\,dx^i \\ &\quad - 2\,x^\mu\,\partial_\mu\beta_i^{(0)}\,dx^i\,dr - 2\,x^\mu\partial_\mu\beta_i^{(0)}\,r^2\,(1-f(r))\,dx^i\,dv - 4\,\frac{x^\mu\,\partial_\mu b^{(0)}}{r^2}\,dv^2\ . \end{aligned} \tag{3.3.24}$$

In order to implement the perturbation programme described in the previous section, we need to find the first order metric $g^{(1)}$ which, when added to (3.3.24), gives a solution to Einstein's equations to first order in derivatives.

The metric (3.3.24) together with $g^{(1)}$ has a background piece (the first line in (3.3.24)) which is simply the metric of a uniform black brane. In addition it has small first derivative corrections, some of which are known (the second line of (3.3.24)), and the remainder of which ($g^{(1)}$) we have to determine. Now note that the background black brane metric preserves a spatial $SO(3)$ rotational symmetry. This symmetry allows us to solve separately for the $SO(3)$ scalars, the $SO(3)$ vector and $SO(3)$ symmetric traceless two tensor ($\mathbf{5}$) components of $g^{(1)}$ and lies at the heart of the separability of the matrix valued linear operator H into a set of ordinary linear operators.

In the following we will discuss each of these sectors separately and determine $g^{(1)}$.

Subsequently, in 3.3.4 we present the full solution to order ϵ and proceed to calculate the stress tensor in 3.3.5.

3.3.1 Scalars of $SO(3)$

The scalar components of $g^{(1)}$ are parameterized by the functions $h_1(r)$ and $k_1(r)$ according to[14]

$$\begin{aligned} g^{(1)}_{ii}(r) &= 3\,r^2\,h_1(r) \\ g^{(1)}_{vv}(r) &= \frac{k_1(r)}{r^2} \\ g^{(1)}_{vr}(r) &= -\frac{3}{2}\,h_1(r). \end{aligned} \tag{3.3.25}$$

Here $g^{(1)}_{ii}$ and $g^{(1)}_{vr}$ are related to each other by our gauge choice $Tr((g^{(0)})^{-1}g^{(1)}) = 0$.

The scalar Einstein's equations (ie. those equations that transform as a scalar of $SO(3)$) may be divided up into constraints and dynamical equations. The constraint equations are obtained by contracting Einstein's equations (the first line of (3.1.1)) with the vector dual to the one form dr. The first scalar constraint is

$$r^2\,f(r)\,E_{vr} + E_{vv} = 0\ , \tag{3.3.26}$$

which evaluates to

$$\partial_v b^{(0)} = \frac{\partial_i \beta^{(0)}_i}{3}\ . \tag{3.3.27}$$

Below, we will interpret (3.3.27) as the expansion of the fluid dynamical stress energy conservation, expanded to first order. The second constraint equation,

$$r^2\,f(r)\,E_{rr} + E_{vr} = 0\ , \tag{3.3.28}$$

leads to

$$12\,r^3\,h_1(r) + (3r^4 - 1)\,h_1'(r) - k_1'(r) = -6\,r^2\,\frac{\partial_i \beta^{(0)}_i}{3}\ . \tag{3.3.29}$$

To this set of constraints we need add only one dynamical scalar equation,[15] the simplest of which turns out to be

$$5\,h_1'(r) + r\,h_1''(r) = 0\ . \tag{3.3.30}$$

[14]In the spatial $\mathbf{R}^3 \subset \mathbf{R}^{3,1}$ we will often for ease of notation, avoid the use of covariant and contravariant indices and adopt a summation convention for repeated indices ie. $g^{(1)}_{ii} = \sum_{i=1}^{3} g^{(1)}_{ii}$.

[15]We have explicitly checked that the equations listed here imply that the second dynamical equation is automatically satisfied.

The LHS of (3.3.30) and (3.3.29) are the restriction of the operator H of (3.2.13) to the scalar sector. The RHS of the same equations are the scalar parts of the source terms s_1. Notice that H is a first order operator in the variables $h_1'(r)$ and $k_1(r)$. Consequently the equation (3.3.30) may be integrated for an arbitrary source term. The resulting solution is regular at all nonzero r provided that the source shares this property, and the growth $h_1(r)$ at infinity is slower than a constant – the behaviour of a non normalizable operator deformation – provided the source in (3.3.30) grows slower than $1/r$ at large r. Once $h_1(r)$ has been obtained $k_1(r)$ may be determined from (3.3.29) by integration, for an arbitrary source term. Once again, the solution will be regular and grows no faster than r^3 at large r, provided the source in that equation is regular and normalizable. The two source terms of this subsection satisfy these regularity and growth requirements, and it seems clear that this result will extend to arbitrary order in perturbation theory (see the next section).

The general solution to the system (3.3.29) and (3.3.30), obtained by the integration described above, is

$$h_1(r) = s + \frac{t}{r^4}, \qquad k_1(r) = \frac{2\, r^3\, \partial_i \beta_i^{(0)}}{3} + 3\, r^4\, s - \frac{t}{r^4} + u\ , \tag{3.3.31}$$

where s, t and u are arbitrary constants (in the variable r). In the solution above, the parameter s multiplies a non normalizable mode (which represents a deformation of the field theory metric) and so is forced to zero by our boundary conditions. A linear combination of the pieces multiplied by t and u is generated by the action of the coordinate transformation $r' = r\,(1 + a/r^4)$ and so is pure gauge, and may be set to zero without loss of generality. The remaining coefficient u corresponds to an infinitesimal temperature variation, and is forced to be zero by our renormalization condition on the stress tensor $u^\mu_{(0)}\, T_{\mu\nu} = 0$ (see the subsection on the stress tensor below). In summary, each of s, t, u may be set to zero and the scalar part of the metric $g^{(1)}$, denoted $g^{(1)}_S$, is

$$\left(g^{(1)}_S\right)_{\alpha\beta} dx^\alpha dx^\beta = \frac{2}{3}\, r\, \partial_i \beta_i^{(0)}\, dv^2. \tag{3.3.32}$$

Two comments about this solution are in order. First note that $k_1(r)$ is manifestly regular at the unperturbed 'horizon' $r = 1$, as we require. Second, it grows at large r like r^3. This is intermediate between the r^0 growth of finite energy fluctuations and the r^4 growth of a field theory metric deformation. As $g^{(0)} + g^{(1)}$ obeys the Einstein's equations to leading order in derivatives, the usual Fefferman-Graham expansion assures us that the sum of first order fluctuations in $g^{(0)} + g^{(1)}$ must (in the appropriate coordinate system) die off like $1/r^4$ compared to terms that appear in the zeroth order metric (this would correspond to $k_1(r)$ constant at infinity). Consequently the unusually slow fall off at infinity of our metric $g^{(1)}$ must be compensated for by an equal but opposite effect from a first order fluctuation piece in the second line of (3.3.24). This indeed turns out to be the case. While an explicit computation of the boundary stress tensor dual to (3.3.24) yields a result that diverges like r^3, this divergence is precisely cancelled when we add $g^{(1)}$ above to the metric, and the correct value of the stress dual to $g^{(0)} + g^{(1)}$ is in fact zero in the scalar sector, in agreement with our renormalization condition $u_{(0)\mu}\, T^{\mu\nu} = 0$.

3.3.2 Vectors of $SO(3)$

In the vector channel the relevant Einstein's equations are the constraint $r^2\, f(r)\, E_{ri} + E_{vi} = 0$ and a dynamical equation which can be chosen to be any linear combination of the Einstein's equations $E_{ri} = 0$ and $E_{vi} = 0$. The constraint evaluates to

$$\partial_i b^{(0)} = \partial_v \beta_i^{(0)} \ , \tag{3.3.33}$$

which we will later interpret as a consequence of the conservation of boundary momentum. In order to explore the content of the dynamical equation (we choose $E_{ri} = 0$), it is convenient to parameterize the vector part of the fluctuation metric by the functions $j_i^{(1)}$, as

$$\left(g_V^{(1)}\right)_{\alpha\beta} dx^\alpha dx^\beta = 2\, r^2\, \left(1 - f(r)\right)\, j_i^{(1)}(r)\, dv\, dx^i. \tag{3.3.34}$$

The dynamical equation for $j_i(r)$ turns out to be

$$\frac{d}{dr}\left(\frac{1}{r^3}\,\frac{d}{dr} j_i^{(1)}(r)\right) = -\frac{3}{r^2}\,\partial_v \beta_i^{(0)}. \tag{3.3.35}$$

The LHS of (3.3.35) is the restriction of the operator H of (3.2.13) to the vector sector, and the RHS of this equation is the projection of s_1 to the vector sector. H is of first order in the variable $j^{(1)'}(r)$ and so may be integrated for an arbitrary source term. The resulting solution is regular and normalizable provided the source is regular and decays at infinity faster than $1/r$. This condition is obeyed in (3.3.35); it seems rather clear that it will continue to be obeyed at arbitrary order in perturbation theory (see the next section).

Returning to (3.3.35), the general solution of this equation is

$$j_i^{(1)}(r) = \partial_v \beta_i^{(0)}\, r^3 + a_i\, r^4 + c_i \tag{3.3.36}$$

for arbitrary constants a_i, c_i. The coefficient a_i multiplies a non-normalizable metric deformation, and so is forced to zero by our choice of boundary conditions. The other integration constant c_i multiplies an infinitesimal shift in the velocity of the brane. It turns out (see below) that a nonzero value for c_i leads to a nonzero value for T_{0i} which violates our 'renormalization' condition, consequently c_i must be set to zero. In summary,

$$\left(g_V^{(1)}\right)_{\alpha\beta} dx^\alpha dx^\beta = 2\, r\, \partial_v \beta_i^{(0)}\, dv\, dx^i. \tag{3.3.37}$$

As in the scalar sector above, this solution grows by a factor of r^3 faster at the boundary than the shear zero mode. This slow fall off leads to a divergent contribution to the stress tensor which precisely cancels an equal and opposite divergence from terms in the expansion of $g^{(0)}$ to first order in derivatives. As we will see below, the full contribution of $g^{(0)}+g^{(1)}$ to the vector part of the boundary stress tensor is just zero, again in agreement with our renormalization conditions.

3.3.3 The symmetric tensors of $SO(3)$

We now turn to $g^{(1)}_T$, the part of $g^{(1)}$ that transforms in the **5**, the symmetric traceless two tensor representation, of $SO(3)$. Let us parameterize our metric fluctuation by

$$\left(g^{(1)}_T\right)_{\alpha\beta} dx^\alpha\, dx^\beta = r^2\, \alpha^{(1)}_{ij}(r)\, dx^i\, dx^j, \tag{3.3.38}$$

where α_{ij} is traceless and symmetric. The Einstein's equation $E_{ij} = 0$ yield

$$\frac{d}{dr}\left(r^5\, f(r)\, \frac{d}{dr}\alpha^{(1)}_{ij}\right) = -6\, r^2\, \sigma^{(0)}_{ij}\ , \tag{3.3.39}$$

where we have defined a symmetric traceless matrix

$$\sigma^{(0)}_{ij} = \partial_{(i}\beta^{(0)}_{j)} - \frac{1}{3}\,\delta_{ij}\,\partial_m\beta^{(0)}_m\ . \tag{3.3.40}$$

The LHS of (3.3.39) is the restriction of the operator H of (3.2.13) to the tensor sector, and the RHS of this equation is the tensor part of the source term s_1. Note that H is a first order operator in the variable $\alpha^{(1)'}_{ij}(r)$ and so may be integrated for an arbitrary source term. The solution to this equation with arbitrary source term $s(r)$ is given by (dropping the tensor indices):

$$\alpha^{(1)} = -\int_r^\infty \frac{dx}{f(x)\, x^5} \int_1^x s(y)\, dy\ . \tag{3.3.41}$$

Note that the lower limit of the inner integral in (3.3.41) has been chosen to be unity. Provided that $s(x)$ is regular at $x = 1$ (this is true of (3.3.39) and will be true at every order in perturbation theory), $\int_1^x s(x)$ has a zero at $x = 1$. It follows that the outer integrand in (3.3.41) is regular at nonzero x (and in particular at $x = 1$) despite the explicit zero in the factor $f(x)$ in the denominator. The solution for $\alpha^{(1)}$ is also normalizable provided the source is regular and grows at infinity slower than r^3. This condition is obeyed in (3.3.39) and is expected to continue to be obeyed at arbitrary order in perturbation theory (see the next section).

Applying (3.3.41) to the source term in (3.3.39) we find that the solution for $\alpha^{(1)}_{ij}$ is given by

$$(g^{(1)}_T)_{\alpha\beta}\, dx^\alpha dx^\beta = 2\, r^2\, F(r)\, \sigma^{(0)}_{ij}\, dx^i dx^j. \tag{3.3.42}$$

with

$$F(r) = \int_r^\infty dx\, \frac{x^2+x+1}{x(x+1)\,(x^2+1)} = \frac{1}{4}\,\left[\ln\left(\frac{(1+r)^2(1+r^2)}{r^4}\right) - 2\,\arctan(r) + \pi\right] \tag{3.3.43}$$

At large r it evaluates to

$$(g_T^{(1)})_{\alpha\beta}\, dx^\alpha dx^\beta = 2\,\left(r - \frac{1}{4\,r^2}\right)\,\sigma_{ij}^{(0)}\, dx^i dx^j. \tag{3.3.44}$$

As in the previous subsections, the first term in (3.3.44) yields a contribution to the stress tensor that diverges like r^3, but precisely cancels the corresponding divergence from first derivative terms in the expansion of $g^{(0)}$. However the second term in this expansion yields an important finite contribution to the stress tensor, as we will see below.

Summary of the first order calculation: In summary, our final answer for $g^{(0)}+g^{(1)}$, expanded to first order in boundary derivatives about $y^\mu = 0$, is given explicitly as

$$\begin{aligned} ds^2 &= 2\,dv\,dr - r^2 f(r)\,dv^2 + r^2\,dx_i\,dx^i \\ &\quad - 2\,x^\mu\,\partial_\mu\beta_i^{(0)}\,dr\,dx^i - 2\,x^\mu\,\partial_\mu\,\beta_i^{(0)} r^2(1-f(r))\,dv\,dx^i - 4\frac{x^\mu\partial_\mu b^{(0)}}{r^2}\,dv^2 \\ &\quad + 2\,r^2\,F(r)\,\sigma_{ij}^{(0)}\,dx^i\,dx^j + \frac{2}{3}\,r\,\partial_i\beta_i^{(0)}\,dv^2 + 2\,r\,\partial_v\beta_i^{(0)}\,dv\,dx^i. \end{aligned} \tag{3.3.45}$$

This metric solves Einstein's equations to first order in the neighbourhood of $x^\mu = 0$ provided the functions $b^{(0)}$ and $\beta_i^{(0)}$ satisfy

$$\begin{aligned} \partial_v b^{(0)} &= \frac{\partial_i\beta_i^{(0)}}{3} \\ \partial_i b^{(0)} &= \partial_v\beta_i^{(0)}. \end{aligned} \tag{3.3.46}$$

3.3.4 Global solution to first order in derivatives

In the previous subsection we have computed the metric $g^{(1)}$ about x^μ assuming that $b^{(0)} = 1$ and $\beta_i^{(0)} = 0$ at the origin. Since it is possible to choose coordinates to set an arbitrary velocity to zero and an arbitrary $b^{(0)}$ to unity at any given point (and since our perturbation procedure is ultralocal), the results of the previous subsection contain enough information to write down the metric $g^{(1)}$ about any point. A simple way to do this is to construct a covariant metric[16], as a function of u_μ and b, which reduces to (3.3.45) when $b^{(0)} = 1$ and $\beta_i^{(0)} = 0$. It is easy to check that

$$\begin{aligned} ds^2 = &-2\, u_\mu\, dx^\mu dr - r^2\, f(b\, r)\, u_\mu u_\nu\, dx^\mu dx^\nu + r^2\, P_{\mu\nu}\, dx^\mu dx^\nu \\ &+ 2\, r^2\, b\, F(b\, r)\, \sigma_{\mu\nu}\, dx^\mu dx^\nu + \frac{2}{3}\, r\, u_\mu u_\nu\, \partial_\lambda u^\lambda\, dx^\mu dx^\nu - r\, u^\lambda \partial_\lambda \left(u_\nu u_\mu\right)\, dx^\mu dx^\nu, \end{aligned} \tag{3.3.47}$$

does the job, up to terms of second or higher order in derivatives. Here we have written the metric in terms of $\sigma_{\mu\nu}$ defined in (3.1.9) and the function $F(r)$ introduced in (3.3.43). Furthermore, it is easy to check that the metric above is the unique choice respecting the symmetries (again up to terms of second or higher order in derivatives). It follows that (3.3.47) is the metric $g^{(0)} + g^{(1)}$. It is also easily verified that the covariant version of (3.3.46) is (3.2.15). We will interpret this as an equation of stress energy conservation in the next subsection.

3.3.5 Stress tensor to first order

Given the solution to the first order equations, we can utilize the AdS/CFT dictionary to construct the boundary stress tensor using the prescription of [38]. For the metric (3.3.47) it is not difficult to compute the stress tensor; all we need to do is compute the extrinsic curvature tensor $K_{\mu\nu}$ to the surface at fixed r. By convention, we choose the unit normal to this surface to be outward pointing, *i.e.* pointing towards the boundary, in the definition of $K_{\mu\nu}$. Using then the definition

$$T^\mu_\nu = -2 \lim_{r\to\infty} r^4 \left(K^\mu_\nu - \delta^\mu_\nu\right), \tag{3.3.48}$$

[16]By abuse of notation, we will refer to expressions transformation covariantly in the boundary metric (chosen here to be $\eta_{\mu\nu}$) as covariant. In particular, we are not interested in full bulk covariance as we will continue to restrict attention to a specific coordinatization of the fifth direction.

on our solution (3.3.47), we find the result is given simply as

$$T^{\mu\nu} = \frac{1}{b^4}\left(4\, u^\mu u^\nu + \eta^{\mu\nu}\right) - \frac{2}{b^3}\,\sigma^{\mu\nu}. \tag{3.3.49}$$

where $\sigma^{\mu\nu}$ was defined in (3.1.9) and all field theory indices are raised and lowered with the boundary metric $\eta_{\mu\nu}$. As explained in the introduction, this stress tensor implies that the ratio of viscosity to entropy density of our fluid is $1/(4\pi)$. Note that as mentioned previously, the expression (3.3.49) is only correct up to first derivative terms in the temperature ($T = 1/b$) and velocities.

3.4 The metric and stress tensor at second order

In order to obtain the metric and stress tensor at second order in the derivative expansion, we follow the method outlined in 3.2 and implemented in detail in 3.3 to leading order. Concretely, we choose coordinates such that $\beta_i^{(0)} = 0$ and $b^{(0)} = 1$ at the point $x^\mu = 0$. The metric $g^{(0)} + g^{(1)}$ given in (3.3.47) may be expanded to second order in derivatives. This involves Taylor expanding $g^{(0)}$ to second order and $g^{(1)}$ to first order, the second order analogue of (3.3.24). As we have explained in 3.2.4, at this stage we also make the substitution $b^{(0)} \to b^{(0)} + b^{(1)}$, and treat $b^{(1)}$ as an order ϵ term, and so retain only those expressions that are of first derivative order in $b^{(1)}$ (and contain no other derivatives). This process is straightforward and we will not record the (rather lengthy) resultant expression here. To this expression we add the as yet undetermined metric fluctuation

$$g^{(2)}_{\alpha\beta}\, dx^\alpha dx^\beta = -3\, h_2(r)\, dv\, dr + r^2\, h_2(r)\, dx_i\, dx^i + \frac{k_2(r)}{r^2}\, dv^2 + 2\, \frac{j_i^{(2)}(r)}{r^2}\, dv\, dx^i + r^2 \alpha^{(2)}_{ij}\, dx^i\, dx^j. \tag{3.4.50}$$

We plug this metric into Einstein's equations and obtain a set of linear second order differential equations that determine $h_2, k_2, j_i^{(2)}, \alpha_{ij}^{(2)}$. As in the previous section, $SO(3)$ symmetry ensures that the equations for the scalars h_2, k_2, the vectors $j_i^{(2)}$, and the tensor $\alpha_{ij}^{(2)}$ do not mix. Moreover, as we have explained in 3.2, the equations that determine these unknown functions are identical to their first order counterparts in the homogeneous terms, but differ from those equations in the sources. As a result, the only new calculation we have to perform in order to obtain the metric at second order is the computation of the source terms. Once these terms are available, the corresponding equations may easily be integrated, as in the previous section.

Recall that the input metric into Einstein's equations includes terms that arise out of the Taylor expansion of $g^{(0)} + g^{(1)}$ that have explicit factors of the coordinates x^μ. Nonetheless, a very simple argument assures us that the source terms in the equations that determine $g^{(2)}$ must all be independent of x^μ. The argument runs as follows: We have explicitly constructed $g^{(1)}$ in the previous section so that $E_{MN}\left(g^{(0)} + g^{(1)}\right) = O_{MN}$ where O_{MN} is a local expression constructed out of second order or higher x^μ derivatives of velocity and temperature fields. It follows that x^μ dependence of sources, which may be obtained by Taylor expanding O_{MN} about $x^\mu = 0$, occurs only at the three derivative level or higher. It follows that source terms at the two derivative level have no x^μ dependence. Clearly, this argument has a direct analogue at arbitrary order in perturbation theory.

A crucial input into the argument of the last paragraph was the fact that $g^{(0)} + g^{(1)}$ satisfies Einstein's equations in a neighbourhood of $x^\mu = 0$ (and not just at that point). As we have seen in the previous section, the fact that the energy conservation equation is obeyed at $x^\mu = 0$ allows us to express all first derivatives of temperature in terms of first derivatives of velocities (see (3.3.33) and (3.3.27)). In addition, $\beta_i^{(0)}$ and $b^{(0)}$ must be chosen so that (3.2.21) is satisfied. The sixteen equations (3.2.21) can be grouped into sets that transform under $SO(3)$ as two scalars, three vectors and one tensor (ie. 5). We will now explain how these constraints may be used to solve for 16 of the independent expressions of second order in derivatives of velocity and temperature fields.

In order to do this, let us first list all two derivative 'source' terms that can be built out of second derivatives of $b^{(0)}$ or $\beta_i^{(0)}$, or out of squares of first derivatives of $\beta_i^{(0)}$. These expressions may be separated according to their transformation properties under $SO(3)$

as scalars, vectors and tensors and higher order terms. The higher order pieces will not be of interest to us. An exhaustive list of these expressions that transform in the **1**, **3** or **5** is given in Table 1.[17] We define the vector ℓ_i as the curl of the velocity ie.

$$\ell_i = \epsilon_{ijk}\,\partial^j \beta^k \ , \tag{3.4.51}$$

and the symmetric traceless tensor σ_{ij} has been previously defined in (3.3.40).

As a simple check on the completeness of expressions in Table 1, notice that the number of degrees of freedom in those of the tabulated expressions that are formed from a product of two single derivatives is 5 (in the scalar sector), $5 \times \mathbf{3}$ (in the vector sector), and $7 \times \mathbf{5}$ in the tensor sector, leading to a total of 55 real parameters. Together with degrees of freedom from the two **7**s and one **9** that can also be formed from the product of two derivatives (but will play no role in our analysis) this gives 78 degrees of freedom. This is in agreement with the expected $\frac{1}{2} \times 12 \times 13$ =78 ways of getting a symmetric object from twelve parameters (the first derivatives of the velocity fields). On the other hand, the genuinely two derivative terms in Table 1 have $3 \times \mathbf{1} + 5 \times \mathbf{3} + 3 \times \mathbf{5} = 33$ degrees of freedom which together with a two derivative term that transforms in the **7** (which however plays no role in our analysis) is the expected number $40 = 10 \times 4$ of two derivative terms arising from temperature and velocity fields.

Assuming that we have already employed the first order conservation equation (3.2.15) to eliminate the first derivatives of b, we have to deal with the constraint equation (3.2.21) at the second order. Using the list of second order quantities given in Table 1, it is possible to show that (3.2.21) take the form of the following linear relations between these two

[17] Note that the tensors are symmetric in their indices. The symmetrization as usual is indicated by parentheses.

derivative terms:

$$
\begin{aligned}
\mathbf{s}_1 &= \frac{1}{3}\,\mathbf{s}_3 - \mathfrak{S}_1 + \frac{1}{9}\,\mathfrak{S}_3 + \frac{1}{6}\,\mathfrak{S}_4 - \frac{1}{3}\,\mathfrak{S}_5 \\
\mathbf{s}_2 &= \mathbf{s}_3 - \mathfrak{S}_1 + \frac{1}{2}\,\mathfrak{S}_4 - \mathfrak{S}_5 \\
\mathbf{v}_{1i} &= \frac{10}{9}\,\mathbf{v}_{4i} + \frac{1}{9}\,\mathbf{v}_{5i} + \frac{1}{3}\,\mathfrak{V}_{1i} - \frac{1}{3}\,\mathfrak{V}_{2i} - \frac{2}{3}\,\mathfrak{V}_{3i} \\
\mathbf{v}_{2i} &= \frac{10}{9}\,\mathbf{v}_{4i} + \frac{1}{9}\,\mathbf{v}_{5i} - \frac{2}{3}\,\mathfrak{V}_{1i} + \frac{1}{6}\,\mathfrak{V}_{2i} - \frac{5}{3}\,\mathfrak{V}_{3i} \\
\mathbf{v}_{3i} &= -\frac{1}{3}\,\mathfrak{V}_{4i} + \mathfrak{V}_{5i} \\
\mathbf{t}_{1ij} &= \mathbf{t}_{3ij} + \mathfrak{T}_{1ij} + \frac{1}{3}\,\mathfrak{T}_{4ij} + \frac{1}{4}\,\mathfrak{T}_{5ij} + \mathfrak{T}_{6ij}\ .
\end{aligned}
\tag{3.4.52}
$$

Given these relations we now proceed to analyze the potential source terms arising from the metric (3.3.47) at $\mathcal{O}\left(\epsilon^2\right)$. The analysis, as before, can be done sector by sector – the computations for the scalar, vector and tensor sectors are given in 3.4.1, 3.4.2 and 3.4.3, respectively.

3.4.1 Solution in the scalar sector

Given the general second order fluctuation (3.4.50), we parameterize scalar components of $g^{(2)}$ in terms of the functions $h_2(r)$ and $k_2(r)$ according to

$$
\begin{aligned}
g^{(2)}_{ii}(r) &= 3\,r^2\,h_2(r) \\
g^{(2)}_{vv}(r) &= \frac{k_2(r)}{r^2} \\
g^{(2)}_{vr}(r) &= -\frac{3}{2}h_2(r)\ .
\end{aligned}
\tag{3.4.53}
$$

As we have explained in the 3.3.1, the constraint Einstein's equations in this sector are given by the r and v component of the one-form formed by contracting the Einstein tensor with the vector dual to the one-form dr. The v component of this constraint, *i.e.* the second order expansion of (3.3.26), evaluates to

$$
\frac{1}{b^{(0)}}\partial_v b^{(1)} = \frac{1}{b^{(1)}}\,\mathfrak{S}_5\ . \tag{3.4.54}
$$

This equation enables us to solve for the first v derivative of $b^{(1)}$ in terms of two derivative terms made up of $\beta_i^{(0)}$, but imposes no further constraints on $b^{(0)}, \beta_i^{(0)}$. (3.4.54) has a simple physical interpretation; it is simply the time component of the conservation equation for the stress tensor (3.3.49), expanded to second order in derivatives. Consequently (3.4.54) is the Navier Stokes equation!

The r component of the constraint, *i.e.* (3.3.28), gives us one relation between the functions $h_2(r)$ and $k_2(r)$ and their derivatives. As in 3.3.1, to this constraint we must add one dynamical equation. We obtain the following equations

$$\begin{aligned} 5\,h_2'(r) + r\,h_2''(r) &= S_h(r) \\ k_2'(r) &= S_k(r) \\ &= 12\,r^3\,h_2(r) + (3\,r^4 - 1)\,h_2'(r) + \widehat{S}_k(r)\ , \end{aligned} \tag{3.4.55}$$

where

$$\begin{aligned} S_h(r) &\equiv \frac{1}{3\,r^3}\,\mathfrak{S}_4 + \frac{1}{2}\,W_h(r)\,\mathfrak{S}_5 \\ \widehat{S}_k(r) &\equiv -\frac{4\,r}{3}\,\mathfrak{s}_3 + 2\,r\,\mathfrak{S}_1 - \frac{2\,r}{9}\,\mathfrak{S}_3 + \frac{1+2\,r^4}{6\,r^3}\,\mathfrak{S}_4 + \frac{1}{2}\,W_k(r)\,\mathfrak{S}_5\ . \end{aligned} \tag{3.4.56}$$

The functions $W_h(r)$ and $W_k(r)$ are given by

$$\begin{aligned} W_h(r) &= \frac{4}{3}\,\frac{\left(r^2+r+1\right)^2 - 2\left(3\,r^2+2r+1\right)\,F(r)}{r\,\left(r+1\right)^2\,\left(r^2+1\right)^2}\ , \\ W_k(r) &= \frac{2}{3}\,\frac{4\,\left(r^2+r+1\right)\,\left(3\,r^4-1\right)\,F(r) - \left(2r^5+2r^4+2r^3-r-1\right)}{r\,\left(r+1\right)\,\left(r^2+1\right)}\ . \end{aligned}$$

As advertised, it is clear that the differential operator acting on the functions $h_2(r)$ and $k_2(r)$ is identical to the one encountered in the first order computation in 3.3.1. The equation (3.4.55) can be explicitly integrated; to do so it is useful to record the leading large r behaviour of the source term $S_h(r)$:

$$S_h(r) \to \frac{1}{r^3}\,S_h^\infty \equiv \frac{1}{r^3}\,\left(\frac{1}{3}\,\mathfrak{S}_4 + \frac{2}{3}\,\mathfrak{S}_5\right)\ . \tag{3.4.57}$$

The first equation in (3.4.55) can be integrated given this asymptotic value to obtain the leading behaviour of the function $h_2(r)$. One finds

$$h_2(r) = -\frac{1}{4\,r^2}\,S_h^\infty + \int_r^\infty \frac{dx}{x^5}\,\int_x^\infty dy\ y^4\,\left(S_h(y) - \frac{1}{y^3}\,S_h^\infty\right)\ . \tag{3.4.58}$$

The integral expression above can be shown to be of $\mathcal{O}\left(r^{-5}\right)$ and hence the asymptotic behaviour of $h_2(r)$ is controlled by s_h. Given $h_2(r)$, one can integrate up the second equation of (3.4.55) for $k_2(r)$. The leading large r behaviour of the source term $S_k(r)$ is given by

$$S_k(r) \to r\, S_k^\infty \equiv r\left(-\frac{4}{3}\,\mathbf{s}_3 + 2\,\mathfrak{S}_1 - \frac{2}{9}\,\mathfrak{S}_3 - \frac{1}{6}\,\mathfrak{S}_4 + \frac{7}{3}\,\mathfrak{S}_5\right) , \tag{3.4.59}$$

and hence we have

$$k_2(r) = \frac{r^2}{2}\, S_k^\infty - \int_r^\infty dx\, \left(S_k(x) - x\, S_k^\infty\right) . \tag{3.4.60}$$

In this case the integral makes a subleading contribution starting at $\mathcal{O}\left(r^{-1}\right)$. As in 3.3.1, we have chosen the coefficients of homogeneous solutions to this differential equation so as to ensure normalizability and vanishing scalar contribution to the stress tensor (according to our renormalization conditions).

3.4.2 Solution in the vector sector

The analysis of the vector fluctuations at second order mimics the computation described in 3.3.2. The vector fluctuation in $g^{(2)}$ is chosen as described in (3.4.50) to be

$$g_{vi}^{(2)} = \frac{j_i^{(2)}}{r^2} . \tag{3.4.61}$$

Once again, the analysis is easily done by looking at the constraint equations which are obtained by contracting the tensor E_{MN} with the vector dual to dr. The i^{th} constraint equation evaluates to

$$18\,\partial_i b^{(1)} = 5\,\mathbf{v}_{4i} + 5\,\mathbf{v}_{5i} + 15\,\mathfrak{V}_{1i} - \frac{15}{4}\,\mathfrak{V}_{2i} - \frac{33}{2}\,\mathfrak{V}_{3i} . \tag{3.4.62}$$

This equation allows us to solve for the spatial derivatives of $b^{(1)}$ in terms of derivatives of $\beta_i^{(0)}$ and $b^{(0)}$. (3.4.62) is simply the expansion to second order in derivatives of the conservation of momentum of the stress tensor (3.3.49).

To complete the solution in the vector channel, we need to solve for $j^{(2)}(r)$, which can be shown to satisfy a dynamical equation

$$\frac{d}{dr}\left(\frac{1}{r^3}\,\frac{d}{dr}j_i^{(2)}(r)\right) = \mathbf{B}_i(r). \tag{3.4.63}$$

Note that the LHS of this expression has the vector part of the operator H acting on $j^{(2)}$. Here $\mathbf{B}_i(r)$ is the source term which is built out of the second derivative terms transforming in the $\mathbf{3}$ of $SO(3)$ given in Table 1.

$$\mathbf{B}(r) = \frac{p(r)\,\mathbf{B}^{\infty} + \mathbf{B}^{\text{fin}}}{18\,r^3\,(r+1)\,(r^2+1)} \tag{3.4.64}$$

with

$$\begin{aligned}\mathbf{B}^{\infty} &= 4\,\left(10\,\mathbf{v}_4 + \mathbf{v}_5 + 3\,\mathfrak{V}_1 - 3\,\mathfrak{V}_2 - 6\,\mathfrak{V}_3\right)\\ \mathbf{B}^{\text{fin}} &= 9\,\left(20\,\mathbf{v}_4 - 5\,\mathfrak{V}_2 - 6\,\mathfrak{V}_3\right),\end{aligned} \tag{3.4.65}$$

and we have introduced the polynomial:

$$p(r) = 2\,r^3 + 2r^2 + 2\,r - 3\ . \tag{3.4.66}$$

Clearly $p(r)$ determines the large r behaviour of the vector perturbation; asymptotically $\mathbf{B}(r) \to \frac{1}{9\,r^3}\,\mathbf{B}^{\infty}$. Hence, integrating (3.4.63) we find that $j^{(2)}(r)$ is given as

$$j_i^{(2)}(r) = -\frac{r^2}{36}\,\mathbf{B}_i^{\infty} + \int_r^{\infty} dx\; x^3\,\int_x^{\infty} dy\,\left(\mathbf{B}_i(y) - \frac{1}{9\,y^3}\,\mathbf{B}_i^{\infty}\right)\ , \tag{3.4.67}$$

where once again we have chosen the coefficients of homogeneous modes in order to maintain normalizability and our renormalization condition. As with the first order computation described in 3.3.2, the solution (3.4.67) makes no contribution to the stress tensor of the field theory.

3.4.3 Solution in the tensor sector

Finally, we turn to the tensor modes at second order where we shall recover the explicit form of the second order contributions to the stress tensor. Our task is now to determine the functions $\alpha^{(2)}_{ij}(r)$ in (3.4.50). As in 3.3.3, in the symmetric traceless sector of $SO(3)$ one has only the dynamical equation given by

$$\frac{1}{2\,r}\frac{d}{dr}\left[r^5\left(1-\frac{1}{r^4}\right)\,\frac{d}{dr}\alpha^{(2)}_{ij}(r)\right]=\mathbf{A}_{ij}(r) \tag{3.4.68}$$

where

$$\mathbf{A}_{ij}(r)=\mathbf{a}_1(r)\,\left(\mathfrak{T}1_{ij}+\frac{1}{3}\mathfrak{T}4_{ij}+\mathbf{t}3_{ij}\right)+\mathbf{a}_5(r)\,\mathfrak{T}5_{ij}+\mathbf{a}_6(r)\,\mathfrak{T}6_{ij}-\frac{1}{4}\,\mathbf{a}_7(r)\,\mathfrak{T}7_{ij}$$

with the coefficient functions

$$\begin{aligned}\mathbf{a}_1(r)&=\frac{3\,p(r)+11}{p(r)+5}-3\,r\,F(r)\\ \mathbf{a}_5(r)&=\frac{1}{2}\left(1+\frac{1}{r^4}\right)\\ \mathbf{a}_6(r)&=\frac{4}{r^2}\,\frac{r^2\,p(r)+3\,r^2-r-1}{p(r)+5}-6\,r\,F(r)\\ \mathbf{a}_7(r)&=2\,\frac{p(r)+1}{p(r)+5}-6\,r\,F(r)\;.\end{aligned} \tag{3.4.69}$$

The functions $F(r)$ and $p(r)$ are defined in (3.3.43) and (3.4.66), respectively.

The desired solution to (3.4.68) can be found by intergrating the right hand side of the equation twice and choosing the solution to the homogenous solution such that we retain regularity[18] at $r=1$ and appropriate normalizability at infinity. The solution with these properties is

$$\alpha^{(2)}_{ij}(r)=-\int_r^\infty\frac{dx}{x\,(x^4-1)}\,\int_1^x dy\;2\,y\,\mathbf{A}_{ij}(y)$$

The quantity of prime interest to us is the leading large r behaviour of $\alpha^{(2)}_{ij}$. This can be inferred from the expressions for the coefficient functions given in (3.4.69) and evaluates to

$$\begin{aligned}\alpha^{(2)}_{ij}(r)=&\;\frac{1}{r^2}\,\left(-\frac{1}{2}\,\mathfrak{T}7_{ij}+\mathfrak{T}6_{ij}-\frac{1}{4}\,\mathfrak{T}5_{ij}\right)\\ &+\frac{1}{2r^4}\,\left[\left(1-\frac{\ln 2}{2}\right)\,\left(\frac{1}{3}\,\mathfrak{T}4_{ij}+\mathfrak{T}1_{ij}+\mathbf{t}3_{ij}\right)+\frac{\ln 2}{4}\,\mathfrak{T}7_{ij}+\mathfrak{T}6_{ij}\right]\end{aligned} \tag{3.4.70}$$

[18]We have imposed the requirement that all metric functions are well behaved in its neighbourhood of $r=1$, a regular point in the spacetime manifold. Note that $r=1$ will not represent the horizon of our perturbed solution, but may well lie very near this horizon manifold.

The leading term here will give a divergent contribution to the stress tensor, which is necessary to cancel the divergence arising from the expansion of $g^{(0)} + g^{(1)}$ to second order. The subleading piece in (3.4.70) is the term that will provide us with the second order stress tensor. Before proceeding to evaluate the stress tensor we present the full solution to second order, appropriately covariantized.

3.4.4 Global solution to second order in derivatives

Consider the following metric

$$ds^2 = -2\,u_\mu\,dx^\mu dr - r^2\,f(b\,r)\,u_\mu u_\nu\,dx^\mu dx^\nu + r^2\,P_{\mu\nu}\,dx^\mu dx^\nu + 3\,b^2\,h_2(b\,r)\,u_\mu\,dx^\mu dr + \mathcal{G}_{\mu\nu}\,dx^\mu dx^\nu\,, \tag{3.4.71}$$

where we have defined a symmetric tensor $\mathcal{G}_{\mu\nu}$ by combing the contributions in the field theory directions from the first and second order metrics $g^{(0)} + g^{(1)}$

$$\begin{aligned}\mathcal{G}_{\mu\nu} = r^2\,\left(2\,b\,F(b\,r)\,\sigma_{\mu\nu} + b^2\,\alpha^{(2)}_{\mu\nu}(b\,r)\right) + \frac{1}{r^2}\left(\frac{2}{3}\,r^3\,\partial_\lambda u^\lambda\,u_\mu\,u_\nu + \frac{k_2(b\,r)}{b^2}\,u_\mu\,u_\nu\right) \\ + r^2\,b^2\,h_2(b\,r)\,P_{\mu\nu} + \frac{1}{r^2}\left(-2\,r^3\,\mathcal{D}u_\alpha + \frac{1}{b^2}\,j^{(2)}_\alpha(b\,r)\right)\,P^\alpha_\nu\,u_\mu\;.\end{aligned} \tag{3.4.72}$$

The covariant expression for $\alpha^{(2)}_{\mu\nu}$ is given by (3.4.68) with the replacements

$$\begin{aligned}&\mathfrak{T}1_{ij} \;\to\; (T_{2d})_{\mu\nu}\;, \qquad \mathfrak{T}4_{ij} \;\to\; (T_{2c})_{\mu\nu}\;, \qquad \mathfrak{T}5_{ij} \;\to\; \ell_\mu\,\ell_\nu - \frac{1}{3}P_{\mu\nu}\,\ell^\alpha\ell_\alpha \\ &\mathfrak{T}6_{ij} \;\to\; (T_{2b})_{\mu\nu}\;, \qquad \mathfrak{T}7_{ij} \;\to\; 2\;(T_{2a})_{\mu\nu}\;, \qquad \mathbf{t}3_{ij} \;\to\; (T_{2e})_{\mu\nu}\;.\end{aligned} \tag{3.4.73}$$

Further, $j^{(2)}_\mu$ given by (3.4.67) with $\mathbf{B}_i(r) \to B_\nu(r)$, where $\mathbf{B}_i(r)$ is given by (3.4.64) and we make the following replacements

$$\begin{aligned}&\mathbf{v}4_i \;\to\; \frac{9}{5}\left[P^\alpha_\nu\,P^{\beta\gamma}\,\partial_\gamma\,\partial_{(\beta}u_{\alpha)} - \frac{1}{3}\,P^{\alpha\beta}\,P^\gamma_\nu\,\partial_\gamma\partial_\alpha u_\beta\right] - P^\mu_\nu\,P^{\alpha\beta}\,\partial_\alpha\partial_\beta u_\mu \\ &\mathbf{v}5_i \;\to\; P^\mu_\nu\,P^{\alpha\beta}\partial_\alpha\partial_\beta u_\mu \\ &\mathfrak{V}1_i \;\to\; \partial_\alpha u^\alpha\,\mathcal{D}u_\nu\;, \qquad \mathfrak{V}2_i \;\to\; \epsilon_{\alpha\beta\gamma\nu}\,u^\alpha\,\mathcal{D}u^\beta\,\ell^\gamma\;, \qquad \mathfrak{V}3_i \;\to\; \sigma_{\alpha\nu}\,\mathcal{D}u^\alpha\;.\end{aligned} \tag{3.4.74}$$

Finally, $h_2(r)$ and $k_2(r)$ are given by (3.4.58) and (3.4.60) respectively, and in the functions $S_h(r)$, $S_k(r)$, S_h^∞ and S_k^∞ defined in (3.4.57) and (3.4.59) we are required to make the replacements

$$
\begin{aligned}
&\mathbf{s}_3 \;\to\; \frac{1}{b^{(0)}} P^{\alpha\beta}\,\partial_\alpha\partial_\beta b^{(0)} \qquad \mathfrak{S}_1 \;\to\; \mathcal{D}u^\alpha\,\mathcal{D}u_\alpha\ , \qquad \mathfrak{S}_2 \;\to\; \ell_\mu\,\mathcal{D}u^\mu \\
&\mathfrak{S}_3 \;\to\; (\partial_\mu u^\mu)^2\ , \qquad \mathfrak{S}_4 \;\to\; \ell_\mu\,\ell^\mu\ , \qquad \mathfrak{S}_5 \;\to\; \sigma_{\mu\nu}\,\sigma^{\mu\nu}\ .
\end{aligned}
\tag{3.4.75}
$$

It may be checked that this metric is the unique (up to terms that differ at third or higher order in derivatives) covariant expression that reduces to two derivative solution determined in the previous subsections, in the neighbourhood of any point y^μ after making the coordinate change that sets $b^{(0)} = 1$ and $\beta_i^{(0)} = 0$ at that point. It follows that (3.4.71) is the desired metric $g^{(0)} + g^{(1)} + g^{(2)}$.

3.4.5 Stress tensor to second order

The stress tensor dual to the solution to second order described in 3.4.4 can be obtained by using the standard formula (3.3.48). To determine the extrinsic curvature at large r, it suffices to know the asymptotic form of the metric since we are interested in terms that have a finite limit as we take $r \to \infty$. Consequently, in order to compute the stress tensor it is sufficient to replace the various functions of r that have appeared in the computation in 3.4.1, 3.4.2 and 3.4.3 by their large r asymptotics. The stress tensor may the be computed in a straightforward fashion, yielding

$$
\begin{aligned}
&(T_2)_{vv} = (T_2)_{vi} = 0\ , \\
&(T_2)_{ij} = -\frac{\ln 2}{4}\,\mathfrak{T}_{7ij} - \mathfrak{T}6_{ij} + \left(-1 + \frac{\ln\,2}{2}\right)\left(\mathbf{t}_{3ij} + \mathfrak{T}_{1ij} + \frac{1}{3}\,\mathfrak{T}_{4ij}\right)\ .
\end{aligned}
\tag{3.4.76}
$$

The vanishing of $(T_2)_{v\mu}$ is actually guaranteed by our renormalization condition. It is easy to check that the covariant form of the expression (3.4.76) is indeed the stress tensor quoted in (3.1.8). This result is the main prediction of our fluctuation analysis to second order in the derivative expansion.

3.5 Second order fluid dynamics

In the previous section we have derived the precise form of the fluid dynamical stress tensor dual to gravity on AdS_5 including all terms with no more than two derivatives. In this section we initiate a study of the physics of this stress tensor. In 3.5.1 below we will demonstrate that our stress tensor transforms homogeneously under Weyl transformations. In 3.5.2 we compute the dispersion relation for low frequency sound and shear waves that follows from our stress tensor.

3.5.1 Weyl transformation of the stress tensor

Thus far we have extracted the stress tensor for a conformal fluid in flat space $\mathbf{R}^{3,1}$. We would like to ensure that the second order stress tensor we have derived transforms homogeneously under Weyl rescaling. In order to check this we perform the obvious minimal covariantization of our stress tensor to generalize it to a fluid stress tensor about an arbitrary boundary metric $g_{\mu\nu}$.[19] and study its Weyl transformation properties.

Consider the Weyl transformation of the boundary metric

$$\begin{aligned} g_{\mu\nu} = e^{2\phi}\, \widetilde{g}_{\mu\nu} \quad &\Rightarrow \quad g^{\mu\nu} = e^{-2\phi} \widetilde{g}^{\mu\nu} \\ \& \quad u^{\mu} = e^{-\phi}\, \widetilde{u}^{\mu}\ , \qquad & T = e^{-\phi} \widetilde{T}. \end{aligned} \tag{3.5.77}$$

It is well known that the first order truncation of the stress tensor (3.1.8) transforms as $T^{\mu\nu} = e^{-6\,\phi}\, \widetilde{T}^{\mu\nu}$ under this transformation (see for instance Appendix D of [39]). We proceed to show that this transformation rule holds for the two derivative stress tensor as well. This transformation property, together with the tracelessness of the stress tensor, ensures Weyl invariance of the fluid dynamical equations $\nabla_\mu T^{\mu\nu}$, appropriate for a conformal fluid.

[19] All metrics in this subsection refer to the metric on the boundary, *i.e.*., the background spacetime on which the fluid is propagating.

It follows from (3.5.77) that $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu = e^{-2\phi}\,\widetilde{P}^{\mu\nu}$. The Christoffel symbols transform as [39]

$$\Gamma^\nu_{\lambda\mu} = \widetilde{\Gamma}^\nu_{\lambda\mu} + \delta^\nu_\lambda\,\partial_\mu\phi + \delta^\nu_\mu\,\partial_\lambda\phi - \widetilde{g}_{\lambda\mu}\,\widetilde{g}^{\nu\sigma}\,\partial_\sigma\phi\,.$$

The transformation of the covariant derivative of u^μ is given by

$$\nabla_\mu u^\nu = \partial_\mu u^\nu + \Gamma^\nu_{\mu\lambda}\,u^\lambda = e^{-\phi}\,\left[\widetilde{\nabla}_\mu\,\widetilde{u}^\nu + \delta^\nu_\mu\,\widetilde{u}^\sigma\,\partial_\sigma\phi - \widetilde{g}_{\mu\lambda}\,\widetilde{u}^\lambda\,\widetilde{g}^{\nu\sigma}\,\partial_\sigma\phi\right]. \tag{3.5.78}$$

This equation can be used to derive the transformation of various quantities of interest in fluid dynamics, such as the acceleration a^μ, shear $\sigma^{\mu\nu}$, *etc.*.

$$\begin{aligned}
\theta &= \nabla_\mu u^\mu = e^{-\phi}\,\left(\widetilde{\nabla}_\mu\widetilde{u}^\mu + 3\,\widetilde{u}^\sigma\,\partial_\sigma\phi\right) = e^{-\phi}\,\left(\widetilde{\theta} + 3\,\widetilde{\mathcal{D}}\phi\right)\ , \\
a^\nu &= \mathcal{D}u^\nu = u^\mu\nabla_\mu u^\nu = e^{-2\phi}\,\left(\widetilde{a}^\nu + \widetilde{P}^{\nu\sigma}\,\partial_\sigma\phi\right), \\
\sigma^{\mu\nu} &= P^{\lambda(\mu}\nabla_\lambda u^{\nu)} - \frac{1}{3}\,P^{\mu\nu}\,\nabla_\lambda u^\lambda = e^{-3\phi}\,\widetilde{\sigma}^{\mu\nu}, \\
\ell^\mu &= u_\alpha\,\epsilon^{\alpha\beta\gamma\mu}\nabla_\beta u_\gamma = e^{-2\phi}\,\widetilde{\ell}^\mu
\end{aligned} \tag{3.5.79}$$

where in the last equation we have accounted for the fact that all epsilon symbols in (3.1.9) should be generalized in curved space to their covariant counterparts. The objects with correct tensor transformation properties scale as metric determinants ie. $\epsilon_{\alpha\beta\gamma\delta} \propto \sqrt{g}$, and $\epsilon^{\alpha\beta\gamma\delta} \propto \frac{1}{\sqrt{g}}$, from which it is easy to infer their scaling behaviour under conformal transformations; in particular, $\epsilon_{\alpha\beta\gamma\delta} = e^{4\phi}\,\widetilde{\epsilon}_{\alpha\beta\gamma\delta}$ and $\epsilon^{\alpha\beta\gamma\delta} = e^{-4\phi}\,\widetilde{\epsilon}^{\alpha\beta\gamma\delta}$.

The Weyl transformation of the two derivative terms that occur in the stress tensor (3.1.9) is given by

$$\begin{aligned}
T^{\mu\nu}_A &= e^{-4\phi}\,\widetilde{T}^{\mu\nu}_A\ , && \text{for } A = \{2a, 2b\} \\
T^{\mu\nu}_B &= e^{-4\phi}\,\left(\widetilde{T}^{\mu\nu}_B + \widetilde{\delta T}^{\mu\nu}_B\right)\ , && \text{for } B = \{2c, 2d, 2e\}
\end{aligned} \tag{3.5.80}$$

where the inhomogeneous terms arising in the Weyl transformation are:

$$
\begin{aligned}
\delta T^{\mu\nu}_{2c} &= 3\,\mathcal{D}\phi\,\left(\nabla^{(\mu}u^{\nu)} + u^{(\mu}\,a^{\nu)} - \frac{1}{3}\,\theta\,P^{\mu\nu}\right)\\
\delta T^{\mu\nu}_{2d} &= 2\,a^{(\mu}\,\nabla^{\nu)}\phi + 2\,u^{(\mu}\,a^{\nu)}\,\mathcal{D}\phi - \frac{2}{3}\,a^{\alpha}\,\nabla_{\alpha}\phi\\
&\qquad + 2\,u^{(\mu}\,\nabla^{\nu)}\phi\,\mathcal{D}\phi + u^{\mu}\,u^{\nu}\,\left(\mathcal{D}\phi\right)^2 - \frac{1}{3}\,P^{\mu\nu}\,\left(\mathcal{D}\phi\right)^2 + \nabla^{\mu}\phi\,\nabla^{\nu}\phi - \frac{1}{3}\,P^{\mu\nu}\,\nabla^{\alpha}\phi\,\nabla_{\alpha}\phi\\
\delta T^{\mu\nu}_{2e} &= -\nabla^{(\mu}u^{\nu)}\,\mathcal{D}\phi - 3\,u^{(\mu}a^{\nu)}\,\mathcal{D}\phi + \frac{1}{3}\,P^{\mu\nu}\,\theta\,\mathcal{D}\phi - 2\,a^{(\mu}\,\nabla^{\nu)}\phi + \frac{2}{3}\,P^{\mu\nu}\,a^{\alpha}\,\nabla_{\alpha}\phi\\
&\qquad - u^{\mu}\,u^{\nu}(\mathcal{D}\phi)^2 + \frac{1}{3}\,P^{\mu\nu}\,(\mathcal{D}\phi)^2 - 2\,u^{(\mu}\,\nabla^{\nu)}\phi\,\mathcal{D}\phi - \nabla^{\mu}\phi\,\nabla^{\nu}\phi + \frac{1}{3}\,P^{\mu\nu}\,\nabla_{\alpha}\phi\,\nabla^{\alpha}\phi
\end{aligned}
\tag{3.5.81}
$$

While the conformal transformation involves the inhomogeneous terms presented in (3.5.81) we need to ensure that the full stress tensor is Weyl covariant. Satisfyingly, these inhomogeneous terms cancel among themselves in the precise combination that occurs in (3.1.8); consequently the linear combination of terms that occurs in the stress tensor transforms covariantly. Note that the cancelation of inhomogeneous terms depends sensitively on the ratio of coefficients of T_{2c}, T_{2d} and T_{2e}; and so provides a check of our results. Note however that T_{2a} and T_{2b} are separately Weyl covariant. In summary, our result for the two derivative stress tensor is a linear combination (with precisely determined coefficients) of three independently Weyl covariant forms, with scaling weight -4 (for upper indices).

Using the transformation of the temperature (3.5.77) it follow that the full stress tensor transforms under Weyl transformation as

$$
T^{\mu\nu} = e^{-6\phi}\,\widetilde{T}^{\mu\nu}\ . \tag{3.5.82}
$$

3.5.2 Spectrum of small fluctuations

Consider a static bath of homogeneous fluid at temperature T. Given the two derivative stress tensor derived above (3.1.9), it is trivial to solve for the spectrum of small oscillations of fluid dynamical modes about this background. As the background is translationally invariant, these fluctuations can be taken to have the form

$$\begin{aligned} \beta_i(v, x^j) &= \delta\beta_i \, e^{i\,\omega\, v + i\, k_j x^j} \\ T(v, x^j) &= 1 + \delta T \, e^{i\,\omega\, v + i\, k_j x^j} \end{aligned} \tag{3.5.83}$$

Plugging (3.5.83) into the equations of fluid dynamics (3.1.7), and working to first order in $\delta\beta_i$ and δT, these equations reduce to a set of four homogeneous linear equations in the amplitudes $\delta\beta_i$ and δT. The coefficients of these equations are functions of ω and k_i. These equations have nontrivial solutions if and only if the matrix formed out of these coefficient functions has zero determinant. Setting the determinant of the matrix of coefficients to zero one can find the following two dispersion relations:

$$\text{Sound mode :} \qquad \boldsymbol{\omega}(\mathbf{k}) = \pm \frac{\mathbf{k}}{\sqrt{3}} + \frac{i\,\mathbf{k}^2}{6} \pm \frac{(3 - \ln 4)}{24\sqrt{3}}\, \mathbf{k}^3 + \mathcal{O}\left(\mathbf{k}^4\right), \tag{3.5.84}$$

$$\text{Shear mode :} \qquad \boldsymbol{\omega}(\mathbf{k}) = \frac{i\,\mathbf{k}^2}{4} + \frac{i}{32}\,(2 - \ln 2)\;\mathbf{k}^4 + \mathcal{O}\left(\mathbf{k}^6\right), \tag{3.5.85}$$

where we have defined the rescaled energy and momenta

$$\boldsymbol{\omega} = \frac{\omega}{\pi\, T}\,, \qquad \mathbf{k} = \frac{k}{\pi\, T}\,. \tag{3.5.86}$$

1 of $SO(3)$	**3** of $SO(3)$	**5** of $SO(3)$
$\mathbf{s}1 = \frac{1}{b}\,\partial_v^2 b$	$\mathbf{v}1_i = \frac{1}{b}\,\partial_i\partial_v b$	$\mathbf{t}1_{ij} = \frac{1}{b}\,\partial_i\partial_j b - \frac{1}{3}\,\mathbf{s}3\,\delta_{ij}$
$\mathbf{s}2 = \partial_v\partial_i\beta_i$	$\mathbf{v}2_i = \partial_v^2\beta_i$	$\mathbf{t}2_{ij} = \partial_{(i}\ell_{j)}$
$\mathbf{s}3 = \frac{1}{b}\,\partial^2 b$	$\mathbf{v}3_i = \partial_v\ell_i$	$\mathbf{t}3_{ij} = \partial_v\sigma_{ij}$
$\mathfrak{S}1 = \partial_v\beta_i\,\partial_v\beta_i$	$\mathbf{v}4_i = \frac{9}{5}\partial_j\sigma_{ji} - \partial^2\beta_i$	$\mathfrak{T}1_{ij} = \partial_v\beta_i\,\partial_v\beta_j - \frac{1}{3}\,\mathfrak{S}1\,\delta_{ij}$
$\mathfrak{S}2 = \ell_i\,\partial_v\beta_i$	$\mathbf{v}5_i = \partial^2\beta_i$	$\mathfrak{T}2_{ij} = \ell_{(i}\,\partial_v\beta_{j)} - \frac{1}{3}\,\mathfrak{S}2\,\delta_{ij}$
$\mathfrak{S}3 = (\partial_i\beta_i)^2$	$\mathfrak{V}1_i = \frac{1}{3}(\partial_v\beta_i)(\partial_j\beta^j)$	$\mathfrak{T}3_{ij} = 2\,\epsilon_{kl(i}\,\partial_v\beta^k\,\partial_{j)}\beta^l + \frac{2}{3}\,\mathfrak{S}2\,\delta_{ij}$
$\mathfrak{S}4 = \ell_i\,\ell^i$	$\mathfrak{V}2_i = -\epsilon_{ijk}\,\ell^j\,\partial_v\beta^k$	$\mathfrak{T}4_{ij} = \partial_k\beta^k\,\sigma_{ij}$
$\mathfrak{S}5 = \sigma_{ij}\,\sigma^{ij}$	$\mathfrak{V}3_i = \sigma_{ij}\,\partial_v\beta^j$	$\mathfrak{T}5_{ij} = \ell_i\,\ell_j - \frac{1}{3}\,\mathfrak{S}4\,\delta_{ij}$
	$\mathfrak{V}4_i = \ell_i\,\partial_j\beta^j$	$\mathfrak{T}6_{ij} = \sigma_{ik}\,\sigma^k_j - \frac{1}{3}\,\mathfrak{S}5\,\delta_{ij}$
	$\mathfrak{V}5_i = \sigma_{ij}\,\ell^j$	$\mathfrak{T}7_{ij} = 2\,\epsilon_{mn(i}\,l^m\,\sigma^n_{j)}$

Table 3.1: An exhaustive list of two derivative terms in made up from the temperature and velocity fields. In order to present the results economically, we have dropped the superscript on the velocities β_i and the inverse temperature b, leaving it implicit that these expressions are only valid at second order in the derivative expansion.

Chapter 4

Event horizon and Entropy current

This chapter is based on [3].

4.1 The Local Event Horizon

As we have explained in the introduction, in this chapter we will study the event horizon of the metrics dual to fluid dynamics presented in the previous chapter. While the explicit dual gravity solution for a generic fluid dynamical state is rather involved, we will see below that the structure of the event horizons of these solutions are insensitive to many of these details. Consequently, in this section we will describe dual metric only in general structural form, and carry out all our computations for an arbitrary spacetime of this form. In § 4.4 we will specialize these calculations to the detailed form of the metric constructed in previous chapter. We start by presenting a geometric interpretation for the coordinate system used in the construction of the dual metric of [40] .

4.1.1 Coordinates adapted to a null geodesic congruence

Consider a null geodesic congruence (*i.e.*, a family of null geodesics with exactly one geodesic passing through each point) in some region of an arbitrary spacetime. Let Σ

be a hypersurface that intersects each geodesic once. Let x^μ be coordinates on Σ. Now ascribe coordinates (ρ, x^μ) to the point at an affine parameter distance ρ from Σ, along the geodesic through the point on Σ with coordinates x^μ. Hence the geodesics in the congruence are lines of constant x^μ. In this chart, this metric takes the form

$$ds^2 = -2\, u_\mu(x)\, d\rho\, dx^\mu + \widehat{\chi}_{\mu\nu}(\rho, x)\, dx^\mu\, dx^\nu, \tag{4.1.1}$$

where the geodesic equation implies that u_μ is independent of ρ. It is convenient to generalize slightly to allow for non-affine parametrization of the geodesics: let r be a parameter related to ρ by $d\rho/dr = \mathcal{S}(r, x)$. Then, in coordinates (r, x), the metric takes the form [1]

$$ds^2 = G_{MN}\, dX^M\, dX^N = -2\, u_\mu(x)\, \mathcal{S}(r, x)\, dr\, dx^\mu + \chi_{\mu\nu}(r, x)\, dx^\mu\, dx^\nu \tag{4.1.2}$$

Note that Σ could be spacelike, timelike, or null. We shall take Σ to be timelike.

This metric has determinant $-\mathcal{S}^2 \chi^{\mu\nu} u_\mu u_\nu \det \chi$, where $\chi^{\mu\nu}$ is the inverse of $\chi_{\mu\nu}$. Hence the metric and its inverse will be smooth if $\mathcal{S}$, u_μ and $\chi_{\mu\nu}$ are smooth, with $\mathcal{S} \neq 0$, $\chi_{\mu\nu}$ invertible, and $\chi^{\mu\nu}\, u_\mu$ timelike. These conditions are satisfied on, and outside, the horizons of the solutions that we shall discuss below.

4.1.2 Spacetime dual to hydrodynamics

The bulk metric of [40] was obtained in a coordinate system of the form (4.1.2) just described, where the role of Σ is played by the conformal boundary and the null geodesics are future-directed and ingoing at the boundary. The key assumption used to derive the solution is that the metric is a slowly varying function of x^μ or, more precisely, that the

[1] We use upper case Latin indices $\{M, N, \cdots\}$ to denote bulk directions, while lower case Greek indices $\{\mu, \nu, \cdots\}$ will refer to field theory or boundary directions. Furthermore, we use lower case Latin indices $\{a, b, i, j, \cdots\}$ to denote the spatial directions in the boundary. Finally, we use (x) to indicate the dependence on the four coordinates x^μ.

metric functions have a perturbative expansion (with a small parameter ϵ):

$$\mathcal{S}(r,x) = 1 - \sum_{k=1}^{\infty} \epsilon^k \, s_a^{(k)} \ , \tag{4.1.3}$$

$$\chi_{\mu\nu}(r,x) = -r^2 \, f(b\,r) \, u_\mu \, u_\nu + r^2 \, P_{\mu\nu} + \sum_{k=1}^{\infty} \epsilon^k \, \left(s_c^{(k)} \, r^2 \, P_{\mu\nu} + s_b^{(k)} \, u_\mu \, u_\nu + \, j_\nu^{(k)} \, u_\mu + j_\mu^{(k)} \, u_\nu + t_{\mu\nu}^{(k)} \right) \ . \tag{4.1.4}$$

The function $f(y)$ above has the form $f = 1 - \frac{1}{y^4}$; however, the only property of f that we will use is that $f(1) = 0$. The remaining functions ($s_a^{(k)}$, $s_b^{(k)} \ldots$) are all local functions of the inverse temperature $b(x)$ and the velocity $u^\mu(x)$ and the coordinate r, whose form was determined in [40] ; we however will not need the specific form of these functions for the present discussion. As far as the calculations in this section are concerned, the expressions $s_a^{(k)}, s_b^{(k)}, s_c^{(k)}, j_\mu^{(k)}$ and $t_{\mu\nu}^{(k)}$ may be thought of as arbitrary functions of r and x^μ. The tensor $P_{\mu\nu} = \eta_{\mu\nu} + u_\mu \, u_\nu$ is a co-moving spatial projector.

In the above formulae, ϵ is a formal derivative counting parameter. Any expression that multiplies ϵ^k in (4.1.3) and (4.1.4) is of k^{th} order in boundary field theory derivatives. Note that any boundary derivative of any of the functions above is always accompanied by an additional explicit power of ϵ. As in [40], all calculations in this chapter will be performed order by order in ϵ which is then set to unity in the final results. This is a good approximation when field theory derivatives are small in units of the local temperature.

As we have explained in the Introduction, the metrics presented in [40] simplify to the uniform black brane metric at late times. This metric describes a fluid configuration with constant u^μ and b. As the derivative counting parameter ϵ vanishes on constant configurations, all terms in the summation in (4.1.3) and (4.1.4) vanish on the uniform black brane configuration. The event horizon of this simplified metric is very easy to determine; it is simply the surface $r = \frac{1}{b}$. Consequently, the event horizon $\mathcal{H}$ of the metric (4.1.2) has a simple mathematical characterization; it is the unique null hypersurface that reduces exactly, at infinite time to $r = \frac{1}{b}$.

In § 4.1.3 we will describe a local construction of a null hypersurface in the metric (4.1.2). Our hypersurface will have the property that it reduces exactly to $r = 1/b$ when u^μ and b are constants, and therefore may be identified with the event horizon for spacetimes of the form (4.1.2) that settle down to constant u^μ and b at late times, as we expect for metrics dual to fluid dynamics. We will evaluate our result for the metrics of [40] in § 4.4 where we will use the explicit expressions for the functions appearing in (4.1.2).

4.1.3 The event horizon in the derivative expansion

When ϵ is set to zero and b and u_μ are constants, the surface $r = \frac{1}{b}$ is a null hypersurface in metrics (4.1.2). We will now determine the corrected equation for this null hypersurface at small ϵ, order by order in the ϵ expansion. As we have explained above, this hypersurface will be physically interpreted as the event horizon $\mathcal{H}$ of the metrics presented in [40].

The procedure can be illustrated with a simpler example. Consider the Vaidya spacetime, describing a spherically symmetric black hole with ingoing null matter:

$$ds^2 = -\left(1 - \frac{2\,m(v)}{r}\right)\,dv^2 + 2\,dv\,dr + r^2\,d\Omega^2\ . \tag{4.1.5}$$

Spherical symmetry implies that the horizon is at $r = r(v)$, with normal $n = dr - \dot{r}\,dv$. Demanding that this be null gives $r(v) = 2\,m(v) + 2\,r(v)\,\dot{r}(v)$, a first order ODE for $r(v)$. Solving this determines the position of the horizon *non-locally* in terms of $m(v)$. However, if we assume that $m(v)$ is slowly varying and approaches a constant for large v, *i.e.*,

$$\dot{m}(v) = \mathcal{O}(\epsilon)\ ,\, m\,\ddot{m} = \mathcal{O}(\epsilon^2),\ \textit{etc.}, \qquad \text{and} \qquad \lim_{v\to\infty} m(v) = m_0 \tag{4.1.6}$$

then we can solve by expanding in derivatives. Consider the ansatz, $r = 2\,m + a\,m\,\dot{m} + b\,m\,\dot{m}^2 + c\,m^2\,\ddot{m} + \ldots$, for some constants $a, b, c, \ldots$; it is easy to show that the solution for the horizon is given by $a = 8$, $b = 64$, $c = 32$, *etc.*. Hence we can obtain a *local* expression for the location of the horizon in a derivative expansion.

Returning to the spacetime of [40], let us suppose that the null hypersurface that we are after is given by the equation

$$S_{\mathcal{H}}(r,x) = 0 \ , \qquad \text{with} \qquad S_{\mathcal{H}}(r,x) = r - r_H(x) \ . \tag{4.1.7}$$

As we are working in a derivative expansion we take

$$r_H(x) = \frac{1}{b(x)} + \sum_{k=1}^{\infty} \epsilon^k \, r_{(k)}(x) \tag{4.1.8}$$

Let us denote the normal vector to the event horizon by ξ^A: by definition,

$$\xi^A = G^{AB} \, \partial_B S_{\mathcal{H}}(r,x) \tag{4.1.9}$$

which also has an ϵ expansion. We will now determine $r_{(k)}(x)$ and $\xi^A_{(k)}(x^\mu)$ order by order in ϵ. In order to compute the unknown functions $r_{(k)}(x)$ we require the normal vector ξ^A to be null, which amounts to simply solving the equation

$$G^{AB} \, \partial_A S_{\mathcal{H}} \, \partial_B S_{\mathcal{H}} = 0 \tag{4.1.10}$$

order by order in perturbation theory. Note that

$$dS_{\mathcal{H}} = dr - \epsilon \, \partial_\mu r_H \; dx^\mu \qquad \text{where} \qquad \epsilon \, \partial_\mu r_H = -\frac{\epsilon}{b^2} \, \partial_\mu b + \sum_{n=1}^{\infty} \epsilon^{n+1} \, \partial_\mu r_{(n)} \ . \tag{4.1.11}$$

In particular, to order ϵ^n, only the functions $r_{(m)}$ for $m \leq n-1$ appear in (4.1.11). However, the LHS of (4.1.10) includes a contribution of two factors of dr contracted with the metric. This contribution is equal to G^{rr} evaluated at the horizon. Expanding this term to order ϵ^n we find a contribution

$$\frac{1}{\kappa_1 \, b} \, r_{(n)}$$

where κ_1 is defined in (4.1.15) below, together with several terms that depend on $r_{(m)}$ for $m \leq n-1$. It follows that the expansion of (4.1.10) to n^{th} order in ϵ yields a simple algebraic expression for $r_{(n)}$, in terms of the functions $r_{(1)}, r_{(2)}, \cdots, r_{(n-1)}$ which are determined from lower order computations.

More explicitly, equation (4.1.10) gives $G^{rr} - 2\,\epsilon\,\partial_\mu r_H\, G^{r\mu} + \epsilon^2\,\partial_\mu r_H\,\partial_\nu r_H\, G^{\mu\nu} = 0$, with the inverse metric G^{MN} given by:

$$G^{rr} = \frac{1}{-\mathcal{S}^2\, u_\mu\, u_\nu\, \chi^{\mu\nu}}\ , \qquad G^{r\alpha} = \frac{\mathcal{S}\,\chi^{\alpha\beta}\, u_\beta}{-\mathcal{S}^2\, u_\mu\, u_\nu\, \chi^{\mu\nu}}\ , \qquad G^{\alpha\beta} = \frac{\mathcal{S}^2\, u_\gamma\, u_\delta\, \left(\chi^{\alpha\beta}\,\chi^{\gamma\delta} - \chi^{\alpha\gamma}\,\chi^{\beta\delta}\right)}{-\mathcal{S}^2\, u_\mu\, u_\nu\, \chi^{\mu\nu}}\ . \tag{4.1.12}$$

where the 'inverse d-metric' $\chi^{\mu\nu}$ is defined via $\chi_{\mu\nu}\,\chi^{\nu\rho} = \delta_\mu^{\ \rho}$. Hence the expression for the location of the event horizon (4.1.10) to arbitrary order in ϵ is obtained by expanding

$$0 = \frac{1}{-\mathcal{S}^2\, u_\mu\, u_\nu\, \chi^{\mu\nu}} \left(1 - 2\,\epsilon\,\mathcal{S}\,\chi^{\alpha\beta}\, u_\beta\,\partial_\alpha r_H - \epsilon^2\,\mathcal{S}^2\,\left(\chi^{\alpha\beta}\,\chi^{\gamma\delta} - \chi^{\alpha\gamma}\,\chi^{\beta\delta}\right)\, u_\gamma\, u_\delta\,\partial_\alpha r_H\,\partial_\beta r_H\right) \tag{4.1.13}$$

to the requisite order in ϵ, using the expansion of the individual quantities $\mathcal{S}$ and r_H specified above, as well as of $\chi^{\mu\nu}$.

4.1.4 The event horizon at second order in derivatives

The equation (4.1.10) is automatically obeyed at order ϵ^0. At first order in ϵ we find that the location of the event horizon is given by $r = r_H^{(1)}$ with[2]

$$r_H^{(1)}(x) = \frac{1}{b(x)} + r_{(1)}(x) = \frac{1}{b} + \kappa_1\,\left(s_b^{(1)} - \frac{2}{b^2}\, u^\mu\,\partial_\mu b\right)\ . \tag{4.1.14}$$

where we define

$$\frac{1}{\kappa_m} = \frac{\partial^m}{\partial r^m}\left(r^2\, f(b\, r)\right)\bigg|_{r=\frac{1}{b}} \tag{4.1.15}$$

At next order, $\mathcal{O}(\epsilon^2)$, we find

$$\begin{split} r_H^{(2)}(x) = \frac{1}{b} + \kappa_1 \Bigg(& s_b^{(1)} + \partial_r s_b^{(1)} r_H^{(1)} - \frac{2}{b^2}\,\left(1 - s_a^{(1)}\right)\, u^\mu\,\partial_\mu b + s_b^{(2)} + 2\, u^\mu\,\partial_\mu r_{(1)} \\ & - \frac{1}{b^2}\, P^{\mu\nu}\,\left(b^2\, j_\mu^{(1)} + \partial_\mu b\right)\,\left(b^2\, j_\nu^{(1)} + \partial_\nu b\right) - \frac{1}{2\,\kappa_2}\, r_{(1)}^2 \Bigg) \end{split} \tag{4.1.16}$$

[2]We have used here the fact that $u^\mu\, j_\mu^{(k)} = 0$ and $u^\mu\, t_{\mu\nu}^{(k)} = 0$ which follow from the solution of [40]. We also restrict to solutions which are asymptotically AdS_5 in this section.

where we have[3]

$$P^{\mu\nu} = u^\mu u^\nu + \eta^{\mu\nu} \qquad \text{and} \qquad \eta^{\mu\nu} = diag(-1,1,1,1) \ .$$

As all functions and derivatives in (4.1.14) and (4.1.16) are evaluated at $r = 1/b$ and the point x^μ and we retain terms to $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$ respectively.

It is now simple in principle to plug (4.1.16) into (4.1.2) to obtain an explicit expression for the metric $H_{\mu\nu}$ of the event horizon.[4] We will choose to use the coordinates x^μ to parameterize the event horizon. The normal vector ξ^A is a vector in the tangent space of the event horizon (this follows since the hypersurface is null), *i.e.*,

$$\xi^A \frac{\partial}{\partial X^A} = n^\mu \frac{\partial}{\partial x^\mu} + n^r \frac{\partial}{\partial r} \ , \tag{4.1.17}$$

which is easily obtained by using the definition (4.1.9) and the induced metric on the event horizon; namely

$$\begin{aligned} n^\mu = &\left(1 + s_a^{(1)} + (s_a^{(1)})^2 + s_a^{(2)}\right) u^\mu - \frac{1}{r^4} (t^{(1)})^{\mu\nu} \left(j_\nu^{(1)} + \frac{\partial_\nu b}{b^2}\right) \\ &+ \frac{1}{r^2} P^{\mu\nu} \left(j_\nu^{(1)} \left(1 + s_a^{(1)} - s_c^{(1)}\right) + \frac{\partial_\nu b}{b^2} \left(1 - s_c^{(1)}\right) + j_\nu^{(2)} - \partial_\nu r_{(1)}\right) \ . \end{aligned} \tag{4.1.18}$$

Before proceeding to analyze the entropy current associated with the local area-form on this event horizon, let us pause to consider the expression (4.1.16). First of all, we see that for generic fluids with varying temperature and velocity, the radial coordinate $r = r_H$ of the horizon varies with x^μ, which, to the first order in the derivative expansion, is given simply by the local temperature. The constraints on this variation are inherited from the

[3] It is important to note that in our expressions involving the boundary derivatives we raise and lower indices using the boundary metric $\eta_{\mu\nu}$; in particular, $u^\mu \equiv \eta^{\mu\nu} u_\nu$ and with this defintion $u^\mu u_\mu = -1$.

[4] There are thus three metrics in play; the bulk metric defined in (4.1.2), the boundary metric which is fixed and chosen to be $\eta_{\mu\nu}$ and finally the metric on the horizon $\mathcal{H}$, $H_{\mu\nu}$, which we do not explicitly write down. As a result there are differing and often conflicting notions of covariance; we have chosen to write various quantities consistently with boundary covariance since at the end of the day we are interested in the boundary entropy current.

equations of relativistic fluid dynamics which govern the behaviour of these temperature and velocity fields, as discussed above. Note that the variation of r_H at a given x^i and as a function of time, can of course be non-monotonic. As we will see in the next section, only the local area needs to increase. This is dual to the fact that while a local fluid element may warm up or cool down in response to interacting with the neighbouring fluid, the local entropy production is always positive. An example of the behaviour of $r_H(x)$ is

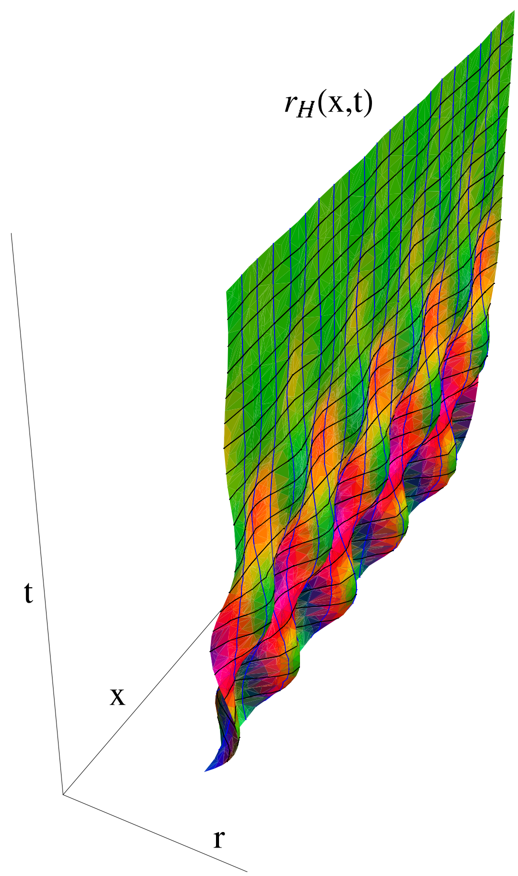

Figure 4.1: The event horizon $r = r_H(x^\mu)$ sketched as a function of the time t and one of the spatial coordinates x (the other two spatial coordinates are suppressed).

sketched in the spacetime diagram of Fig. 4.1, with time plotted vertically and the radial coordinate as well as one of the spatial x^i coordinates plotted horizontally.

4.2 The Local Entropy Current

Having determined the location of the event horizon, it is a simple matter to compute the area of the event horizon to obtain the area of the black brane. However, as we wish to talk about the spatio-temporal variation of the entropy, we will first describe entropy production in a local setting. This will allow us to derive an expression for the boundary entropy current in § 4.4.

4.2.1 Abstract construction of the area $(d-1)$-form

In this brief subsection we present the construction of the area $d-1$ form on the spatial section of any event horizon of a $d+1$ dimensional solution of general relativity.

First, recall that the event horizon is a co-dimension one null submanifold of the $d+1$ dimensional spacetime. As a result its normal vector lies in its tangent space. The horizon generators coincide with the integral curves of this normal vector field, which are in fact null geodesics[5] that are entirely contained within the event horizon. Let us choose coordinates (λ, α^a), with $a = 1, \cdots, d-1$, on the event horizon such that α^a are constant along these null geodesics and λ is a future directed parameter (not necessarily affine)

[5]This follows from the fact that the event horizon is the boundary of the past of future infinity $\mathcal{I}^+$ together with the fact that boundaries of causal sets are generated by null geodesics [41]. We pause here to note a technical point regarding the behaviour of the horizon generators: While by definition these null geodesics generating the event horizon have no future endpoints [42], they do not necessarily remain on the event horizon when extended into the past. This is because in general dynamical context, these geodesics will have non-zero expansion, and by Raychaudhuri's equation they must therefore caustic in finite affine parameter when extended into the past. Hence, although the spacetime, and therefore the event horizon, are smooth, the horizon generators enter the horizon at points of caustic. However, since the caustic locus forms a set of measure zero on the horizon, in the following discussion we will neglect this subtlety.

along the geodesics. As ∂_λ is orthogonal to every other tangent vector on the manifold including itself, it follows that the metric restricted on the event horizon takes the form

$$ds^2 = g_{ab}\, d\alpha^a\, d\alpha^b \tag{4.2.19}$$

Let g represent the determinant of the $(d-1)\times(d-1)$ metric g_{ab}. We define the entropy $(d-1)$-form as the appropriately normalized area form on the spatial sections of the horizon[6]

$$a = \frac{1}{4\, G_{d+1}}\, \sqrt{g}\, d\alpha^1 \wedge d\alpha^2 \wedge \ldots \wedge d\alpha^{d-1} \tag{4.2.20}$$

The area increase theorems of general relativity[7] are tantamount to the monotonicity of the function g, *i.e.*,

$$\frac{\partial g}{\partial \lambda} \geq 0 \tag{4.2.21}$$

which of course leads to

$$da = \frac{\partial_\lambda \sqrt{g}}{4\, G_{d+1}}\, d\lambda \wedge d\alpha^1 \wedge d\alpha^2 \ldots \wedge d\alpha^{d-1} \geq 0\ . \tag{4.2.22}$$

We have chosen here an orientation on the horizon $\mathcal{H}$ by declaring a d-form to be positive if it is a positive multiple of the d-form $d\lambda \wedge d\alpha^1 \wedge d\alpha^2 \ldots \wedge d\alpha^{d-1}$.

4.2.2 Entropy $(d-1)$-form in global coordinates

The entropy $(d-1)$-form described above was presented in a special set of α^a coordinates which are well adapted to the horizon. We will now evaluate this expression in terms of a more general set of coordinates. Consider a set of coordinates x^μ for the spacetime in the neighbourhood of the event horizon, chosen so that surfaces of constant $x^0 = v$ intersect the horizon on spacelike slices Σ_v. The coordinates x^μ used in (4.1.2) provide an example

[6]This definition is consistent with the Noether charge derivation of entropy currents, *a la* Wald, *cf.*, [43] for a discussion for dynamical horizons.

[7]We assume here that the null energy condition is satisfied. This is true of the Lagrangian used in [40] to construct the gravitation background (4.1.2).

of such a coordinate chart (as we will see these are valid over a much larger range than the neighbourhood of the horizon).

As surfaces of constant v are spacelike, the null geodesics that generate the event horizon each intersect any of these surfaces exactly once. Consequently, we may choose the coordinate v as a parameter along geodesics. Then we can label the geodesics by α^a, the value of x^a at which the geodesic in question intersects the surface $v = 0$. The coordinate system $\{v, \alpha^a\}$ is of the form described in § 4.2.1; as a result in these coordinates the entropy $(d-1)$-form is given by (4.2.20). We will now rewrite this expression in terms of the coordinates x^μ at $v = 0$; for this purpose we need the formulas for the change of coordinates from x^μ to $\{v, \alpha^a\}$, in a neighbourhood of $v = 0$. It is easy to verify that

$$\begin{aligned} x^a &= \alpha^a + \frac{n^a}{n^v}\, v + \frac{v^2}{2\, n^v}\, n^\mu\, \partial_\mu \left(\frac{n^a}{n^v}\right) + \mathcal{O}(v^3) \cdots \\ dx^a &= d\alpha^a + v\, d\alpha^k\, \partial_k \left(\frac{n^a}{n^v}\right) + dv\, \left(\frac{n^a}{n^v} + \frac{v}{n^v}\, n^\mu\, \partial_\mu \left(\frac{n^a}{n^v}\right)\right) + \mathcal{O}(v^2) \end{aligned} \tag{4.2.23}$$

The coordinate transformation (4.2.23) allows us to write an expression for the metric on the event horizon in terms of the coordinates $\{v, \alpha^a\}$, in a neighbourhood of $v = 0$. Let $H_{\mu\nu}\, dx^\mu\, dx^\nu = G_{MN}\, dx^M\, dx^N|_{\mathcal{H}}$ denote the metric restricted to the event horizon in the x^μ coordinates.

$$\begin{aligned} ds^2_{\mathcal{H}} &= H_{\mu\nu}(x)\, dx^\mu\, dx^\nu \equiv g_{ab}\, d\alpha^a\, d\alpha^b \\ &= h_{ij}\left(v, \alpha^i + \frac{n^i}{n^v}\right) \left(d\alpha^i + v\, d\alpha^k\, \partial_k \left(\frac{n^i}{n^v}\right)\right) \left(d\alpha^j + v\, d\alpha^k\, \partial_k \left(\frac{n^j}{n^v}\right)\right) + \mathcal{O}(v^2) \end{aligned} \tag{4.2.24}$$

where $h_{ij}(v, x)$ is the restriction of the metric $H_{\mu\nu}$ onto a spatial slice Σ_v, which is a constant-v slice. Note that since the horizon is null, all terms with explicit factors of dv cancel from (4.2.24) in line with the general expectations presented in § 4.2.1. It follows that the determinant of the induced metric, $\sqrt{g}$ of (4.2.20), is given as

$$\sqrt{g} = \sqrt{h} + \frac{v}{n^v}\, \left(n^i\, \partial_i \sqrt{h} + \sqrt{h}\, n^v\, \partial_i \frac{n^i}{n^v}\right) + \mathcal{O}(v^2)\ , \tag{4.2.25}$$

where h is the determinant of the metric on Σ_v, in x^μ coordinates (restricted to $v = 0$).

We are now in a position to evaluate the area $(d-1)$-form

$$a = \frac{\sqrt{h}}{4\, G_{d+1}}\, d\alpha^1 \wedge d\alpha^2 \ldots \wedge d\alpha^{d-1} \ , \tag{4.2.26}$$

at $v = 0$. Clearly, for this purpose we can simply set to zero all terms in (4.2.23) with explicit powers of v, which implies that $d\alpha^a = dx^a - \frac{n^a}{n^v}\, dv$ and

$$a = \frac{\sqrt{h}}{4\, G_{d+1}} \left(dx^1 \wedge dx^2 \ldots \wedge dx^{d-1} - \sum_{i=1}^{d-1} \frac{n^i}{n^v}\, d\lambda \wedge dx^1 \wedge \ldots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \ldots \wedge dx^{d-1} \right) \tag{4.2.27}$$

From (4.2.27) we can infer that the area-form can be written in terms a current as

$$a = \frac{\epsilon_{\mu_1 \mu_2 \ldots \mu_d}}{(d-1)!}\, J_S^{\mu_1}\, dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_d} \tag{4.2.28}$$

where J_S^μ is given by

$$J_S^\mu = \frac{\sqrt{h}}{4\, G_N^{(d+1)}}\, \frac{n^\mu}{n^v} \tag{4.2.29}$$

and our choice of orientation leads to $\epsilon_{v12\cdots(d-1)} = 1$. We can further establish that

$$da = \frac{1}{(d-1)!} \epsilon_{\mu_1 \mu_2 \ldots \mu_d}\, \partial_\alpha J_S^\alpha\, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_d} \tag{4.2.30}$$

so that da is simply the flat space Hodge dual of $\partial_\mu J_S^\mu$. While the appearance of the flat space Hodge dual might be puzzling at first sight, given the non-flat metric on $\mathcal{H}$, its origins will become clear once we recast this discussion in terms of the fluids dynamical variables.

4.2.3 Properties of the area-form and its dual current

Having derived the expression for the area-form we pause to record some properties which will play a role in interpreting J_S^μ as an entropy current in hydrodynamics.

Non-negative divergence: Firstly, we note that the positivity of da (argued for on general grounds in § 4.2.1) guarantees the positivity of $\partial_\mu J^\mu_S$; hence we have $\partial_\mu\, J^\mu_S \geq 0$. This in fact may be verified algebraically from (4.2.25), as

$$\frac{1}{4G_{d+1}}\partial_v(\sqrt{g}) = \partial_\mu J^\mu_S \ . \tag{4.2.31}$$

The positivity of $\partial_v(\sqrt{g})$ thus guarantees that of $\partial_\mu J^\mu_S$ as is expected on general grounds.

Lorentz invariance: The final result for our entropy current, (4.2.30), is invariant under Lorentz transformations of the coordinate x^μ (a physical requirement of the entropy current for relativistic fluids) even though this is not manifest. We now show that this is indeed the case.

Let us boost to coordinates $\hat{x}^\mu = \Lambda_\nu{}^\mu x^\nu$; denoting the horizon metric in the new coordinates by $\hat{h}_{\mu\nu}$ and the boosted normal vector by $\hat{n}^\mu$ we find

$$h_{ij} = A_i{}^m\, A_j{}^n\, \hat{h}_{mn}, \qquad A_i{}^m = \Lambda_i{}^m - \frac{\Lambda_i{}^v\, \hat{n}^m}{\hat{n}^v} \tag{4.2.32}$$

(where we have used $\hat{n}^\mu \hat{h}_{\mu\nu} = 0$). It is not difficult to verify that

$$\det A = \frac{\left(\Lambda^{-1}\right)_\mu{}^v\, n^\mu}{\hat{n}^v} = \frac{n^v}{\hat{n}^v}$$

from which it follows that $\frac{\sqrt{h}}{n^v} = \frac{\sqrt{\hat{h}}}{\hat{n}^v}$, thereby proving that our area-form defined on the a spatial section of the horizon is indeed Lorentz invariant.

4.3 The Horizon to Boundary Map

4.3.1 Classification of ingoing null geodesics near the boundary

Our discussion thus far has been an analysis of the causal structure of the spacetime described by the metric in (4.1.2) and the construction of an area-form on spatial sections of the horizon in generic spacetimes. As we are interested in transporting information about the entropy from the horizon to the boundary (where the fluid lives), we need to define a map between the boundary and the horizon. The obvious choice is to map the point on the boundary with coordinates x^μ to the point on the horizon with coordinates $(r_H(x), x^\mu)$. More geometrically, this corresponds to moving along the geodesics $x^\mu =$ constant. However, congruences of null geodesics shot inwards from the boundary of AdS are far from unique. Hence, we digress briefly to present a characterization of the most general such congruence. In § 4.3.2 we will then see how the congruence of geodesics with constant x^μ fits into this general classification.

We will find it simplest to use Fefferman-Graham coordinates to illustrate our point. Recall that any asymptotically AdS_{d+1} spacetime may be put in the form

$$ds^2 = \frac{du^2 + \left(\eta_{\mu\nu} + u^d\,\phi_{\mu\nu}(w)\right) dw^\mu\, dw^\nu}{u^2}\ , \tag{4.3.33}$$

in the neighbourhood of the boundary. The collection of null geodesics that intersect the boundary point $(w^\mu, u = 0)$ are given by the equations

$$\frac{dw^A}{d\lambda} = u^2\,\left(t^A + \mathcal{O}\left(u^d\right)\right) \tag{4.3.34}$$

where A runs over the $d+1$ variables $\{u, w^\mu\}$ and the null tangent vector must obey $t^A\,t_A = 0$. It is always possible to re-scale the affine parameter to set $t^u = 1$; making this choice, our geodesics are labelled by a d-vector t^μ satisfying $\eta_{\mu\nu}\,t^\mu\,t^\nu = -1$. With these conventions t^μ may be regarded as a d-velocity. In summary, the set of ingoing null geodesics that emanate from any given boundary point are parameterized by the $d-1$ directions in which they can go – this parameterization is conveniently encapsulated in terms of a unit normalized timelike d-vector t^μ which may, of course, be chosen as an arbitrary function of x^μ. Consequently, congruences of ingoing null geodesics are parameterized by an arbitrary d-velocity field, $t^\mu(x)$ on the boundary of AdS.

4.3.2 Our choice of $t^\mu(x)$

It is now natural to ask what $t^\mu(x)$ is for the congruence defined by $x^\mu = const$ in the coordinates of [40]. The answer to this question is easy to work out, and turns out to be satisfyingly simple: for this choice of congruence, $t^\mu(x) = u^\mu(x)$ where $u^\mu(x)$ is the velocity field of fluid dynamics![8]

While metrics dual to fluid dynamics are automatically equipped with a velocity field, it is in fact also possible to associate a velocity field with a much larger class of asymptotically AdS spacetimes. Recall that any such spacetime has a boundary stress tensor $T_{\mu\nu}$.[9] For most such spacetimes there is a natural velocity field associated with this stress tensor; the velocity $u^\mu(x)$ to which one has to boost in order that T^{0i} vanish at the point x. More invariantly, $u^\mu(x)$ is chosen to be the unique timelike eigenvector of the matrix $T^\mu_\nu(x)$.[10] That is, we choose $u^\mu(x)$ to satisfy

$$(\eta_{\mu\nu} + u_\mu\, u_\nu)\, T^{\nu\kappa}\, u_\kappa = 0 \tag{4.3.36}$$

This definition of $u^\mu(x)$ coincides precisely with the velocity field in [40] (this is the so-called Landau frame). The null congruence given by $t^\mu(x) = u^\mu(x)$ is now well defined for an arbitrary asymptotically AdS spacetime, and reduces to the congruence described earlier in this section for the metrics dual to fluid dynamics.

[8] In order to see this note that

$$\begin{aligned} u_\mu \frac{dx^\mu}{d\lambda} &= u_\mu \frac{dw^\mu}{d\lambda} + \frac{du}{d\lambda} \\ P_{\nu\mu} \frac{dx^\mu}{d\lambda} &= \mathcal{P}_{\nu\mu} \frac{dw^\mu}{d\lambda} \end{aligned} \tag{4.3.35}$$

whereas indicated quantities on the LHS of (4.3.35) refer to the coordinate system of [40], the quantities on the RHS refer to the Fefferman-Graham coordinates (4.3.33). It follows from these formulae that the geodesic with $t^A = (1, u^\mu)$ maps to the null geodesic $\frac{dx^\mu}{d\lambda} = 0$ in the coordinates used to write (4.1.2).

[9] In a general coordinate system the stress tensor is proportional to the extrinsic curvature of the boundary slice minus local counter-term subtractions. In the Fefferman-Graham coordinate system described above, the final answer is especially simple; $T_{\mu\nu} \propto \phi_{\mu\nu}(x^\mu)$.

[10] This prescription breaks down when u^μ goes null - *i.e.*, if there exist points at which the energy moves at the speed of light.

4.3.3 Local nature of the event horizon

As we have seen in § 4.1 above, the event horizon is effectively local for the metrics dual to fluid dynamics such as (4.1.2). In particular, the position of the event horizon $r_H(x^\mu)$ depends only on the values and derivatives of the fluid dynamical variables in a neighbourhood of x^μ and not elsewhere in spacetime. Given the generic teleological behaviour of event horizons (which requires knowledge of the entire future evolution of the spacetime), this feature of our event horizons is rather unusual. To shed light on this issue, we supply an intuitive explanation for this phenomenon, postponing the actual evaluation of the function $r_H(x^\mu)$ to § 4.4.1.

The main idea behind our intuitive explanation may be stated rather simply. As we have explained above, the metric of [40] is tube-wise well approximated by tubes of the metric of a uniform black brane at constant velocity and temperature. Now consider a uniform black brane whose parameters are chosen as $u_\mu = (-1, 0, 0, 0)$ and $b = 1/(\pi T) = 1$ by a choice of coordinates. In this metric a radial outgoing null geodesic that starts at $r = 1 + \delta$ (with $\delta \gg \epsilon$) and $v = 0$ hits the boundary at a time $\delta v = \int \frac{dr}{r^2 f(r)} \approx -4 \ln \delta$. Provided this radial outgoing geodesic well approximates the path of a geodesic in the metric of [40] throughout its trajectory, it follows that the starting point of this geodesic lies outside the event horizon of the spacetime.

The two conditions for the approximation described above to be valid are:

1. That geodesic in question lies within the tube in which the metric of [40] is well approximated by a black brane with constant parameters throughout its trajectory. This is valid when $\delta v \approx -4 \ln \delta \ll 1/\epsilon$.

2. That even within this tube, the small corrections to the metric of [40] do not lead to large deviations in the geodesic. Recall that the radial geodesic in the metric of [40] is given by the equation

$$\frac{dv}{dr} = -\frac{G_{rv} + \mathcal{O}(\epsilon)}{G_{vv} + \mathcal{O}(\epsilon)} = \frac{2 + \mathcal{O}(\epsilon)}{f(r) + \mathcal{O}(\epsilon)}.$$

This geodesic well approximates that of the uniform black brane provided the $\mathcal{O}(\epsilon)$ corrections above are negligible, a condition that is met provided $f(r) \gg \epsilon$, *i.e.*, when $|r-1| = \delta \gg \epsilon$.

Restoring units we conclude that a point at $r = \frac{1}{b}(1+\delta)$ necessarily lies outside the event horizon provided $\delta \gg \epsilon$ (this automatically ensures $\delta v \approx -4 \ln \delta \ll 1/\epsilon$. when ϵ is small).

In a similar fashion it is easy to convince oneself that all geodesics that are emitted from $r = \frac{1}{b}(1-\delta)$ hit the singularity within the regime of validity of the tube approximation provided $\delta \gg \epsilon$. Such a point therefore lies inside the event horizon. It follows that the event horizon in the solutions of [40] is given by the hypersurface $r = \pi T (1 + \mathcal{O}(\epsilon))$.

4.4 Specializing to Dual Fluid Dynamics

We will now proceed to determine the precise form of the event horizon manifold to second order in ϵ using the results obtained in § 4.1. This will be useful to construct the entropy current in the fluid dynamics utilizing the map derived in § 4.3.

4.4.1 The local event horizon dual to fluid dynamics

The metric dual to fluid flows given in [40] takes the form (4.1.2) with explicitly determined forms of the functions in that metric . We list the properties and values of these functions that we will need below: [11]

$$
\begin{aligned}
&f(1) = 0, \qquad s_a^{(1)} = 0, \qquad s_c^{(1)} = 0, \\
&s_b^{(1)} = \frac{2}{3}\,\frac{1}{b}\,\partial_\mu u^\mu \ , \qquad \partial_r s_b^{(1)} = \frac{2}{3}\,\partial_\mu u^\mu \ , \\
&j_\mu^{(1)} = -\frac{1}{b}\,u^\nu\,\partial_\nu u_\mu \ , \qquad t_{\mu\nu}^{(1)} = \frac{1}{b}\left(\frac{3}{2}\ln 2 + \frac{\pi}{4}\right)\sigma_{\mu\nu} \equiv F\,\sigma_{\mu\nu} \\
&s_a^{(2)} = \frac{3}{2}\,s_c^{(2)} = \frac{b^2}{16}\left(2\,\mathfrak{S}_4 - \mathfrak{S}_5\,\left(2 + 12\,\mathcal{C} + \pi + \pi^2 - 9\,(\ln 2)^2 - 3\pi\,\ln 2 + 4\,\ln 2\right)\right) \\
&s_b^{(2)} = -\frac{2}{3}\,\mathbf{s}_3 + \mathfrak{S}_1 - \frac{1}{9}\,\mathfrak{S}_3 - \frac{1}{12}\,\mathfrak{S}_4 + \mathfrak{S}_5\,\left(\frac{1}{6} + \mathcal{C} + \frac{\pi}{6} + \frac{5\,\pi^2}{48} + \frac{2}{3}\ln 2\right) \\
&j_\mu^{(2)} = \frac{1}{16}\mathbf{B}^\infty - \frac{1}{144}\mathbf{B}^{\text{fin}}
\end{aligned}
\tag{4.4.37}
$$

where $\mathcal{C}$ is the Catalan number. We encounter here various functions (of the boundary coordinates) which are essentially built out the fluid velocity u^μ and its derivatives. These have been abbreviated to symbols such as $\mathbf{s}_3$, $\mathfrak{S}_1$, *etc.*, and are defined as

$$
\begin{aligned}
&\mathbf{s}_3 = \frac{1}{b}P^{\alpha\beta}\,\partial_\alpha\partial_\beta b \qquad \mathfrak{S}_1 = \mathcal{D}u^\alpha\,\mathcal{D}u_\alpha \ , \qquad \mathfrak{S}_2 = \ell_\mu\,\mathcal{D}u^\mu \\
&\mathfrak{S}_3 = (\partial_\mu u^\mu)^2 \ , \qquad \mathfrak{S}_4 = \ell_\mu\,\ell^\mu \ , \qquad \mathfrak{S}_5 = \sigma_{\mu\nu}\,\sigma^{\mu\nu} \ .
\end{aligned}
\tag{4.4.38}
$$

Likewise $\mathbf{B}^\infty$ and $\mathbf{B}^{\text{fin}}$ are defined as

$$
\begin{aligned}
\mathbf{B}^\infty &= 4\,\left(10\,\mathbf{v}_4 + \mathbf{v}_5 + 3\,\mathfrak{V}_1 - 3\,\mathfrak{V}_2 - 6\,\mathfrak{V}_3\right) \\
\mathbf{B}^{\text{fin}} &= 9\,\left(20\,\mathbf{v}_4 - 5\,\mathfrak{V}_2 - 6\,\mathfrak{V}_3\right),
\end{aligned}
\tag{4.4.39}
$$

Using the equation for the conservation of stress tensor ($\partial_\mu T^{\mu\nu} = 0$) up to second order in derivatives one can simplify the expression for r_H (4.1.16). Conservation of stress tensor gives

$$
\partial_\nu\left[\frac{1}{b^4}\,\left(\eta^{\mu\nu} + 4u^\mu u^\nu\right)\right] = \partial_\nu\left[\frac{2}{b^3}\sigma^{\mu\nu}\right] \tag{4.4.40}
$$

[11] Since we require only the values of the functions appearing in the metric (4.1.3) and (4.1.4) at $r = 1/b$ to evaluate (4.1.16), we present here the functions evaluated at this specific point.

Projection of (4.4.40) into the co-moving and transverse directions, achieved by contracting it with u_μ and $P_{\mu\nu}$ respectively, we find

$$\begin{aligned} s_b^{(1)} - \frac{2}{b^2}\, u^\mu\, \partial_\mu b &= \frac{1}{3}\, \sigma_{\mu\nu}\, \sigma^{\mu\nu} = \mathcal{O}\left(\epsilon^2\right) \\ P^{\mu\nu}\left(b^2\, j^{(1)}_\mu + \partial_\mu b\right) &= -\frac{b^2}{2}\, P_\mu^{\ \nu}\left(\partial_\alpha \sigma^{\alpha\mu} - 3\, \sigma^{\mu\alpha}\, u^\beta\, \partial_\beta u_\alpha\right) + \mathcal{O}\left(\epsilon^3\right) \end{aligned} \tag{4.4.41}$$

Inserting (4.4.41) into (4.1.14) we see that $r_{(1)}$ of (4.1.14) simply vanishes for the spacetime dual to fluid dynamics, and so, to first order in ϵ, $r_H^{(1)} = \frac{1}{b}$. At next order this formula is corrected to

$$r_H^{(2)} = \frac{1}{b(x)} + r_{(2)}(x) = \frac{1}{b} + \frac{b}{4}\left(s_b^{(2)} + \frac{1}{3}\, \sigma_{\mu\nu}\, \sigma^{\mu\nu}\right) \tag{4.4.42}$$

In order to get this result we have substituted into (4.1.16) the first of (4.4.41), utilized the fact that $r_{(1)} = 0$ and the observation (from the second line of (4.4.41)) that

$$P^{\mu\nu}\, \left(b^2 j^{(1)}_\mu + \partial_\mu b\right)\, \left(b^2 j^{(1)}_\nu + \partial_\nu b\right) = \mathcal{O}(\epsilon^4)$$

In this special case the components of normal vector in the boundary directions (4.1.18) (accurate to $\mathcal{O}\left(\epsilon^2\right)$) are given by

$$n^\mu = \left(1 + s_a^{(2)}\right) u^\mu - \frac{b^2}{2}\, P^{\mu\nu}\, \left(\partial^\alpha \sigma_{\alpha\nu} - 3\, \sigma_{\nu\alpha}\, u^\beta\, \partial_\beta u^\alpha\right) + b^2\, P^{\mu\nu}\, j^{(2)}_\nu\ . \tag{4.4.43}$$

4.4.2 Entropy current for fluid dynamics

We will now specialize the discussion of § 4.2.2 to the metric of [40], using the formulae derived in § 4.4.1. In the special case of the metric of [40] we have

$$\begin{aligned} \sqrt{g} &= \frac{1}{b^3}\left(1 - \frac{b^4}{4}\, F^2\, \sigma_{\mu\nu}\, \sigma^{\mu\nu} + 3\, b\, r_{(2)} + s_a^{(2)}\right) \\ &= \frac{1}{b^3}\left(1 - \frac{b^4}{4} F^2\, \sigma_{\mu\nu}\, \sigma^{\mu\nu} + \frac{b^2}{4} \sigma_{\mu\nu}\, \sigma^{\mu\nu} + \frac{3\, b^2}{4}\, s_b^{(2)} + s_a^{(2)}\right)\ , \end{aligned} \tag{4.4.44}$$

where the various quantities are defined in (4.4.37). We conclude from (4.2.29) that

$$\begin{aligned} 4\, G_N^{(5)}\, b^3\, J^\mu_S = u^\mu\, &\left(1 - \frac{b^4}{4}\, F^2\, \sigma_{\alpha\beta}\, \sigma^{\alpha\beta} + \frac{b^2}{4} \sigma_{\alpha\beta}\, \sigma^{\alpha\beta} + \frac{3\, b^2}{4}\, s_b^{(2)} + s_a^{(2)}\right) \\ &+ b^2\, P^{\mu\nu}\left[-\frac{1}{2}\left(\partial^\alpha \sigma_{\alpha\nu} - 3\, \sigma_{\nu\alpha}\, u^\beta\, \partial_\beta u^\alpha\right) + j^{(2)}_\nu\right]\ . \end{aligned} \tag{4.4.45}$$

This is the expression for the fluid dynamical entropy current which we derive from the gravitational dual.

4.5 Divergence of the Entropy Current

In previous sections, we have presented a gravitational construction of an entropy current which, we have argued, is guaranteed to have non-negative divergence at each point. We have also presented an explicit construction of the entropy current to order ϵ^2 in the derivative expansion. In this section we directly compute the divergence of our entropy current and verify its positivity. We will find it useful to first start with an abstract analysis of the most general Weyl invariant entropy current in fluid dynamics and compute its divergence, before specializing to the entropy current constructed above.

4.5.1 The most general Weyl covariant entropy current and its divergence

The entropy current in d-dimensions has to be a Weyl covariant vector of weight d. We will work in four dimensions ($d = 4$) in this section, and so will consider currents that are Weyl covariant vector of weight 4. Using the equations of motion, it may be shown that there exists a 7 dimensional family of two derivative weight 4 Weyl covariant vectors that have the correct equilibrium limit for an entropy current. In the notation of [44], (reviewed in § 4.6), this family may be parameterized as

$$\begin{aligned}(4\pi\,\eta)^{-1}\,J_S^\mu = 4\,G_N^{(5)}\,b^3\,J_S^\mu = &\left[1 + b^2\left(A_1\,\sigma_{\alpha\beta}\,\sigma^{\alpha\beta} + A_2\,\omega_{\alpha\beta}\,\omega^{\alpha\beta} + A_3\,\mathcal{R}\right)\right]u^\mu \\ &+ b^2\,\left[B_1\,\mathcal{D}_\lambda\sigma^{\mu\lambda} + B_2\,\mathcal{D}_\lambda\omega^{\mu\lambda}\right] \\ &+ C_1\;b\;\ell^\mu + C_2\;b^2 u^\lambda\,\mathcal{D}_\lambda\ell^\mu + \dots\end{aligned} \tag{4.5.46}$$

where $b = (\pi T)^{-1}, \eta = (16\pi\,G_N^{(5)}\,b^3)^{-1}$ and the rest of the notation is as in [44] (see also § 4.6).

In § 4.6 we have computed the divergence of this entropy current (using the third order equations of motion derived and expressed in Weyl covariant language in [44]). Our final result is

$$
\begin{aligned}
4\, G_N^{(5)}\, b^3\, \mathcal{D}_\mu J_S^\mu = \frac{b}{2}\, \Bigg[& \sigma_{\mu\nu} + b\, \left(2\, A_1 + 4\, A_3 - \frac{1}{2} + \frac{1}{4}\, \ln 2\right)\, u^\lambda \mathcal{D}_\lambda \sigma^{\mu\nu} + 4\, b\, (A_2 + A_3)\, \omega^{\mu\alpha} \omega_\alpha{}^\nu \\
& + b\, (4\, A_3 - \frac{1}{2})(\sigma^{\mu\alpha}\, \sigma_\alpha{}^\nu) + b\, C_2\, \mathcal{D}^\mu \ell^\nu \Bigg]^2 \\
& + (B_1 - 2A_3)\, b^2\, \mathcal{D}_\mu \mathcal{D}_\lambda \sigma^{\mu\lambda} + (C_1 + C_2)\, b^2\, \ell_\mu \mathcal{D}_\lambda \sigma^{\mu\lambda} + \ldots
\end{aligned}
\tag{4.5.47}
$$

Note that the leading order contribution to the divergence of the arbitrary entropy current is proportional to $\sigma_{\mu\nu}\, \sigma^{\mu\nu}$. This term is of second order in the derivative expansion, and is manifestly non-negative. In addition the divergence has several terms at third order in the derivative expansion.

Within the derivative expansion the second order piece dominates all third order terms whenever it is nonzero. However it is perfectly possible for $\sigma_{\mu\nu}$ to vanish at a point – $\sigma_{\mu\nu}$ are simply 5 of several independent Taylor coefficients in the expansion of the velocity field at a point (see § 4.7 for details). When that happens the third order terms are the leading contributions to $\mathcal{D}_\mu J_S^\mu$. Since such terms are cubic in derivatives they are odd orientation reversal ($x^\mu \to -x^\mu$), and so can be non-negative for all velocity configurations only if they vanish identically. We conclude that positivity requires that the RHS of (4.5.47) vanish upon setting $\sigma_{\mu\nu}$ to zero.

As is apparent, all terms on the first two lines of (4.5.47) are explicitly proportional to $\sigma_{\mu\nu}$. The two independent expressions on the third line of that equation are in general nonzero even when $\sigma_{\mu\nu}$ vanishes. As a result $\mathcal{D}_\mu J_S^\mu \geq 0$ requires that the second line of (4.5.47) vanish identically; hence, we obtain the following constraints on coefficients of the second order terms in the entropy current

$$
B_1 = 2\, A_3 \qquad\qquad C_1 + C_2 = 0 \tag{4.5.48}
$$

for a non-negative divergence entropy current.

These two conditions single out a 5 dimensional submanifold of non-negative divergence entropy currents in the 7 dimensional space (4.5.46) of candidate Weyl covariant entropy currents.

Since a local notion of entropy is an emergent thermodynamical construction (rather than a first principles microscopic construct), it seems reasonable that there exist some ambiguity in the definition of a local entropy current. We do not know, however, whether this physical ambiguity is large enough to account for the full 5 parameter non uniqueness described above, or whether a physical principle singles out a smaller sub family of this five dimensional space as special. Below we will see that our gravitational current - which is special in some respects - may be generalized to a two dimensional sub family in the space of positive divergence currents.

4.5.2 Positivity of divergence of the gravitational entropy current

It may be checked (see § 4.6) that our entropy current (4.4.45) may be rewritten in the form (4.5.46) with the coefficients

$$\begin{gathered}A_1 = \frac{1}{4} + \frac{\pi}{16} + \frac{\ln 2}{4}; \qquad A_2 = -\frac{1}{8}; \qquad A_3 = \frac{1}{8} \\ B_1 = \frac{1}{4}; \qquad B_2 = \frac{1}{2} \\ C_1 = C_2 = 0\end{gathered} \tag{4.5.49}$$

It is apparent that the coefficients listed in (4.5.49) obey the constraints of positivity (4.5.48). This gives a direct algebraic check of the positivity of the divergence of (4.4.45).

The fact that it is possible to write the current (4.4.45) in the form (4.5.46) also demonstrates the Weyl covariance of our current (4.4.45).

4.5.3 A two parameter class of gravitational entropy currents

As we have seen above, there exists a five parameter set of non-negative divergence conformally covariant entropy currents that have the correct equilibrium limit. An example of such a current was first constructed in [44].

Now let us turn to an analysis of possible generalizations of the gravitational entropy current presented in this chapter. Our construction admits two qualitatively distinct, reasonable sounding, generalizations that we now discuss.

Recall that we constructed our entropy $(d-1)$-form via the pullback of the area-form on the event horizon. While the area-form is a very natural object, all its physically important properties (most importantly the positivity of divergence) appear to be retained if we add to it the exterior derivative of a $(d-2)$-form. This corresponds to the addition of the exterior derivative of a $(d-2)$-form to the entropy current J_S^μ. Imposing the additional requirement of Weyl invariance at the two derivative level this appears to give us the freedom to add a multiple of $\frac{1}{b}\mathcal{D}_\lambda \omega^{\lambda\sigma}$ to the entropy current in four dimensions.

In addition, we have the freedom to modify our boundary to horizon map in certain ways; our construction of the entropy current (4.4.45) depends on this map and we have made the specific choice described in § 4.3. Apart from geometrical naturalness and other aesthetic features, our choice had two important properties. First, under this map $r_H(x^\mu)$ (and hence the local entropy current) was a local function of the fluid dynamical variables at x^μ. Second, our map was Weyl covariant; in particular, the entropy current obtained via this map was automatically Weyl covariant. We will now parameterize all boundary to horizon maps (at appropriate order in the derivative expansion) that preserve these two desirable properties.

Any one to one boundary to horizon map may be thought of as a boundary to boundary diffeomorphism compounded with the map presented in § 4.3. In order to preserve the locality of the entropy current, this diffeomorphism must be small (*i.e.*, of sub-leading order in the derivative expansion). At the order of interest, it turns out to be sufficient to study diffeomorphisms parameterized by a vector $\delta\zeta$ that is of at most first order in the derivative expansion. In order that our entropy current have acceptable Weyl transformation properties under this map, $\delta\zeta$ must be Weyl invariant. Up to terms that vanish by the equations of motion, this singles out a two parameter set of acceptable choices for $\delta\zeta$;

$$\delta\zeta^\mu = 2\ \delta\lambda_1\ b\ u^\mu + \delta\lambda_2\ b^2\ \ell^\mu \tag{4.5.50}$$

To leading order the difference between the $(d-1)$-forms obtained by pulling the area $(d-1)$-form a back under the two different maps is given by the Lie derivative of the pull-back s of a

$$\delta s = \mathcal{L}_{\delta\zeta}\, s = d(\delta\zeta_\mu\, s^\mu) + \delta\zeta_\mu\, (ds)^\mu.$$

Taking the boundary Hodge dual of this difference we find

$$\begin{aligned} \delta J_S^\mu &= \mathcal{L}_{\delta\zeta} J_S^\mu - J_S^\nu\, \nabla_\nu \delta\zeta^\mu \\ &= \mathcal{D}_\nu \left[J_S^\mu\, \delta\zeta^\nu - J_S^\nu\, \delta\zeta^\mu \right] + \delta\zeta^\mu\, \mathcal{D}_\nu J_S^\nu \end{aligned} \tag{4.5.51}$$

Similarly

$$\begin{aligned} \delta\partial_\mu J_s^\mu &= \delta\zeta^\mu\, \partial_\mu\partial_\nu J_s^\nu + \partial_\mu\delta\zeta^\mu\, \partial_\nu J_s^\nu = \mathcal{L}_{\delta\zeta}\, \partial_\mu J_s^\mu + \partial_\mu\delta\zeta^\mu\, \partial_\nu J_s^\nu \\ &= \mathcal{L}_{\delta\zeta}\, \mathcal{D}_\mu J_s^\mu + \mathcal{D}_\mu\delta\zeta^\mu\, \mathcal{D}_\nu J_s^\nu \end{aligned} \tag{4.5.52}$$

Using the fluid equations of motion it turns that the RHS of (4.5.51) is of order ϵ^3 (and so zero to the order retained in this chapter) for $\zeta^\mu \propto b^2\, l^\mu$. Consequently, to second order we find a one parameter generalization of the entropy current – resulting from the diffeomorphisms (4.5.50) with $\delta\lambda_2$ set to zero.

Note that, apart from the diffeomorphism shift, the local rate of entropy production changes in magnitude (but not in sign) under redefinition (4.5.51) by a factor proportional to the Jacobian of the coordinate transformation parameterized by $\delta\zeta$. In § 4.6 we have explicitly computed the shift in the current (4.4.45) under the operation described in (4.5.51) (with $\delta\zeta$ of the form (4.5.50)) and also explicitly verified the invariance of the positivity of divergence under this map.

In summary we have constructed a two parameter generalization of our gravitational entropy current (4.4.45). One of these two parameters arose from the freedom to add an exact form to the area form on the horizon. The second parameter had its origin in the freedom to generalize the boundary to horizon map.

4.6 Weyl covariant formalism

In this section, we present the various results related to Weyl covariance in hydrodynamics that are relevant to this chapter. The conformal nature of the boundary fluid dynamics strongly constrains the form of the stress tensor and the entropy current [44,45]. An efficient way of exploiting this symmetry is to employ a manifestly Weyl-covariant formalism for hydrodynamics that was introduced in the reference [44].

In brief, for an arbitrary tensor with weight w, one defines a Weyl-covariant derivative[12]

$$\begin{aligned}\mathcal{D}_\lambda\, Q^{\mu\ldots}_{\nu\ldots} \equiv{}& \nabla_\lambda\, Q^{\mu\ldots}_{\nu\ldots} + w\, \mathcal{A}_\lambda Q^{\mu\ldots}_{\nu\ldots}\\ &+ \left[g_{\lambda\alpha}\mathcal{A}^\mu - \delta^\mu_\lambda \mathcal{A}_\alpha - \delta^\mu_\alpha \mathcal{A}_\lambda\right] Q^{\alpha\ldots}_{\nu\ldots} + \ldots\\ &- \left[g_{\lambda\nu}\mathcal{A}^\alpha - \delta^\alpha_\lambda \mathcal{A}_\nu - \delta^\alpha_\nu \mathcal{A}_\lambda\right] Q^{\mu\ldots}_{\alpha\ldots} - \ldots\end{aligned} \tag{4.6.53}$$

where the Weyl-connection $\mathcal{A}_\mu$ is related to the fluid velocity via the relation

$$\mathcal{A}_\mu = u^\lambda \nabla_\lambda u_\mu - \frac{\nabla_\lambda u^\lambda}{d-1} u_\mu \tag{4.6.54}$$

We shall exploit the manifest Weyl covariance of this formalism to establish certain results concerning the entropy current that are relevant to the discussion in the main text.

In § 4.6.1, we write down the most general Weyl-covariant entropy current and compute its divergence. This computation leads us directly to an analysis of the constraints on the

[12] In contrast to the analysis in the main text, we find it convenient here to work with an arbitrary background metric, whose associated torsion-free connection is used to define the covariant derivative ∇_μ.

entropy current imposed by the second law of thermodynamics. This analysis generalizes and completes the analysis in [44] where a particular example of an entropy current which satisfies the second law was presented. Following that, in § 4.6.2, we rewrite the results of this chapter in a Weyl-covariant form and show that the expression for the entropy current derived in this chapter satisfies the constraint derived in § 4.6.1. This is followed by a discussion in § 4.6.3 on the ambiguities in the definition of the entropy current.

4.6.1 Constraints on the entropy current: Weyl covariance and the second law

We begin by writing down the most general derivative expansion of the entropy current in terms Weyl-covariant vectors of weight 4.[13] After taking into account the equations of motion and various other identities, the most general entropy current consistent with Weyl covariance can be written as:

$$\begin{aligned}(4\pi\,\eta)^{-1}\,J_S^\mu = 4\,G_N^{(5)}\,b^3\,J_S^\mu = &\left[1 + b^2\left(A_1\,\sigma_{\alpha\beta}\,\sigma^{\alpha\beta} + A_2\,\omega_{\alpha\beta}\,\omega^{\alpha\beta} + A_3\,\mathcal{R}\right)\right]u^\mu \\ &+ b^2\,\left[B_1\,\mathcal{D}_\lambda\sigma^{\mu\lambda} + B_2\,\mathcal{D}_\lambda\omega^{\mu\lambda}\right] \\ &+ C_1\;b\;\ell^\mu + C_2\;b^2 u^\lambda\,\mathcal{D}_\lambda\ell^\mu + \ldots\end{aligned} \tag{4.6.55}$$

where $b = (\pi\,T)^{-1}$ and we have already assumed the leading order result for the entropy density $s = 4\,\pi\,\eta = (4\,G_N^{(5)}\,b^3)^{-1}$ and $\ell^\mu = \epsilon^{\alpha\beta\nu\mu}\omega_{\alpha\beta}u_\nu$.[14]

[13] We will restrict attention to fluid dynamics in $3+1$ dimensions.

[14] We shall follow the notations of [44] in the rest of this appendix. In particular, we recall the following definitions

$$\begin{aligned}\mathcal{A}_\mu = a_\mu - \frac{\vartheta}{3}u_\mu\;; &\qquad \mathcal{F}_{\mu\nu} = \nabla_\mu\mathcal{A}_\nu - \nabla_\nu\mathcal{A}_\mu \\ \mathcal{R} = R - 6\nabla_\lambda\mathcal{A}^\lambda + 6\mathcal{A}_\lambda\mathcal{A}^\lambda\;; &\qquad \mathcal{D}_\mu u_\nu = \sigma_{\mu\nu} + \omega_{\mu\nu} \\ \mathcal{D}_\lambda\sigma^{\mu\lambda} = \nabla_\lambda\sigma^{\mu\lambda} - 3\mathcal{A}_\lambda\sigma^{\mu\lambda}\;; &\qquad \mathcal{D}_\lambda\omega^{\mu\lambda} = \nabla_\lambda\omega^{\mu\lambda} - \mathcal{A}_\lambda\omega^{\mu\lambda}\end{aligned} \tag{4.6.56}$$

Note that in a flat spacetime, R is zero but $\mathcal{R}$ is not. Though we will always be working in flat spacetime, we will keep the R-terms around to make our expressions manifestly Weyl-covariant.

Now, we want to derive the constraints imposed by the second law on the A,B and C coefficients appearing above. To this end, we take the divergence of the entropy current above to get

$$\begin{aligned}4\,G_N^{(5)}\,b^3\,\mathcal{D}_\mu J_S^\mu =& -3b^{-1}\,u^\mu\,\mathcal{D}_\mu b - 2\,C_1\;\ell^\mu\mathcal{D}_\mu b\\ &+ b^2\,\mathcal{D}_\mu\left[\left(A_1\,\sigma_{\alpha\beta}\,\sigma^{\alpha\beta} + A_2\,\omega_{\alpha\beta}\,\omega^{\alpha\beta} + A_3\,\mathcal{R}\right)\,u^\mu\right.\\ &\left.+\left(B_1\,\mathcal{D}_\lambda\sigma^{\mu\lambda} + B_2\,\mathcal{D}_\lambda\omega^{\mu\lambda} + C_2\;u^\lambda\,\mathcal{D}_\lambda\ell^\mu\right)\right] + \ldots\end{aligned} \tag{4.6.57}$$

where we have used the facts that $\mathcal{D}_\mu\ell^\mu = 0$ and that $\mathcal{D}_\mu b$ gets non-zero contributions only at second order (4.6.59). Further, $u^\lambda\,\mathcal{F}_{\mu\lambda}$ gets non-zero contributions only at third order (the equations of motion force $u^\lambda\mathcal{F}_{\mu\lambda} = 0$ at second order).

In order to simplify the expression further, we need the equations of motion. Let us write the stress tensor in the form

$$T^{\mu\nu} = (16\pi\,G_N^{(5)}\,b^4)^{-1}\,(\eta^{\mu\nu} + 4\,u^\mu u^\nu) + \pi^{\mu\nu} \tag{4.6.58}$$

where $\pi_{\mu\nu}$ is transverse – $u^\nu\pi_{\mu\nu} = 0$. This would imply

$$\begin{aligned}0 = b^4\,u_\mu\,\mathcal{D}_\nu T^{\mu\nu} &= b^4\,\mathcal{D}_\nu(u_\mu T^{\mu\nu}) - b^4\,(\mathcal{D}_\nu u_\mu)\,T^{\mu\nu}\\ \implies\; 4\,&\left(\frac{3}{b}\,u^\mu\mathcal{D}_\mu b - \frac{b}{4\,\eta}\,\sigma_{\mu\nu}\pi^{\mu\nu}\right) = 0\end{aligned} \tag{4.6.59}$$

where we have multiplied the equation by $16\pi\,G_N^{(5)}$ in the second line to express things compactly. Similarly, we can write $2\,\ell^\mu\,\mathcal{D}_\mu b = -b^2\,\ell_\mu\mathcal{D}_\lambda\sigma^{\mu\lambda}$ which is exact upto third order in the derivative expansion. Note that these are just the Weyl-covariant forms of the equations that we have already encountered in (4.4.41).

We further invoke the following identities(which follow from the identities proved in the Appendix A of [44]) [15]

$$
\begin{aligned}
\mathcal{D}_\mu(\sigma_{\alpha\beta}\,\sigma^{\alpha\beta}\,u^\mu) &= 2\,\sigma_{\mu\nu}\,u^\lambda\mathcal{D}_\lambda\sigma^{\mu\nu} \\
\mathcal{D}_\mu(\omega_{\alpha\beta}\,\omega^{\alpha\beta}\,u^\mu) &= 4\sigma^{\mu\nu}\,\omega_\mu{}^\alpha\,\omega_{\alpha\nu} - 2\,\mathcal{D}_\mu\mathcal{D}_\lambda\omega^{\mu\lambda} \\
\mathcal{D}_\mu(\mathcal{R}\,u^\mu) &= -2\,\sigma_{\mu\nu}\,\mathcal{R}^{\mu\nu} + \mathcal{D}_\mu\left[-2\,\mathcal{D}_\lambda\sigma^{\mu\lambda} + 2\,\mathcal{D}_\lambda\omega^{\mu\lambda} + 4\,u_\lambda\,\mathcal{F}^{\mu\lambda}\right] \\
-2\,\sigma_{\mu\nu}\mathcal{R}^{\mu\nu} &= 4\,\sigma_{\mu\nu}\,\left[u^\lambda\mathcal{D}_\lambda\sigma^{\mu\nu} + \omega^{\mu\alpha}\,\omega_\alpha{}^\nu + \sigma^{\mu\alpha}\,\sigma_\alpha{}^\nu - C^{\mu\alpha\nu\beta}\,u_\alpha u_\beta\right] \\
\mathcal{D}_\mu(u^\lambda\,\mathcal{D}_\lambda\ell^\mu) &= \mathcal{D}_\mu(\ell^\lambda\,\mathcal{D}_\lambda u^\mu) - \mathcal{F}_{\mu\nu}\,\ell^\mu u^\nu \\
\mathcal{D}_\mu(\ell^\lambda\,\mathcal{D}_\lambda u^\mu) &= \sigma_{\mu\nu}\,\mathcal{D}^\mu\ell^\nu + \ell_\mu\,\mathcal{D}_\lambda\sigma^{\mu\lambda}
\end{aligned}
\tag{4.6.60}
$$

to finally obtain

$$
\begin{aligned}
4\,G_N^{(5)}\,b^3\,\mathcal{D}_\mu J_S^\mu \;&=\; b^2\,\sigma_{\mu\nu}\,\left[-\frac{\pi^{\mu\nu}}{4\,\eta\,b} + 2\,A_1\,u^\lambda\,\mathcal{D}_\lambda\sigma^{\mu\nu} + 4\,A_2\,\omega^{\mu\alpha}\,\omega_\alpha{}^\nu - 2\,A_3\,\mathcal{R}^{\mu\nu} + C_2\,\mathcal{D}^\mu\ell^\nu\right] \\
&\quad +(B_1 - 2A_3)\,b^2\,\mathcal{D}_\mu\mathcal{D}_\lambda\sigma^{\mu\lambda} + (C_1 + C_2)\,b^2\,\ell_\mu\,\mathcal{D}_\lambda\sigma^{\mu\lambda} + \ldots \\
&= b^2\sigma_{\mu\nu}\left[-\frac{\pi^{\mu\nu}}{4\,\eta\,b} + (2A_1 + 4A_3)\,u^\lambda\mathcal{D}_\lambda\sigma^{\mu\nu} + 4\,(A_2 + A_3)\,\omega^{\mu\alpha}\,\omega_\alpha{}^\nu + 4\,A_3\,\sigma^{\mu\alpha}\,\sigma_\alpha{}^\nu + C_2\,\mathcal{D}^\mu\ell^\nu\right] \\
&\quad +(B_1 - 2A_3)\,b^2\,\mathcal{D}_\mu\mathcal{D}_\lambda\sigma^{\mu\lambda} + (C_1 + C_2)\,b^2\,\ell_\mu\mathcal{D}_\lambda\sigma^{\mu\lambda} + \ldots
\end{aligned}
\tag{4.6.61}
$$

Substituting the value of $\pi^{\mu\nu}$ as calculated from the known stress tensor, we find

$$
\begin{aligned}
4\,G_N^{(5)}\,b^3\,\mathcal{D}_\mu J_S^\mu \;&=\; b^2\sigma_{\mu\nu}\left[\frac{\sigma^{\mu\nu}}{2\,b} + \left(2\,A_1 + 4\,A_3 - \frac{1}{2} + \frac{1}{4}\,\ln 2\right)\,u^\lambda\mathcal{D}_\lambda\sigma^{\mu\nu}\right. \\
&\quad \left. + 4\,(A_2 + A_3)\,\omega^{\mu\alpha}\,\omega_\alpha{}^\nu + (4\,A_3 - \frac{1}{2})\,(\sigma^{\mu\alpha}\,\sigma_\alpha{}^\nu) + C_2\,\mathcal{D}^\mu\ell^\nu\right] \\
&\quad +(B_1 - 2A_3)\,b^2\,\mathcal{D}_\mu\mathcal{D}_\lambda\sigma^{\mu\lambda} + (C_1 + C_2)\,b^2\,\ell_\mu\,\mathcal{D}_\lambda\sigma^{\mu\lambda} + \ldots
\end{aligned}
\tag{4.6.62}
$$

This expression can in turn be rewritten in a more useful form by isolating the terms that are manifestly non-negative:

$$
\begin{aligned}
4\,G_N^{(5)}\,b^3\,\mathcal{D}_\mu J_S^\mu \;&=\; \frac{b}{2}\,\left[\sigma_{\mu\nu} + b\,\left(2\,A_1 + 4\,A_3 - \frac{1}{2} + \frac{1}{4}\,\ln 2\right)\,u^\lambda\mathcal{D}_\lambda\sigma^{\mu\nu} + 4\,b\,(A_2 + A_3)\,\omega^{\mu\alpha}\omega_\alpha{}^\nu\right. \\
&\quad \left. + \,b\,(4\,A_3 - \frac{1}{2})(\sigma^{\mu\alpha}\,\sigma_\alpha{}^\nu) + b\,C_2\,\mathcal{D}^\mu\ell^\nu\right]^2 \\
&\quad +(B_1 - 2A_3)\,b^2\,\mathcal{D}_\mu\mathcal{D}_\lambda\sigma^{\mu\lambda} + (C_1 + C_2)\,b^2\,\ell_\mu\mathcal{D}_\lambda\sigma^{\mu\lambda} + \ldots
\end{aligned}
\tag{4.6.63}
$$

[15] Since we are only interested in the case where boundary is conformally flat, we will consistently neglect terms proportional to the Weyl curvature in the following.

The second law requires that the right hand side of the above equation be positive semi-definite at every point in the boundary. First, we note from (4.6.63) that the first two lines are positive semi-definite whereas the terms in the third line are not – given a velocity configuration in which the third line evaluates to a particular value, as argued in the main text, we can always construct another configuration to get a contribution with opposite sign. Consider, in particular, points in the boundary where $\sigma_{\mu\nu} = 0$ – at such points, the contribution of the first two lines become subdominant in the derivative expansion to the contribution from the third line. The entropy production at these points can be positive semi-definite only if the combination the coefficients appearing in the third line vanish identically.

Hence, we conclude that the second law gives us two constraints relating A,B and C, *viz.*,

$$B_1 = 2\,A_3 \qquad\qquad C_1 + C_2 = 0 \tag{4.6.64}$$

Any entropy current which satisfies the above relations constitutes a satisfactory proposal for the entropy current from the viewpoint of the second law.

One simple expression for such an entropy current which satisfies the above requirements was proposed in [44]. The J_s^λ proposed there is given by

$$(4\pi\,\eta)^{-1}\,J_S^\lambda = u^\lambda - \frac{b^2}{8}\,\left[(\ln 2\,\sigma^{\mu\nu}\sigma_{\mu\nu} + \omega^{\mu\nu}\,\omega_{\mu\nu})\;u^\lambda + 2\,u_\mu\,(\mathcal{G}^{\mu\lambda} + \mathcal{F}^{\mu\lambda}) + 6\,\mathcal{D}_\nu\omega^{\lambda\nu}\right] + \dots \tag{4.6.65}$$

Now, using the identity

$$u_\mu\,(\mathcal{G}^{\mu\lambda} + \mathcal{F}^{\mu\lambda}) = -\frac{\mathcal{R}}{2}\,u^\lambda - \mathcal{D}_\nu\sigma^{\lambda\nu} - \mathcal{D}_\nu\omega^{\lambda\nu} + 2\,u_\mu\mathcal{F}^{\lambda\mu} \tag{4.6.66}$$

and the equations of motion, we can rewrite the above expression in the form appearing in (4.6.55) to get the value of A,B and C coefficients as

$$
\begin{gathered}
A_1 = -\frac{\ln 2}{8}; \qquad A_2 = -\frac{1}{8}; \qquad A_3 = \frac{1}{8} \\
B_1 = \frac{1}{4}; \qquad B_2 = -\frac{1}{2} \\
C_1 = C_2 = 0
\end{gathered} \tag{4.6.67}
$$

It can easily be checked that these values satisfy the constraints listed in (4.6.64). Further, for these values, the divergence of the entropy current simplifies considerably and we get

$$
4\, G_N^{(5)}\, b^3\, \mathcal{D}_\mu J_S^\mu = \frac{b}{2}\, \sigma_{\mu\nu}\, \sigma^{\mu\nu} \tag{4.6.68}
$$

However, as the analysis in this section shows, this proposal is just one entropy current among a class of entropy currents that satisfy the second law. This is not surprising, since (as was noted in [44]) the second law alone cannot determine the entropy current uniquely.

4.6.2 Entropy current and entropy production from gravity

We now calculate the coefficients A_i's and B_i's for the actual entropy current calculated from gravity in (4.4.45) and check whether the they obey the constraints in (4.6.64). Unlike the proposal in [44] , the entropy current derived in § 4.4 takes into account the detailed microscopic dynamics(of which hydrodynamics is an effective description) encoded in the dual gravitational description.

In order to cast the entropy current in the form given by (4.6.55) , we have to first rewrite the quantities appearing in this chapter in a Weyl-covariant form. We have the following relations in the flat spacetime which identify the Weyl-covariant forms appearing in the second-order metric of [40] –

$$
\begin{gathered}
\mathfrak{S}4 = 2\, \omega_{\alpha\beta}\, \omega^{\alpha\beta}; \qquad \mathfrak{S}5 = \sigma_{\alpha\beta}\, \sigma^{\alpha\beta}; \\
-\frac{4}{3}\, \mathbf{s}3 + 2\, \mathfrak{S}1 - \frac{2}{9}\, \mathfrak{S}3 = \frac{2}{3}\, \sigma_{\alpha\beta}\, \sigma^{\alpha\beta} - \frac{2}{3}\, \omega_{\alpha\beta}\omega^{\alpha\beta} + \frac{1}{3}\, \mathcal{R} \\
\frac{5}{9}\, \mathbf{v}4_\mu + \frac{5}{9}\, \mathbf{v}5_\mu + \frac{5}{3}\, \mathfrak{V}1_\mu - \frac{5}{12}\, \mathfrak{V}2_\mu - \frac{11}{6}\, \mathfrak{V}3_\mu = P_\mu^\nu\, \mathcal{D}_\lambda \sigma_\nu{}^\lambda \\
\frac{15}{9}\, \mathbf{v}4_\mu - \frac{1}{3}\, \mathbf{v}5_\mu - \mathfrak{V}1_\mu - \frac{1}{4}\, \mathfrak{V}2_\mu + \frac{1}{2}\, \mathfrak{V}3_\mu = P_\mu^\nu\, \mathcal{D}_\lambda \omega_\nu{}^\lambda
\end{gathered} \tag{4.6.69}
$$

These can be used to obtain

$$
\begin{aligned}
\mathbf{B}^{\infty}_{\mu} &= 18\, P^{\nu}_{\mu}\, \mathcal{D}_{\lambda}\sigma^{\lambda}_{\nu} + 18\, P^{\nu}_{\mu}\, \mathcal{D}_{\lambda}\omega^{\lambda}_{\nu} \\
&= 18\, \left(-\sigma_{\alpha\beta}\, \sigma^{\alpha\beta} + \omega_{\alpha\beta}\, \omega^{\alpha\beta}\right)\, u_{\mu} + 18\, \mathcal{D}_{\lambda}\sigma_{\mu}{}^{\lambda} + 18\, \mathcal{D}_{\lambda}\omega_{\mu}{}^{\lambda} \\
\mathbf{B}^{\text{fin}}_{\mu} &= 54\, P^{\nu}_{\mu}\, \mathcal{D}_{\lambda}\sigma^{\lambda}_{\nu} + 90\, P^{\nu}_{\mu}\, \mathcal{D}_{\lambda}\omega^{\lambda}_{\nu} \\
&= \left(-54\, \sigma_{\alpha\beta}\, \sigma^{\alpha\beta} + 90\, \omega_{\alpha\beta}\, \omega^{\alpha\beta}\right)\, u_{\mu} + 54\, \mathcal{D}_{\lambda}\sigma_{\mu}{}^{\lambda} + 90\, \mathcal{D}_{\lambda}\omega_{\mu}{}^{\lambda}
\end{aligned}
\tag{4.6.70}
$$

Hence, all the second-order scalar and the vector contributions to the metric can be written in terms of three Weyl-covariant scalars $\sigma_{\alpha\beta}\, \sigma^{\alpha\beta}$, $\omega_{\alpha\beta}\, \omega^{\alpha\beta}$ and $\mathcal{R}$ and two Weyl-covariant vectors $\mathcal{D}_{\lambda}\sigma_{\mu}{}^{\lambda}$ and $\mathcal{D}_{\lambda}\omega_{\mu}{}^{\lambda}$.

Using the above expressions, we can rewrite the second order scalar and the vector contributions to the entropy current appearing in (4.4.37) as

$$
\begin{aligned}
s^{(2)}_{a} &= \frac{3}{2}\, s^{(2)}_{c} = -\frac{b^2}{4}\, \left(\frac{1}{2} + \ln\, 2 + 3\, \mathcal{C} + \frac{\pi}{4} + \frac{5\pi^2}{16} - \left(\frac{3}{2}\ln\, 2 + \frac{\pi}{4}\right)^2\right)\, \sigma_{\alpha\beta}\, \sigma^{\alpha\beta} - \frac{b^2}{4}\, \omega_{\alpha\beta}\, \omega^{\alpha\beta} \\
s^{(2)}_{b} &= \left(\frac{1}{2} + \frac{2}{3}\ln\, 2 + \mathcal{C} + \frac{\pi}{6} + \frac{5\,\pi^2}{48}\right)\, \sigma_{\alpha\beta}\, \sigma^{\alpha\beta} - \frac{1}{2}\, \omega_{\alpha\beta}\, \omega^{\alpha\beta} + \frac{1}{6}\, \mathcal{R}
\end{aligned}
\tag{4.6.71}
$$

while the vector contribution is given as

$$
j^{(2)}_{\mu} = P^{\nu}_{\mu}\, \left[\frac{3}{4}\, \mathcal{D}_{\lambda}\sigma_{\nu}{}^{\lambda} + \frac{1}{2}\, \mathcal{D}_{\lambda}\omega_{\nu}{}^{\lambda}\right] = \left(-\frac{3}{4}\, \sigma_{\alpha\beta}\, \sigma^{\alpha\beta} + \frac{1}{2}\, \omega_{\alpha\beta}\, \omega^{\alpha\beta}\right)\, u_{\mu} + \frac{3}{4}\, \mathcal{D}_{\lambda}\sigma_{\mu}{}^{\lambda} + \frac{1}{2}\, \mathcal{D}_{\lambda}\omega_{\mu}{}^{\lambda}
\tag{4.6.72}
$$

Now, we use (4.4.42), (4.4.43) and (4.4.44) to write r_H, n^{μ} and $\sqrt{g}$ in Weyl covariant form as follows:

$$
r_H = \frac{1}{b}\left(1 + \frac{b^2}{4}\left[\left(\frac{5}{6} + \frac{2}{3}\ln\, 2 + \mathcal{C} + \frac{\pi}{6} + \frac{5\,\pi^2}{48}\right)\, \sigma_{\alpha\beta}\, \sigma^{\alpha\beta} - \frac{1}{2}\, \omega_{\alpha\beta}\, \omega^{\alpha\beta} + \frac{1}{6}\, \mathcal{R}\right]\right)
\tag{4.6.73}
$$

$$
\begin{aligned}
n^{\mu} \;\; &= \;\; \left(1 - \frac{b^2}{4}\left[\frac{1}{2} + \ln\, 2 + 3\, \mathcal{C} + \frac{\pi}{4} + \frac{5\,\pi^2}{16} - \left(\frac{3}{2}\ln\, 2 + \frac{\pi}{4}\right)^2\right]\, \sigma_{\alpha\beta}\, \sigma^{\alpha\beta} - \frac{b^2}{4}\, \omega_{\alpha\beta}\, \omega^{\alpha\beta}\right)\, u^{\mu} \\
&\qquad + b^2\, P^{\nu}_{\mu}\, \left(\frac{1}{4}\, \mathcal{D}_{\lambda}\sigma_{\nu}{}^{\lambda} + \frac{1}{2}\, \mathcal{D}_{\lambda}\omega_{\nu}{}^{\lambda}\right)
\end{aligned}
\tag{4.6.74}
$$

$$\sqrt{g} = \frac{1}{b^3}\left(1 + \frac{b^2}{4}\left[\left(2 + \ln 2 + \frac{\pi}{4}\right)\sigma_{\alpha\beta}\,\sigma^{\alpha\beta} - \frac{5}{2}\,\omega_{\alpha\beta}\,\omega^{\alpha\beta} + \frac{1}{2}\,\mathcal{R}\right]\ .\right) \tag{4.6.75}$$

Putting all of these together we can finally obtain the expression for the entropy current:

$$\begin{aligned} 4\,G_N^{(5)}\,b^3\,J_S^\mu &= \left(1 + b^2\,\left[\left(\frac{1}{2} + \frac{1}{4}\ln 2 + \frac{\pi}{16}\right]\,\sigma_{\alpha\beta}\,\sigma^{\alpha\beta} - \frac{5}{8}\,\omega_{\alpha\beta}\omega^{\alpha\beta} + \frac{1}{8}\,\mathcal{R}\right]\right)u^\mu \\ &\quad + b^2\,P_\mu^\nu\,\left(\frac{1}{4}\,\mathcal{D}_\lambda\sigma_\nu{}^\lambda + \frac{1}{2}\,\mathcal{D}_\lambda\omega_\nu{}^\lambda\right) \\ &= \left(1 + b^2\,\left[\left(\frac{1}{4} + \frac{1}{4}\ln 2 + \frac{\pi}{16}\right)\,\sigma_{\alpha\beta}\,\sigma^{\alpha\beta} - \frac{1}{8}\,\omega_{\alpha\beta}\,\omega^{\alpha\beta} + \frac{1}{8}\,\mathcal{R}\right]\right)\,u^\mu \\ &\quad + b^2\,\left(\frac{1}{4}\,\mathcal{D}_\lambda\sigma^{\mu\lambda} + \frac{1}{2}\,\mathcal{D}_\lambda\omega^{\mu\lambda}\right) \end{aligned} \tag{4.6.76}$$

from which we can read off the coefficients A, B and C appearing in the general current (4.6.55)

$$\begin{gathered} A_1 = \frac{1}{4} + \frac{\pi}{16} + \frac{\ln 2}{4}; \qquad A_2 = -\frac{1}{8}; \qquad A_3 = \frac{1}{8} \\ B_1 = \frac{1}{4}; \qquad B_2 = \frac{1}{2} \\ C_1 = C_2 = 0 \end{gathered} \tag{4.6.77}$$

These coefficients manifestly obey the constraints laid down in (4.6.64) and hence, the entropy current derived from gravity obeys the second law. Further, we get the divergence of the entropy current as

$$\begin{aligned} 4\,G_N^{(5)}\,b^3\,J_S^\mu &= b^2\,\sigma_{\mu\nu}\,\left[\frac{\sigma^{\mu\nu}}{2\,b} + 2\,\left(\frac{1}{4} + \frac{\pi}{16} + \frac{3}{8}\ln 2\right)\,u^\lambda\mathcal{D}_\lambda\sigma^{\mu\nu}\right] + \ldots \\ &= \frac{b}{2}\,\left[\sigma^{\mu\nu} + b\,\left(\frac{1}{4} + \frac{\pi}{16} + \frac{3}{8}\ln 2\right)\,u^\lambda\mathcal{D}_\lambda\sigma^{\mu\nu}\right]^2 + \ldots \end{aligned} \tag{4.6.78}$$

which can alternatively be written in the form

$$T\,\mathcal{D}_\mu J_S^\mu = 2\,\eta\,\left[\sigma^{\mu\nu} + \frac{(\pi + 4 + 6\,\ln 2)}{16\pi\,T}\,u^\lambda\mathcal{D}_\lambda\sigma^{\mu\nu}\right]^2 + \ldots \tag{4.6.79}$$

which gives the final expression for the rate of entropy production computed via holography.

4.6.3 Ambiguity in the holographic entropy current

We now examine briefly the change in the coefficients A, B and C parametrizing the arbitrary entropy current, under the ambiguity shift discussed in § 4.5.3, see Eq. (4.5.51). In particular, we want to verify explicitly that under such a shift, the entropy production still remains positive semi-definite.

The first kind of ambiguity in the entropy current arises due to the addition of an exact form to the entropy current. The only Weyl covariant exact form that can appear in the entropy current at this order is given by

$$4\, G_N^{(5)}\, b^3\, \delta J_S^\mu = \delta\lambda_0 b^2 \mathcal{D}_\nu \omega^{\mu\nu} \tag{4.6.80}$$

which induces a shift in the above coefficients $B_2 \longrightarrow B_2 + \delta\lambda_0$.

The second kind shift in the entropy current (due to the arbitrariness in the boundary to horizon map) is parametrised by a vector $\delta\zeta^\mu$(which is Weyl-invariant) and is given by

$$\begin{aligned} \delta J_S^\mu &= \mathcal{L}_{\delta\zeta} J_S^\mu - J_S^\nu\, \nabla_\nu \delta\zeta^\mu \\ &= \mathcal{D}_\nu \left[J_S^\mu\, \delta\zeta^\nu - J_S^\nu\, \delta\zeta^\mu \right] + \delta\zeta^\mu\, \mathcal{D}_\nu J_S^\nu \end{aligned} \tag{4.6.81}$$

where in the last line we have rewritten the shift in a manifestly Weyl-covariant form.

If we now write down a general derivative expansion for $\delta\zeta^\mu$ as

$$\delta\zeta^\mu = 2\ \delta\lambda_1\ b\ u^\mu + \delta\lambda_2\ b^2\ \ell^\mu + \dots \tag{4.6.82}$$

the shift in the entropy current can be calculated using the above identities as

$$4\, G_N^{(5)}\, b^3\, \delta J_S^\mu = \delta\lambda_1\ b^2\ \sigma_{\alpha\beta}\, \sigma^{\alpha\beta}\, u^\mu + \dots \tag{4.6.83}$$

which implies a shift in the above coefficients given by $A_1 \longrightarrow A_1 + \delta\lambda_1$.

Note that both these shifts maintain the constraints listed in (4.6.64) and hence, the positive semi-definite nature of the entropy production is unaffected by these ambiguities as advertised.

4.7 Independent data in fields up to third order

There are 16, 40 and 80 independent components at first, second and third orders in the Taylor expansion of velocity and temperature.[16] These pieces of data are not all independent; they are constrained by equations of motion. The relevant equations of motion are the conservation of the stress tensor and its first and second derivatives[17] (at our spacetime point) which are $4, 16$ and 40 respectively in number.[18] The terms that appear in the three kinds of equations listed above start at first, second and third order respectively. Consequently these equations may be used to cut down the independent data in Taylor series coefficients of the velocity and temperature at first second and third order to 12, 24 and 40 components respectively. We will now redo this counting keeping track of the $SO(3)$ transformation properties of all fields.

Let us list degrees of freedom by the vector (a, b, c, d, e) where a represents the number of $SO(3)$ scalars ($\mathbf{1}$), b the number of $SO(3)$ vectors ($\mathbf{3}$), *etc.*. Working up to third order we encounter terms transforming in at most the $\mathbf{9}$ representation of $SO(3)$. In this notation, the number of degrees of freedom in Taylor coefficients are $(2, 3, 1, 0, 0)$, $(3, 5, 3, 1, 0)$, and $(4, 7, 5, 3, 1)$ at first, second and third order respectively. The number of equations of motion are $(1, 1, 0, 0, 0)$, $(2, 3, 1, 0, 0)$ and $(3, 5, 3, 1, 0)$ respectively (note that the number of equations of motion at order $n + 1$ is the same as the number of variables at order n). It follows from subtraction that the number of unconstrained variables at zeroth, first, second and third order respectively can be chosen to be $(1, 1, 0, 0, 0)$, $(1, 2, 1, 0, 0)$, $(1, 2, 2, 1, 0)$ and $(1, 2, 2, 1, 1)$. This choice is convenient in checking the statements about the non-negativity of the divergence of the entropy current at third order explicitly.

[16] For each independent function we count the number of independent partial derivatives at a given order; for the temperature we have $\partial_\mu T$, $\partial_\mu \partial_\nu T$, *etc.*.

[17] The relevant equations are just the moments of the conservation equation which arise as local constraints at higher orders.

[18] As $T^{\mu\nu}$ is not homogeneous in the derivative expansion, these equations of motion mix terms of different order in this expansion.

Bibliography

[1] S. Bhattacharyya and S. Minwalla, *Weak Field Black Hole Formation in Asymptotically AdS Spacetimes*, *JHEP* **09** (2009) 034, [arXiv:0904.0464].

[2] S. Bhattacharyya, V. E. Hubeny, S. Minwalla, and M. Rangamani, *Nonlinear Fluid Dynamics from Gravity*, *JHEP* **02** (2008) 045, [arXiv:0712.2456].

[3] S. Bhattacharyya *et. al.*, *Local Fluid Dynamical Entropy from Gravity*, *JHEP* **06** (2008) 055, [arXiv:0803.2526].

[4] G. Policastro, D. T. Son, and A. O. Starinets, *The shear viscosity of strongly coupled $n = 4$ supersymmetric yang-mills plasma*, *Phys. Rev. Lett.* **87** (2001) 081601, [hep-th/0104066].

[5] R. A. Janik and R. Peschanski, *Asymptotic perfect fluid dynamics as a consequence of ads/cft*, *Phys. Rev.* **D73** (2006) 045013, [hep-th/0512162].

[6] R. A. Janik and R. Peschanski, *Gauge / gravity duality and thermalization of a boost- invariant perfect fluid*, *Phys. Rev.* **D74** (2006) 046007, [hep-th/0606149].

[7] S. Nakamura and S.-J. Sin, *A holographic dual of hydrodynamics*, *JHEP* **09** (2006) 020, [hep-th/0607123].

[8] S. Bhattacharyya, S. Lahiri, R. Loganayagam, and S. Minwalla, *Large rotating AdS black holes from fluid mechanics*, 0708.1770.

[9] G. Policastro, D. T. Son, and A. O. Starinets, *From ads/cft correspondence to hydrodynamics. ii: Sound waves*, *JHEP* **12** (2002) 054, [hep-th/0210220].

[10] G. Policastro, D. T. Son, and A. O. Starinets, *From ads/cft correspondence to hydrodynamics*, *JHEP* **09** (2002) 043, [hep-th/0205052].

[11] P. Kovtun, D. T. Son, and A. O. Starinets, *Holography and hydrodynamics: Diffusion on stretched horizons*, *JHEP* **10** (2003) 064, [hep-th/0309213].

[12] P. Kovtun, D. T. Son, and A. O. Starinets, *Viscosity in strongly interacting quantum field theories from black hole physics*, *Phys. Rev. Lett.* **94** (2005) 111601, [hep-th/0405231].

[13] P. K. Kovtun and A. O. Starinets, *Quasinormal modes and holography*, *Phys. Rev.* **D72** (2005) 086009, [hep-th/0506184].

[14] D. T. Son and A. O. Starinets, *Viscosity, black holes, and quantum field theory*, arXiv:0704.0240.

[15] P. M. Chesler and L. G. Yaffe, *Horizon formation and far-from-equilibrium isotropization in supersymmetric Yang-Mills plasma*, *Phys. Rev. Lett.* **102** (2009) 211601, [arXiv:0812.2053].

[16] M. W. Choptuik, *Universality and scaling in gravitational collapse of a massless scalar field*, *Phys. Rev. Lett.* **70** (1993) 9–12.

[17] R. A. Janik, *Viscous plasma evolution from gravity using ads/cft*, *Phys. Rev. Lett.* **98** (2007) 022302, [hep-th/0610144].

[18] M. P. Heller and R. A. Janik, *Viscous hydrodynamics relaxation time from ads/cft*, *Phys. Rev.* **D76** (2007) 025027, [hep-th/0703243].

[19] S. Coleman, *Aspects of symmetry*, *Cambridge University Press, New York* (1985).

[20] J. M. Maldacena, *Lectures on AdS/CFT*, hep-th/0309246.

[21] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, *Large N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183–386, [hep-th/9905111].

[22] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231–252, [hep-th/9711200].

[23] J. Polchinski, *String theory, volume 1*, *Cambridge University Press, New York* (2005).

[24] J. Polchinski, *String theory, volume 2*, *Cambridge University Press, New York* (2005).

[25] J. Polchinski, *Dirichlet-Branes and Ramond-Ramond Charges*, *Phys. Rev. Lett.* **75** (1995) 4724–4727, [hep-th/9510017].

[26] J. Polchinski, *Lectures on D-branes*, hep-th/9611050.

[27] I. R. Klebanov, *World-volume approach to absorption by non-dilatonic branes*, *Nucl. Phys.* **B496** (1997) 231–242, [hep-th/9702076].

[28] S. S. Gubser, I. R. Klebanov, and A. A. Tseytlin, *String theory and classical absorption by three-branes*, *Nucl. Phys.* **B499** (1997) 217–240, [hep-th/9703040].

[29] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253–291, [hep-th/9802150].

[30] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B428** (1998) 105–114, [hep-th/9802109].

[31] K. Skenderis, *Lecture notes on holographic renormalization*, *Class. Quant. Grav.* **19** (2002) 5849–5876, [hep-th/0209067].

[32] E. Witten, *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505–532, [hep-th/9803131].

[33] S. W. Hawking and D. Page, *Thermodynamics of black holes in Anti-de Sitter space*, *Commun. Math. Phys.* **87** (1983) 577.

[34] K. Skenderis, *Asymptotically anti-de Sitter spacetimes and their stress energy tensor*, *Int. J. Mod. Phys.* **A16** (2001) 740–749, [hep-th/0010138].

[35] G. T. Horowitz and V. E. Hubeny, *Quasinormal modes of AdS black holes and the approach to thermal equilibrium*, *Phys. Rev.* **D62** (2000) 024027, [hep-th/9909056].

[36] A. Awad, S. R. Das, A. Ghosh, J. Oh, and S. Trivedi, *To appear*, .

[37] A. Muronga, *Causal theories of dissipative relativistic fluid dynamics for nuclear collisions*, *Phys. Rev.* **C69** (2004) 034903, [nucl-th/0309055].

[38] V. Balasubramanian and P. Kraus, *A stress tensor for anti-de sitter gravity*, *Commun. Math. Phys.* **208** (1999) 413–428, [hep-th/9902121].

[39] R. Wald, *General relativity*, *Chicago, University of Chicago Press, 1984, 504 p.* (1984).

[40] S. Bhattacharyya, V. E. Hubeny, S. Minwalla, and M. Rangamani, *Nonlinear Fluid Dynamics from Gravity*, *JHEP* **02** (2008) 045, [0712.2456].

[41] S. Hawking and G. Ellis, *The large structure of space-time*, *Cambridge Monographs on Mathematical Physics* (1973).

[42] R. Penrose, *Structure of space-time*, *Battelle Recontres* (1967) 121–235.

[43] V. Iyer and R. M. Wald, *Some properties of Noether charge and a proposal for dynamical black hole entropy*, *Phys. Rev.* **D50** (1994) 846–864, [gr-qc/9403028].

[44] R. Loganayagam, *Entropy Current in Conformal Hydrodynamics*, 0801.3701.

[45] R. Baier, P. Romatschke, D. T. Son, A. O. Starinets, and M. A. Stephanov, *Relativistic viscous hydrodynamics, conformal invariance, and holography*, 0712.2451.