BULL. AUSTRAL. MATH. SOC. 90C30
VOL. 12 (1975), 391-397.

Nonlinear fractional programming

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If an optimal solution exists for a nonlinear fractional programming problem, then this solution is shown to be obtainable by solving two associated programming problems whose objective functions are no longer fractional. A certain restriction is assumed on the constraint sets of the latter problems. This result includes various known theorems as special cases.

Consider the fractional programming problem

$$\text{P1} \quad \text{Maximize} \quad f(x)/g(x)$$

(1) $$\text{subject to} \quad h(x) \le 0 ,$$

where f and g are mappings from R^n into R, and h is a mapping from R^n into R^m. It is assumed that f and g do not simultaneously become zero. There has been a great deal of interest in various special cases of the above problem. In particular, if f, g, and h are linear, Charnes and Cooper [1] showed that optimal solutions can be determined from optimal solutions of two associated linear programming problems. Charnes and Cooper's result was extended to the ratio of two quadratic functions subject to linear constraints by Swarup [5]. He considered P1 with f and g quadratic, and h linear, and showed that an optimal solution, if it exists, can be obtained from the solutions of two associated quadratic programming problems, each with linear constraints and one quadratic constraint. Sharma [4] considered P1 with f and g polynomials, and h

Received 24 January 1975. This paper was presented at the XX International Meeting of the Institute of Management Sciences at Tel Aviv in June 1973, but through an editorial mishap was omitted from the Proceedings of the Conference.

linear. He showed that an optimal solution, if it exists, can be obtained from the solutions of two associated programming problems where the objective function is a polynomial and the constraints are all linear except for one polynomial constraint. Mond and Craven [3] considered P1 with a larger class of functions f and g, with h linear. They showed how a solution, if it exists, can be obtained from the solution of two associated problems where the objective function is no longer fractional and all but one of the constraints are linear.

Here we consider the more general problem P1 where h, as well as f and g, may be nonlinear, and obtain a theorem that includes, as special cases, the corresponding results of [1], [3], [4], and [5].

Notation and definitions

Let $\phi_0 : \mathrm{R} \to \mathrm{R}$ denote a monotone strictly increasing function, with $\phi_0(t) > 0$ for $t > 0$. For $i = 1, 2, \ldots, m$ let $\phi_i : \mathrm{R} \to \mathrm{R}$ denote a positive function; that is, $t \geq 0$ implies $\phi_i(t) \geq 0$. Define the functions F, G, and H_i, $i = 1, 2, \ldots, m$, for $t \in \mathrm{R}_+$ and $y \in \mathrm{R}^n$, by

(2)
$$F(y, t) = f(y/t)\phi_0(t) ,$$
$$G(y, t) = g(y/t)\phi_0(t) ,$$
$$H_i(y, t) = h_i(y/t)\phi_i(t) , \quad i = 1, 2, \ldots, m .$$

Define

(3)
$$F(y, 0) = \lim_{t \to 0} F(y, t) ,$$
$$G(y, 0) = \lim_{t \to 0} G(y, t) ,$$
$$H_i(y, 0) = \lim_{t \to 0} H_i(y, t) , \quad i = 1, 2, \ldots, m ,$$

whenever these limits exist. Assume that $G(0, 0) = 0$ whenever it exists. Let $H(y, t)$ denote the m-dimensional vector whose i-th component is $H_i(y, t)$.

Results

Let us introduce the transformation $y = tx$, where for specified function ϕ_0 and non-zero constant $\Delta \in \mathrm{R}$, we require

(4) $$G(y, t) = \Delta .$$

On multiplying f and g by ϕ_0, h_i by ϕ_i, $i = 1, \ldots, m$, and using (2) we obtain the associated problem

P2 Maximize $F(y, t)$

(5) subject to $G(y, t) = \Delta$

(6) $H(y, t) \leq 0$

(7) $t \geq 0$.

THEOREM 1. *If*

(i) *the point* $(y, 0)$ *is not feasible for* P2,

(ii) $0 < \operatorname{sgn} \Delta = \operatorname{sgn} g(x^*)$ *for an optimal solution* x^* *of* P1, *and*

(iii) (y^*, t^*) *is an optimal solution of* P2,

then y^*/t^* *is an optimal solution of* P1.

Proof. Assume that the theorem is false, that is, that there exists an optimal x^* such that

(8) $$f(x^*)/g(x^*) > f(y^*/t^*)/g(y^*/t^*) .$$

By Condition *(ii)*, $g(x^*) = \theta\Delta$ for some $\theta > 0$.

Consider $t = \phi_0^{-1}(1/\theta)$ and $y = \phi_0^{-1}(1/\theta)x^*$. Then

$$\phi_0(t)g(x^*) = G(y, t) = \Delta ,$$

$$H_i(y, t) = h_i(y/t)\phi_i(t) = h_i(x^*)\phi_i(t) \leq 0 , \quad i = 1, \ldots, m .$$

Thus (y, t) is a feasible solution for P2. Now

(9) $$f(x^*)/g(x^*) = \phi_0(t)f(x^*)/[\phi_0(t)g(x^*)]$$
$$= F(y, t)/G(y, t) = F(y, t)/\Delta .$$

Also

https://doi.org/10.1017/S0004972700024047 Published online by Cambridge University Press

(10) $$f(y^*/t^*)/g(y^*/t^*) = \phi_0(t^*)f(y^*/t^*)/[\phi_0(t^*)g(y^*/t^*)] = F(y^*, t^*)/G(y^*, t^*) = F(y^*, t^*)/\Delta .$$

Hence, for feasible (y, t), (8), (9), and (10) show that

$$F(y, t) > F(y^*, t^*) ,$$

contradicting the assumption that (y^*, t^*) is optimal for P2.

If $\operatorname{sgn} g(x^*) < 0$, for x^* an optimal solution of P1, then replacing f by $-f$ and g by $-g$ the objective function is unaltered and for the new denominator we have $-g(x^*) > 0$.

Thus, if P1 has a solution, it can be obtained by solving, for suitably selected functions $\phi_i(t)$, $i = 0, 1, \ldots, m$, the two programming problems

P3 Maximize $F(y, t)$

subject to $G(y, t) = 1$

$H(y, t) \le 0$

$t \ge 0$

and

P4 Maximize $-F(y, t)$

subject to $-G(y, t) = 1$

$H(y, t) \le 0$

$t \ge 0$.

If, in addition to (1), P1 has constraints of the form $x \ge 0$, then [for example, by taking the corresponding $\phi_i(t) = t$] one obtains in P2, P3, and P4 the additional constraints $y \ge 0$.

Special cases

If f, g, and h are linear and all $\phi_i(t) = t$, then our theorem gives the results of Charnes and Cooper [1]. If f and g are quadratic, h is linear, $\phi_0(t) = t^2$, and all other $\phi_i(t) = t$, then the results of Swarup [5] are obtained. If f and g are polynomials of degree k, h

is linear, $\phi_0(t) = t^k$, and all other $\phi_i(t) = t$, then we obtain the result of Sharma [4]. If h is linear, f and g unrestricted, and $\phi_i(t) = t$, $i \neq 0$, the results of Mond and Craven [3] are obtained.

Remark 1

The assumption *(i)* of Theorem 1 is always satisfied if the constraint set of P1 is non-empty and bounded, h is linear, and $\phi_i(t) = t$, $i = 1, \ldots, m$ ([1] or [3]).

An example of nonlinear constraints where *(i)* of Theorem 1 is satisfied is the following:

Assume that the constraint set of P1 is non-empty and bounded and that (1) consists of constraints of the form

$$(11) \qquad h_i(x) \equiv x^tCx + \alpha \leq 0 , \quad i = 1, \ldots, m ,$$

where C is a positive semi-definite symmetric matrix, $\alpha \in R$ and $\phi_i(t) = t^2$, $i = 1, \ldots, m$. Thus

$$(12) \qquad H_i(y, t) = y^tCy + \alpha t^2 \leq 0 , \quad i = 1, \ldots, m .$$

If now $(y, 0)$ satisfies (12), $y^tCy \leq 0$ implies, for C positive semi-definite, that $y^tCy = 0$, and hence $Cy = 0$ (by [2], Appendix (g)).

From $G(y, t) = \Delta \neq 0$ and $G(0, 0) = 0$ it follows that $(y, 0)$ feasible implies $y \neq 0$.

Hence if x satisfies (11), for any scalar k ,

$$\begin{aligned} h_i(x+ky) &\equiv (x+ky)^tC(x+ky) + \alpha \\ &= x^tCx + k^2y^tCy + 2kx^tCy + \alpha \\ &= x^tCx + 0 + 0 + \alpha \\ &\leq 0 . \end{aligned}$$

Thus, for any scalar k , $x + ky$ $(y \neq 0)$ is feasible for P1, contradicting the assumption of boundedness.

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If (1) is non-empty and bounded, and (1) consists of linear constraints and constraints of the form (11), then by a combination of the arguments here and in [1] it follows that *(i)* of Theorem 1 is always satisfied.

Remark 2

As noted in [3], even if $f(x)$ is concave with respect to x, $F(y, t)$ need not be concave with respect to the vector variable (y, t). Even if G and H are convex with respect to (y, t), the constraint set of P3 is not necessarily convex. Instead of P3, therefore, it is sometimes more convenient to deal with the following

(13) P3' Maximize $F(y, t)$

$$\text{subject to } G(y, t) \le 1, \quad H(y, t) \le 0, \quad t \ge 0 .$$

If (y^*, t^*) is optimal for P3', $t^* > 0$, and $G(y^*, t^*) = 0$, then $x^* = y^*/t^*$ is feasible for P1 but $g(x^*) = 0$. In this case P1 need not possess a maximum. If $G(y^*, t^*) = \Delta_1$, where $0 < \Delta_1 < \infty$, then (y^*, t^*) is also optimal for P2 with $\Delta = \Delta_1$, so Theorem 1 holds.

Note that *(i)* of Theorem 1 may no longer be satisfied under the conditions given in Remark 1, since $y = 0$, $t = 0$ might now be feasible. In [3], an example is given of P3', with linear constraints and non-empty bounded constraint set, such that the optimum occurs at $(0, 0)$. In such a case, nothing can be deduced from P3' concerning the optimum of P1.

Observe that if $F(y, t)$ is concave, and G and H convex with respect to the vector variable (y, t), then P3' is a concave programming problem with a convex constraint set, and may be solved by any standard technique for concave programming.

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https://doi.org/10.1017/S0004972700024047 Published online by Cambridge University Press