NONLINEAR GEOMETRIC OPTICS FOR HYPERBOLIC SYSTEMS OF CONSERVATION LAWS

BY

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NONLINEAR GEOMETRIC OPTICS FOR HYPERBOLIC SYSTEMS OF CONSERVATION LAWS

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I. Introduction

Here we give a detailed discussion of recent developments, both formal and rigorous, in the theory of weakly nonlinear geometric optics for constructing asymptotic solutions of quasi-linear hyperbolic systems in one and several space variables. This method is the main perturbation technique used in analyzing nonlinear wave motion for hyperbolic systems. The ideas for this method originated in the late 1940's and early 1950's in pioneering work of Landau [8],

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Lighthill [10], and Whitham [18]. However, it is only in very recent work [4], [5], [7], [1] that these methods have been developed through systematic self-consistent perturbation schemes for resonant and nonresonant wave problems in one and several space dimensions. Sections II and III of this paper give a detailed discussion and description of these formal perturbation methods applied to problems in 1-D and multi-D, respectively. The reader can consult the survey in [16] which reviews the literature on weakly nonlinear hyperbolic waves before 1981 and compare this treatment with the one described in sections II and III to see the recent progress in the field in constructing such formal perturbation expansions.

One of the goals of the use of such asymptotic methods is to reduce extremely complex problems to simpler but often non-trivial problems which are more readily understood. In fact one of the main themes of this paper is that for weakly nonlinear hyperbolic wave motions, extremely complicated wave motions for systems in 1-D and multi-D are well approximated through solutions of much simpler equations such the inviscid Burgers equation. As we will see in section 2, the tacit assumptions used in the formal derivation of the expansions of weakly nonlinear geometric optics are that solutions of the underlying hyperbolic system remain smooth; nevertheless, in a variety of applied contexts these methods are often used after shock waves have formed in general weak solutions of conservation laws and yield a very good approximation. In section 4 we give a leisurely discussion of the recent paper of DiPerna and the author [3] which contains rigorous results on the use of weakly nonlinear geometric optics for systems in 1-D. These results indicate that the method is even better for weak solutions then could be anticipated from the formal predictions of the perturbation theory!! For example, within errors that are of order ε^2 uniformly for all time, the weak solution of a general initial value problem for a general $M \times M$ system of genuinely nonlinear conservation laws with initial data with amplitude ε is approximated by the weak solutions of completely decoupled Burgers equations.

There are several recent multi-D applications of the formal methods presented here to complex physical problems including the development of simplified models in reacting gas flow [17], the regular reflection of weak shocks [6], and the formation of Mach stems in reacting shock fronts [13], [14]. We will not discuss any of these applications in detail here but briefly mention one accessible open theoretical problem which arises from the work in [13], [14]. An application of weakly nonlinear geometric optics to the (free-surface) shock fronts in multi-D reacting gases yields the simplified governing integro-differential conservation law (see [13], [14]) for a scalar quantity, $\sigma(x,t)$, given by

$$\sigma_t + a_1\left(\frac{1}{2}\sigma^2\right)_x + a_2\left[\int_0^\infty \sigma(x+\beta s)\sigma_x(x+s)ds\right]_x = 0 \tag{1.1}$$

with $a_1 - a_2 \neq 0$ and the parameter β satisfying $\beta > I$. We remark that if $\beta \equiv I$, the equation in (1.1) reduces to the inviscid Burgers equation. The numerical experiments with (1.1) strongly predict that smooth solutions of (1.1) develop shocks (see [14]) although the nonlocal terms have some prominent and unusual effects in this breakdown process in various parameter regimes. The accessible open problem for (1.1) which we present here is the following:

Problem: Find some (or any) initial data for (1.1) for which one can prove rigorously that shock waves occur.

A brief discussion of a number of other accessible open problems in the rigorous theory of weakly nonlinear geometric optics is given at the end of section 4 of this paper. The applications of the ideas presented here to understanding complex multi-D problems is only beginning and possible applications to complex Mach bifurcations, instability in supersonic jets, and the detailed structure of multi-D reacting shock fronts are currently being developed.

Acknowledgement: The author thanks his scientific collaborators, R. Rosales, J. Hunter, and R. DiPerna for their explicit and implicit contributions to this paper.

II. The Formal Theory: Weakly Nonlinear Hyperbolic Waves in 1-D

We consider solutions of the general $M \times M$ strictly hyperbolic system in a single space dimension given by the M equations,

$$(F_0(u))_t + (F_1(u))_x = 0, \quad -\infty < x < \infty \quad t > 0. \tag{2.1}$$

The requirement of strict hyperbolicity at a constant vector, $u_0 \in \mathbb{R}^M$, with t, a time-like direction, means that

$$\det(A_0(u_0)) \neq 0 \tag{2.2a}$$

and the generalized eigenvalue problems

$$\begin{aligned} (A_1(u_0) - \lambda A_0(u_0)) \cdot r &= 0 \\ \ell \cdot (A_1(u_0) - \lambda A_0(u_0)) &= 0 \end{aligned} \tag{2.2b}$$

have M-distinct real eigenvalues $\lambda_1 < \lambda_2 < \ldots < \lambda_M$ with corresponding right eigenvectors, r_j, $1 \leq j \leq M$ and left eigenvectors, ℓ_K, $1 \leq K \leq M$ satisfying the normalization conditions,

$$\ell_K \cdot A_0 r_j = \delta_{Kj} \tag{2.2c}$$

with δ_{Kj} the Kronecker delta symbol. Here $A_0(u)$, $A_1(u)$ are the $M \times M$ Jacobian matrices of the corresponding smooth mappings, $F_0(u)$, $F_1(u)$, i.e. $A_j(u) = \frac{\partial F_j(u)}{\partial u}$, $j = 0,1$. We also assume that $F_j(u)$ admits the Taylor expansion at u_0 given by

$$F_j(u_0 + \varepsilon v) = F_j(u_0) + \varepsilon A_j v + \frac{\varepsilon^2}{2} B_j(v,v) + O(\varepsilon^3) \quad j = 0,1 \tag{2.3}$$

where the B_j are the corresponding Hessian matrices of $F_j(u)$ at u_0.

Our objective here is to construct formal asymptotic approximate solutions of the hyperbolic system in (2.1) with small amplitude rapidly oscillating initial data with the form,

$$u_0^\varepsilon(x) = u_0 + \varepsilon u_1^0\left(x, \frac{x}{\varepsilon}\right) \tag{2.4}$$

with $u_1^0(x,\tilde{x})$ a smooth structure function. To clarify and simplify the presentation we will often assume that $u_1^0(x,\tilde{x}) = u_1(\tilde{x})$ and either

Hypothesis #1 $u_1^0(\tilde{x})$ has compact support

or

Hypothesis #2 $u_1^0(\tilde{x})$ is a periodic function with period one and mean zero, i.e.,

$$\int_0^1 u_1^0(\tilde{x} + a)d\tilde{x} = 0 \quad \text{for any} \quad a \in R_0 .$$

Under Hypothesis #1, the basic asymptotic expansions which we construct below are non-resonant and to the formal degree of approximation, nonlinear wave fields can be superimposed and do not interact as in the linear case. On the other hand, Hypothesis #2 describes the simplest situation where different nonlinear wave fields can interact and resonate to leading order.

To summarize, in the remainder of this section we will build formal asymptotic approximate solutions of the hyperbolic system in (2.1). These solutions approximate the solution, $u^\varepsilon(x,t)$, which satisfies

$$\begin{aligned} &F_0(u^\varepsilon)_t + F_1(u^\varepsilon)_x = 0, \ -\infty < x < \infty, \ 0 < t \\ &u^\varepsilon(x,0) = u_0 + \varepsilon u_1^0(x, \tfrac{x}{\varepsilon}) \ . \end{aligned} \tag{2.5}$$

The methods which we present below in developing the formal constructions are less explicit and somewhat more complicated than those already used in [15]. The advantages of these methods is that they generalize readily to many interesting situations in several space dimensions which we describe in the next section. The fourth section of this paper contains a discussion of rigorous work by DiPerna and the author which justifies many of the expansions presented in this section, even after shock waves have formed in the given weak solution in (2.5). Nevertheless, many interesting theoretical questions remain regarding these approximations. Finally, we refer the reader to [15] for a complete discussion of the formal constructions presented below for more general initial data via explicit solution of the second order perturbation equations.

2.1: The Single Wave Expansion in 1-D

Here we develop the simplest asymptotic approximation for the solution in

(2.5). The result of this expansion will be a formal asymptotic solution, $\tilde{u}_w^\varepsilon(x,t)$, with the special initial data,

$$u_0^\varepsilon(x) = u_0 + \varepsilon\, \sigma_j^0(x, \tfrac{x}{\varepsilon})r_j \tag{2.6}$$

for any fixed j with $1 \le j \le M$. Here r_j is the corresponding right eigenvector and $\sigma_j^0(x,\tilde{x})$ is a smooth function which is bounded with bounded first derivatives.

By analogy with linear geometric optics, we attempt to construct, formal single wave asymptotic solutions of the equations in (2.1) with the form,

$$\tilde{u}_w^\varepsilon(x,t) = u_0 + \varepsilon u_1(x,t,\tfrac{\phi}{\varepsilon}) + \varepsilon^2 u_2(x,t,\tfrac{\phi}{\varepsilon}) \tag{2.7}$$

where $\phi(x,t)$ is a phase function to be determined. We require $\tilde{u}_w^\varepsilon(x,t)$ defined in (2.7) to be a smooth formal solution of the strictly hyperbolic system in (2.1) to two orders in ε, i.e., $\tilde{u}_w^\varepsilon(x,t)$ should satisfy

$$\frac{\partial}{\partial t} F_0(\tilde{u}_w^\varepsilon) + \frac{\partial}{\partial x} F_1(\tilde{u}_w^\varepsilon) = o(\varepsilon). \tag{2.8}$$

The requirement in (2.8) means that after inserting the ansatz from (2.7) into (2.1), we must be able to choose $u_1(x,t,\frac{\phi}{\varepsilon})$, $u_2(x,t,\frac{\phi}{\varepsilon})$ so that

(2.9)
A) The terms of order zero in ε vanish
B) The terms of order one in ε vanish.

Obviously, in order to have the function, $u_1(x,t,\frac{\phi}{\varepsilon})$, consistently describe the leading order asymptotic behavior on some region G, in space time, we need to require additionally that

$$|u_2(x,t,\tfrac{\phi}{\varepsilon})| = o(\varepsilon^{-1}) \quad \text{for} \quad (x,t) \in G, \tag{2.10}$$

i.e. $u_2(x,t,\theta)$ should grow <u>sublinearly</u> in θ for $x,t \in G$ and $\theta = \frac{\phi}{\varepsilon}$. The requirement that the initial data have the special form in (2.6) for these given single wave approximate solutions will arise after we impose the requirements in (2.9) and (2.10) on the solution $\tilde{u}_w^\varepsilon(x,t)$ with the single wave ansatz in (2.7).

By substituting the ansatz from (2.7) into (2.1) and expanding the nonlinear terms via (2.3), we compute that the requirements in (2.9a) are satisfied provided that we can choose $u_1(x,t,\theta)$, $\phi(x,t)$ so that,

Order zero

(2.11)
$$(A_0\phi_t + A_1\phi_x)\frac{\partial}{\partial\theta}u_1(x,t,\theta)\Big|_{\theta=\frac{\phi}{\varepsilon}} = 0 .$$

Similarly, the requirement in (2.9b) is satisfied provided that we can choose $u_2(x,t,\theta)$ so that

Order one

(2.12)
$$-(A_0\phi_t + A_1\phi_x)\frac{\partial}{\partial\theta}u_2(x,t,\theta)\Big|_{\theta=\frac{\phi}{\varepsilon}} =$$

$$[A_0(u_1)_t + A_1(u_1)_x + \phi_t B_0((u_1),(u_1)_\theta) + \phi_x B_1((u_1),(u_1)_\theta)]\Big|_{\theta=\frac{\phi}{\varepsilon}} .$$

We use the familiar method of multiple scales and regard the variable θ in (2.11) and (2.12) as an additional independent variable linked with (x,t) only in the final asymptotic form through the evaluation, $\theta = \frac{\phi}{\varepsilon}$.

The equation in (2.11) is satisfied provided that

(2.13a)
ϕ^j satisfies the eikonal equation,
$$\phi^j_t + \lambda_j\phi^j_x = 0 \quad \text{for some } j, \quad 1 \leq j \leq M$$

with λ_j an eigenvalue from (2.2b) and

(2.13b)
$u_1(x,t,\theta)$ satisfies
$$u_1(x,t,\theta) = \sigma_j(x,t,\theta)r_j$$

with r_j the corresponding right eigenvector from (2.2c). We recognize the equation in (2.13a) as the familiar eikonal equation of linear geometric optics. For a single space variable, we choose the simplest phase function solution of this equation, namely

(2.13c)
$$\phi^j = x - \lambda_j(u_0)t, \quad \text{any } j, \quad 1 \leq j \leq M.$$

At this state in the argument, $\sigma_j(x,t,\theta)$ is an arbitrary amplitude function.

This completes the solution of the order zero equations.

Now, we turn to the equations in (2.12) which represent the order one contribution in powers of ε. First, we substitute the form of $u_1(x,t,\theta)$ as determined in (2.13b) into the right hand side of (2.12) and regard the equation in (2.12) to be satisfied for all θ following the ideas of the method of multiple scales. With ϕ^j satisfying (2.13), we see from (2.2c) that $\ell_j \cdot (A_0\phi_t^j + A_1\phi_x^j) = 0$ and the ordinary differential equation in θ defined by (2.12) with the form

(2.14a)
$$(A_0\phi_t^j + A_1\phi_x^j)\,\frac{\partial}{\partial\theta}\,u_2 = F(\theta)$$

has a solution, $u_2(\theta)$, if and only if

(2.14b)
$$\ell_j \cdot F(\theta) = 0.$$

By imposing this condition on the explicit inhomogeneous terms defined by the right hand side of (2.12), we obtain the much simpler differential equation for $\sigma_j(x,t,\theta)$,

(2.15)
$$(\sigma_j)_t + \lambda_j(\sigma_j)_x + b_j\left(\frac{1}{2}\,\sigma_j^2\right)_\theta = 0$$

with b_j given by the formula,

(2.16)
$$b_j = \ell_j \cdot (-\lambda_j B_0(r_j,r_j) + B_1(r_j,r_j))$$

for any j with $j = 1,\ldots,M$.

The final step in the formal derivation of the leading order asymptotics is to find a solution, $u_2(x,t,\theta)$, satisfying the ordinary differential equation in (2.14) with $F(\theta)$ defined in (2.12) and with sub-linear growth in θ for (x,t) in a space-time region G; i.e. we need to satisfy the requirement in (2.10). For these special simple wave forms this is easily achieved. We seek the solution, $u_2(\theta)$, for (2.14) in the form, $u_2(\theta) = \sum_{p\neq j} \tilde{\sigma}_p r_p$ and compute that (2.14) is satisfied if and only if

(2.17)
$$\frac{\partial\tilde{\sigma}_p}{\partial\theta} = (\lambda_j - \lambda_p)^{-1} c_{pj}(\sigma_j^2)_\theta$$

for $p \neq j$, $1 \leq p \leq M$ with c_{pj} explicit constants. We remark that one consequence of the conservation form in (2.1) is that the inhomogeneous right hand side of the equations in (2.17) is an exact θ derivative. Thus, u_2 has the explicit form,

$$u_2 = \sum_{p \neq j} (\lambda_j - \lambda_p)^{-1} c_{pj} \sigma_j^2 r_j$$

and u_2 is <u>bounded</u> <u>automatically</u> on any space-time region G where σ_j is smooth and bounded - thus, the sublinear growth condition in (2.10) is trivially satisfied in this case.

What has been achieved by this expansion? To summarize, we have demonstrated that solutions u^ε of the general quasi-linear hyperbolic initial value problem

$$(2.18) \qquad \begin{aligned} &F_0(u^\varepsilon)_t + F_1(u^\varepsilon)_x = 0, \quad x \in R^1, \ t > 0 \\ &u^\varepsilon(x,0) = u_0 + \varepsilon \sigma_j^0(x, \tfrac{x}{\varepsilon}) r_j \end{aligned}$$

are uniformly approximated in regions of smoothness within terms of order ε^2 by the solution u_w^ε with the form,

$$(2.19) \qquad u_w^\varepsilon = u_0 + \varepsilon \sigma_j(x, t, \frac{\phi_j}{\varepsilon}) r_j$$

where σ_j satisfies the much simpler scalar conservation law,

$$(2.20) \qquad \begin{aligned} &(\sigma_j)_t + \lambda_j (\sigma_j)_x + b_j(\tfrac{1}{2}\sigma_j^2)_\theta = 0 \\ &\sigma_j(x,t,\theta)|_{t=0} = \sigma_j^0(x,\theta). \end{aligned}$$

First, we remark that if the initial data has the form in Hypothesis #1 or #2, i.e. σ_j^0 is a function of θ alone, then the solution of (2.20) is independent of x and reduces to the famous inviscid Burgers' equation,

$$(2.21) \qquad \begin{aligned} &(\sigma_j)_t + b_j(\tfrac{1}{2}\sigma_j^2)_\theta = 0 \\ &\sigma_j(t,\theta)|_{t=0} = \sigma_j^0(\theta) \end{aligned}$$

provided that $b_j \neq 0$. We remark that in the special case that $F_0(u) = u$, the condition

(2.22) $b_j \neq 0$ is equivalent to Lax's genuine nonlinearity condition at u_0

(see [12]). The more general problem in (2.20) is also easily solved exactly by reduction to the inviscid Burgers' equation through the use of characteristic coordinates for the operator, $\frac{\partial}{\partial t} + \lambda_j \frac{\partial}{\partial x}$.

2.2: Multi-Wave Non-Resonant Asymptotic Solutions in 1-D

Here we consider the asymptotic approximation of the solution $u^{\varepsilon}(x,t)$ to the initial value problem with general initial data,

$$
\begin{aligned}
F_0(u^{\varepsilon})_t + F_1(u^{\varepsilon})_x &= 0 \ , \ x \in R^1, \quad t > 0 \\
u^{\varepsilon}(x,0) &= u_0 + \varepsilon u_1^0\left(\frac{x}{\varepsilon} \right).
\end{aligned}
\tag{2.23}
$$

If the problem in (2.23) were linear, we would simply decompose the data, $u_1^0(\frac{x}{\varepsilon})$, according to the right eigenvectors r_j and superimpose the single wave asymptotic solutions defined in (2.19) and (2.12). Thus, to solve the asymptotic problem in (2.23), our first naive guess is to define $u_w^{\varepsilon}(x,t)$ by superposition as

$$
u_w^{\varepsilon}(x,t) = u_0 + \varepsilon \sum_{j=1}^{M} \sigma_j\left(t,\frac{\phi^j}{\varepsilon}\right) r_j
\tag{2.24}
$$

where $\phi^j = x - \lambda_j t$ and each σ_j solves the much simpler initial value problem for the inviscid Burgers' or linear advection equation given by

$$
\begin{aligned}
(\sigma_j)_t + b_j\left(\tfrac{1}{2} \sigma_j^2\right)_\theta &= 0, \quad \theta \text{ d } R^1, \quad t > 0 \\
\sigma_j(t,\theta)\big|_{t=0} &= \ell_j \cdot u_1^0(\theta)
\end{aligned}
\tag{2.25}
$$

for $1 < j < M$. <u>When such a simple multi-wave expansion</u> from (2.24) and (2.25) <u>generates a self-consistent formal asymptotic solution</u>, we say that the multi-wave approximation is <u>nonresonant</u>. Of course the requirements of generating a formal asymptotic solution are analagous to those already discussed in (2.8) - (2.10). We must find a function, $u_2(x,t,\vec{\theta})$ with $\vec{\theta} \in R^M$ so that with $\tilde{u}_w^{\varepsilon}$ given by

$$
\tilde{u}_w^{\varepsilon} = u_0 + \varepsilon \sum_{j=1}^{M} \sigma_j\left(t,\frac{\phi^j}{\varepsilon}\right) r_j + \varepsilon^2 u_2\left(x,t,\frac{\vec{\phi}}{\varepsilon}\right)
\tag{2.26}
$$

both of the conditions in (2.9) are satisfied and in addition,

$$|u_2(x,t,\frac{\vec{\phi}}{\varepsilon})| = o(\varepsilon^{-1}) \tag{2.27}$$

for $(x,t) \in G$, some space-time domain; then the multi-wave approximation from (2.24) and (2.25) is uniformly valid on G and we have achieved a remarkable simplification by approximating the general small amplitude initial value problem in (2.23) by the much simpler equations in (2.24) and (2.25) with exact solutions. Obviously, with the form in (2.24) and (2.13), we have already guaranteed that the terms of order zero in powers of ε vanish; the crux of the matter is to choose $u_2(x,t,\vec{\theta})$ so that simultaneously the terms of order one in ε vanish, i.e. (2.9b) is satisfied, and u_2 satisfies the sublinear growth conditions. This will not be possible always and resonances can occur, this is the topic of the next subsection. Here we will derive some simple sufficient conditions for the form validity of the nonresonant wave expansions including a careful formal analysis of the behavior of u_2; these sufficient conditions always apply under hypothesis #1, i.e. when $u_0^1(x)$ is smooth with compact support.

By following the calculations in (2.12) it is tedious but straightforward to compute with the ansatz in (2.26) that the order one terms in ε vanish provided that

Order one

$$-\sum_{\ell=1}^{M} \mathcal{A}_\ell \frac{\partial}{\partial\theta_\ell} u_2(x,t,\vec{\theta})\Big|_{\vec{\theta}=\frac{\vec{\phi}}{\varepsilon}} = G(t,\vec{\theta})\Big|_{\vec{\theta}=\frac{\vec{\phi}}{\varepsilon}} . \tag{2.28}$$

Here the matrix coefficients, $\mathcal{A}_\ell$ are given by

$$\mathcal{A}_\ell = \phi_t^\ell A_0 + \phi_x^\ell A_1 = -\lambda_\ell A_0 + A_1 \tag{2.29}$$

for $1 \leq \ell \leq M$. The term $G(t,\vec{\theta})$ is a sum of simple and binary wave interaction terms,

$$G(t,\theta) = \sum_{j=0,1}^{M} g_j(t,\theta_j) + \sum_{\substack{1 \leq p,q \leq M \\ p \neq q}} g_{pq}(t,\theta_p,\theta_q) \tag{2.30a}$$

with the single wave terms given by

(2.30b) $$g_j(t,\theta_j) = A_0((\sigma_j)_t\, r_j + [-\lambda_j B_0(r_j,r_j) + B_1(r_j,r_j)]\sigma_j(\sigma_j)_{\theta_j}$$

and the binary wave interaction terms given by

(2.30c) $$g_{pq}(t,\theta_p,\theta_q) = \frac{1}{2}\,[-\lambda_p B_0(r_p,r_q) + B_1(r_p,r_q)]\sigma_q(t,\theta_q)(\sigma_p(t,\theta_p))_{\theta_p}\,.$$

Below, we use the notation $\sigma'(t,\theta)$ for $\sigma(t,\theta)_\theta$. Clearly, the single wave contributions in (2.30b) are handled exactly as we discussed earlier in (2.14)-(2.16) provided that the Burgers equations from (2.25) are satisfied. To handle the contributions from the binary wave interaction terms in (2.28), we use the method of multiple scales and regard the $\vec{\theta}$ variables as independent variables. By superposition, we only need to construct a solution $u_2(x,t,\theta_p,\theta_q)$ satisfying the sublinear growth condition from (2.27) and the linear P.D.E.,

(2.31) $$(-\lambda_p A_0 + A_1)\,\frac{\partial}{\partial\theta_p}\,u_2 + (-\lambda_p A_0 + A_1)\,\frac{\partial}{\partial\theta_q}\,u_2 = g_{pq}(\theta_p,\theta_q)$$

for $p \neq q$, $1 \leq p,q \leq M$. Once again, in a single space variable, we solve for u_2 explicitly through the basis expansion,

$$u_2 = \sum_{j=1}^{M} \tilde{\sigma}_j r_j.$$

The problem in (2.31) is equivalent to the M scalar advection equations,

(2.32) $$(\lambda_j - \lambda_p)\,\frac{\partial}{\partial\theta_p}\,\tilde{\sigma}_j + (\lambda_j - \lambda_q)\,\frac{\partial}{\partial\theta_q}\,\tilde{\sigma}_j = \ell_j \cdot g_{pq}(\theta_p,\theta_q)$$

for $p \neq q$, $1 \leq j \leq M$.

Thus, we only need to decide the conditions on the inhomogeneous terms $\ell_j \cdot g_{pq}(\theta_p,\theta_q)$ with g_{pq} given explicitly in (2.30c) which guarantee that the scalar advection equations in (2.32) have solutions $\tilde{\sigma}_j$ with sublinear growth in $|\theta_p| + |\theta_q|$. The cases when $j = p,q$ are special and we handle these first. From (2.30c), we compute that when $j = p$, (2.32) becomes with some constant, c_p^{pq},

(2.33) $$(\lambda_p - \lambda_q)\,\frac{\partial}{\partial\theta_q}\,\tilde{\sigma}_p = c_p^{pq}\,\sigma_q(t,\theta_q)\,\frac{\partial}{\partial\theta_p}\,(\sigma_p(t,\theta_p))$$

and the solution $\tilde{\sigma}_p$ within a function of θ_p is given by integration,

(2.33b) $$\tilde{\sigma}_p(\theta_p,\theta_q) = (\lambda_p - \lambda_q)^{-1} c_p^{pq}\sigma'(t,\theta_p)\int_{\theta_q^0}^{\theta_q} \sigma_q(t,\theta)d\theta.$$

We see that under Hypothesis #1 or Hypothesis #2 on the initial data, $\sigma_p(\theta_p,\theta_q)$ is always bounded on any region where $\sigma_p'(t,\theta_p)$ is bounded and

$$(2.33c)\qquad |\tilde{\sigma}_p(t,\theta_p,\theta_q)| < |c_p^{pq}||\lambda_p - \lambda_q|^{-1}|\sigma_p'(t,\theta)|_\infty \times \max_{-\infty<\theta_q<\infty} \left| \int_{\theta_q^0}^{\theta_q} \sigma_q(t,\theta)d\theta \right|$$

with $|f(t,\theta)|_\infty = \max_{\theta\in R^1}|f(t,\theta)|$. Similarly, when $j = q$, $\tilde{\sigma}_q(t,\theta_p,\theta_q)$ can be chosen explicitly as a bounded function of (θ_p,θ_p) under Hypothesis #1 or #2 in any region where $\sigma_p(t,\theta)$, $\sigma_q(t,\theta)$ remain smooth and $\tilde{\sigma}_q$ satisfies the estimate,

$$(2.34)\qquad |\tilde{\sigma}_q(t,\theta_p,\theta_q)| < |c_p^{pq}||\lambda_p - \lambda_q|^{-1}|\sigma_p(t,\theta)|_\infty|\sigma_p(t,\theta)|_\infty.$$

The treatment of the scalar advection equation in (2.32) is quite different when $j \neq p$ and $j \neq q$, i.e. for $M > 3$. Here the nonresonant expansion will always be valid under Hypotheis #1 on the initial data; on the other hand, this will not be true for periodic data satisfying Hypothesis #2 due to the appearance of resonant wave interactions (see the next subsection). Within a bounded function, the solution of the inhomogeneous scalar advection equation in (2.32) is given by

$$(2.35)\qquad \tilde{\sigma}_j = c_j^{pq}(\lambda_j - \lambda_p)^{-1} \int_0^{\theta_p} \sigma_p'(t,s)\sigma_q(t,\theta_q - h_j^{pq}(\theta_p - s))ds$$

with $h_j^{pq} = \dfrac{\lambda_j - \lambda_q}{\lambda_j - \lambda_p}$. Next, we give some simple explicit conditions which guarantee that $\tilde{\sigma}_j$ is uniformly bounded. For a smooth function $\sigma_p(t,\theta)$, the θ-variation of σ_p is the L^1-norm of the first derivative defined by

$$(2.36)\qquad \mathrm{Var}(\sigma_p(t,\theta)) = \int_{-\infty}^{\infty} |\sigma_p'(t,s)|ds.$$

With this definition we see that $\tilde{\sigma}_j(t,\theta_p,\theta_q)$ satisfies the estimate,

$$(2.37)\qquad |\tilde{\sigma}_j(t,\theta_p,\theta_q)| < c_j^{pq}|\lambda_j - \lambda_p|^{-1}|\sigma_p(t,\theta)|_\infty \mathrm{Var}\,\sigma$$

for $1 < j < M$, $j \neq p,q$.

Let's summarize the conditions which guarantee that we can find $u_2(x,t,\vec{\theta})$ which is bounded on some space time region G and so that $u_2(x,t,\vec{\theta})$ satisfies (2.28). From (2.17), (2.34), and (2.37), we need

(2.38a) $$\sum_{j=1}^{M}(|\sigma_j(t,\theta)|_\infty + \mathrm{Var}\sigma_j(t,\theta)) \le C_0$$

while the conditions in (2.33) require

(2.38b) $$\sum_{\substack{q,j=1\\ q\neq j}}^{M} (|\sigma_j'(t,\theta)|_\infty \times \max_{-\infty<\theta_q<\infty} |\int_0^{\theta_q}\sigma_q(t,\theta)d\theta|) \le C_0 .$$

Finally, we check the self-consistency of the asymptotic expansion. The functions $\sigma_j(t,\theta)$ satisfy the simple scalar conservation laws in (2.25) with initial data, $\ell_j\cdot u_1^0(\theta)$ for each $j = 1,\dots,M$. Since scalar conservation laws do not increase the maximum norm, the variation of a solution, or the functional, $\max_{\theta_1,\theta_2\in R^1} |\int_{\theta_1}^{\theta_2} f(s)ds|$, we see that all of the quantities in (2.38) except $|\sigma'(t,\theta)|_\infty$ are <u>uniformly bounded in time</u> provided that initially, we have

(2.39) $$|u_1^0(x)|_\infty + \mathrm{Var}\ u_1^0 + \max_{-\infty<x_1<x_2<\infty} |\int_{x_1}^{x_2} u^0(s)ds| < \infty.$$

Thus, under the assumption in (2.39) for the smooth initial data, the formal nonresonant expansion is valid with a formal error of order ε^2 for all (x,t) with $0 \le t \le T$ where T is any interval of existence of smooth solutions for the initial value problem for the Burgers equations in (2.25) where

(2.40) $$\max_{1\le j\le M} |\sigma_j'(t,\theta)|_\infty < \infty$$

for $0 \le t \le T$, i.e. until shock waves have formed in these simple solutions. For many (rarefaction) initial data, $T \equiv +\infty$!!

We have "proved" the following:

<u>Proposition 1</u>: (<u>Formal Validity of the Nonresonant Multi-Wave Expansion</u>)

Assume that the initial data for the initial value problem in (2.23) satisfies the condition in (2.39), then the nonresonant multi-wave expansion defined in (2.24) and (2.25) is a formal uniformly valid approximate solution of (2.23) with errors uniformly of order ε^2 on any space-time region G defined by $-\infty < x < \infty$ and $0 < t < T$ with T restricted by (2.40). In particular,

A) For smooth data satisfying hypothesis #2 so that $u_1^0(x)$ has compact support, the general solution of the general hyperbolic system in (2.23) is always formally approximated uniformly within terms of order ε^2 in regions of smoothness by the much simpler decoupled inviscid Burgers' solutions satisfying (2.24) and (2.25).

B) The same remark as in A) applies for solutions with periodic initial data provided that $M < 2$.

In the above, we have carried out the formal assessment of the validity of the nonresonant asymptotic expansions in great detail. In section 4 we will develop a rigorous analysis which both confirms these formal calculations in regions of smoothness and somewhat surprisingly extends the validity of the asymptotics to regions where shock waves have formed!! This is not predicted by the formal asymptotic analysis but is a consequence of the rigorous analysis of nonlinear wave interactions.

2.3 Periodic Resonant Wave Asymptotics in 1-D

Here we consider the problem of constructing uniformly valid formal asymptotic approximate solutions for the initial value problem,

$$
\begin{aligned}
&F_0(u^\varepsilon)_t + (F_1(u^\varepsilon))_x = 0, \quad x \in R^1, \quad t > 0 \\
&u^\varepsilon(x,0) = u_0 + \varepsilon u_1^0\left(\frac{x}{\varepsilon}\right)
\end{aligned}
\tag{2.41}
$$

where $u_1^0(x)$ is a smooth periodic function of period one with mean zero, i.e. Hypothesis #2 is satisfied. We will discover that typically for $M > 3$, the simple decoupled nonresonant multi-wave expansions are not a uniformly valid asymptotic approximation; in fact, the inviscid Burgers equations need to be coupled through terms incorporating nonlocal resonant wave interactions in order to develop a uniform approximation. We will develop the asymptotic expansions by a more general but less explicit approach than the one used earlier in [15]. The advantage of the more general approach that we develop here is that the same method can be used in studying the multi-dimensional planar oblique wave interactions which we describe in section three (see [7]).

In a familiar fashion, we begin with the ansatz

$$\tilde{u}_w^\varepsilon = u_0 + \sum_{j=1}^{M} \varepsilon\sigma_j(t,\frac{\phi_j}{\varepsilon})r_j + \varepsilon^2 u_2(t,\frac{\vec{\phi}}{\varepsilon}) \tag{2.42}$$

with $\phi_j = x - \lambda_j t$ and $\sigma_j(0,\theta) = \ell_j \cdot u_1^0(\theta)$ for $1 \leq j \leq M$. As in (2.13), the form of the leading terms automatically guarantees that the terms of order zero in ε from (2.9a) vanish but here we do not impose the equations in (2.25) at the outset. The equation for the terms of order one in ε is the same one as given in (2.28) with the same definitions for the coefficients as given in (2.29) and (2.30). Now, in order to guarantee that a solution $u_2(t,\frac{\vec{\phi}}{\varepsilon})$ of (2.28) can be found with sublinear growth for the periodic case, we need to investigate (as in (2.31)) the auxilliary constant coefficient P.D.E.,

$$(-\lambda_p A_0 + A_1)\frac{\partial u}{\partial\theta_p} + (-\lambda_q A_0 + A_1)\frac{\partial u}{\partial\theta_q} = \tilde{g}(\theta_p,\theta_q) \tag{2.43}$$

with $\tilde{g}$ double periodic. We remark that the use of $\tilde{g}(\theta_p,\theta_q)$ on the right hand side of (2.43) is not a typographical error since we will ultimately use solutions of (2.43) when $\tilde{g}(\theta_p,\theta_q)$ is <u>not</u> the function defined on the right hand side of (2.31) from our earlier argument. We have the following preliminary fact:

<u>Lemma 1</u>: Assume that $\tilde{g}(\theta_p,\theta_q)$ is a doubly periodic function with $\int_0^1 \tilde{g}(s,\theta_q)ds = \int_0^1 \tilde{g}(\theta_p,s)ds = 0$, then the P.D.E. in (2.43) has a solution $u(\theta_p,\theta_q)$ with sublinear growth in $|\theta_p| + |\theta_q|$ if and only if

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} \ell_j \cdot \tilde{g}((\theta_p,\theta_q) + \vec{a}_{pq}^{\,j}s)ds = 0 \tag{2.44}$$

for all $j \neq p,q$ and $1 \leq j \leq M$ with

$$\vec{a}_{pq}^{\,j} = (\lambda_j - \lambda_p,\ \lambda_j - \lambda_q)$$

<u>Remark</u>: Since $\tilde{g}(\theta_p,\theta_q)$ is a periodic function of two variables, $\tilde{g}((\theta_p,\theta_q) + \vec{a}_{pq}^{\,j}s)$ is an almost periodic function of the scalar variable s. Standard properties of almost periodic functions (see [22]) guarantee that the mean of the terms on the right hand side exists for a general doubly periodic function and is uniform for

arbitrary translations of the argument s. We postpone the proof of Lemma 1 until the end of this section and continue the argument. With the explicit binary wave interaction terms from (2.30c) defining $g_{pq}(t,\theta_p,\theta_q)$ in the periodic case, we can apply the above lemma to obtain sufficient (and essentially necessary) conditions for the nonresonant multi-wave expansions from the last subsection to be uniformly valid for periodic waves. These conditions require that

$$0 = \Gamma_{pq}^{j} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sigma_p'(t,\theta_p + u_j - \lambda_p)s)\sigma_q(t,\theta_q + (\lambda_j - \lambda_q)s)ds \tag{2.45}$$

$$j \neq p \neq q$$

with Γ_{pq}^{j}, the asymmetric binary interaction coefficients defined by

$$\Gamma_{pq}^{j} = \ell_j \cdot (-\lambda_p B_0(r_q,r_p) + B_1(r_q,r_p)). \tag{2.46}$$

In general, as the reader can see by inspection for periodic wave patterns, the conditions in (2.45) are not satisfied and the extremely simple nonresonant multi-wave pattern from (2.24) cannot describe the leading order asymptotics through decoupled inviscid Burgers equations.

The lemma and the conditions in (2.45) suggest an obvious strategy for satisfying the additional requirements in (2.45) by incorporating additional non-local terms in the periodic case. With g_{pq} given by the formula in (2.30c), we consider the function $\bar{g}_{pq}(t,\theta_p,\theta_q)$ uniquely defined by the conditions,

$$\begin{aligned} &\ell_j \cdot g_{pq}(t,\theta_p,\theta_q) = 0, \qquad j = p,q \\ &\ell_j \cdot g_{p_q} = \Gamma_{pq}^{j} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sigma_p'(t,\theta_p+(\lambda_q-\lambda_p)s)\sigma_q(t,\theta_q+(\lambda_j-\lambda_q)s) \\ &\qquad j \neq p,q \quad , \qquad 1 < j < M \end{aligned} \tag{2.47}$$

The function $\tilde{g}(t,\theta_p,\theta_q)$ defined by

$$\tilde{g}(t,\theta_p,\theta_q) = g_{pq}(t,\theta_p,\theta_q) - \bar{g}_{pq}(t,\theta_p,\theta_q) \tag{2.48}$$

satisfies all of the conditions from Lemma #1 and has a solution with sublinear growth. The terms, $g_{pq}(t,\theta_p,\theta_q)$ yield new contributions to the leading order asymptotics besides those from the single wave contributions in (2.30b) already discussed earlier in (2.14)-(2.16). Straightforward addition of these terms results in the following leading order asymptotic equations

$$0 = (\sigma_j)_t + b_j\left(\frac{1}{2}\sigma_j^2\right)_{\theta_j} + \sum_{\substack{p\neq j\\ q\neq j}} I^j_{pq} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sigma_p'(t,\theta_p + (\lambda_j-\lambda_p)s)\sigma_q(t,\theta_p + (\lambda_j-\lambda_q)s)ds \; . \tag{2.49}$$

Unfortunately, such a straightforward approach fails for the following reason: The tacit assumption in the ansatz from (2.42) is that σ_j is a function of only two variables, (t,θ_j) while, when $\Gamma^j_{pq} \neq 0$, the integro-differential terms on the right hand side of (2.49) are functions of the additional variables (θ_p,θ_q) so these equations are <u>not</u> a self-consistent closed system of equations. How can we treat this difficulty and obtain a closed self-consistent system of equations? Here we present an idea which we call the principle of "exchange of phase functions" which uses some additional flexibility in the method of multiple scales which is usually ignored in other applications of this method ----- this idea will enable us to obtain a closed system of equations with a similar form as given in (2.47).

The idea behind the principle of "exchange of phase functions" is an extremely simple one. The order one perturbation equation from (2.28) only needs to be satisfied when $\vec{\theta}$ is restricted to the values, $\vec{\theta} = \frac{\vec{\phi}}{\varepsilon}$; thus, to satisfy the equation in (2.28), we can replace $G(t,\vec{\theta})$ by any other function $\tilde{G}(t,\vec{\theta})$ satisfying

$$\tilde{G}(t,\vec{\theta})\big|_{\vec{\theta}=\frac{\vec{\phi}}{\varepsilon}} = G(t,\vec{\theta})\big|_{\vec{\theta}=\frac{\vec{\phi}}{\varepsilon}} \tag{2.50}$$

in order to continue the asymptotics. We consider $\hat{h}_{pq}(t,\theta_j)$ the function of θ_j alone defined by

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} \sigma_p'(t,\theta_j + (\lambda_j - \lambda_p)s)\sigma_q(t,\sigma_j + (\lambda_j - \lambda_q)s)ds = \hat{h}_{pq}(t,\theta_j). \tag{2.51}$$

We denote the nonlocal averages appearing on the right hand side of (2.49) by

$h_{pq}(t,\theta_p,\theta_q)$, i.e.

$$(2.52)\quad h_{pq}(t,\theta_p,\theta_q) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} \sigma_p'(t,\theta_p + (\lambda_j - \lambda_p)s)\sigma_q(t,\theta_q + (\lambda_j - \lambda_q)s)ds.$$

The reader can readily verify that with $\phi^\ell = x - \lambda_\ell t$, $\ell = 1,\ldots,M$,

$$(2.53)\qquad \hat{h}_{pq}(t,\frac{\phi_j}{\varepsilon}) = h_{pq}(t,\frac{\phi_p}{\varepsilon},\frac{\phi_q}{\varepsilon}).$$

Therefore, if we define $\bar{\hat{g}}_{pq}$ by

$$(2.54)\qquad \begin{aligned} \ell_j \cdot \bar{\hat{g}}_{pq} &= 0, \quad j = p,q \\ \ell_j \cdot \bar{\hat{g}}_{pq} &= \Gamma_{pq}\hat{h}_{pq}, \quad j \neq p,\ j \neq q, \end{aligned}$$

we use the terms involving $\bar{\hat{g}}_{pq}(t,\theta_j)$ in the leading order asymptotics but invoke the principle of "exchange of phases" guaranteed by (2.53) to replace $\bar{\hat{g}}_{pq}(t,\theta_j)$ by $g_{pq}(t,\theta_p,\theta_q)$ in (2.48) to guarantee that $u_2(t,\theta_p,\theta_q)$ can be found with sublinear growth. Thus, we replace the non-local terms on the right hand side of (2.49) by those in (2.51) and obtain for the M-functions, $\{\sigma_j(t,\theta)\}_{j=1}^M$, the closed system of <u>Periodic Resonant Wave Asymptotic Equations</u>

$$(2.55a)\qquad 0 = (\sigma_j)_t + b_j(\tfrac{1}{2}\sigma_j^2)_\theta + \sum_{\substack{p\neq q\\ p\neq j\\ q\neq j}} \Gamma_{pq}^j \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} \sigma_p'(t,\theta+(\lambda_j-\lambda_p)s)\sigma_q(t,\theta+(\lambda_j-\lambda_q)s)ds$$

for $j = 1,\ldots,M$ with the initial data

$$(2.55b)\qquad \sigma_j(t,\theta)|_{y=0} = \ell_j \cdot u_1^0(\theta).$$

In the above argument we have developed the following:

<u>Proposition 2 (Formal Validity of the Resonant Asymptotic Equations for Periodic Waves)</u>

The solution u^ε of the general small amplitude periodic initial value problem in (2.41) has a formal uniformly valid asymptotic approximation within terms of order $o(\varepsilon)$ given by

$$(2.56) \qquad u_w^\varepsilon = u_0 + \varepsilon \sum_{j=1}^{M} \sigma_j(t, \frac{\phi^j}{\varepsilon}) r_j$$

where $\{\sigma_j(t,\theta)\}_{j=1}^{M}$ solves the initial value problem for the coupled system of resonant asymptotic equations in (2.55) with Γ_{pq}^j defined in (2.46). This expansion has a region of formal validity within terms that are $o(\varepsilon)$ at least on any region of the form $R^1 \times [0,T]$ where the solution in (2.55) remains smooth.

In the next subsection, we will display these equations explicitly for the equations of compressible fluid flow and describe some ongoing research and further developments using these equations. We still owe the reader a sketch of the proof of Lemma 1. We expand the solution u of (2.43) as $u = \sum_{j=1}^{M} \tilde{u}_j r_j$; for $j \neq p$ and $j \neq q$, we obtain the scalar convection equation;

$$(-\lambda_p + \lambda_j) \frac{\partial \tilde{u}_j}{\partial \theta_p} + (-\lambda_q + \lambda_j) \frac{\partial \tilde{u}_j}{\partial \theta_q} = \ell_j \cdot \tilde{g}.$$

With the characteristic co-ordinate, $(\tilde{\theta}_1, \tilde{\theta}_2)$, defined by

$$\begin{aligned} \theta_p &= (\lambda_j - \lambda_p)\tilde{\theta}_1 \\ \theta_q &= \tilde{\theta}_2 + (\lambda_j - \lambda_q)\tilde{\theta}_1 \end{aligned}$$

the above equation reduces to

$$\frac{\partial}{\partial \tilde{\theta}_1} \tilde{u}_j = \ell_j \cdot g(\vec{a}_{pq}^j \, \tilde{\theta}_1 + (0, \tilde{\theta}_2)).$$

The Lemma follows immediately from direct integration of this equation and standard properties of almost periodic functions. In particular, one property of almost periodic functions needed to complete the proof of the lemma and already used implicitly in the proof of (2.53) is the fact that for any almost periodic function, $\sigma(s)$ the limit,

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \sigma(s + h) ds = M$$

is a number M independent of $h \in R^I$ and the convergence is uniform in h (see Chapter 6 of [22]).

2.4: An Application to Periodic Resonant Waves for Compressible Fluid Flows in 1-D

In this section; we apply the theory of periodic resonant waves developed in the previous subsection to the inviscid 1-D compressible fluid equations. These equations are the system of three conservation laws, expressing conservation of mass, momentum, and total energy, given by

$$
\begin{aligned}
\rho_t + (\rho v)_x &= 0 \\
(\rho v)_t + (\rho v^2 + p)_x &= 0 \qquad (2.57)\\
(\rho e + \tfrac{1}{2}\rho v^2)_t + (\rho e v + \tfrac{1}{2}\rho v^3 + p v)_x &= 0
\end{aligned}
$$

where ρ is the mass density, v is the flow velocity, S is the entropy, and $p(\rho,S)$ is the pressure given as a specified function of (ρ,S) through thermodynamic considerations - for an ideal gas, $p = R\rho^\gamma \exp(s/c_v)$ for some γ with $\gamma > 1$. We assume that at the constant background value of interest, (ρ_0,S_0), $p_\rho(\rho_0,S_0) > 0$ so that the system in (2.57) is strictly hyperbolic. To conform with the general notation in the last sections, we set $u = {}^t(u_1,u_2,u_3) = {}^t(\rho,v,S)$ and we concentrate on the asymptotic approximation of solutions of (2.57) with small amplitude periodic initial data satisfying Hypothesis #2 with the form,

$$
u_0^\varepsilon(x) = \begin{pmatrix} \rho_0 \\ 0 \\ S_0 \end{pmatrix} + \varepsilon u_0^1(o\ \frac{x}{\varepsilon}) \qquad (2.58)
$$

with ρ_0, S_0 given constants.

Much is known rigorously about the behavior of small amplitude weak solutions of the 1-D compressible fluid equations for data with compact support and finite total variation as a consequence of ideas based on Glimm's method including shock formation, structure of singularities, and large-time behavior (see the bibliography of [12] for some recent references). On the other hand, very little is known rigorously about the behavior of weak solutions of the compressible fluid equations with fixed small amplitude periodic initial data (the transformation $x' = \frac{x}{\varepsilon}$, $t' = \frac{t}{\varepsilon}$ converts the problem in (2.57) and (2.58) into this problem). In fact, even at a formal level, very little is known regarding the important conceivable large time asymptotic behavior. The compressible fluid equations in 1-D are the simplest system with $M > 3$ where resonance occurs and the formal asymptotic theory from the previous subsection applies. With the rescaling $x' = \frac{x}{\varepsilon}$, $t' = \frac{t}{\varepsilon}$, Proposition 2 guarantees that for fixed small amplitude periodic initial data, the resonant asymptotic equations in (2.55) yield a formally valid approximation for large times of order $O(\varepsilon^{-1})$. Thus, from a theoretical viewpoint, one might investigate the structure and large-time asymptotic behavior for the approximating system of equations in (2.55) and use this information to formulate conjectures regarding the rigorous behavior of solutions of (2.57) and (2.58) - incidentally, the validity of such an approach has been confirmed rigorously for initial data of compact support in 1-D through the recent theoretical work of DiPerna and the author [3] described in section 4. From a more applied point of view, the problem in (2.57) and (2.58) is the simplest model problem in 1-D where natural resonances couple and drive the behavior of solutions of conservation laws; the asymptotic equations in (2.55) provide a uniformly valid leading order approximation for these effects. For an example of a multi-dimensional problem of extreme complexity where similar effects as we describe through asymptotics below seem to play a prominent role, we refer the reader to the recent beautiful numerical simulations by Woodward [19], [30] of instabilities in supersonic jets.

In the remainder of this subsection, we record the resonant asymptotic equations from (2.55) for the specific system in (2.57) and then briefly describe some of the work in progress by Rosales, Schonbek, and the author ([21]) in analyzing this sytem. For the full details of the explicit calculations and more information, we refer the reader to [21]. At the background state, ${}^t(\rho_0, 0, S_0)$, the system of equations in (2.57) has the three eigenvalues,

$$\lambda_1 = -c_0, \quad \lambda_2 = 0, \quad \lambda_3 = c_0 \tag{2.59}$$

with corresponding right eigenvectors,

$$r_1 = \begin{pmatrix} \rho_0 \\ -c_0 \\ 0 \end{pmatrix}, \quad r_2 = \begin{pmatrix} (p_S)_0 \\ 0 \\ -c_0^2 \end{pmatrix}, \quad r_3 = \begin{pmatrix} \rho_0 \\ c_0 \\ 0 \end{pmatrix} \tag{2.60}$$

where $c_0 = (p_\rho(\rho_0,S_0))^{1/2}$ is the speed of sound and the subscript zero, here and subsequently, denotes evalution at the background state, (ρ_0,S_0). The uniformly valid asymptotic approximation from the last subsection has the form,

$$u_W^\varepsilon = \begin{pmatrix} \rho_0 \\ 0 \\ S_0 \end{pmatrix} + \varepsilon \sum_{j=1}^{3} \sigma_j(t, \frac{\phi^j}{\varepsilon})r_j \tag{2.61}$$

where $\phi^j = x - \lambda_j t$ and $\{\sigma_j(t,\theta)\}_{j=1}^3$ satisfy the resonant wave asymptotic equations from (2.55) for the special system in (2.57). For this physical system, these equations are remarkably simple and are given by

<u>The Resonant Wave Equations for Periodic 1-D Compressible Flow</u>

$$\begin{aligned} &(\sigma_1)_t + \alpha(\tfrac{1}{2}\sigma_1^2)_\theta + \beta \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sigma_2'(\tfrac{1}{2}\theta + \tfrac{1}{2}s)\sigma_3(t,s)ds = 0 \\ &(\sigma_2)_t = 0 \\ &(\sigma_3)_t - \alpha(\tfrac{1}{2}\sigma_3^2)_\theta - \beta \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sigma_2'(\tfrac{1}{2}\theta + \tfrac{1}{2}s)\sigma_1(t,s)ds. \end{aligned} \tag{2.62}$$

With $T(\rho,S)$ the temperature, the coefficients α and β are given by

$$\begin{aligned} \alpha &= -\frac{1}{2\rho_0 c_0}[(\rho^2 c^2)_\rho]_0 \\ \beta &= \rho_0^2 c_0^3[(c^{-2}T_\rho)_\rho]_0. \end{aligned} \tag{2.63}$$

Since the second wave field of gas dynamics associated with the eigenvalue λ_2 is linearly degenerate, it is not surprising that the second equation in (2.62) is trivial and $\sigma_2(t,\theta) = \sigma_2^0(\theta)$, the initial amplitude for the entropy wave. On the other hand, for gas dynamics, the two sound waves are typically genuinely nonlinear and the requirement $\alpha \neq 0$ is a disguised version of this condition at the state (ρ_0,S_0). Since $\sigma_2(t,\theta) = \sigma_2^0(\theta)$, the resonant asymptotic equations in (2.62) reduce to two Burgers equations coupled through a skew-symmetric linear integral operator with a known kernel given by the derivative of the entropy field. Thus, for the 1-D compressible fluid equations, all leading order resonances can be described by the coupling of two Burgers equations by a known linear integral operator - a remarkable simplification through asymptotics. Next, we simplify the integral operators on the right-hand side of (2.62) even further assuming that Hypothesis #2 is satisfied.

It will be convenient to expand the known derivative of the smooth initial entropy field $\sigma_0'(\theta)$ in the real Fourier series,

$$\sigma_0'(\theta) = \sum_{n=1}^{\infty} (a_n e^{2\pi i n\theta} + \bar{a}_n e^{-2\pi i n\theta}) \tag{2.64}$$

the projection of $\sigma_0'(\theta)$ onto the <u>even</u> harmonics, $P\sigma_0'(\theta)$, is given by

$$P\sigma_0'(\theta) = \sum_{n=1}^{\infty} (a_{2n} e^{4\pi i n\theta} + \bar{a}_{2n} e^{-4\pi i n\theta}). \tag{2.65}$$

We substitute (2.64) into the nonlocal averages on the right hand side of (2.62) and compute explicitly (by expanding $\sigma(t,s)$ in a Fourier series) that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sigma_0'(\tfrac{1}{2}\theta + \tfrac{1}{2}s)\sigma(t,s)ds = \int_0^1 P\sigma_0'(\tfrac{1}{2}\theta + \tfrac{1}{2}s)\sigma(t,s)ds. \tag{2.66}$$

Thus, if we denote by $K(\theta)$, the 1-periodic function defined by $K(\theta) = P\sigma_0'(\frac{1}{2}\theta)$, we see that the equations in (2.62) can be reduced to the

<u>Simplified Resonant Wave Equations</u>

$$\begin{aligned} (\sigma_1)_t + \alpha(\tfrac{1}{2}\sigma_1^2)_\theta + \beta \int_0^1 K(\theta + s)\sigma_3(t,s)ds &= 0 \\ (\sigma_3)_t - \alpha(\tfrac{1}{2}\sigma_3^2)_\theta - \beta \int_0^1 K(\theta + s)\sigma_1(t,s)ds & \end{aligned} \tag{2.67}$$

with $K(\theta) = P\sigma_0'(\frac{1}{2}\theta)$. Thus, we see that only <u>even harmonics</u> in the initial entropy field are expected to <u>produce resonances</u> while an <u>initial entropy distribution</u>, $\sigma_0(\theta)$, while only <u>odd harmonics will never resonate to leading order</u>. In section four, we will show that resonant planar oblique wave interactions for isentropic compressible fluid flow in 2-D will yield a similar set of equations as in (2.67) although the physical interpretation of the variables $\{\sigma_j\}_{j=1}^{3}$ will be quite different. In the calculations by Woodward already mentioned, there is a preference of odd harmonics over the even ones in producing resonances in that extremely complex physical problem; perhaps, the preference for half the harmonics in producing resonances in the simpler problem discussed here is not merely a coincidence.

We conclude this section with a few remarks about the simplied asymptotic coupled system in (2.67) which will appear in detail in a forthcoming paper of Rosales, Schonbek, and the author [21]. First, global existence of weak solutions is easily established as a consequence of two formal estimates for solutions of the system in (2.67)

$$\#1)\quad \mathrm{Var}(\sigma_1(\cdot,t),\sigma_3(\cdot,t))) \le \exp(Ct\|K(\theta)\|_1)\mathrm{Var}(\sigma_1^0(\theta),\sigma_3^0(\theta))$$

$$\#2)\quad \frac{\partial}{\partial t}\int_0^1 (\sigma_1^2(\theta,t) + \sigma_3^2(\theta,t))d\theta \le 0.$$

(Actually, only #1) is needed for existence but #2) is an important fact.) Secondly, in [21], numerical computations for the system in (2.67) will be presented which elucidate the large-time asymptotic behavior of solutions of the system in (2.67). Due to the effects of artificial viscosity, such meaningful numerical computations for small amplitude periodic waves via a direct simulation of the compressible 1-D fluid equations in (2.57) would be a rather difficult undertaking. However, it is not difficult to design an efficient numerical scheme without artificial viscosity for the numerical solution of the asymptotic equations in (2.67). The scheme that has been developed for the calculations to be reported in [21] combines, through fractional steps, the deterministic Glimm scheme for the periodic inviscid Burgers solutions to avoid any artificial viscosity with a Fourier-spectral step which efficiently solves the nonlocal integral operators exactly.

III. The Formal Theory: Weakly Nonlinear Hyperbolic Waves in Several Space Dimensions

In this section, we present some of the ideas and results which extend the formal theory described in section II to several space variables. We both briefly review some of the ideas of Choquet-Bruhat [1] and Hunter and Keller [5] and describe several new results on resonantly interacting waves in multi-D which will appear in a forthcoming paper of Hunter, Rosales, and the author [7]. Unlike the detailed treatment presented in section II, here we will only discuss the ideas and make several remarks - most details will be omitted. However, many of the details only involve a mild extension of the ideas already developed in detail in section II and we will attempt to point out the essential differences and new difficulties. We end this section with a discussion of oblique resonant wave interactions for isentropic 2-D compressible fluid flow. The asymptotics will yield essentially the same simplified system as we discussed earlier in (2.67).

Here we will discuss the asymptotic approximation of solutions to the initial value problem for an $M \times M$ system of hyperbolic conservation laws in N+1 variables (N-space variables) given by

$$(3.1a)\qquad \frac{\partial}{\partial x_0} F_0(u^\varepsilon) + \sum_{j=1}^{N} \frac{\partial}{\partial x_j} F_j(u^\varepsilon) = 0, \quad \begin{matrix} x_0 > 0 \\ x' \in R^N \end{matrix}$$

with $x = (x_0, x')$, $x' = (x_1, \ldots, x_N)$, and small amplitude rapidly oscillating initial data,

$$(3.1b)\qquad u^\varepsilon(x_0, x')\big|_{x_0=0} = u_0 + \varepsilon u_1^0\left(x', \frac{\vec{\phi}^0(x')}{\varepsilon}\right).$$

Here x_0 denotes the time variable, $u = {}^t(u_1, \ldots, u_M) \in R^M$, and $\{F_j(u)\}_{j=0}^N$ are smooth functions of u defined on an open set containing the constant vector, u_0, with $A_j = \frac{\partial F_j}{\partial u}\Big|_{u_0}$, $0 \leq j \leq N$, the corresponding $M \times M$ Jacobian matrices at u_0.

We assume that $F_j(u)$ has the Taylor expansion,

$$(3.2)\qquad F_j(u_0 + \varepsilon v) = F_j(u_0) + \varepsilon A_j v + \varepsilon^2 \frac{B_j(v,v)}{2} + O(\varepsilon^3)$$

for $j = 0, 1, \ldots, N$ where $B_j(v,w)$ is the symmetric M-vector valued bilinear form defined by the Hessian matrix of each component evaluated at u_0. The phase functions in the initial data, $\vec{\phi}^\circ(x')$, belong to R^ℓ so that there are ℓ distinct phase functions initially. We always assume that x_0 is a time-like direction for (3.1) and for simplicity in exposition, we require that the system in (3.1a) is strictly hyperbolic at u_0, thus,

(3.3)

A) $\det A_0(u_0) \neq 0$

B) $\det(-\lambda A_0(u_0) + \sum_{j=1}^{N} A_j(u_0)\xi_j) = 0$ has M distinct roots, $\{\lambda_\rho(\xi)\}_{\rho=1}^{M}$, for any $\xi \in R^N$ with $\xi \neq 0$.

We denote by $R_\rho(\xi)$, $L_K(\xi)$, the corresponding non-zero left and right eigenvectors satisfying

(3.4a)
$$(-\lambda_\rho(\xi)A_0 + \sum_{j=1}^{N} A_j\xi_j)R_\rho(\xi) = 0$$
$$L_K(\xi)\cdot(-\lambda_K(\xi)A_0 + \sum_{j=1}^{N} A_j\xi_j) = 0$$

and the normalization,

(3.4b)
$$L_K(\xi)\cdot A_0 R_\rho(\xi) = \delta_{K\rho}$$

with $\delta_{K\rho}$, the Kronecker delta. For the initial data, $u_1(x', \frac{\vec{\phi}^\circ(x')}{\varepsilon})$, we require that $u_1^0(x',\vec{\theta})$ is a smooth function of $\vec{\theta} \in R^\ell$ and that each separate generalized mean of u_1^0 with respect to θ_j vanishes, i.e.

(3.5)
$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} u_1^0(x',\theta_1,\ldots,\theta_{j-1},s,\theta_{j+1},\ldots,\theta_\ell) = 0$$

for any j with $1 \leq j \leq \ell$.

Before beginning a discussion of the formal asymptotic solutions, we remark that equations of the form in (3.1) with nonlinear source terms, $S(u)$, are also easily handled by the methods described below. For the simple wave expansions which we discuss below in subsection 3.1, it is inessential that (3.5) is satisfied and in the general case, there is a mean field correction of order ε (see [5]). Furthermore, the background state can be any smooth function $u_0(x)$. On the other hand, for the general multi-dimensional theory of planar oblique binary wave interactions as presented in subsection 3.2, at the present time, it is quite essential that u_0 be a constant state and that each $\phi^j = \vec{x}\cdot\vec{\omega}^j$ be a plane wave phase function for the formal asymptotics to be justified in the fashion presented here. The theory as presented requires only minor changes to incorporate hyperbolic systems with roots of constant multiplicity (see [5]) - this is an important remark for applications to the complete 3-D Euler equations of compressible fluid flow.

3.1: The Single Wave Expansion in Multi-D

Here we describe the construction of asymptotic solutions of the equation in (3.1a) with the single-wave ansatz,

$$\tilde{u}_w^\varepsilon = u_0 + \varepsilon u_1(x,\tfrac{\phi}{\varepsilon}) + \varepsilon^2 u_2(x,\tfrac{\phi}{\varepsilon}) \tag{3.6}$$

where ϕ is a single phase function. Later, we will identify the initial data compatible with this ansatz. In an analogous fashion as in a single space variable, we must be able to choose $u_1(x,\theta), u_2(x,\theta)$ so that simultaneously both (2.9) and (2.10) are satisfied. The terms of order zero in ε vanish provided that

Order zero

$$\left(\sum_{j=0}^{N} A_j \phi_{x_j}\right) \frac{\partial}{\partial\theta} u_1(x,\theta)\Big|_{\theta=\frac{\phi}{\varepsilon}} = 0. \tag{3.7}$$

The requirement that the power of order one in ε vanish in the formal solution of (3.1) requires that we choose u_2 so that

Order one

$$-\left(\sum_{j=0}^{N} A_j \phi_{x_j}\right) \frac{\partial u_2}{\partial\theta}(x,\theta)\Big|_{\theta=\frac{\phi}{\varepsilon}} = \left(\sum_{j=0}^{N} A_j (u_1)_{x_j} + \sum_{j=0}^{N} \phi_{x_j} B_j(u_1,(u_1)_\theta)\right)\Big|_{\theta=\frac{\phi}{\varepsilon}} . \tag{3.8}$$

The solution of the equation in (3.7) is identical to the solution of the leading terms of the equations of geometric optics at the background state, u_0. With the notation, $\nabla\phi = \left(\frac{\partial\phi}{\partial x_1},\ldots,\frac{\partial\phi}{\partial x_N}\right)$, the equation in (3.7) is satisfied only if there exists a p so that for some wave speed λ_p,

$$\begin{aligned} &\phi_{x_0} + \lambda_p(\nabla\phi) = 0, \quad 0 < x_0 < X_0 \\ &\phi(x_0,x')\big|_{x_0=0} = \phi^\circ(x'), \end{aligned} \tag{3.9a}$$

and $u_1(x,\theta)$ has the form,

$$u_1(x,\theta) = \sigma(x,\theta)R_p(\nabla\phi) \tag{3.9b}$$

where $R_p(\xi)$ is the right eigenvector from (3.4a) associated with the wave speed, $\lambda_p(\xi)$ here $\sigma(x,\theta)$ is an arbitrary scalar multiplier at this stage in the argument. In satisfying the equation of order one in (3.8), as in (2.14) above, we see that

$$L_p(\nabla\phi) \cdot \sum_{j=0}^{N} A_j\phi_{x_j} = 0 \tag{3.10}$$

with $L_p(\xi)$ the left eigenvector from (3.4) and the equation in (3.8) has a solution, $u_2(x,\theta)$, by the Fredholm alternative, if and only if the inner product of the right hand side of (3.8), regarded as a function of θ, vanishes with $u_1(x,\theta)$ defined in (3.9b). This yields the scalar nonlinear differential equation for $\sigma(x,\theta)$ given by

$$D_p\sigma(x,\theta) + b_p\left(\frac{1}{2}\sigma^2(x,\theta)\right)_\theta = 0, \qquad 0 < x_0 \tag{3.11a}$$

with initial data,

$$\sigma(x,\theta)\big|_{x_0=0} = \sigma_0(x',\theta) \tag{3.11b}$$

and $\sigma_0(x',\theta)$ a given initial wave-form at $x_0 = 0$. Here D_p is the linear transport operator of geometric optics associated with the phase function, ϕ, defined by

$$D_p = \frac{\partial}{\partial x_0} + \vec{a}_p \cdot \nabla + c_p \tag{3.12a}$$

with $\vec{a}_p = (a_p^j(x))$, $c_p(x)$, given by

$$\text{(3.12b)} \qquad a_p^j = L_p(\nabla\phi)\cdot A_j R_p(\nabla\phi), \quad 1 \leq j \leq N$$

$$c_p(x) = L_p(\nabla\phi)\cdot \sum_{j=0}^{N} A_j (R_p)_{x_j}.$$

The coefficient, b_p, is a multi-dimensional generalization of the coefficient in (2.16) and is defined by

$$\text{(3.13)} \qquad b_p = L_p(\nabla\phi)\cdot\left(\sum_{j=0}^{N} \phi_{x_j} B_j(R_p,R_p)\right).$$

We briefly recall the reasons for calling the operator in (3.12), the transport operator of linear geometric optics associated with the phase function, ϕ. If one takes the identity

$$\left(-\lambda_p(\xi)A_0 + \sum_{j=1}^{N} A_j \xi_j\right)R_p(\xi) = 0,$$

differentiates with respect to ξ and computes the dot product with $L_p(\xi)$, after evaluation at $\xi = \nabla\phi$, the result is the formula,

$$\nabla_\xi \lambda_p(\xi)\big|_{\nabla\phi} = \vec{a}_p(x) .$$

If one considers the integral curves $X(x_0,x') \in R^N$ defined by the O.D.E.

$$\frac{dX}{dx_0} = \vec{a}_p(x_0,X), \quad X(x_0,x')\big|_{x_0=0} = x',$$

we use the above formula and (3.9a) to compute that

$$\frac{d}{dx_0} \phi(x_0,X(x_0,x')) = \phi_{x_0} + \lambda_p(\nabla\phi) = 0,$$

i.e. x_0, $X(x_0,x')) = \phi_0(x')$. Thus, the direction of differentiation defined by the transport operator in (3.12) is always tangent to the surfaces of constant phase for the phase function, ϕ. With the condition in (3.5), it is completely straightforward to follow the reasoning in section (2.1) to verify that $u_2(x,\theta)$ be chosen with sublinear growth in θ once the Fredholm alternative condition from (3.10) is satisfied.

To summarize, we have verified the following:

Proposition 3: For any initial wave form, $\sigma_0(x',\theta)$ with $\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sigma_0(x',s)ds = 0$ and any initial plane function, $\phi_0(x')$ with $\nabla\phi_0 \neq 0$ on the x-support of $\sigma_0(x',\theta)$, there is a uniformly valid formal asymptotic solution of (3.1) with the initial data,

$$u_0 + \varepsilon\sigma_0(x',\frac{\phi_0}{\varepsilon})R_p(\nabla\phi_0)$$

for any p with $1 \leq p \leq M$. The leader order approximation is given by

$$u_W^\varepsilon = u_0 + \varepsilon\sigma(x,\frac{\phi}{\varepsilon})R_p(\nabla\phi) \tag{3.14}$$

where ϕ solves the eikonal equations of geometric optics from (3.9) while the amplitude, σ, solves the much simpler nonlinear transport equation defined by the scalar equation,

$$\begin{aligned} &D_p\sigma + b_p(\tfrac{1}{2}\sigma^2)_\theta = 0 \qquad x_0 > 0 \\ &\sigma(x_0,x',\theta)\big|_{x_0=0} = \sigma_0(x',\theta) \end{aligned} \tag{3.15}$$

with the coefficients for D_p and b_p defined in (3.12) and (3.13). The region of formal validity of this approximate solution is restricted in another way beyond the conditions imposed in the 1-D case described in section 2.1 - the expansion obviously ceases to be valid in regions where the eikonal solutions in (3.9a) form multi-dimensional caustics.

Remark: The equation in (3.15) is a scalar equation in several space variables. However, since the coefficients of D_p and b_p do not depend on θ, once characteristic coordinates are used to straighten out this linear operator, the equation in (3.15) essentially simplifies to a parametrized family of inviscid Burgers equation with lower order terms with the form,

$$\sigma_\tau + c_p(\tau)\sigma + b_p(\tau)(\sigma^2)_\theta = 0. \tag{3.16}$$

One significant difference in this equation when compared with the asymptotic equations in one space dimension is the appearance of the linear lower order term, $c_p(\tau)$. When the linear geometric optic light rays are expanding in multi-D, we have $c_p(\tau) > 0$ while $c_p(\tau) < 0$ corresponds to local focusing of these light rays. Thus, the equations account for both linear geometric distortion and nonlinear distortion of the wave form and amplitude.

For brevity, we will not discuss non-resonant wave patterns in multi-D although the conditions described in section 2.2 can be generalized easily to yield sufficient conditions for non-resonance and superposition of the simple wave patterns from (3.14). Instead, we describe some new work of Hunter, Rosales, and the author to appear in [7] where we study some very interesting multi-D problems with resonant wave interactions.

3.2: Resonant Oblique Small Amplitude Plane Wave Interactions in Multi-D

We consider two different highly oscillatory small amplitude wave packets which do not interact for times $x_0 < X_0$, i.e. so that the simple wave expansions discussed in section 3.1 are uniformly valid approximations for $0 < x_0 < X_0$. This is easily arranged by guaranteeing that for example, the bicharacteristic rays emanating from the supports of the initial wave forms do not intersect until times, $x_0 > X_0$. Thus,

$$u_w^\varepsilon(x) = u_0 + \varepsilon\sigma_1(x,\frac{\phi_1}{\varepsilon})R_1(\nabla\phi_1) + \varepsilon\sigma_2(x,\frac{\phi_2}{\varepsilon})|R_2(\nabla\phi_2) + o(\varepsilon) \tag{3.17}$$

is a uniformly valid approximation for times x_0 with $0 < x_0 < X_0$) where the functions ϕ_j, $\sigma_j(x,\theta)$ satisfy the respective decoupled asymptotic equations in (3.9), (3.11), (3.12). We assume that for times $x_0 > X_0$, the bicharacteristic rays associated with ϕ_1 and ϕ_2 intersect; then generally one might expect, for periodic or almost periodic wave forms, that the simple wave expansion from (3.17) ceases to be uniformly valid for times $x_0 > X_0$ because new oblique waves are generated from the collision and interaction of these two different wave fronts. Thus, the following problem arises naturally:

(3.18) Describe the quantitative locations and strengths of the different oblique wave patterns generated after the collision time, X_0.

The problem posed in (3.18) explains the meaning of the title of this section.

Below, we describe the uniformly valid formal asymptotic solution for the problem of oblique wave interactions posed in (3.18) under two hypotheses:

(3.19)

A) The two wave fronts in (3.18) are plane wave fronts with linear phase functions, $\phi_j = \sum_{r=0}^{N} x_r w_r^j = x \cdot \vec{w}^j$, $j = 1,2$ with $\det(\sum_{K=0}^{N} A_k \omega_K) = 0$

B) The generic geometric condition to be defined in (3.24) is satisfied for $\vec{w}^1$ and $\vec{w}^2$ with $\vec{w}^j = ((\phi_j)_{x_0}, \nabla\phi_j)$.

With the hypotheses in (3.19), the problem given in (3.18) is solved by finding new directions, $\{\vec{w}^j\}_{j=1}^{\ell}$, with ℓ a well-determined number satisfying $\ell \leq M$ ($\ell < M$ quite often for $M \geq 4$) so that for times beyond the interaction time, X_0, the asymptotic solution in (3.17) is continued by the more complex uniformly valid asymptotic solution given by

$$u_w^\varepsilon(x) = u_0 + \varepsilon \sum_{j=1}^{\ell} \sigma_j(x,\frac{\phi_j}{\varepsilon})R_j + \varepsilon^2 u_2(x,\frac{\vec{\phi}}{\varepsilon}) \tag{3.20}$$

with the linear phase functions $\phi_j = \vec{w}^j \cdot x$, $1 \leq j \leq \ell$ and $|u_2(x,\frac{\vec{\phi}}{\varepsilon})| = o(\varepsilon^{-1})$. For times $x_0 \geq X_0$, the nonlinear amplitude functions from (3.20), $\{\sigma_j(x,\theta)\}_{j=1}^{\ell}$, no longer decouple but instead solve the $\ell \times \ell$ system,

<u>The Resonant Plane Wave Asymptotic Interaction Equations</u>

$$D^j\sigma_j + b_j \frac{\partial}{\partial\theta}(\frac{1}{2}\sigma_j) + \sum_{\substack{1 \leq p,q \leq \ell \\ p \neq q \\ q \neq p}} \Gamma_{pq}^j \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sigma_q'(x,s)\sigma_p(x,s+h_{pq}^j\theta)ds = 0, \tag{3.21}$$

$$1 \leq j \leq \ell$$

with $\sigma_q'(x,s) = \frac{\partial\sigma_q}{\partial s}(x,s)$. The coefficients Γ_{pq}^j and h_{pq}^j have an explicit form generalizing the formulae in (2.46) and (2.55) but these expressions are too cumbersome and not important enough for the subsequent presentation to display here. The transport operators, D^j, and the coefficients b_j are given by the formulae in (3.12) and (3.13) specialized to the planar phase functions, $\phi_j = x \cdot \vec{w}^j$. Of course, to solve the wave interaction problem from (3.18), the equations in (3.21) should be supplemented by the initial conditions,

$$\sigma_j(x,\theta)\big|_{x_0=X_0} = 0, \quad j = 3,\dots,\ell \tag{3.22}$$

while the initial data for $\sigma_1(x,\theta)$, $\sigma_2(x,\theta)$ on $x_0 = X_0$ are computed by evaluating the leading order terms from (3.17) on $x_0 = X_0$.

How are these new oblique wave directions $\vec{w}^j$ computed? In general, given the two vectors $\vec{w}^1$ and $\vec{w}^2$, there are ℓ distinct directions with $\ell \leq M$ and parametrized by linearly independent vectors, $h^j = (h_1^j, h_2^j)$, $1 \leq j \leq \ell$, so that $h^1 = (1,0)$, $h^2 = (0,1)$, and the $\vec{w}^j$ from (3.20) and (3.21) are defined by the requirements,

$$\vec{w}^j = h_1^j\vec{w}^1 + h_2^j\vec{w}^2, \quad 1 \leq j \leq \ell \tag{3.23a}$$

and

$$\det(A\cdot(h_1(\vec{w}^1) + h_2(\vec{w}^2))) = 0 \tag{3.23b}$$

if and only if (h_1,h_2) is a multiple of some (h_1^j,h_2^j) with $1 \leq j \leq \ell$.

Here we use the shorthand notation, $A\cdot\vec{w} = \sum^N A_\ell\vec{w}^\ell$. We define $Q(n_1,n_2)$ by $Q(n_1,n_2) = \det(A\cdot(n_1\vec{w}^1 + n_2\vec{w}^2))$. In addition, we assume that

$$\nabla_n Q(n_1,n_2)\big|_{(n_1^j,n_2^j)} \neq 0 \quad \text{for} \quad j = 1,\dots,\ell. \tag{3.24}$$

This is the generic geometric assumption which we mentioned earlier in (3.19b). We note that when the condition in (3.24) is violated, a resonant caustic appears. The $\vec{w}^j$ are obviously the resonant directions since they are all the linear combinations of the two fundamental directions, $\vec{w}^1$ and $\vec{w}^2$ which are also characteristic directions. In 1-D, the coefficients Γ_{pq}^j in one space dimension in (2.46), (2.55) agree to leading order with the celebrated wave interaction coefficients of Glimm when $F_0(u) = I$ - the coefficients Γ_{pq}^j from (3.21) are a multi-D generalization of Glimm's interaction coefficients. This completes our description of the solution of the problem of multi-D oblique planar wave interactions which is given in full detail in [7].

We end this section with a brief discussion of the derivation of (3.21). First, we used the augmented ansatz in (3.20) beyond the collision time, X_0, in an attempt to include all conceivable oblique resonant wave directions as determined by the geometric conditions. With such an ansatz, the strategy of the argument follows in outline the one presented in section 2.3 for resonant periodic wave interactions in one space dimension. By the method of multiple scales, simultaneously we need to obtain a solution $u_2(x,\vec{\theta})$ of an auxilliary system of linear P.D.E.'s with sublinear growth in $\vec{\theta}$ and also a closed system of leading order asymptotic equations. As in (2.43), we need to find necessary and sufficient conditions on an almost periodic function $\tilde{g}(\theta_p,\theta_q)$, so that the auxilliary constant coefficient P.D.E.'s,

$$\mathcal{A}_p \frac{\partial u}{\partial \theta_p} + \mathcal{A}_q \frac{\partial u}{\partial \theta_q} = \tilde{g}(\theta_p,\theta_q), \quad 1 \leq p,\ q \leq \ell, \quad p \neq q \tag{3.25}$$

have a solution $u(\theta_p,\theta_q)$ with sublinear growth. Here $\mathcal{A}_j$ is given by the formula,

$$\mathcal{A}_j = A\cdot\vec{w}^j, \quad 1 \leq j \leq \ell \tag{3.26}$$

with $\vec{w}^j$ defined in (3.23). One important difference between 1-D and multi-D is expressed by the following easily proved

<u>LEMMA</u>: The <u>auxilliary system of equations</u> in (3.25) is <u>hyperbolic</u> for every $p \neq q$, $1 \leq p,\ q \leq \ell$ if and only if

(3.27) <u>the plane spanned by $\vec{w}^1$ and $\vec{w}^2$ contains a time-like direction</u> for the operator $\sum_{\ell=0}^{N} A_\ell \partial_{x_\ell}$.

Furthermore, $\ell < M$ unless (3.27) is satisfied and then $\ell = M$.

Thus, in 1-D, the auxilliary system is always hyperbolic while in multi-D, for $M \geq 4$ this system is often <u>not</u> hyperbolic; we invite the reader to construct simple examples using the slowness surfaces of a system with $M = 4$ with two circular characteristic cones. For isentropic compressible flow in 2-D, the system in (3.25) is always hyperbolic provided that the condition in (3.24) is satisfied. In general, the geometric condition in (3.24) guarantees that the operator in (3.25) has real principal type - with this condition the auxilliary problem posed in (3.25) can be solved (see [7]). With this information, one way to complete the asymptotic derivation and to obtain the closed system of equations in (3.21) is to use the principle of "exchange of phase functions" in a similar argument as we already presented in detail for 1-D in (2.47)-(2.54).

3.3: The Asymptotic Equations for Resonant Oblique Waves for 2-D Compressible Flow

Here we record the equations from (3.21) which describe resonant oblique plane wave interactions for the equations of 2-D isentropic fluid flow. One might argue with some justification that the asymptotic solution described in section 3.2 has replaced one difficult system of equations by another - the general resonant wave equations in (3.21). Here we report on algebraic calculations presented in detail in [7] which yield the equations in (3.21) for resonant plane wave interactions for 2-D isentropic compressible flow. We will show that after several reductions, essentially the same simplified resonant system as described in section 3.4 in (2.67) occurs in describing oblique interacting waves in a 2-D isentropic fluid - this is a remarkable fact!

The 3×3 hyperbolic system describing isentropic compressible fluid flow in two-space variables is given by

$$(3.29)\qquad \begin{aligned} &\rho_t + \operatorname{div} \vec{m} = 0 \\ &(m_i)_t + \operatorname{div}\left(\frac{m_i}{\rho} \vec{m}\right) + p(\rho)_{x_i} = 0, \quad i = 1,2. \end{aligned}$$

Here ρ is the density, $\vec{m} = {}^t(m_1, m_2)$ is the momentum vector, and $p(\rho)$ defines the pressure as a specified function of ρ through an isentropic equation of state - $p = A\rho^\gamma$, $\gamma > 1$ for an ideal gas. In this subsection we will use conventional physical notation with the variable t denoting the time variable rather than x_0 from the previous sections.

We set $u = {}^t(\rho, \vec{m})$ and consider small amplitude perturbations around the constant state

$$u_0 = {}^t(\rho_0,0,0) \tag{3.30}$$

with ρ_0 a constant reference density and $c_0 = (p_\rho(\rho_0))^{1/2}$, the corresponding sound speed - of course, we assume $p_\rho(\rho_0) > 0$ to guarantee hyperbolicity. General planar <u>sound waves</u> move at velocity c_0 at the background state ${}^t(\rho_0,0,0)$ and have an associated linear phase function and right eigenvector defined by

$$\phi = x\cdot\vec{w} - c_0 t, \quad R(\vec{w}) = {}^t(1,c_0\vec{w}) \tag{3.31}$$

for any $\vec{w} = (w_1,w_2)$ with $|\vec{w}| = 1$. Any direction $\vec{w}^0 = (w_1^0,w_2^0)$ defines a <u>vorticity wave</u> (steady shear layer solution) at the background state from (3.30) with phase function and right eigenvector defined by

$$\phi^\circ = x\cdot w^\circ, \quad R^\circ = {}^t(0,-w_2^\circ,w_1^\circ) \tag{3.32}$$

(for convenience, we don't normalize w° with $|\vec{w}^\circ| = 1$). It is not difficult to check that a given sound wave direction and a given vorticity wave direction always satisfy the nondegeneracy condition in (3.24) unless

$$\vec{w}^\circ\cdot\vec{w} = 0$$

i.e. the propagating sound wave and vorticity wave are at exactly right angles. On the other hand, two distinct vorticity waves provide an example where the geometric condition in (3.24) is always violated since their linear span is a characteristic hyperplane.

It is well-known that to leading order two oblique sound waves do not resonate and the simplified asymptotic solution with the form in (3.17) continues the solution to leading order beyond the interaction time, X_0 (see [7]). Thus, here we treat the oblique wave interaction of a planar sound wave and a vorticity wave. The equations in (3.29) have obvious rotational invariance so without loss of generality, we consider the interaction of a sound wave with phase function,

$$\phi^1 = x_1 - c_0 t \tag{3.33}$$

and a vorticity wave with phase function

$$\phi^2 = x_1(\cos\tilde{\theta} - 1) + x_2 \sin\tilde{\theta}. \tag{3.34}$$

When the angle, $\tilde{\theta}$, varies with $0 < \tilde{\theta} < 2\pi$, we generate all oblique wave patterns which satisfy the condition in (3.24). According to the recipe in (3.23), new oblique sound waves are generated through the interaction of the vorticity and sound waves associated with (3.34) and (3.33) along the new direction with phase function, ϕ^3, given by

$$\phi^3 = \cos\tilde{\theta}\, x_1 + \sin\tilde{\theta}\, x_2 - c_0 t. \tag{3.35}$$

the equations in (3.20) and (3.21) forxthe resonant oblique interactions in this special case have the following form: For (3.20),

$$u(x,t,\varepsilon) = \begin{pmatrix} \rho_0 \\ 0 \\ 0 \end{pmatrix} + \varepsilon\sigma_1(x,t,\frac{\phi_1}{\varepsilon}) \begin{pmatrix} I \\ c_0 \\ 0 \end{pmatrix} + $$

$$+ \varepsilon\sigma_2(x,t,\frac{\phi^2}{\varepsilon}) \begin{pmatrix} 0 \\ -\sin\tilde{\theta} \\ \cos\tilde{\theta}-1 \end{pmatrix} + \varepsilon\sigma_3(x,t,\frac{\phi^3}{\varepsilon}) \begin{pmatrix} 1 \\ c_0\cos\tilde{\theta} \\ c_0\sin\tilde{\theta} \end{pmatrix}$$

$$+ 0(\varepsilon).$$

For (3.21), the vorticity wave is linearly degenerate and the amplitude satisfies the trivial equation,

$$\frac{\partial}{\partial t}\sigma_2(x,t,\theta) = 0 \tag{3.37a}$$

i.e., $\sigma_2(x,t,\theta) \equiv \sigma_0(x,\theta)$. The amplitude of the incident sound wave satisfies the resonant equation

$$(\sigma_1)_t + c_0(\sigma_1)x_1 + b(\frac{1}{2}\sigma^2)_\theta + D_1(\tilde{\theta}) \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sigma_0'(x,s-\theta)\sigma_3(x,t,s)ds = 0. \tag{3.37b}$$

The amplitude of the generated sound wave satisfies the resonant equation,

$$(\sigma_3)_t + c_0 \cos\theta(\sigma_3)_{x_1} + c_0 \sin\theta(\sigma_3)_{x_2} + b(\tfrac{1}{2}\sigma^2)_\theta +$$

$$+ D_2(\tilde{\theta}) \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sigma_0'(x,-s-\theta)\sigma_1(x,t,s)ds = 0 \quad .$$

Here $\sigma_0' = \frac{\partial}{\partial s}\sigma_0(x,s)$ has the physical interpretation as the <u>vortex</u> <u>strength</u> in the perturbed initial <u>shear</u> <u>layer</u> while the coefficients b, $D_1(\tilde{\theta})$, $D_2(\tilde{\theta})$ are given by

$$b = \frac{1}{2}\frac{p_{\rho\rho}}{c_0} + \frac{c_0}{\rho_0}$$

(3.38)

$$D_1(\tilde{\theta}) = \frac{\cos\tilde{\theta}\sin\tilde{\theta}}{\rho_0}$$

$$D_2(\tilde{\theta}) = \frac{-(\cos^2\tilde{\theta} + 1)\sin\theta}{2\rho_0} \quad .$$

Of course, the requirement, $b \neq 0$, is the genuine nonlinearity condition at ρ_0. Thus, once again we arrive at a coupled system of multi-D scalar conservation laws coupled through convolutions with a known kernel defined essentially by the vortex strength.

We assume that the interaction time is at $x_0 = 0$ and also that the initial conditions for $\vec{\sigma} = (\sigma_1,\sigma_2,\sigma_3)$ are given as periodic functions of θ alone of mean zero and period one - in this case, σ_1 and σ_3 can be chosen as functions of (t,θ) alone and the equations in (3.37) specialize to the

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$$(\sigma_1)_t + b(\sigma_1^2)_\theta + \int_0^1 K_1(s-\theta)\sigma_3(t,s)ds = 0$$

(3.39)

$$(\sigma_3)_t + b(\sigma_3^2)_\theta + \int_0^1 K_2(-s-\theta)\sigma_1(t,s)ds = 0$$

with the kernels, $K_1(s)$, $K_2(s)$ given by

$$K_1(s) = D_1(\tilde{\theta})\sigma_0'(s)$$

(3.40)

$$K_2(s) = D_2(\tilde{\theta})\sigma_0'(s)$$

with σ_0' the vortex strength. This system has essentially the same structure as the system we discussed earlier in section 2.4. One difference is that the integral operator is linear but not skew-symmetric in the Euclidean inner product - however, the key estimate #1 needed for existence of weak solutions is still valid. The numerical experiments which are in progress for the simplified system in (3.39) should indicate any essential differences in the wave behavior for the system in (3.39) as compared to the corresponding behavior for the system in (2.67).

IV. The Rigorous Theory of Nonlinear Geometric Optics for Weak Solutions of Conservation Laws in 1-D

As we have mentioned earlier in the introduction, the formal methods of weakly nonlinear geometric optics, as described in sections II and III, have been used in a wide variety of applied contexts. In particular, the simplified asymptotic methods involving exact solutions of the inviscid Burgers equation have been used after shock waves have formed in these solutions in a variety of applied contexts; qualitatively reliable predictions for weak solutions have been made through this approach. As described in detail in section II of this paper, the tacit assumptions used in constructing the uniformly valid approximation require that these approximate solutions remain smooth for $M > 2$ (especially see (2.33c) where the norm $|\sigma_j'(t,\theta)|_\infty$ is used). Thus, the following theoretical problem is important for understanding these formal asymptotic methods:

PROBLEM: Assess the validity of the approximation of weakly nonlinear geometric
(4.1) optics after shocks have formed in weak solutions.

Recently, DiPerna and the author [3] have provided a rigorous solution of the basic problem in (4.1) for general genuinely nonlinear systems of conservation laws in one space variable and initial data which is generally of compact support - other results are given in [3] for periodic initial data with the restriction, $M < 2$. Section 4.1 contains a description of these results while a sketch of the proof for a scalar convex conservation law is presented in section 4.2. This work indicates in a striking fashion that the methods of weakly nonlinear geometric optics are even better than the predictions of the formal asymptotic theory. These results may surprise some readers who are experts in perturbation theory. Nevertheless, several accessible open problems remain in studying the problem from (4.1) in 1-D and some of these problems are given in the final subsection.

Below, we consider the general weak solution (given by Glimm's method) of the $M \times M$ system of conservation laws with small amplitude initial data,

$$
\begin{aligned}
&F_0(u^\varepsilon)_t + F_1(u^\varepsilon)_x = 0, \quad x \in R^1, \quad t > 0 \\
&u^\varepsilon(x,0) = u_0 + \varepsilon u_1^0(x)
\end{aligned}
\tag{4.2}
$$

When $u_1^0(x)$ has compact support or $u_1^0(x)$ is periodic with $M \leq 2$, in Proposition 1 from section 2.2 we have constructed a formal uniformly valid weakly nonlinear approximation to u^ε in (4.2) in regions of smoothness. This approximation is given by

$$
u_w^\varepsilon = u_0 + \varepsilon \sum_{j=1}^{M} \sigma_j(\varepsilon t, \phi_j) r_j
\tag{4.3a}
$$

where $\tau = \varepsilon t$, $\phi_j = x - \lambda_j(u_0)t$, and $\sigma_j(\tau,\theta)$ satisfies the inviscid Burgers equation,

$$
\begin{aligned}
&(\sigma_j)_\tau + b_j\left(\tfrac{1}{2}\sigma_j^2\right)_\theta = 0, \quad \tau > 0, \ \theta \in R^1 \\
&\sigma_j(\tau,\theta)\big|_{\tau=0} = \ell_j \cdot u_1^0(\theta)
\end{aligned}
\tag{4.3b}
$$

for $1 \leq j \leq M$ with b_j given in (2.16). In this context, the problem in (4.1) becomes the question,

(4.4) How close are the weak solutions, u^ε, of (4.2) and the asymptotic solutions, u_w^ε, from (4.3) after shock waves have formed?

We remark that in presenting the results from section 2 in the equations from (4.2) and (4.3), we have used the rescaling $x = \frac{x'}{\varepsilon}$, $t = \frac{t'}{\varepsilon}$ this should not cause confusion for the reader.

4.1: The Validity of Nonlinear Geometric Optics for Weak Solutions of 1-D Conservation Laws

Why should the simplified asymptotic solutions from (4.3) continue to approximate the weak solutions of (4.2) after shock waves have formed? First, a wide variety of explicit simple wave solutions of (4.2) and (4.3) were analyzed rigorously in Chapter 1 of [12] with explicit error estimates independent of the first derivatives; second, in the case where the M×M system in (4.2) is genuinely nonlinear so that $b_j \neq 0$, $1 \leq j \leq M$, the earlier work of DiPerna [2] and Liu [11] on the detailed large time asymptotic behavior of solutions of (4.2) for initial data, $u_1^0(x)$, with compact support gave a rigorous proof that the solution of (4.2) asymptotically approaches decoupled N-waves as $T \to \infty$. These decoupled N-waves are not exact solutions of the equations in (4.2) but instead are weak solutions of the decoupled formal asymptotic Burgers equations in (4.3b). Finally, as long as the solutions of (4.2) and (4.3) remain smooth, i.e. until times, $T = o(\varepsilon^{-1})$, it is not difficult to justify the asymptotics via the classical method of characteristics. Since both the short-time and large-time behavior of solutions of (4.2) is well-approximated by solutions of the simplified asymptotic approximation in (4.3), it is reasonable to conjecture that u_w^ε approximates u^ε for weak solutions and perhaps for all time!

The behavior conjectured above has been rigorously proved recently in [3]. In describing this theorem, we use the L^1-norm, $|u|_1 = \int_{-\infty}^{\infty} |u|dx$, and also the L^1-norm of a periodic function with period p, $|u|_1 = \frac{1}{p}\int_0^p |u|dx$. The following result is proved in [3].

Theorem: (Justification of Nonlinear Geometric Optics)

Assume that the general $M \times M$ hyperbolic system in (4.2) is genuinely nonlinear at u_0 in all wave fields, i.e. $b_j \neq 0$ in (4.3) for all j with $1 \leq j \leq M$.

A) Assume that $u_1^0(x)$ is an arbitrary function of bounded variation with compact support. Consider the (any) weak solution, $u^\varepsilon(x,t)$, constructed by Glimm's method for the initial value problem in (4.2) and the corresponding weakly nonlinear geometric optics approximation, $u_w^\varepsilon(x,t)$ defined in (4.3), then we have the estimate,

$$\max_{0<t<+\infty} |u^\varepsilon(\cdot,t) - u^\varepsilon(\cdot,t)|_1 \leq C\varepsilon^2 \tag{4.5}$$

uniformly for all times where C depends only on the support of $u_1^0(x)$ and the derivatives of $F_j(u)$, $j = 0,1$.

B) Assume that $u_1^0(x)$ is a periodic function with bounded variation per period. For a scalar convex conservation law so that $M = 1$

$$\max_{0<t<+\infty} |u^\varepsilon(\cdot,t) - u_w^\varepsilon(\cdot,t)|_1 \leq C\varepsilon^3 . \tag{4.6}$$

On the other hand, for a pair of conservation laws, i.e. $M = 2$ and periodic initial data, we have the weaker estimate,

$$|u^\varepsilon(\cdot,t) - u_w^\varepsilon(\cdot,t)|_1 \leq tC\varepsilon^2 \tag{4.7}$$

In B), the constants, C, in (4.6) and (4.7) depend on derivatives of $F_j(u)$, $j = 0,1$, the L^∞ norm of $u_1^0(x)$, and the period, p.

The estimates in (4.5) and (4.6) are much stronger than anticipated by the formal theory and provide very strong supporting evidence for the use of the formal approximations from sections II and III for discontinuous weak solution. Even the somewhat weaker estimate in(4.7) for $M = 2$ and periodic waves is still sufficient to justify u_w^ε as the leading order asymptotic term for discontinuous weak solutions until times of order, $O(\varepsilon^{-1})$.

Why do we use the L^1 norm? Next, we give an elementary example which illustrates this point. We consider small amplitude solutions about the zero background state for the scalar convex conservation law,

$$u_t + (f(u))_x = 0 \tag{4.8a}$$

with $f'(u) = a(u)$, $f''(u) > 0$ and conveniently normalized with $f'(0) = 0$, $f''(0) = 1$. For discontinuous Riemann initial data with the form

$$u^{\varepsilon}(x,0) = \begin{cases} \varepsilon, & x < 0 \\ -\varepsilon, & x > 0 \end{cases} \tag{4.8b}$$

we have the explicit solution of (4.8) given by

$$u^{\varepsilon}(x,t) = \begin{cases} \varepsilon, & x < s^{\varepsilon}t \\ -\varepsilon, & x > s^{\varepsilon}t \end{cases}$$

with $s^{\varepsilon} = (2\varepsilon)^{-1}(f(\varepsilon) - f(-\varepsilon))$. Provided that $f'''(0) \neq 0$, s^{ε} satisfies $C^{-1}\varepsilon^{2} < |s^{\varepsilon}| < C\varepsilon^{2}$ with some $C > 0$. For this example, u_{w}^{ε} is the solution of the inviscid Burgers equation with the initial data in (4.8b), i.e.

$$u_{w}^{\varepsilon}(x,t) = \begin{cases} \varepsilon, & x < 0,\ t > 0 \\ -\varepsilon. & x > 0,\ t > 0. \end{cases}$$

Let's compute the error, $u_{w}^{\varepsilon} - u^{\varepsilon}$, in various norms. In the maximum norm, the deviation of u_{w}^{ε} from u^{ε} is always only $O(\varepsilon)$ due to the difference in shock location. On the other hand, if we compute the deviation in the L^{1} norm, we have

$$C^{-1}t\varepsilon^{3} < |u^{\varepsilon}(\cdot,t) - u_{w}^{\varepsilon}(\cdot,t)|_{1} < Ct\varepsilon^{3}.$$

This example explains both why we need the L^{1}-norm to prove the above Theorem for weak solutions rather than the L^{∞} norm and also why estimates as presented in (4.7) grow in time due to phase shifts in the shock location. For a more detailed explanation of the factor of t in the estimate from (4.7) in the periodic case with $M = 2$ which we conjecture in [3] is due to the appearance of different phase shifts in the different wave speeds, we refer the reader to section 6 of [3]. The remarkable fact that the estimates in (4.5) and (4.6) are uniform for all time is a consequence of the large time cancellation of shocks and rarefactions of the same family.

We end this subsection with a brief discussion of the proof - the proof does not mimic the derivation of the asymptotic approximations in a straightforward fashion. There are four main aspects to the proof:

1) The use of structural symmetries of the Burgers equation solutions from (4.3)

2) The use of L^1-stability of the total variation to control the error up to intermediate asymptotic times of order ε^{-1}

(4.9) 3) The large-time asymptotic decoupling of general weak solutions of (4.2) into solutions of scalar conservation laws with forcing terms that are suitably small Borel measures following earlier ideas of DiPerna and Liu.

4) The rapid decay of total variation for large time for solutions of scalar convex conservation laws.

In the next subsection, we sketch the proof of the theorem for scalar convex conservation laws and periodic initial data. This gives us the opportunity to illustrate the main ingredients in 1), 2), and 4) of the general proof of the theorem.

We remark that the theorem in [3] is actually only formulated and proved for conservation laws with $F_0(u) = u$; however, since t is a time-like direction for (4.2), we use (2.2a), introduce $w = F_0(u)$ as a new dependent variable, and $F(w) = F_1(F_0^{-1}(w))$ as a new flux function and apply the theorem above to (4.2) in the form

$$w_t + F(w)_x = 0.$$

We leave the elementary details to the reader.

4.2 The Proof of the Theorem for Scalar Convex Conservation Laws with Periodic Initial Data

Here we sketch the proof of the estimate in (4.6) for scalar convex conservation laws and periodic initial data. We will describe the main steps 1), 2), and 4) in this special case and their use in the proof of this estimate.

Our first step is based on the following considerations: Consider the solution, $\tilde{u}^\varepsilon$, of the initial value problem for the inviscid Burgers equation,

$$\tilde{u}^\varepsilon_t + b\left(\frac{1}{2}(\tilde{u}^\varepsilon)^2\right)_x = 0$$

(4.10)

$$\tilde{u}^\varepsilon(x,0) = u_0 + \varepsilon u_1^0(x).$$

This equation arises as the leading order asymptotic equation from (4.3) for any general genuinely nonlinear system of conservation laws - thus, we might anticipate that the weakly nonlinear approximation from (4.3) is exact without any errors for weak solutions of the special initial value problem in (4.10). How can we justify weakly nonlinear geometric optics for weak solutions of the inviscid Burgers equation in (4.10)? We observe by direct calculation that if $\sigma(\tau,\theta)$ satisfies the inviscid Burgers equation,

(4.11a)
$$\sigma_\tau + b\left(\frac{1}{2}\sigma^2\right)_\theta = 0$$

then for any $\varepsilon > 0$ and constant u_0,

(4.11b)
$$\tilde{u}^\varepsilon(x,t) = u_0 + \varepsilon\sigma(\varepsilon t, x - bu_0 t)$$

is also a solution of the inviscid Burgers equation

(4.11c)
$$\tilde{u}^\varepsilon_t + b\left(\frac{1}{2}(\tilde{u}^\varepsilon)^2\right)_x = 0.$$

Thus, solutions of the inviscid Burgers equation are invariant under an additional two-parameter family of structural symmetries besides space-time dilations. If we choose as initial data for σ in (4.11a), the initial value $u_1^0(\theta)$, we observe that the facts in (4.11) justify weakly nonlinear geometric optics exactly for weak solutions of the Burgers equation - this is step 1) from (4.9). Actually, below we use the following mild extension of (4.11):

If $\sigma(\tau,\theta)$ satisfies (4.11a), then $\tilde{u}^\varepsilon = u_0 + \varepsilon\sigma(\varepsilon t, x - (c + bu_0)t)$ is a solution of

(4.12)
$$\tilde{u}^\varepsilon_t + c\tilde{u}^\varepsilon_x + b\left(\frac{1}{2}(\tilde{u}^\varepsilon)^2\right)_x = 0$$

so that weakly nonlinear geometric optics is justified exactly for weak solution of the conservation law in (4.12).

The facts in the preceding paragraph reduce the proof of the estimate in (4.6) of the Theorem to a direct L^1 estimate for the difference, $u^\varepsilon - \tilde{u}^\varepsilon$, where u^ε, $\tilde{u}^\varepsilon$ satisfy

(4.13a) $$u_t^\varepsilon + (f(u^\varepsilon))_x = 0, \quad u^\varepsilon(x,0) = u_0 + \varepsilon u_1^0(x)$$

and

(4.13b) $$\tilde{u}_t^\varepsilon + c\tilde{u}_x^\varepsilon + b\left(\frac{1}{2}(\tilde{u}^\varepsilon)^2\right)_x = 0, \quad \tilde{u}^\varepsilon(x,0) = u_0 + \varepsilon u_1^0(x)$$

with $b = f''(u_0)$ and $c = f'(u_0) - f''(u_0)u_0$. We expand $f(u)$ by

$$f(u) = Qf(u,u_0) + R(u,u_0)$$

with $Qf(u,u_0)$, the quadratic part at u_0 given by

(4.14a) $$QF(u,u_0) = f(u_0) + f'(u_0)(u - u_0) + \frac{f''(u_0)}{2}(u - u_0)^2$$

and the remainder $R(u,u_0)$ satisfying

(4.14b) $$R(u,u_0) = O(u - u_0)^3.$$

Thus, $\tilde{u}^\varepsilon$ from (4.13b) satisfies the scalar law,

(4.15a) $$\tilde{u}_t^\varepsilon + (Qf(\tilde{u}^\varepsilon,u_0))_x = 0$$

while, by using the chain rule for BV functions (see section 2 and 4 of [3]), u^ε from (4.13a) satisfies

(4.15b) $$u_t^\varepsilon + (Qf(u^\varepsilon,u_0))_x = \nu$$

where ν is a finite Borel measure on the period strip, $S_{0,T} = \{(x,t) \mid 0 \leq x \leq p, 0 \leq x \leq T\}$ with total mass on this strip, $|\nu|(S_{0,T})$ estimated by

(4.16) $$|\nu|(S_{0,T}) \leq C \int_0^T \int_0^p |u^\varepsilon - u_0|^2 |\partial_x u^\varepsilon| dx\, dt$$

$$\leq C \int_0^T (\mathrm{Var}_p(u^\varepsilon(\cdot,s)))^3 ds\ .$$

Here $Var_p(t(x))$ is the total variation on a period strip; the first estimate in (4.16) is given formally by differentiating (4.14b) and is justified rigorously by using the calculus for BV functions (see sections 2 and 4 of [3]) - the second estimate in (4.16) follows from the fact that the oscillation is dominated by the total variation.

We observe that $\tilde{u}^\varepsilon$ and u^ε are two solutions of the same inhomogeneous scalar conservation law with right hand side given by different finite Borel measures but the same initial data. The second key ingredient, 2) of (4.9) is the general L^1 stability estimate with inhomogeneous terms defined by finite Borel measures proved in section 3 of [3] for scalar convex conservation laws. Since this L^1-stability estimate generalizes the well-known earlier L^1-stability estimates for scalar laws of Volpert-Kruzkhov-Keyfitz, we won't state this stability estimate in detail here. However, a direct consequence is the estimate,

$$|u^\varepsilon(\cdot,t) - \tilde{u}^\varepsilon\cdot,t)|_1 \leq 2|v|(S_{0,T}) \tag{4.17}$$

for the two weak solutions with the same initial data satisfying (4.15). By combining (4.16) and (4.17), we obtain

$$|u^\varepsilon(\cdot,t) - \tilde{u}^\varepsilon((\cdot,t)|_1 \leq C \int_0^T (Var_p(u^\varepsilon(\cdot,s))^3 ds. \tag{4.18}$$

The second ingredient in the stability estimates for 2) in (4.9) is the well-known fact that

$$Var_p u^\varepsilon(\cdot,t) \leq Var_p u_0^\varepsilon 0 = \varepsilon. \tag{4.19}$$

A direct substitution of the estimate from (4.19) into (4.18) yields

$$|u^\varepsilon(\cdot,t) - u_w^\varepsilon(\cdot,t)|_1 \leq ct\varepsilon^3 \tag{4.20}$$

since $u_w^\varepsilon = \tilde{u}^\varepsilon$. This estimate is strong enough to justify the weakly nonlinear approximation for large times of order $o(\varepsilon^{-2})$. However, we can do better if we exploit the ideas mentioned in 4) of (4.9). It is well-known that for periodic waves, the total variation decays rapidly in time, in fact,

$$Var_p u^\varepsilon(\cdot,t) \leq \frac{C}{1+t} Var_p u_0^\varepsilon = \frac{C}{1+t}\varepsilon \tag{4.21}$$

and substitution of (4.21) into (4.18) improves the estimate in (4.20) to

$$|u^{\varepsilon}(\cdot,t) - u_{W}^{\varepsilon}(\cdot,t)|_1 < C\varepsilon^3 \int_0^t (1 + s)^{-3} ds < C_1\varepsilon^3 .$$

This completes our sketch of the proof of the estimate in (4.6).

4.3: Some Accessible Open Problems in the Rigorous Theory of Weakly Nonlinear Geometric Optics

Problem #1: As described in section 4.1, the main theorem in [3] is proved under the hypothesis that every wave field for the hyperbolic system in (4.2) is genuinely nonlinear. Extend the main theorem in [3] for initial data of compact support to systems with eigenvalues that are either genuinely nonlinear or linearly degenerate. The equations of compressible fluid flow from section 2.4 are the prototypical example. In particular, assess whether the estimate in (4.5) remains valid in this case.

Problem #2: It is not difficult to extend the formal theory of weakly nonlinear approximations as developed in section 2 to general mixed initial boundary value problems for the hyperbolic system in (4.2) on the space-time quarter-space, $x > 0$, $t > 0$ with suitable general boundary conditions on $x = 0$ and initial data of compact support. Also, the use of Glimm's method to prove existence of global small amplitude weak solutions has been carried out in detail in [23]. Formulate and prove a suitable analogue of the main theorem in [3] for data of compact support for the general mixed problem described above.

Problem #3: Extend and generalize the main theorem of [3] to data with compact support and suitable systems like those occurring in nonlinear elasticity with eigenvalues that are neither genuinely nonlinear nor linearly degenerate. Here the formal asymptotic solutions from section 2 needs to be modified suitably - see section 1.5 and 1.6 of [12] for the modifications of single wave expansions in the formal asymptotics. The system in 1-D with $M = 2$ describing nonlinear hyperelasticity is the simplest starting point for systems. The scalar case under the assumptions, $f^{(1)}(u_0) = a(u_0)$, $f^{(j)}(u_0) = 0$, $2 < j < p - 1$, $f^{(p)}(u_0) \neq 0$ can be handled by the same proof as we gave in section 4.2 with minor modifications - in fact, uniform decay estimates as in (4.21) are also known in these situations.

Problem 4: Study the validity of the resonant asymptotic approximations discussed in sections 2.3 and 2.4 for general periodic weak solutions of conservation laws. This is the most difficult problem of this section.

Problem 5: Study the large time asymptotic behavior of periodic waves for general pairs of conservation laws with genuinely nonlinear wave fields - i.e. $M = 2$. The asymptotics from section 4.2 yield a conjectured behavior involving explicit periodic solutions of the Burgers equation. Can one prove that the estimate in (4.6) is sharp for periodic waves? More importantly, if we regard the weakly nonlinear approximation u_w^ε as an approximation to solutions u^ε, then the examples in (4.8) and in [3] indicate that the resolution of these methods, according to the ideas of P.D. Lax (see [9]), is even better than the integral estimates from (4.6). Can this be established rigorously?

BIBLIOGRAPHY

[1] Choquet-Bruhat, V., "Ondes asmptotiques et approchées pour systèmes d'équations aux dérivées partielles nonlinéaires, J. Math. Pures et Appl., 48, 117-158 (1969).

[2] DiPerna, R., "Decay and asymptotic behavior of solutions to nonlinear hyperbolic systems of conservation laws", Indiana Univ. Math. J., 24, 1047-1071 (1975).

[3] DiPerna, R. and Majda, A., "The validity of nonlienar geometric optics for weak solutions of conservation laws", Commun. Math. Physics, April 1985, 35 pp.

[4] Hunter, J., "A ray method for slowly modulated nonlinear waves" (preprint, March 1984).

[5] Hunter, J. and Keller, J.B., "Weakly nonlinear high frequency waves", Commun. Pure Appl. Math.

[6] Hunter, J. and Keller, J.B., "Weak shock diffraction" (in press, Wave Motion).

[7] Hunter, J., Majda, A., and Rosales, R., "Resonantly interacting weakly nonlinear hyperbolic waves, II: several space variables" (in preparation).

[8] Landau, L.D., "On shock waves at large distances from their place of origin", J. Phys. USSR 9, 495-500 (1945).

[9] Lax, P.D., "Accuracy and resolution in the computation of solutions of linear and nonlinear equations", in Recent Advances in Numerical Analysis, pp. 107-118, DeBoor, C. and Golub, G. ed. New York Academic Press (1978).

[10] Lighthill, M.J., "A method for rendering approximate solutions to physical problems uniformly valid", Phil. Mag. 40, 1179-1201 (1949).

[11] Liu, T.P., "Decay to N-waves of solutions of general systems of nonlinear hyperbolic conservation laws", Comm. Pure Appl. Math. 30, 585-610 (1977).

[12] Majda, A., Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables, Appl. Math. Science Series #53, Springer-Verlag, New York, 1984.

[13] Majda, A. and Rosales, R., "A theory for spontaneous Mach system formation in reacting shock fronts, I: the basic perturbation analysis", S.I.A.M. J. Appl. Math. 43, 1310-1334 (1983).

[14] Majda, A. and Rosales, R., "A theory for spontaneous Mach stem formation in reacting shock fronts, II: steady wave bifurcations and the evidence for breakdown", Stud. Appl. Math. 71, 117-148 (1984).

[15] Majda, A. and Rosales, R., "Resonantly interacting weakly nonlinear hyperbolic waves, I: a single space variable", Stud. Appl. Math. 71, 149-179 (1984).

[16] Nayfeh, A., "A comparison of perturbation methods for nonlinear hyperbolic waves", in Singular Perturbations and Asymptotics, pp. 223-276, Meyer, R. and Parter, S. editors, Academic Press, New York, 1980.

[17] Rosales, R., and Majda, A., "Weakly nonlinear detonation waves", SIAM J. Appl. Math. 43, 1086-1118 (1983).

[18] Whitham, G., "The flow pattern of a supersonic projectile", Commun. Pure Appl. Math. 5, 301-348 (1952).

[19] Woodward, P., "Piecewise-parabolic methods for astrophysical fluid dynamics", L.L.L. preprint November 1983.

[20] Woodward, P., "Simulation of the Kelvin-Helmholtz instability of a supersonic slip surface with the piecewise parabolic method", L.L.L. preprint, March 1984.

[21] Majda, A., Rosales, R., and Schonbek, M., "A system of integro-differential equations arising in resonant wave asymptotics", (to appear).

[22] Katznelson, Y., An introduction to Harmonic Analysis, Dover, New York, 1976.

[23] Goodman, J., "The Mixed Initial Boundary Value Problem for Conservation Laws", Ph.D. Thesis Stanford, August, 1982.

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