Nonlinear guiding center theory of perpendicular diffusion: General properties and comparison with observation

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[1] A nonlinear guiding center (NLGC) theory for diffusion of charged particles perpendicular to the mean magnetic field was recently proposed. Here, we draw attention to a number of attractive features of this theory: (1) The theory provides a natural mechanism to connect the perpendicular mean free path with the parallel mean free path. In fact, the parallel mean free path is the only particle property required to determine uniquely the perpendicular mean free path. (2) Under a broad range of conditions, the theory predicts that the perpendicular mean free path will be of order one percent or a few percent of the parallel mean free path, in agreement with numerical simulations of particle transport. (3) For conditions representative of the inner heliosphere, the theory predicts values of the perpendicular mean free path in agreement with observational determinations from Jovian electrons and from modeling Ulysses observations of Galactic protons. *INDEX TERMS:* 2114 Interplanetary Physics: Energetic particles, heliospheric (7514); 2116 Interplanetary Physics: Energetic particles, planetary; 2104 Interplanetary Physics: Cosmic rays; 7807 Space Plasma Physics: Charged particle motion and acceleration. **Citation:** Bieber, J. W., W. H. Matthaeus, A. Shalchi, and G. Qin (2004), Nonlinear guiding center theory of perpendicular diffusion: General properties and comparison with observation, *Geophys. Res. Lett.*, *31*, L10805, doi:10.1029/2004GL020007.

1. Introduction

[2] Understanding how energetic charged particles diffuse perpendicular to a large-scale guide field has long been a challenging problem of space physics and astrophysics. Approaches based upon hard-sphere scattering [*Gleeson*, 1969] and related extensions based upon the Boltzmann equation [*Jones*, 1990] provide a suitable description for elastic scattering, but they are probably inapplicable to most space plasmas, where field line random walk (FLRW) is expected to play a key role [*Jokipii*, 1966; *Forman et al.*, 1974]. However a pure FLRW description also fails in the presence of parallel diffusion, because the particles partially retrace their paths after they backscatter. In the case of pure slab turbulence (e.g., a field composed of Alfvén waves propagating parallel to the large-scale field), this backscattering completely thwarts diffusion in the normal sense. Instead, particle displacements perpendicular to the mean field scale subdiffusively [*Urch*, 1977; *Kóta and Jokipii*, 2000; *Qin et al.*, 2002a].

[3] Recent numerical simulations have provided important new insights into perpendicular diffusion [*Giacalone and Jokipii*, 1999; *Mace et al.*, 2000; *Qin et al.*, 2002a, 2002b]. Turbulence geometry is emerging as a crucial factor in perpendicular diffusion, just as it is in parallel diffusion [*Bieber et al.*, 1994]. When the turbulent magnetic field has sufficient perpendicular structure, the subdiffusive tendency seen in pure slab turbulence can be overcome, and a regime of "second diffusion" [*Qin et al.*, 2002b] emerges.

[4] The process of perpendicular diffusion is thus a combination of field line random walk, backscatter from parallel diffusion, and transfer of particles across field lines owing to the magnetic field's perpendicular complexity. All of these factors are considered in a recently proposed theory of particle diffusion, the "nonlinear guiding center" (NLGC) theory [*Matthaeus et al.*, 2003]. In this work, we discuss the general properties of the NLGC theory and point out several attractive features of the theory, most notably its excellent agreement with observation.

2. Parallel Mean Free Path: Governing Factor in Perpendicular Diffusion

[5] We find it convenient to recast the parallel and perpendicular diffusion coefficients, $K_\parallel$ and $K_\perp$, in terms of equivalent mean free paths, $\lambda_\parallel$ and $\lambda_\perp$,

$$\lambda_\parallel = \frac{3}{V} K_\parallel; \qquad \lambda_\perp = \frac{3}{V} K_\perp, \tag{1}$$

where V is the particle speed. In terms of these mean free paths, equation (7) of *Matthaeus et al.* [2003] can be written

$$\lambda_\perp = \lambda_\parallel \frac{a^2}{B_0^2} \int d^3k \frac{S_{xx}(\mathbf{k})}{1 + k_\perp^2 \lambda_\perp \lambda_\parallel / 3 + k_z^2 \lambda_\parallel^2 / 3}, \tag{2}$$

where a^2 is a constant approximately equal to 1/3 according to numerical simulations [*Matthaeus et al.*, 2003], B_0 is the magnitude of the large-scale magnetic field, $\mathbf{k}$ is the wave vector of a turbulent magnetic fluctuation, $S_{xx}(\mathbf{k})$ is the modal (three-dimensional) power spectrum of one of the perpendicular components of the fluctuating field, $k_\perp = (k_x^2 + k_y^2)^{1/2}$ is perpendicular wave number, and k_z is parallel wave number. Following *Matthaeus et al.* [2003], we assume the turbulent field is transverse to the large-scale field and axisymmetric with respect to the large-scale field.

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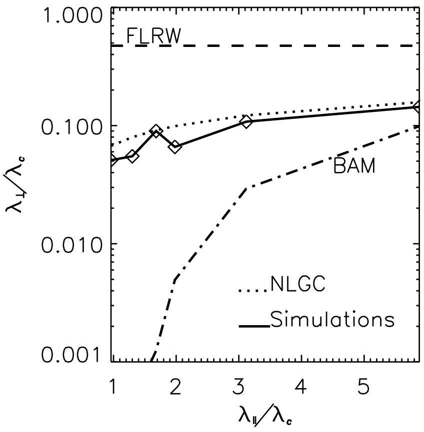

Figure 1. Particle perpendicular mean free path $\lambda_\perp$ plotted against the numerically determined parallel mean free path $\lambda_\parallel$ (both scaled to the slab correlation length $\lambda_c = 2\pi C(\nu) l_{slab}$). This numerical simulation used a composite turbulence geometry with 80% energy in 2D modes and 20% in slab modes, with $\delta B/B_0 = 1$, and with $l_{2D} = l_{slab}/10$. Solid line indicates simulation result, dotted line the NLGC theory, equation (5), dashed line the FLRW limit, and dash-dotted line the BAM theory.

The theory is nonlinear and does not require the fluctuations to be small in comparison with the mean field. Here we consider NLGC theory in the magnetostatic limit, setting $\gamma(\mathbf{k}) = 0$ in the nomenclature of *Matthaeus et al.* [2003].

[6] In order to derive quantitative results, we specify the following two-component (2D/slab) spectrum model:

$$S_{xx}(\mathbf{k}) = \frac{2}{\pi} C(\nu) l_{2D} \delta B_{2D}^2 \left(1 + k_\perp^2 l_{2D}^2\right)^{-\nu} \frac{k_y^2}{k_\perp^3} \delta(k_z) + C(\nu) l_{SLAB} \delta B_{SLAB}^2 \left(1 + k_z^2 l_{SLAB}^2\right)^{-\nu} \delta(k_x) \delta(k_y), \quad (3)$$

where

$$C(\nu) = \frac{\Gamma(\nu)}{2\pi^{1/2} \Gamma(\nu - 1/2)}. \quad (4)$$

Here δB_{2D}^2 is the total (sum of the two perpendicular variances) variance in the 2D component, δB_{SLAB}^2 is the total variance of the slab component, l_{2D} and l_{SLAB} define the break points of the 2D and slab spectra, and 2ν is the inertial range spectral index. With this spectrum model, equation (2) can now be recast in the form

$$\lambda_\perp = \lambda_\parallel \frac{2a^2 C(\nu)}{B_0^2} \left\{ \delta B_{SLAB}^2 \int_0^\infty \mathrm{d}x \frac{\left(1 + x^2\right)^{-\nu}}{1 + x^2 \lambda_\parallel^2 / \left(3 l_{SLAB}^2\right)} + \delta B_{2D}^2 \int_0^\infty \mathrm{d}x \frac{\left(1 + x^2\right)^{-\nu}}{1 + x^2 \lambda_\perp \lambda_\parallel / \left(3 l_{2D}^2\right)} \right\}. \quad (5)$$

[7] The feature of equations (2) and (5) we wish to stress is that the only properties of the particle that appear on the right hand side are the parallel ($\lambda_\parallel$) and perpendicular ($\lambda_\perp$) mean free paths. If $\lambda_\parallel$ is specified (whether from theory or observation), then equations (2) and (5) become integral equations for $\lambda_\perp$. Thus, in the NLGC theory of perpendicular diffusion in magnetostatic turbulence, the perpendicular mean free path is governed entirely by the parallel mean free path and properties of the turbulent field.

3. Scaling of $K_\perp$ With $K_\parallel$

[8] In applications of cosmic ray transport such as modulation modeling, the perpendicular diffusion coefficient is often taken to be a fixed fraction of the parallel diffusion coefficient, i.e.,

$$K_\perp = b K_\parallel, \quad (6)$$

with b a constant typically taken to be in the range 0.005–0.05 [e.g., *Jokipii et al.*, 1995; *Ferrando*, 1997; *Burger et al.*, 2000; *Ferreira et al.*, 2001] It has been something of a puzzle that equation (6) worked so well, because in the field line random walk picture of perpendicular transport the two diffusion coefficients should scale inversely. An increase in turbulence level decreases the parallel diffusion coefficient, because resonant scattering is more intense, but it increases the perpendicular diffusion coefficient, because the rate of field line diffusion is increased. In order to produce scaling of $K_\perp$ with $K_\parallel$, factors beyond field line random walk must be considered [e.g., *Chuvilgin and Ptuskin*, 1993].

[9] Approximate scaling of $K_\perp$ with $K_\parallel$ is also indicated by numerical simulations of particle transport. *Giacalone and Jokipii* [1999] and *Qin* [2002] both found that $K_\perp/K_\parallel$ is in the range 0.01–0.05 for representative solar wind conditions, and depends rather weakly upon particle rigidity.

[10] Figure 1 illustrates this property of perpendicular diffusion using our particle trajectory tracing code [*Qin*, 2002; *Qin et al.*, 2002a, 2002b]. Simulation results are shown by data points connected by the solid line, and these closely follow the NLGC prediction shown by the dotted line. The ratio $K_\perp/K_\parallel$ ranges from 0.05 at the left edge of the graph to about 0.02 at the right edge. Neither the field line random walk (FLRW) limit nor the *Bieber and Matthaeus* [1997] (hereinafter referred to as BAM) prediction provide a satisfactory description of the simulation results.

[11] *Shalchi et al.* [2004] have provided analytic approximations to the NLGC result, which shed light on the scaling of $K_\perp$ with $K_\parallel$. For either pure 2D or composite 2D/slab geometry, the ratio $K_\perp/K_\parallel$ is constant or nearly constant in a regime defined by $\lambda_\perp \lambda_\parallel/(3 l_{2D}^2) \ll 1$. A higher energy regime defined by $\lambda_\perp \lambda_\parallel/(3 l_{2D}^2) \gg 1$ and where also $\lambda_\parallel \gg 3^{1/2} l_{SLAB}$ may be applicable to most cosmic ray energies at 1 AU. Here *Shalchi et al.* [2004] find $\lambda_\perp \propto \lambda_\parallel^{1/3}$. (We observe that these conditions are marginally satisfied on the left edge of Figure 1, and well satisfied on the right edge.) Thus, a perpendicular mean free path that increases monotonically with the parallel mean free path is a robust property of the NLGC model, in contrast with the FLRW picture, but in accord with simulations and with common practice in modulation modeling.

4. NLGC Mean Free Path in Comparsion With Observation

[12] Figure 2a compares the NLGC prediction for electrons (red curve) and protons (blue curve) with selected

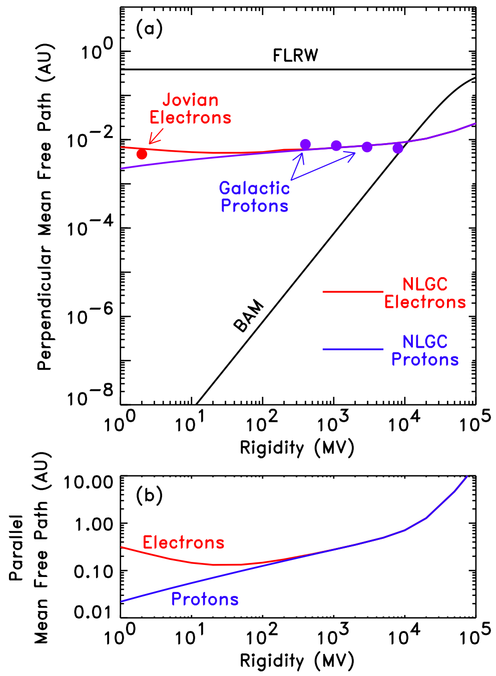

Figure 2. (a) Perpendicular mean free paths at 1 AU from several theoretical models (curves) compared with observations (circles). The NLGC theory predicts different mean free paths for electrons (red) and protons (blue) of the same rigidity. The NLGC theory is clearly favored by observational determinations from Jovian electrons [*Chenette et al.*, 1977] and Ulysses measurements of Galactic protons [*Burger et al.*, 2000]. (b) Parallel mean free path used in computing the perpendicular mean free path shown in upper panel.

mean free paths derived from observation, and with the FLRW and BAM models of $\lambda_\perp$. For the theoretical models, we used turbulence parameters representative of the solar wind at 1 AU: $\delta B_{2D}^2 = 21.1$ nT^2, $\delta B_{SLAB}^2 = 5.28$ nT^2, $B_0 = 4.12$ nT, $l_{SLAB} = 0.023$ AU, $l_{2D} = l_{SLAB}/10$ (motivated by *Robinson and Rusbridge* [1971]), and $\nu = 5/6$. For the BAM model we assumed the ultrascale is $l_{ULTRA} = 10 l_{SLAB}$ [*Matthaeus et al.*, 1999]. The energy of turbulence in the 2D and slab components is in the ratio 80:20, in accord with observation [*Bieber et al.*, 1996].

[13] In order to plot $\lambda_\perp$ as a function of rigidity in Figure 2a, it is necessary to specify the rigidity dependence of $\lambda_\parallel$. For this purpose we used the rigidity dependence shown in Figure 2b. The curves correspond to the "damping model" of dynamical slab turbulence [*Bieber et al.*, 1994] with $\alpha = 0.1$ and with a turbulence amplitude corresponding to the 20% slab portion in our turbulence model. Possible nonresonant or nonlinear contributions of the 2D component to $\lambda_\parallel$ were ignored. Technically we should employ an expression for $\lambda_\perp$ that also includes dynamical effects [*Matthaeus et al.*, 2003], but explicit computation shows that it would shift the curves in Figure 2a by less than 0.1%.

[14] It is important for our purpose that the parallel mean free path we use to compute $\lambda_\perp$ be compatible with observation, and the curves in Figure 2b are indeed representative of the observational results for $\lambda_\parallel$ reported by *Dröge* [2000]. Scattering in dynamical turbulence has the property that the parallel mean free path depends explicitly upon particle speed as well as rigidity; hence low-rigidity electrons and protons have different parallel mean free paths at the same value of rigidity. Considering the results of Section 2 above, this translates into a corresponding electron-proton difference in perpendicular mean free paths.

[15] The NLGC theory predicts that $\lambda_\perp$ shows remarkably little variation over the 5 decades of rigidity shown in Figure 2. All of Figure 2 is in the "higher energy" regime discussed in the previous section, where $\lambda_\perp \propto \lambda_\parallel^{1/3}$. In the case of protons, for which $\lambda_\parallel$ varies with rigidity P approximately as $P^{1/3}$ below 10 GV, this implies $\lambda_\perp \propto P^{1/9}$, a very flat rigidity dependence indeed. Electrons have a more complex rigidity dependence, but their perpendicular mean free path displays even less variation than the protons, less than a factor of 2 between 1 MV and 10 GV. The ratio $\lambda_\perp/\lambda_\parallel$ remains in the range 0.01–0.04 for electrons and 0.01–0.10 for protons between 1 MV and 10 GV.

[16] The very weak rigidity dependence of the perpendicular mean free path is supported by Ulysses measurements of the latitude gradient of Galactic cosmic ray protons. The perpendicular mean free path deduced from the Ulysses observations is shown as blue data points in Figure 2a [*Burger et al.*, 2000, Figure 2a]. Agreement with the NLGC model is excellent.

[17] Jovian electrons and their 13-month modulation with the Jovian synodic year provide a particularly sensitive method for determining $\lambda_\perp$ observationally, because the particles are emitted from a known point source. The principal mechanism for them to travel from the Jupiter-Sun to the Earth-Sun magnetic field line is via perpendicular diffusion. Thus the width of the 13-month modulation envelope provides a relatively model-independent measure of the perpendicular mean free path. As shown in Figure 2a, the Jovian electron result [*Chenette et al.*, 1977] (red data point) is in a low rigidity region where the different theoretical models diverge widely. The FLRW and BAM predictions are incorrect by orders of magnitude, but the NLGC result displays essentially perfect agreement with the Jovian electrons.

5. Can $\lambda_\perp/\lambda_\parallel$ Approach Unity?

[18] There are recent reports that $\lambda_\perp/\lambda_\parallel$ is occasionally very large, even approaching or exceeding unity [*Dwyer et al.*, 1997; *Zhang et al.*, 2003]. It is beyond the scope of this Letter to address this issue in detail. However, we should like to point out that the NLGC theory can predict rather large values of $\lambda_\perp/\lambda_\parallel$ in certain parameter regimes.

[19] For instance, the *Dwyer et al.* [1997] result is for relatively low rigidity ions. It is a property of the NLGC theory (somewhat counter-intuitively) that ions have the largest $\lambda_\perp/\lambda_\parallel$ ratios at smaller values of $\lambda_\parallel$. In Figure 2a, the ratio for protons is only 0.01 at 10 GV but is 0.10 at 1 MV. Further, in a regime defined by the two conditions, $\lambda_\parallel \ll 3^{1/2} l_{SLAB}$ and $\lambda_\perp \lambda_\parallel/(3 l_{2D}^2) \ll 1$, *Shalchi et al.* [2004] show that

$$\frac{\lambda_\perp}{\lambda_\parallel} \approx \frac{a^2}{2} \frac{\delta B_{2D}^2}{B_0^2}. \quad (7)$$

With sufficiently strong turbulence (and recalling that a^2 = 1/3) the ratio $\lambda_\perp/\lambda_\parallel$ could indeed approach unity in this regime.

6. Summary

[20] The nonlinear guiding center (NLGC) theory shows promise of solving the long-standing puzzle of how cosmic rays diffuse perpendicular to a large-scale guide field. Key features of the theory include:

[21] 1. The parallel mean free path is the only property of the particle required to specify the perpendicular mean free path.

[22] 2. In two-component (2D/slab) turbulence, the perpendicular mean free path increases monotonically with the parallel mean free path, in accord with indications from modulation modeling.

[23] 3. The NLGC theory is in excellent agreement with numerical simulation results.

[24] 4. For conditions representative of the solar wind at 1 AU, the perpendicular mean free path has an extremely weak dependence on particle rigidity, in accord with conclusions derived from Ulysses observations of Galactic protons.

[25] 5. For conditions representative of the solar wind at 1 AU, the ratio $\lambda_\perp/\lambda_\parallel$ is in the range 0.01–0.10 between 1 MV and 10 GV.

[26] 6. The perpendicular mean free path derived from the 13-month modulation of Jovian electrons applies to a low rigidity regime where the theoretical models diverge widely. The Jovian electron result decisively favors the NLGC model of perpendicular diffusion.

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