Nonlinear Hamiltonian Waves with Constant Frequency and Surface Waves on Vorticity Discontinuities

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Waves with constant, nonzero linearized frequency form an interesting class of nondispersive waves whose properties differ from those of nondispersive hyperbolic waves. We propose an inviscid Burgers-Hilbert equation as a model equation for such waves, and give a dimensional argument to show that it models Hamiltonian surface waves with constant frequency. Using the method of multiple scales, we derive a cubically nonlinear, quasilinear, nonlocal asymptotic equation for weakly nonlinear solutions. We show that exactly the same asymptotic equation describes surface waves on a planar discontinuity in vorticity in two-dimensional inviscid, incompressible fluid flows. Thus, the Burgers-Hilbert equation provides an effective equation for these waves. We describe the Hamiltonian structure of the Burgers-Hilbert and asymptotic equations, and show that the asymptotic equation may be also be derived by means of a near-identity transformation. We derive a semi-classical approximation of the asymptotic equation, and show that spatially periodic, harmonic traveling waves are linearly and modulationaly stable. Numerical solutions of the Burgers-Hilbert and asymptotic equations are in excellent agreement in the appropriate regime. In particular, the lifespan of small-amplitude smooth solutions of the Burgers-Hilbert equation is given by the cubically nonlinear timescale predicted by the asymptotic equation.

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1. INTRODUCTION

In this paper, we study a class of nonlinear waves whose linearized frequency is nonzero and independent of the magnitude of their wavenumber. We call such waves *constant-frequency waves*; they arise in wave motions that depend only upon space-time parameters with the dimensions of time (but, for example, no length, velocity or acceleration parameters). A specific physical example of such a wave motion is provided by the surface waves that propagate on a vorticity discontinuity in a two-dimensional, inviscid, incompressible fluid flow.

Constant-frequency waves are nondispersive, but their qualitative behavior is different from that of the more familiar nondispersive waves whose linearized phase speed is independent of the magnitude of their wavenumber. We refer to the latter waves as nondispersive hyperbolic waves, since they are often described by hyperbolic partial differential equations (without lower-order terms).

A linear constant-frequency wave has an arbitrary spatial profile that oscillates periodically in time, whereas a linear nondispersive hyperbolic wave has an arbitrary profile that propagates with constant speed. Weakly nonlinear effects lead to a slow distortion of these profiles; but, as we will explain, the resonant nonlinear effects are cubic for constant-frequency waves, whereas they are quadratic for hyperbolic waves.

A useful model equation for unidirectional, constant-frequency, Hamiltonian waves is provided by

$$u_t + \left(\frac{1}{2}u^2\right)_x = \mathbf{H}[u]. \tag{1}$$

In (1), and below, we use $\mathbf{H}$ to denote the spatial Hilbert transform, defined by

$$\mathbf{H}\left[e^{ikx}\right] = -i(\operatorname{sgn} k)e^{ikx}$$

where sgn is the usual sign-function. We summarize our notation and definitions in the Appendix.

Equation (1) consists of an inviscid Burgers equation for $u(x,t)$ with a lower-order, nonlocal, linear, oscillatory term. Positive wavenumber solutions of the linearized equation $u_t = \mathbf{H}[u]$ have frequency equal to 1, while negative wavenumber solutions have frequency equal to -1. Thus, (1) describes a single, real-valued nonlinear wave whose linearized frequency is constant. We call (1) the inviscid Burgers-Hilbert equation, or, for brevity, the Burgers-Hilbert equation. Marsden and Weinstein [12] wrote down (1) as a quadratic approximation for the motion of the boundary of a vortex patch, but they did not analyze it.

Weakly nonlinear solutions of (1) have the form

$$u(x,t) \sim \psi(x,t)e^{-it} + \psi^*(x,t)e^{it}, \tag{2}$$

where $\psi(x,t)$ is a small, complex-valued amplitude-function that varies slowly in time[1] and * denotes the complex conjugate. The amplitude ψ contains only positive wavenumber spatial components, meaning that

$$\mathbf{P}\psi = \psi, \tag{3}$$

where $\mathbf{P} = \frac{1}{2}\left(\mathbf{I} + i\mathbf{H}\right)$ denotes the projection on to positive wavenumber components. As we will show, both by the method of multiple scales and by a symplectic near-identity transformation of the Hamiltonian of (1), $\psi(x,t)$ satisfies the following cubically nonlinear, nonlocal asymptotic equation:

$$\psi_t = \mathbf{P}\partial_x\Big[\psi\left|\partial_x\right|n - n\left|\partial_x\right|\psi\Big] \qquad n = |\psi|^2. \tag{4}$$

Here, ∂_x denotes the partial derivative with respect to x and $|\partial_x| = \mathbf{H}\partial_x$ has symbol $|k|$. We remark that, since $\mathbf{P}\psi = \psi$, we may write $|\partial_x|\,\psi = -i\partial_x\psi$ in (4), but there is no similar identity for $|\partial_x|\,n$, since n contains both positive and negative wavenumber components. This equation for the complex wave amplitude ψ may also be written in an equivalent real form, which is given in (51) below.

The spectral form of (4) for the spatial Fourier transform $\hat{\psi}(k,t)$ of $\psi(x,t)$ is

$$\hat{\psi}_t(k,t) + ik\int \delta(k + k_2 - k_3 - k_4)\Lambda\left(k,k_2,k_3,k_4\right) \hat{\psi}^*(k_2,t)\hat{\psi}(k_3,t)\hat{\psi}(k_4,t)\,dk_2dk_3dk_4 = 0 \tag{5}$$

for $k > 0$, with $\hat{\psi}(k,t) = 0$ for $k \leq 0$. In (5), δ denotes the delta-function, the integrals are taken over $k_j > 0$, and the interaction coefficient Λ is given by

$$\Lambda(k_1,k_2,k_3,k_4) = 2\min\left\{k_1,k_2,k_3,k_4\right\}. \tag{6}$$

An alternative, but equivalent, expression for Λ is given in (49).

An interesting physical example of constant-frequency waves arises in two-dimensional incompressible, inviscid fluid flows. A planar discontinuity in vorticity that separates two linear shear flows with constant, but different, vorticities is linearly stable.[2] Small disturbances of the vorticity

[1] We omit the explicit introduction of a small parameter in (2).

[2] This stability contrasts with the Kelvin-Helmholtz instability of a vortex sheet, across which the tangential fluid velocity is discontinuous.

discontinuity, with fluid velocity perturbations that decay away from the discontinuity, oscillate with a constant frequency that is proportional to the jump in vorticity. The constant frequency of these waves is a consequence of the fact that the only dimensional parameters on which the wave motion depends are the vorticities of the half-spaces, and these have the dimension of frequency.

Remarkably, when weakly nonlinear effects are taken into account, the complex amplitude of the displacement of the vorticity discontinuity satisfies an asymptotic equation of exactly the same form as the equation (4) arising from the Burgers-Hilbert equation. Thus, the Burgers-Hilbert equation, with a suitably renormalized nonlinear coefficient given in (34) below, captures not only the nonresonant quadratically nonlinear dynamics of a vorticity discontinuity but also the resonant cubically nonlinear dynamics. This apparent coincidence is explained, in part, by dimensional analysis, which shows that (1) is an appropriate model equation for constant-frequency *surface* waves, just as the inviscid Burgers equation is an appropriate model for constant-velocity *bulk* waves.

A planar vorticity discontinuity may be thought of as providing a local approximation to the curved boundary of a vortex patch, so the asymptotic equation (4) is relevant to the dynamics of vortex patches [3, 11, 13]. We note that a spectral form of a related asymptotic equation was derived by Dritschel [5] for weakly nonlinear deformations of a circular vortex patch.

Equation (4) has a great deal of structure. In this paper, we describe its Hamiltonian form, and derive some conserved quantities. The only explicit solution we know of is the harmonic, spatially-periodic traveling-wave

$$\psi(x,t) = Ae^{ikx-i\omega t} \tag{7}$$

where $A \in \mathbb{C}$, $k > 0$, and the frequency ω satisfies the nonlinear dispersion relation

$$\omega = |A|^2k^2. \tag{8}$$

As we show, this solution is linearly and modulationaly stable.

Numerical computations show that solutions of the asymptotic equation (4) develop singularities in finite time in which the derivative ψ_x blows up. This singularity formation corresponds to the breakdown of smooth solutions of the inviscid Burgers-Hilbert equation. A numerical comparison shows that there is excellent agreement between solutions of the asymptotic equation and small-amplitude solutions of the Burgers-Hilbert equation, and that the singularity formation time of the asymptotic equation provides an accurate approximation for the singularity formation time of the Burgers-Hilbert equation.

Thus, smooth solutions of the Burgers-Hilbert equation with small initial data have a cubically nonlinear lifespan, which is longer than the quadratically nonlinear lifespan of smooth solutions of the inviscid Burgers equation. This extension in lifespan is a result of the averaging to zero of the quadratic Burgers nonlinearity over a period of the temporal oscillations induced by the Hilbert transform.

We will consider the nature of the singularities in solutions of the asymptotic equation and the continuation of smooth solutions by weak solutions after a singularity forms in subsequent work.

We conclude the introduction by outlining the contents of the paper. In Section 2, we discuss constant-frequency waves in the context of the general theory of nonlinear waves. In Section 3, we carry out a dimensional analysis of Hamiltonian wave motions which shows that the Burgers-Hilbert equation (1) provides an appropriate model equation for unidirectional, constant-frequency surface waves. In Section 4, which can be read directly after this introduction, we use the method of multiple scales to derive the asymptotic equation (4) from (1). In Section 5, we derive (4) for surface waves on a vorticity discontinuity. In Section 6, we study the Hamiltonian structure of (4), and write it in equivalent spectral and real forms. In Section 7 we show that (4) can also be derived from (1) by the use of a near-identity transformation. In Section 8, we derive a semi-classical approximation for (4), and show that harmonic solutions are both linearly and modulationaly stable. Finally, in Section 9, we present and compare some numerical solutions of (1) and (4).

2. CONSTANT-FREQUENCY WAVES

In this section, we discuss constant-frequency waves in the context of the general theory of nonlinear waves in a uniform medium ([14], [15]). For simplicity, we consider a unidirectional wave motion with a single mode propagating freely in space. Abusing notation slightly, we write the linearized dispersion relation between the frequency ω and the wavenumber k of the waves as $\omega = \omega(k)$, where $\omega : \mathbb{R} \to \mathbb{R}$ is an odd function for real-valued wave-fields.

We begin by recalling some well-known facts about dispersive and nondispersive hyperbolic waves. Waves are dispersive if $\omega'' \neq 0$, where the prime denotes a derivative with respect to k. According to the linearized theory, a small-amplitude wave spreads out into a locally harmonic wave-train. Over longer times, weakly nonlinear effects come into play, and the complex amplitude of the wave typically satisfies the cubically nonlinear Schrödinger (NLS) equation.

Nondispersive hyperbolic waves have a dispersion relation of the form $\omega = c_0 k$ where c_0 is a constant wave-speed. According to the linearized theory, a wave consists of an arbitrary spatial wave-profile that propagates at constant velocity without distortion. Over longer times, weakly nonlinear effects distort the wave-profile, and the profile of a bulk wave typically satisfies the quadratically nonlinear inviscid Burgers equation [2, 7].

The degree of nonlinearity in the weakly nonlinear equations is a consequence of the resonances allowed by the linearized dispersion relation. The quadratically nonlinear effects on nondispersive hyperbolic waves arise from three-wave resonances among harmonics, of the form

$$\omega_1 = \omega_2 + \omega_3, \qquad k_1 = k_2 + k_3.$$

Since $\omega = c_0 k$, this resonance condition is satisfied for any k_j such that $k_1 = k_2 + k_3$. As a result, the quadratically nonlinear self-interaction of an initially spatially-harmonic wave with wavenumber k generates higher harmonics nk, for all integers n.[3] The resulting harmonic 'cascade' is well-described by the inviscid Burgers equation.

By contrast, three-wave resonances of dispersive waves do not occur,[4] but four-wave resonances of the form

$$\omega = \omega + \omega - \omega, \qquad k = k + k - k$$

always occur. These resonances lead to cubically nonlinear effects on the amplitude of a single harmonic, although they do not generate new harmonics. The combined effect of this nonlinear self-interaction with weak dispersion is described by the cubic NLS equation.

Constant-frequency waves have a linearized dispersion relation of the form

$$\omega = \omega_0 \operatorname{sgn} k \tag{9}$$

where ω_0 is a nonzero constant.[5] They are nondispersive, since $\omega'' = 0$ for $k \neq 0$, and, according to the linearized theory, a wave consists of an arbitrary spatial profile that oscillates with frequency ω_0. Over long times, weakly nonlinear effects distort this profile. Three-wave resonances do not occur, since $\omega_0 \neq \omega_0 + \omega_0$, but four-wave resonances occur among any spatial harmonics with positive wavenumbers k_1, k_2, k_3, k_4 such that

$$k_1 = k_2 + k_3 - k_4. \tag{10}$$

The corresponding condition for the frequencies is satisfied automatically. Thus, cubically nonlinear interactions among different spatial harmonics

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[3]We assume throughout this discussion that all relevant nonlinear interaction coefficients are nonzero.

[4]Except in non-generic cases, such as second-harmonic resonance, which we do not consider.

[5]The general nondispersive case, in which $\omega = \omega_0 + c_0 k$, can be reduced to (9) by means of a Galilean transformation.

generate new spatial harmonics, leading to a wide wavenumber spectrum, but a narrow frequency spectrum. As a result, the complex wave-amplitude function that describes the spatial profile of the wave typically satisfies a cubically nonlinear asymptotic equation, such as (4), which may be spatially nonlocal. The nonlinear dynamics of constant-frequency waves is therefore fundamentally different from that of either dispersive waves or nondispersive hyperbolic waves.

3. DIMENSIONAL ANALYSIS

From the perspective of dimensional analysis, constant-frequency waves arise in systems that are invariant under space-time scalings $(x,t) \mapsto (\lambda x, t)$, which implies that the wave motion does not depend on any length or velocity parameters, only time parameters. By contrast, nondispersive hyperbolic waves arise in systems that are invariant under scalings $(x,t) \mapsto (\lambda x, \lambda t)$, which implies that the wave motion does not depend on any length or time parameters, only velocity parameters.

In this section, we carry out a dimensional analysis of general Hamiltonian constant-frequency waves which explains why (1) provides an appropriate model equation for surface waves, such as waves on a vorticity discontinuity.

We consider a Hamiltonian wave motion [15] that depends upon two dimensional parameters: a frequency ω_0 and a density ρ_0. We use M, L, T to denote the dimensions of mass, length, time, respectively, and denote the dimensions of a quantity X by $[X]$. Then $[\omega_0] = 1/T$, $[\rho_0] = M/L^n$ where n is the number of space dimensions.

We denote canonically conjugate wave amplitudes by $\{a(k), a^*(k)\}$, and suppose that they are parametrized by a wavenumber vector $k \in \mathbb{R}^d$. Thus, $d = n$ for bulk waves, and $d = n-1$ for waves that propagate along a surface of codimension one.

Taylor expanding the Hamiltonian $\mathcal{H}$ in powers of the amplitude a, we get

$$\begin{aligned}\mathcal{H}(a,a^*) = &\int \omega_0 a(k) a^*(k)\, dk \\ &+\frac{1}{2}\int \delta(k_1 - k_2 - k_3)\, V(k_1,k_2,k_3)\, a^*(k_1) a(k_2) a(k_3)\, dk_1 dk_2 dk_3 \\ &+\frac{1}{2}\int \delta(k_1 + k_2 - k_3)\, V^*(k_1,k_2,k_3)\, a(k_1) a^*(k_2) a^*(k_3)\, dk_1 dk_2 dk_3 \\ &+O\left(|a|^4\right),\end{aligned}$$

where $V(k_1,k_2,k_3) = V(k_1,k_3,k_2)$. We omit "creation" and "annihilation" terms in $\mathcal{H}$ proportional to $a^*(k_1)a^*(k_2)a^*(k_3)$ and $a(k_1)a(k_2)a(k_3)$, respectively. These terms are nonresonant for the linearized dispersion relation (9) and they can be removed by a near-identity transformation.

Since $\mathcal{H}$ is an energy, we have $[\mathcal{H}] = ML^2/T^2$, which implies that

$$[a] = \left(\frac{M}{T}\right)^{1/2} L^{1+d/2}, \qquad [V] = \left(\frac{M}{T}\right)^{1/2} L^{3(1+d/2)+n}.$$

It follows that

$$V\left(k_1, k_2, k_3\right) = \left(\rho_0 \omega_0\right)^{1/2} W\left(k_1, k_2, k_3\right),$$

where $[W] = L^\nu$ with

$$\nu = \frac{n-d+2}{2}.$$

The inverse wavenumber provides the only lengthscale for a constant-frequency wave-motion, so W is a homogeneous function of degree $(-\nu)$, meaning that

$$W\left(\lambda k_1, \lambda k_2, \lambda k_3\right) = \lambda^{-\nu} W\left(k_1, k_2, k_3\right) \qquad \text{for all } \lambda > 0.$$

In particular, we have $\nu = 1$ for bulk waves and $\nu = 3/2$ for surface waves.

Next, we specialize to unidirectional waves and consider some examples of bulk and surface waves that are consistent with this dimensional scaling argument.

3.1. Unidirectional waves

A unidirectional wave may be described by complex-canonical variables

$$\left\{a(k), a^*(k) \mid 0 < k < \infty\right\}.$$

We consider a cubic Hamiltonian,

$$\begin{aligned}
\mathcal{H}(a, a^*) \;=\; & \int_0^\infty \omega_0 a(k) a^*(k)\, dk \\
& + \frac{1}{2} \int_0^\infty \int_0^k V\left(k, k-\xi, \xi\right) a^*(k) a(k-\xi) a(\xi)\, dk d\xi \\
& + \frac{1}{2} \int_0^\infty \int_0^\infty V^*\left(k, \xi, k+\xi\right) a(k) a^*(\xi) a^*(k+\xi)\, dk d\xi,
\end{aligned}$$

and suppose that all higher-degree terms are zero.

The complex canonical form of Hamilton's equation,

$$i a_t = \frac{\delta \mathcal{H}}{\delta a^*}, \tag{11}$$

leads to the following equation for $a(k,t)$,

$$\begin{aligned} i a_t(k,t) &= \omega_0 a(k,t) + \frac{1}{2}\int_0^k V(k, k-\xi, \xi) a(k-\xi,t) a(\xi,t)\, d\xi \\ &\quad + \int_0^\infty V^*(k,\xi,k+\xi) a(k+\xi,t) a^*(\xi,t)\, d\xi, \end{aligned}$$

where $k > 0$. Defining $a(-k,t) = a^*(k,t)$, and $V(-k_1,k_2,k_3) = V^*(k_1,k_2,k_3)$, we may write this equation in convolution form as

$$i(\operatorname{sgn} k) a_t(k,t) = \omega_0 a(k,t) + \int_{-\infty}^{\infty} V(k,k-\xi,\xi) a(k-\xi,t) a(\xi,t)\, d\xi,$$

where $-\infty < k < \infty$.

3.2. Bulk waves

The three-wave interaction coefficients of constant-frequency bulk waves are symmetric, homogeneous functions of degree 1. A simple choice for them is

$$V(k_1,k_2,k_3) = (\rho_0\omega_0)^{1/2} (k_1 k_2 k_3)^{1/3}$$

where $k_j > 0$. The corresponding Hamiltonian equation is

$$\begin{aligned} & i(\operatorname{sgn} k) a_t(k,t) = \omega_0 a(k,t) \\ &\quad + (\rho_0\omega_0)^{1/2} |k|^{1/3} \int_{-\infty}^{\infty} |k-\xi|^{1/3} |\xi|^{1/3} a(k-\xi,t) a(\xi,t)\, d\xi. \end{aligned}$$

Introducing a non-canonical variable

$$u(x,t) = (\rho_0\omega_0)^{1/2} \int_{-\infty}^{\infty} |k|^{1/3} a(k,t) e^{ikx}\, dk,$$

we get

$$u_t + \frac{1}{2}\mathbf{H} |\partial_x|^{2/3} \left(u^2\right) = \omega_0 \mathbf{H}[u].$$

This equation therefore provides a model Hamiltonian equation for constant-frequency bulk waves. We observe that the nonlinear term is nonlocal.

3.3. Surface waves

The three-wave interaction coefficients of constant-frequency surface waves are symmetric, homogeneous functions of degree 3/2. A simple choice for them is

$$V(k_1,k_2,k_3) = (\rho_0\omega_0)^{1/2} (k_1 k_2 k_3)^{1/2}.$$

The corresponding Hamilton's equation is

$$\begin{aligned} i(\operatorname{sgn} k) a_t(k,t) &= \omega_0 a(k,t) \\ &\quad + (\rho_0 \omega_0 |k|)^{1/2} \int_{-\infty}^{\infty} |k-\xi|^{1/2} |\xi|^{1/2} a(k-\xi,t) a(\xi,t)\, d\xi. \end{aligned}$$

Introducing the non-canonical variable

$$u(x,t) = (\rho_0 \omega_0)^{1/2} \int_{-\infty}^{\infty} |k|^{1/2} a(k,t) e^{ikx}\, dk,$$

we get the Burgers-Hilbert equation,

$$u_t + \left(\frac{1}{2}u^2\right)_x = \omega_0 \mathbf{H}[u]. \tag{12}$$

The Hamiltonian form of (12) is

$$u_t = \mathbf{J}\left[\frac{\delta \mathcal{H}}{\delta u}\right], \tag{13}$$

where the constant Hamiltonian operator $\mathbf{J}$ is given by

$$\mathbf{J} = -\partial_x \tag{14}$$

and the Hamiltonian functional $\mathcal{H}$ is

$$\mathcal{H}[u] = \int \left\{ \frac{1}{2}\omega_0 u |\partial_x|^{-1}[u] + \frac{1}{6}u^3 \right\} dx. \tag{15}$$

We may normalize $\omega_0 = 1$ in (12) by the rescaling $t \mapsto \omega_0 t$. By making a spatial reflection $x \mapsto -x$, $u \mapsto -u$, if necessary, which transforms $\omega_0 \mapsto -\omega_0$ in (12), we may ensure that this rescaling preserves the time-direction.

As shown in [2], a similar dimensional analysis of nondispersive hyperbolic waves depending on a wave speed c_0 implies that the bulk-wave interaction coefficients are homogeneous of degree $3/2$, leading to an inviscid Burgers equation

$$u_t + c_0 u_x + \left(\frac{1}{2}u^2\right)_x = 0.$$

This analysis explains why the local inviscid Burgers nonlinearity is appropriate for *surface* waves depending on a frequency ω_0 (for example, nonlinear waves on a vorticity discontinuity), and *bulk* waves depending on a speed c_0 (for example, nonlinear sound waves).

4. THE BURGERS-HILBERT EQUATION

In this section, we use the method of multiple scales to show that weakly nonlinear solutions of (1) satisfy the asymptotic equation (4). Before doing so, we briefly compare (1) with some other nonlinear, nonlocal wave equations that have been studied previously.

The inviscid Burgers equation with a lower-order source term consisting of a spatial convolution with an odd, integrable function $g : \mathbb{R} \to \mathbb{R}$,

$$u_t + \left(\frac{1}{2}u^2\right)_x = g * u, \tag{16}$$

provides a useful model for nonlinear waves with general dispersion relations of the form $\omega = i\hat{g}(k)$ (see §13.14 of Whitham [14], [8, 9, 10], and §3.4.1 of [7], for example). One difference between (16) and (1) is that convolution with an integrable function is a bounded linear operator on L^1, whereas the Hilbert transform is a singular integral operator that is unbounded on L^1. A more significant difference for the problems considered here, however, is that (16) is dispersive, whereas (1) is nondispersive. As a result, the long-time behavior of small-amplitude solutions of (1) is different from that of (16).

Equation (1) also differs from the Benjamin-Ono equation,

$$u_t + \left(\frac{1}{2}u^2\right)_x + \mathbf{H}\left[u\right]_{xx} = 0,$$

which contains a higher-order, nonlocal, linear dispersive term.

The nonlinear, nonlocal equation

$$u_t = u\mathbf{H}[u],$$

introduced in [4] as a one-dimensional model for vortex stretching, includes a Hilbert transform in the nonlinearity, as does the equation

$$u_t = \left(\frac{1}{2}u^2\right)_{xx} + \mathbf{H}\left[u\mathbf{H}\left[u\right]_{xx}\right]$$

introduced in [6] as a model for nonlinear Rayleigh waves, and derived in [1] for surface waves on a tangential discontinuity in magnetohydrodynamics.

4.1. Linearized equation

The linearization of (1) is

$$u_t = \mathbf{H}[u]. \tag{17}$$

For definiteness, we consider solutions $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; with appropriate modifications, similar results apply to spatially periodic solutions $u : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$.

The dispersion relation of (17) is $\omega = \operatorname{sgn} k$, and the general solution is

$$u(x,t) = \int_0^\infty \left[\hat{\psi}(k) e^{ikx-it} + \hat{\psi}^*(k) e^{-ikx+it}\right] dk,$$

where $\hat{\psi} : (0,\infty) \to \mathbb{C}$ is arbitrary. Equivalently, we have

$$u(x,t) = \psi(x) e^{it} + \psi^*(x) e^{-it}$$

where $\psi : \mathbb{R} \to \mathbb{C}$ is given by

$$\psi(x) = \int_0^\infty \hat{\psi}(k) e^{ikx}\, dk.$$

It follows that $\mathbf{P}\psi = \psi$, where $\mathbf{P}$ is the projection onto positive wavenumbers defined in (A.5). Writing

$$\psi(x) = \frac{f(x) + ig(x)}{2}$$

where f, g are real-valued functions, we find that $g = \mathbf{H}[f]$, and

$$u(x,t) = f(x)\cos t + \mathbf{H}[f](x)\sin t.$$

Thus, the spatial profile of the solution of (17) oscillates in time between f and $\mathbf{H}[f]$, where f is an arbitrary function.

Another way to understand this solution is to write $v = \mathbf{H}[u]$ and take the Hilbert transform of (17). Using the fact that $\mathbf{H}^2 = -\mathbf{I}$, we find that

$$u_t = v, \qquad v_t = -u.$$

Thus, the linearized wave-motion consists of simple harmonic oscillators at each point of space. Oscillations at different points are coupled together only by the fact their velocity is the spatial Hilbert transform of their displacement. Although the phase velocity $\omega/k = 1/|k|$ of the waves is nonzero, the group velocity ω' is zero, and the waves do not transport energy.

4.2. Weakly nonlinear waves

Next, we consider weakly nonlinear solutions of (1). Using the method of multiple scales, we look for an asymptotic solution of the the form

$$u(x,t;\varepsilon) = \varepsilon u_1(x,t,\varepsilon^2 t) + \varepsilon^2 u_2(x,t,\varepsilon^2 t) + \varepsilon^3 u_3(x,t,\varepsilon^2 t) + O\left(\varepsilon^4\right). \quad (18)$$

We use (18) in (1), expand the result in power series with respect to ε, and equate coefficients of ε, ε^2, ε^3. We find that $u_1(x,t,\tau)$, $u_2(x,t,\tau)$, $u_3(x,t,\tau)$ satisfy

$$u_{1t} = \mathbf{H}\left[u_1\right], \tag{19}$$

$$u_{2t} + \left(\frac{1}{2}u_1^2\right)_x = \mathbf{H}\left[u_2\right], \tag{20}$$

$$u_{3t} + u_{1\tau} + \left(u_1 u_2\right)_x = \mathbf{H}\left[u_3\right]. \tag{21}$$

The solution of (19) for u_1 is

$$u_1(x,t,\tau) = F(x,\tau)e^{-it} + F^*(x,\tau)e^{it}, \tag{22}$$

where $F(\cdot,\tau) : \mathbb{R} \to \mathbb{C}$ satisfies

$$\mathbf{P}[F] = F. \tag{23}$$

We use (22) in (20) and solve the resulting equation for u_2. We find that

$$u_2(x,t,\tau) = G(x,\tau)e^{-2it} + M(x,\tau) + G^*(x,\tau)e^{2it}, \tag{24}$$

where

$$G = -\left(\frac{1}{2}iF^2\right)_x, \qquad M = -i\mathbf{H}\left[FF^*\right]. \tag{25}$$

We obtain an equation for F from the requirement that (21) is solvable for u_3. To state this requirement, we consider the following equation for $u(x,t)$:

$$u_t = \mathbf{H}[u] + B(x)e^{-int}, \tag{26}$$

where $n = 0, 1, 2, \ldots$ and $B : \mathbb{R} \to \mathbb{C}$.

Proposition 4.1. *If $n^2 \neq 1$, then (26) is uniquely solvable for every $B \in L^2(\mathbb{R})$. If $n = 1$, then (26) is solvable for $B \in L^2(\mathbb{R})$ if and only if*

$$\mathbf{P}[B] = 0 \tag{27}$$

where $\mathbf{P}$ is defined in (A.5).

Proof. A solution of (26) has the form

$$u(x,t) = A(x)e^{-int}$$

where A satisfies

$$inA + \mathbf{H}[A] = -B. \tag{28}$$

Taking the Hilbert transform of this equation, and using the fact that $\mathbf{H}^2 = -\mathbf{I}$, we get

$$A - in\mathbf{H}[A] = \mathbf{H}[B]. \tag{29}$$

If $n^2 \neq 1$, the unique solution of (28)–(29) is

$$A = \frac{inB - \mathbf{H}\left[B\right]}{n^2 - 1}.$$

If $n = 1$, the system (28)–(29) is solvable if and only if $\mathbf{H}[B] = iB$. From (A.7), this means that B satisfies (27). In that case, the solution is

$$A = \frac{1}{2}iB + C$$

where $C(x)$ is an arbitrary function such that $\mathbf{P}\left[C\right] = C$. ∎

We use (22)–(25) in (21) and compute the coefficient of the nonhomogeneous term proportional to e^{-it}. Imposing the solvability condition in Proposition 4.1 and simplifying the result, we find that that (21) is solvable for u_3 if and only if $F(x,\tau)$ satisfies

$$F_\tau = \mathbf{P}\Big[F\mathbf{H}\left[FF^*\right]_x + iFF^*F_x\Big]_x. \tag{30}$$

Equation (30) with $F = \psi$ and $\tau = t$ is equivalent to (4).

5. SURFACE WAVES ON A VORTICITY DISCONTINUITY

In this section, we study surface waves on an interface between two half-spaces with constant vorticities in a two-dimensional, inviscid, incompressible fluid shear flow. We suppose that the unperturbed interface is located at $y = 0$, and denote the vorticities in $y > 0$ and $y < 0$ by $-\alpha_+$ and $-\alpha_-$, respectively, where $\alpha_+ \neq \alpha_-$.

We will derive the following asymptotic solution for the perturbed location $y = \eta(x,t;\varepsilon)$ of the vorticity discontinuity:

$$\eta(x,t;\varepsilon) = \varepsilon\left\{F(x,\varepsilon^2 t)e^{-i\omega_0 t} + F^*(x,\varepsilon^2 t)e^{i\omega_0 t}\right\} + O\left(\varepsilon^2\right) \tag{31}$$

as $\varepsilon \to 0$, where $F(x,\tau)$ satisfies $\mathbf{P}[F] = F$ and

$$F_\tau = \gamma_0\mathbf{P}\left[F\mathbf{H}\left[FF^*\right]_x + iFF^*F_x\right]_x. \tag{32}$$

The frequency parameters ω_0, γ_0 in (31)–(32) are given by

$$\omega_0 = \frac{\alpha_+ - \alpha_-}{2}, \qquad \gamma_0 = \frac{\alpha_+^2 + \alpha_-^2}{\alpha_+ - \alpha_-}. \tag{33}$$

Equation (32) with $F = \psi$, and $\gamma_0 \tau = t$ is equivalent to the asymptotic equation (4).

After accounting for the coefficients, we find that (32) is identical to the asymptotic equation derived in Section 4 for the Burgers-Hilbert equation

$$\eta_t + \left(\frac{1}{2}\beta_0 \eta^2\right)_x = \omega_0 \mathbf{H}[\eta] \tag{34}$$

where

$$\beta_0^2 = \frac{\alpha_+^2 + \alpha_-^2}{2}.$$

We note that either choice of sign for β_0 in (34) leads to the same asymptotic equation (32). This is because the transformation $\eta \mapsto -\eta$ corresponds to a half-period phase shift in the linearized oscillations, which does not affect the averaged, long-time dynamics.

Since (34) leads to the same asymptotic equation as the one derived from the primitive fluid equations, it provides an effective equation for the motion of a planar vorticity discontinuity with slope of the order ε over times of the order $\omega_0^{-1}\varepsilon^{-2}$. The coefficient β_0 in (34) is not equal to the coefficient α_0 of the quadratic nonlinearity in the equations of motion for a vorticity discontinuity, which, from (43) below, is given by

$$\alpha_0 = \frac{\alpha_+ + \alpha_-}{2}. \tag{35}$$

Instead, β_0 describes the combined effect of both quadratic and cubic nonlinearities. For example, if the shear flow is symmetric and $\alpha_+ = -\alpha_-$, then $\beta_0 = \alpha_+$, even though $\alpha_0 = 0$; while if the flow in the lower-half space is irrotational and $\alpha_- = 0$, then $\beta_0 = \alpha_+/\sqrt{2}$ and $\alpha_0 = \alpha_+/2$.

In the remainder of this section, we derive (32).

5.1. Formulation of the problem

First, we formulate equations that describe surface waves on a vorticity discontinuity. The equations are summarized in (41) below.

Consider a velocity perturbation (u, v) of an unperturbed shear flow $(\alpha y, 0)$, where (x, y) are spatial coordinates and α is a constant shear rate. The two-dimensional incompressible Euler equations for (u, v) and the pressure p are

$$\begin{aligned} &u_t + (\alpha y + u)\, u_x + v\,(\alpha + u_y) + p_x = 0,\\ &v_t + (\alpha y + u)\, v_x + v v_y + p_y = 0,\\ &u_x + v_y = 0. \end{aligned}$$

Since the vorticity is advected by the perturbed flow, and the unperturbed vorticity is constant (equal to $-\alpha$), it is consistent to assume that the flow perturbations are irrotational.

We therefore introduce a velocity potential $\varphi(x, y, t)$ such that

$$u = \varphi_x, \qquad v = \varphi_y$$

and a streamfunction $\psi(x, y, t)$ such that

$$u = \psi_y, \qquad v = -\psi_x.$$

Then the incompressibility condition implies that

$$\Delta\varphi = 0, \tag{36}$$

and an integration of the momentum equations gives

$$\varphi_t + \alpha\,(y\varphi_x - \psi) + \frac{1}{2}\,|\nabla\varphi|^2 + p = 0. \tag{37}$$

Next, we consider perturbations of a shear flow given by $(\alpha_+ y, 0)$ in $y > 0$ and $(\alpha_- y, 0)$ in $y < 0$, where the shear rates $\alpha_\pm$ are distinct constants. The unperturbed flow has a jump in vorticity across $y = 0$. We assume that the flow perturbations are irrotational, and write the perturbed location of the interface where the vorticity jumps as

$$y = \eta(x, t).$$

We assume that the interface is a graph, and do not attempt to continue the solution past any time where the interface 'breaks'. We use the notation

$$\alpha = \begin{cases} \alpha_+ & \text{if } y > \eta(x,t),\\ \alpha_- & \text{if } y < \eta(x,t), \end{cases}$$

with similar notation for other quantities that jump across the interface.

The kinematic boundary condition states that the interface moves with the fluid, meaning that

$$\eta_t + (\alpha\eta + \varphi_x)\,\eta_x - \varphi_y = 0 \qquad \text{on } y = \eta(x,t)^{\pm}. \tag{38}$$

The dynamic boundary condition states that the pressure is continuous across the interface, meaning that

$$[p] = 0 \tag{39}$$

where

$$[f]\,(x,t) = f\left(x, \eta(x,t)^{+}, t\right) - f\left(x, \eta(x,t)^{-}, t\right)$$

denotes the jump in a quantity $f(x,y,t)$ across $y = \eta(x,t)$. Using (37) in (39), we get

$$\left[\varphi_t + \alpha\left(y\varphi_x - \psi\right) + \frac{1}{2}\left|\nabla\varphi\right|^2\right] = 0. \tag{40}$$

Finally, we require that the flow perturbation and the pressure decay to zero away from the interface. This condition, together with (36), (38), (40), gives

$$\begin{aligned}
&\Delta\varphi = 0 && \text{in } y > \eta(x,t),\ y < \eta(x,t),\\
&\eta_t + \left(\alpha\eta + \varphi_x\right)\eta_x - \varphi_y = 0 && \text{on } y = \eta(x,t)^{\pm},\\
&\left[\varphi_t + \alpha\left(y\varphi_x - \psi\right) + \frac{1}{2}\left|\nabla\varphi\right|^2\right] = 0, &&\\
&\varphi(x,y,t) \to 0 && \text{as } y \to \pm\infty,
\end{aligned} \tag{41}$$

where ψ is the harmonic conjugate of φ such that $\psi \to 0$ as $y \to \pm\infty$.

5.2. Linearized equations

Linearizing (41) around the unperturbed solution $\varphi = 0$, $\eta = 0$, we get

$$\begin{aligned}
&\Delta\varphi = 0 && \text{in } y > 0 \text{ and } y < 0,\\
&\eta_t - \varphi_y = 0 && \text{on } y = 0^{\pm},\\
&\left[\varphi_t - \alpha\psi\right] = 0, &&\\
&\varphi(x,y,t) \to 0 && \text{as } |y| \to \infty,
\end{aligned}$$

where $[\cdot]$ now denotes a jump across $y = 0$, and

$$\alpha = \begin{cases} \alpha_+ & \text{if } y > 0,\\ \alpha_- & \text{if } y < 0. \end{cases}$$

Taking the jump of the kinematic boundary condition $\eta_t - \varphi_y = 0$ across $y = 0$, we get

$$[\varphi_y] = 0.$$

The Fourier solutions of $\Delta\varphi = 0$ that decay as $y \to \pm\infty$ are

$$\varphi(x, y, t) = \begin{cases} \hat{A}_+(k)e^{ikx-|k|y-i\omega t} & \text{in } y > 0, \\ \hat{A}_-(k)e^{ikx+|k|y-i\omega t} & \text{in } y < 0, \end{cases}$$

where $k \in \mathbb{R}$. We write these solutions as

$$\varphi(x, y, t) = \hat{A}(k)e^{ikx-\sigma|k|y-i\omega t},$$

where $\hat{A} = \hat{A}_\pm$, and

$$\sigma = \begin{cases} +1 & \text{if } y > 0, \\ -1 & \text{if } y < 0. \end{cases}$$

The corresponding streamfunction and interface displacement are

$$\begin{aligned} \psi(x, y, t) &= \hat{B}(k)e^{ikx-\sigma|k|y-i\omega t}, \\ \eta(x, t) &= \hat{F}(k)e^{ikx-i\omega t}, \end{aligned}$$

where

$$\hat{A}(k) = i\sigma \operatorname{sgn}(k)\hat{B}(k), \qquad \hat{F}(k) = \frac{k}{\omega}\hat{B}(k). \tag{42}$$

Using these solutions in the jump conditions and eliminating $\hat{A}$, we find that

$$\left[\{\sigma\omega \operatorname{sgn}(k) - \alpha\}\, \hat{B}\right] = 0, \qquad \left[\hat{B}\right] = 0.$$

It follows that $\hat{B}$ is continuous across $y = 0$, and

$$\omega = \omega_0 \operatorname{sgn}(k),$$

where the frequency ω_0 is given in (33).

Superposing Fourier solutions, we get the linearized solution with general spatial dependence,

$$\begin{aligned} \varphi(x, y, t) &= A(x, y)e^{-i\omega_0 t} + A^*(x, y)e^{i\omega_0 t} \\ \psi(x, y, t) &= B(x, y)e^{-i\omega_0 t} + B^*(x, y)e^{i\omega_0 t}, \\ \eta(x, t) &= F(x)e^{-i\omega_0 t} + F^*(x)e^{i\omega_0 t}. \end{aligned}$$

Here, the complex amplitudes A, B, F consist of positive wavenumbers,

$$\begin{aligned} A(x, y) &= \int_0^\infty \hat{A}(k)e^{ikx-\sigma ky}\, dk, \\ B(x, y) &= \int_0^\infty \hat{B}(k)e^{ikx-\sigma ky}\, dk, \\ F(x) &= \int_0^\infty \hat{F}(k)e^{ikx}\, dk, \end{aligned}$$

and, from (42),

$$A(x,y) = i\,\mathrm{sgn}(y)B(x,y), \qquad F(x) = \frac{i}{\omega_0}B_x(x,0).$$

We remark that the linearized tangential fluid velocity U on the interface is the sum of the x-velocity components of the unperturbed shear flow and the flow perturbation, so

$$U = \alpha\eta + \varphi_x|_{y=0}\,.$$

We compute from the linearized solution that U is continuous across the interface, as it must be, and is given by

$$U(x,t) = \alpha_0\eta(x,t) \tag{43}$$

where α_0 is defined in (35). Thus, since $\eta(x,t)$ has zero mean with respect to t, the time-averaged translation of the interface in the x-direction is zero according to the linearized theory. In the symmetric case $\alpha_0 = 0$, the linearized x-velocity is identically zero. In the general case, the advection of the interface in one direction for positive displacements is canceled by its advection in the opposite direction for negative displacements. There is, however, a nonlinear Stokes drift of the interface that is second-order in the amplitude.

For use in the asymptotic expansion, we state a solvability condition for the nonhomogeneous linearized problem,

$$\begin{aligned}
&\Delta\varphi = 0 && \text{in } y > 0 \text{ and } y < 0,\\
&[\varphi_y] = f(x)e^{-in\omega_0 t} && \text{on } y = 0,\\
&[\varphi_t - \alpha\psi] = g(x)e^{-in\omega_0 t} && \text{on } y = 0,\\
&\varphi(x,y,t) \to 0 && \text{as } |y| \to \infty.
\end{aligned} \tag{44}$$

PROPOSITION 5.1. *If $n^2 \neq 1$, then (44) is solvable for φ for any functions $f, g \in L^2(\mathbb{R})$. If $n = 1$, then (44) equation is solvable if and only if*

$$\mathbf{P}\left[\alpha_0 f + g\right] = 0, \tag{45}$$

where α_0 is defined in (35) and $\mathbf{P}$ is defined in (A.5).

Proof. We write $\varphi(x,y,t) = \Phi(x,y)e^{-in\omega_0 t}$, take the Fourier transform of (44) with respect to x, and solve the resulting equations. The details are omitted. ▌

5.3. Weakly nonlinear waves

We look for an asymptotic solution of (41), depending on a small parameter ε, of the form

$$\begin{aligned} \varphi^\varepsilon(x,y,t) &= \varepsilon\varphi_1(x,y,\varepsilon^2 t,t) + \varepsilon^2\varphi_2(x,y,\varepsilon^2 t,t) + \varepsilon^3\varphi_3(x,y,\varepsilon^2 t,t) + \dots, \\ \psi^\varepsilon(x,y,t) &= \varepsilon\psi_1(x,y,\varepsilon^2 t,t) + \varepsilon^2\psi_2(x,y,\varepsilon^2 t,t) + \varepsilon^3\psi_3(x,y,\varepsilon^2 t,t) + \dots, \\ \eta^\varepsilon(x,t) &= \varepsilon\eta_1(x,\varepsilon^2 t,t) + \varepsilon^2\eta_2(x,\varepsilon^2 t,t) + \varepsilon^3\eta_3(x,\varepsilon^2 t,t) + \dots. \end{aligned}$$

The velocity potentials φ_n and streamfunctions ψ_n satisfy

$$\Delta\varphi_n = 0, \qquad \psi_{ny} = \varphi_{nx}, \quad \psi_{nx} = -\varphi_{ny}.$$

Expansion and simplification of the boundary conditions yields the following perturbation equations: at the order ε,

$$\begin{aligned} &\eta_{1t} - \varphi_{1y} = 0, \\ &[\varphi_{1t} - \alpha\psi_1] = 0; \end{aligned}$$

at the order ε^2,

$$\begin{aligned} &\eta_{2t} - \varphi_{2y} + \alpha\eta_1\eta_{1x} + \eta_{1x}\varphi_{1x} - \eta_1\varphi_{1yy} = 0, \\ &\left[\varphi_{2t} - \alpha\psi_2 + \eta_1\varphi_{1ty} + \frac{1}{2}\left|\nabla\varphi_1\right|^2\right] = 0; \end{aligned}$$

and, at the order ε^3,

$$\begin{aligned} &\eta_{3t} - \varphi_{3y} + \eta_{1\tau} + \alpha\left(\eta_1\eta_{2x} + \eta_2\eta_{1x}\right) + \eta_{1x}\varphi_{2x} + \eta_{2x}\varphi_{1x} + \eta_1\eta_{1x}\varphi_{1xy} \\ &\qquad - \eta_1\varphi_{2yy} - \eta_2\varphi_{1yy} - \frac{1}{2}\eta_1^2\varphi_{1yyy} = 0, \\ &\Big[\varphi_{3t} - \alpha\psi_3 + \varphi_{1\tau} + \eta_2\varphi_{1ty} + \eta_1\varphi_{2ty} + \frac{1}{2}\eta_1^2\varphi_{1tyy} \\ &\qquad - \frac{1}{2}\alpha\eta_1^2\psi_{1xx} + \nabla\varphi_1\cdot\nabla\varphi_2 + \eta_1\nabla\varphi_1\cdot\nabla\varphi_{1y}\Big] = 0. \end{aligned}$$

Here, all boundary conditions and jumps are evaluated at $y = 0$.

The solution of the first-order equations is

$$\begin{aligned} \varphi_1(x,y,\tau,t) &= A(x,y,\tau)e^{-i\omega_0 t} + A^*(x,y,\tau)e^{i\omega_0 t}, \\ \psi_1(x,y,\tau,t) &= B(x,y,\tau)e^{-i\omega_0 t} + B^*(x,y,\tau)e^{i\omega_0 t}, \\ \eta_1(x,\tau,t) &= F(x,\tau)e^{-i\omega_0 t} + F^*(x,\tau)e^{i\omega_0 t}, \end{aligned}$$

where $B(x,y,\tau)$ is continuous across $y = 0$ with $\mathbf{P}[B(x,0,\tau)] = 0$, and

$$A(x,y,\tau) = i\sigma B(x,y,\tau), \qquad F(x,\tau) = -\frac{i}{\omega_0}B_x(x,0,\tau).$$

Using these expressions in the second-order equations and solving the result, we get

$$\begin{aligned}
\varphi_2(x,y,\tau,t) &= C(x,y,\tau)e^{-2i\omega_0 t} + M(x,y,\tau) + A^*(x,y,\tau)e^{2i\omega_0 t},\\
\psi_2(x,y,\tau,t) &= D(x,y,\tau)e^{-2i\omega_0 t} + N(x,y,\tau) + D^*(x,y,\tau)e^{2i\omega_0 t},\\
\eta_2(x,\tau,t) &= G(x,\tau)e^{-2i\omega_0 t} + G^*(x,y,\tau)e^{2i\omega_0 t},
\end{aligned}$$

where

$$\begin{aligned}
D(x,y,\tau) &= -\frac{\alpha}{2\omega_0^2}B_x^2(x,y,\tau),\\
N(x,y,t) &= -\frac{\beta}{\omega_0^2}B_x(x,y,\tau)B_x^*(x,y,\tau)\\
G(x,\tau) &= \frac{i\alpha_0}{2\omega_0^3}\left\{B_x^2(x,0,\tau)\right\}_x,
\end{aligned}$$

with

$$\beta = \begin{cases} \alpha_- & \text{if } y>0,\\ \alpha_+ & \text{if } y<0,\end{cases} \qquad C = i\sigma D, \qquad M_x = N_y, \quad M_y = -N_x.$$

We use these expression in the third-order equations, collect terms proportional to $e^{-i\omega_0 t}$, and impose the solvability condition (45). After writing this condition in terms of F and simplifying the result, we find that $F(x,\tau)$ satisfies (32).

6. THE ASYMPTOTIC EQUATION

In this section, we describe the formal Hamiltonian structure of the asymptotic equation (4), and derive equivalent spectral and real forms.

6.1. Hamiltonian structure

We denote by $\psi, \psi^* : \mathbb{R} \to \mathbb{C}$ complex-conjugate functions that satisfy the constraints

$$\mathbf{P}[\psi] = \psi, \qquad \mathbf{Q}[\psi^*] = \psi^*, \tag{46}$$

where $\mathbf{P}$, $\mathbf{Q}$ are the projections onto positive, negative wavenumber components defined in (A.5), (A.6), respectively. The constraints may also be written as $\mathbf{Q}[\psi] = 0$, $\mathbf{P}[\psi^*] = 0$.

If $\mathcal{H}(\psi,\psi^*)$ is a functional of (ψ,ψ^*), then the functional derivative of $\mathcal{H}$ with respect to ψ^* is given by

$$\frac{\delta\mathcal{H}}{\delta\psi^*} = \mathbf{P}[h]$$

where $h : \mathbb{R} \to \mathbb{C}$ is any function such that

$$\frac{d}{d\epsilon}\mathcal{H}\left(\psi, \psi^* + \epsilon\varphi^*\right)\Big|_{\epsilon=0} = \int h\varphi^*\, dx$$

for all $\varphi^* : \mathbb{R} \to \mathbb{C}$ with $\mathbf{Q}[\varphi^*] = \varphi^*$. Hamilton's equation is

$$\psi_t = \mathbf{J}\left[\frac{\delta\mathcal{H}}{\delta\psi^*}\right] \tag{47}$$

where $\mathbf{J} = -\partial_x$ is the Hamiltonian operator in (14).

Proposition 6.1. *Equation (4) has the Hamiltonian form (47) where $\mathcal{H}$ is given by*

$$\mathcal{H}\left(\psi, \psi^*\right) = \int \left\{ \frac{i}{4}\psi\psi^* \left(\psi\psi_x^* - \psi^*\psi_x\right) - \frac{1}{2}\psi\psi^* \left|\partial_x\right| \left[\psi\psi^*\right] \right\} dx. \tag{48}$$

Proof. Computing the functional derivative of (48) with respect to ψ^* and using the result in (47), we get (4). ▮

6.2. Spectral form

Equation (4) has a particularly simple spectral form (5). This equation describes the evolution of a nonlinear wave due to four-wave resonant interactions (10) with the interaction coefficient Λ in (6).

Proposition 6.2. *The function $\psi(x,t)$ satisfies (3)–(4) if and only if its Fourier transform $\hat{\psi}(k,t)$ is equal to zero for $k \leq 0$ and satisfies (5) for $k > 0$ where Λ is defined in (6).*

Proof. Suppressing an explicit indication of the t-variable, we have

$$\psi(x) = \int_0^\infty \hat{\psi}(k)e^{ikx}\, dx, \qquad \psi^*(x) = \int_0^\infty \hat{\psi}^*(k)e^{-ikx}\, dx.$$

Using these expressions in (48), and evaluating the x-integral, we get

$$\begin{aligned}
\mathcal{H}(\hat{\psi}, \hat{\psi}^*) &= \int \left\{ \frac{1}{4}\left(k_2 + k_4\right) - \frac{1}{2}\left|k_2 - k_4\right| \right\} \hat{\psi}(k_1)\hat{\psi}(k_2)\hat{\psi}^*(k_3)\hat{\psi}^*(k_4) \\
&\qquad e^{i(k_1+k_2-k_3-k_4)x}\, dx dk_1 dk_2 dk_3 dk_4 \\
&= 2\pi \int \left\{ \frac{1}{4}\left(k_2 + k_4\right) - \frac{1}{2}\left|k_2 - k_4\right| \right\} \hat{\psi}(k_1)\hat{\psi}(k_2)\hat{\psi}^*(k_3)\hat{\psi}^*(k_4) \\
&\qquad \delta\left(k_1 + k_2 - k_3 - k_4\right)\, dk_1 dk_2 dk_3 dk_4.
\end{aligned}$$

The integrals here are taken over the range $k_j > 0$. Symmetrizing this expression, we get

$$\mathcal{H} = \pi \int \delta\left(k_1 + k_2 - k_3 - k_4\right) \Lambda\left(k_1, k_2, k_3, k_4\right) \hat{\psi}(k_1)\hat{\psi}(k_2)\hat{\psi}^*(k_3)\hat{\psi}^*(k_4)\, dk_1 dk_2 dk_3 dk_4$$

where $k_1, k_2, k_3, k_4 > 0$ and

$$\Lambda(k_1, k_2, k_3, k_4) = \frac{1}{2}\Big\{k_1 + k_2 + k_3 + k_4 - \big(|k_1 - k_3| + |k_1 - k_4| + |k_2 - k_3| + |k_2 - k_4|\big)\Big\}. \tag{49}$$

Only the values of $\Lambda(k_1, k_2, k_3, k_4)$ on $k_1 + k_2 = k_3 + k_4$ are relevant, and by symmetry we may suppose $k_1 \geq k_3 \geq k_4 \geq k_2$. In that case,

$$\Lambda(k_1, k_2, k_3, k_4) = \frac{1}{2}\left(3k_2 - k_1 + k_3 + k_4\right) = 2k_2.$$

Hence, we see that Λ is given by (6).

Writing (47) in terms of $\hat{\psi}$, we get

$$\hat{\psi}_t = -ik \frac{1}{2\pi} \frac{\delta \mathcal{H}}{\delta \hat{\psi}^*}.$$

We therefore get the spectral form of the asymptotic equation given in (5). One can also verify this result directly by taking the Fourier transform of (4). ▮

6.3. Real form

Next, we write the complex equation (4) for ψ in an equivalent real form. We define

$$v(x,t) = \psi(x,t)e^{-it} + \psi^*(x,t)e^{it}. \tag{50}$$

The function v corresponds to the leading-order approximation for u in the expansion (18), and to the leading order approximation for the displacement η of a vorticity discontinuity in (31).

Proposition 6.3. *A complex-valued function $\psi(x,t)$ such that $\mathbf{P}[\psi] = \psi$ satisfies (4) if and only if the real-valued function $v(x,t)$ in (50) satisfies*

$$v_t + \partial_x\left\{\frac{1}{6}\left|\partial_x\right|\left[v^3\right] - \frac{1}{2} v \left|\partial_x\right|\left[v^2\right] + \frac{1}{2} v^2 \left|\partial_x\right|\left[v\right]\right\} = \mathbf{H}[v]. \tag{51}$$

Proof. Since $\mathbf{P}[\psi] = \psi$, we have from (50) that

$$\psi e^{-it} = \mathbf{P}[v], \qquad \psi^* e^{it} = \mathbf{Q}[v], \tag{52}$$

where the projections $\mathbf{P}$, $\mathbf{Q}$ are defined in (A.5), (A.6).

Differentiating (50) with respect to t, we get

$$v_t = (\psi_t - i\psi)\, e^{-it} + (\psi_t^* + i\psi^*)\, e^{it}. \tag{53}$$

Defining

$$n = \psi\psi^*, \tag{54}$$

and using (A.7), we may write (4) as

$$\psi_t = \frac{1}{2}\,(\mathbf{I} + i\mathbf{H})\left[\psi \mathbf{H}[n]_x + in\psi_x\right]. \tag{55}$$

We use (55) in (53), use (52) to eliminate ψ, ψ^* in terms of v, and simplify the result. We find that

$$v_t = \mathbf{H}[v] + \frac{1}{2}\partial_x\left\{v\mathbf{H}[n]_x - \mathbf{H}[nv_x] - n\mathbf{H}[v]_x - \mathbf{H}\Big[\mathbf{H}[v]\mathbf{H}[n]_x\Big]\right\}. \tag{56}$$

Using the convolution identity (A.3) with $w = n_x$ in (56) we get

$$v_t = \mathbf{H}[v] + \frac{1}{2}\partial_x\left\{2v\mathbf{H}[n]_x - \mathbf{H}[nv]_x + n_x\mathbf{H}[v] - n\mathbf{H}[v]_x\right\}. \tag{57}$$

It follows from (52), (54), (A.3), and (A.7) that

$$n = \frac{1}{4}\left(v^2 + \mathbf{H}[v]^2\right), \qquad \mathbf{H}[n] = \frac{1}{2}\left(\mathbf{H}\left[v^2\right] - v\mathbf{H}[v]\right).$$

Using these expressions in (57) and simplifying the result, we get

$$\begin{aligned} v_t \;&=\; \mathbf{H}[v] - \frac{1}{8}\partial_x^2\left\{\mathbf{H}\left[v^3 + v\mathbf{H}[v]^2\right] + v^2\mathbf{H}[v] - \frac{1}{3}\mathbf{H}[v]^3\right\} \\ &\qquad + \frac{1}{2}\partial_x\left\{v\mathbf{H}\left[v^2\right]_x - v^2\mathbf{H}[v]_x\right\}. \end{aligned} \tag{58}$$

Using the identity (A.4) in (58), writing $\mathbf{H}\partial_x = |\partial_x|$, and simplifying the result, we get (51).

Conversely, given a real-valued function $v(x,t)$ that satisfies (51), we define a complex-valued function $\psi(x,t) = e^{it}\mathbf{P}v(x,t)$. Then v satisfies (50) and $\mathbf{P}\psi = \psi$. Reversing the steps above, we find that ψ satisfies (4). ▮

The Hamiltonian form of (51) is

$$v_t = \mathbf{J}\left[\frac{\delta\mathcal{K}}{\delta v}\right] \tag{59}$$

where $\mathbf{J}$ is given in (14) and

$$\mathcal{K}(v) = \int \left\{ \frac{1}{2} v|\partial_x|^{-1}[v] + \frac{1}{6} v^3 |\partial_x| [v] - \frac{1}{8} v^2 |\partial_x| \left[v^2\right] \right\} dx. \tag{60}$$

6.4. Symmetries of the equation

The Hamiltonian in (48) has the following obvious symmetries and associated conserved quantities.

1. Time translation invariance, $\psi(x,t) \mapsto \psi(x, t+\epsilon)$, generated by the Hamiltonian $\mathcal{H}$.

2. Space translation invariance, $\psi(x,t) \mapsto \psi(x+\epsilon, t)$, generated by the momentum

$$\mathcal{P} = \int \psi\psi^* \, dx.$$

3. Phase translation invariance, $\psi(x,t) \mapsto e^{-i\epsilon}\psi(x,t)$, generated by the action

$$\mathcal{S} = \int \psi^* \left|\partial_x\right|^{-1} \psi \, dx.$$

We may check directly that these quantities are conserved. For the momentum, using (4), we find after some computations that

$$\begin{aligned} \partial_t n \;=\; & \partial_x \big\{ 2n \left|\partial_x\right| n + in\left(\psi^* \partial_x \psi - \psi \partial_x \psi^*\right) - n \left|\partial_x\right| n \big\} \\ & -\psi \mathbf{P} \partial_x \left(\psi^* \left|\partial_x\right| n - in\partial_x \psi^*\right) - \psi^* \mathbf{Q} \partial_x \left(\psi \left|\partial_x\right| n + in\partial_x \psi\right) \\ & +n\partial_x \left|\partial_x\right| n. \end{aligned}$$

Assuming that the appropriate boundary terms vanish, the spatial integral of the right-hand side of this equation is zero, since $\partial_x \left|\partial_x\right|$ is skew-symmetric and, from (46) and (A.9),

$$\int \psi \mathbf{P}[f] \, dx = \int \mathbf{Q}[\psi] f \, dx = 0, \qquad \int \psi^* \mathbf{Q}[f] \, dx = \int \mathbf{P}[\psi^*] f \, dx = 0$$

for an arbitrary function f. Thus, the momentum $\mathcal{P}$ is conserved.

For the action, we write

$$\psi\left|\partial_x\right|^{-1}\psi^* + \psi^*\left|\partial_x\right|^{-1}\psi = i\psi^*\partial_x^{-1}\psi - \psi\partial_x^{-1}\psi^*,$$

and compute that

$$\begin{aligned}\partial_t\left(\psi^*\partial_x^{-1}\psi - \psi\partial_x^{-1}\psi^*\right) &= 2in\partial_x n + \partial_x\left\{\left(\psi^*\partial_x^{-1}\psi - \psi\partial_x^{-1}\psi^*\right)\left|\partial_x\right| n\right.\\ &\quad \left. - in\left[\left(\partial_x^{-1}\psi\right)\left(\partial_x\psi^*\right) + \left(\partial_x\psi\right)\left(\partial_x^{-1}\psi^*\right)\right]\right\}\\ &\quad + \psi\mathbf{P}\left(\psi^*\left|\partial_x\right| n - in\partial_x\psi^*\right)\\ &\quad - \psi^*\mathbf{Q}\left(\psi\left|\partial_x\right| n + in\partial_x\psi\right)\\ &\quad - \left(\partial_x^{-1}\psi\right)\mathbf{P}\partial_x\left(\psi^*\left|\partial_x\right| n - in\partial_x\psi^*\right)\\ &\quad + \left(\partial_x^{-1}\psi^*\right)\mathbf{Q}\partial_x\left(\psi\left|\partial_x\right| n + in\partial_x\psi\right).\end{aligned}$$

The right-hand side integrates to zero as before, and we conclude that $\mathcal{S}$ is conserved.

Equation (47) also has a less obvious translational symmetry, which is stated in the following proposition.

Proposition 6.4. *If $\psi_0 \in \mathbb{C}$, then*

$$\mathcal{H}\left[\psi_0 + \psi\right] = \mathcal{H}\left[\psi\right].$$

Moreover, if $\psi(x,t)$ is a solution of (4), then so is

$$\tilde{\psi}(x,t) = \psi_0 + \psi(x,t).$$

Proof. Let

$$\psi = \psi_0 + \tilde{\psi}, \qquad n = n_0 + \psi_0^*\tilde{\psi} + \psi_0\tilde{\psi}^* + \tilde{n},$$

where

$$n_0 = \psi_0\psi_0^*, \qquad \tilde{n} = \tilde{\psi}\tilde{\psi}^*.$$

Then we find that

$$\begin{aligned}\psi\left|\partial_x\right| n + in\partial_x\psi &= \psi_0^2\left|\partial_x\right|\psi^* + n_0\left[\left|\partial_x\right|\tilde{\psi} + i\partial_x\tilde{\psi}\right] + \psi_0^*\left[\left|\partial_x\right|\tilde{\psi} + i\partial_x\tilde{\psi}\right]\\ &\quad + \psi_0\left[\left|\partial_x\right|\tilde{n} + i\tilde{\psi}^*\partial_x\tilde{\psi} + \tilde{\psi}\left|\partial_x\right|\tilde{\psi}^*\right]\\ &\quad + \tilde{\psi}\left|\partial_x\right|\tilde{n} + i\tilde{n}\partial_x\tilde{\psi}.\end{aligned}$$

The term proportional to ψ^* projects to zero under $\mathbf{P}$. Moreover, since $\mathbf{Q}\tilde{\psi} = 0$, we have $|\partial_x|\,\tilde{\psi} + i\partial_x\tilde{\psi} = 0$, and

$$|\partial_x|\,\tilde{n} + i\tilde{\psi}^*\partial_x\tilde{\psi} + \tilde{\psi}\,|\partial_x|\,\tilde{\psi}^* = |\partial_x|\,\tilde{n} + i\partial_x\tilde{n}.$$

This term also projects to zero, so that

$$\mathbf{P}\partial_x\left(\psi\,|\partial_x|\,n + in\partial_x\psi\right) = \mathbf{P}\partial_x\left(\tilde{\psi}\,|\partial_x|\,\tilde{n} + i\tilde{n}\partial_x\tilde{\psi}\right).$$

Thus, $\tilde{\psi}$ satisfies (4) if and only if ψ does. ▌

Similarly, the real Hamiltonian $\mathcal{K}(v)$ in (60) is invariant under translations $v(x,t) \mapsto v_0 + v(x,t)$.

The symmetry in Proposition 6.4 depends critically upon a cancelation in the Hamiltonian (48); it does not hold for Hamiltonians whose densities involve different linear combinations of the terms $i\psi\psi^*(\psi\psi_x^* - \psi^*\psi_x)$ and $\psi\psi^*\,|\partial_x|\,[\psi\psi^*]$.

6.5. The initial value problem

Using $|\partial_x|\,\psi = -i\partial_x\psi$, we write the initial-value problem for (4) as

$$\psi_t = \mathbf{P}\partial_x\Big[\psi\,|\partial_x|\,n + in\partial_x\psi\Big] \tag{61}$$

$$\psi(x,0) = \psi_0(x). \tag{62}$$

We require that the initial data ψ_0 satisfies the constraint

$$\mathbf{P}[\psi_0] = \psi_0. \tag{63}$$

Acting on (61) by $\mathbf{Q}$, and using the fact that $\mathbf{QP} = 0$, we get $\partial_t\mathbf{Q}[\psi] = 0$. Thus, if the initial data ψ_0 satisfies (63), then a smooth solution $\psi(\cdot,t)$ satisfies the same constraint for all times t.

Energy methods imply the short-time existence of smooth solutions of (61)–(63), but we will not give the analysis here.

We may interpret (61) as analogous to the Leray projection of the incompressible Euler equations, with $\mathbf{P}$ corresponding to the orthogonal projection onto divergence-free fields and $\mathbf{Q}$ corresponding to the orthogonal projection onto gradients. Introducing a complex-valued constraint function $\pi(x,t)$, analogous to the pressure gradient, we may write (61)–(62) as

$$\begin{aligned}
&\psi_t = \partial_x\Big[\psi\,|\partial_x|\,n + in\partial_x\psi\Big] + \pi,\\
&\mathbf{P}[\psi] = \psi, \qquad \mathbf{Q}[\pi] = \pi,\\
&\psi(x,0) = \psi_0(x)
\end{aligned}$$

where ψ_0 satisfies (63). It follows from these equations that ψ satisfies (61), and π is given in terms of ψ by

$$\pi = -\mathbf{Q}\partial_x\Big[\psi\,|\partial_x|\,n + in\partial_x\psi\Big].$$

7. NEAR-IDENTITY TRANSFORMATION

In this section, we derive the real form (51) of the asymptotic equation (4) from the Burgers-Hilbert equation (1) by use of a near-identity transformation. This transformation removes the quadratically nonlinear terms from (1), which is possible because they are nonresonant, leading to the cubically nonlinear terms in (51).

We denote the Poisson bracket associated with $\mathbf{J} = -\partial_x$ in (14) by

$$\{\mathcal{F}, \mathcal{G}\} = \int \frac{\delta\mathcal{F}}{\delta u}\mathbf{J}\left[\frac{\delta\mathcal{G}}{\delta u}\right]\,dx.$$

Introducing an amplitude parameter ε, and normalizing $\omega_0 = 1$, we write the Hamiltonian (15) for the Burgers-Hilbert equation (12) as

$$\mathcal{H} = \mathcal{H}_2 + \varepsilon\mathcal{H}_3,$$

where $\mathcal{H}_j$ is homogeneous of degree j in u, and

$$\mathcal{H}_2(u) = \int \frac{1}{2}u|\partial_x|^{-1}[u]\,dx, \qquad \mathcal{H}_3(u) = \int \frac{1}{6}u^3\,dx. \tag{64}$$

We define a symplectic, near-identity transformation $v \mapsto u(\varepsilon)$ by

$$u_\varepsilon = \mathbf{J}\left[\frac{\delta\mathcal{F}}{\delta u}\right], \qquad u(0) = v,$$

where

$$\mathcal{F} = \mathcal{F}_3 + \varepsilon\mathcal{F}_4,$$

with $\mathcal{F}_j$ homogeneous of degree j in u.

Then, since

$$\mathcal{H}_\varepsilon = \{\mathcal{H}, \mathcal{F}\},$$

a Taylor expansion with respect to ε gives

$$\begin{aligned}\mathcal{H}\left[u(\varepsilon)\right] \;=\;& \mathcal{H}_2(v) + \varepsilon\Big(\mathcal{H}_3 + \{\mathcal{H}_2, \mathcal{F}_3\}\Big)(v) \\ &+\varepsilon^2\left(\{\mathcal{H}_2, \mathcal{F}_4\} + \{\mathcal{H}_3, \mathcal{F}_3\} + \frac{1}{2}\left\{\{\mathcal{H}_2, \mathcal{F}_3\}, \mathcal{F}_3\right\}\right)(v) + O(\varepsilon^3).\end{aligned}$$

To eliminate the cubically nonlinear terms from $\mathcal{H}(u)$, we choose

$$\mathcal{F}_3(v) = -\frac{1}{6}\int \mathbf{H}[v]^3\,dx.$$

Then, using (A.3) and the fact that the Hilbert transform is a skew-adjoint isometry on L^2, we compute that

$$\{\mathcal{H}_2, \mathcal{F}_3\} = -\mathcal{H}_3,$$

and

$$\mathcal{H}\left(u(\varepsilon)\right) = \mathcal{H}_2(v) + \varepsilon^2 \mathcal{K}_4(v) + O\left(\varepsilon^3\right), \tag{65}$$

where

$$\mathcal{K}_4 = \{\mathcal{H}_2, \mathcal{F}_4\} + \widetilde{\mathcal{H}}_4, \tag{66}$$

with

$$\widetilde{\mathcal{H}}_4(v) = \frac{1}{12}\int \left\{\frac{2}{3}v^3\,|\partial_x|\,[v] - \left(\mathbf{H}[v]^2 + \frac{1}{2}v^2\right)|\partial_x|\left[v^2\right]\right\}dx. \tag{67}$$

To simplify the forth-degree terms, we choose

$$\mathcal{F}_4(v) = \int \left\{\frac{1}{6}v_x \mathbf{H}[v]\mathbf{H}\left[v^2\right] - \frac{1}{8}\mathbf{H}[v]^2\left(v^2\right)_x\right\}dx.$$

After some algebra, we compute that

$$\begin{aligned}\{\mathcal{H}_2, \mathcal{F}_4\}(v) \;=\; &\int\left\{\frac{1}{12}\left(\mathbf{H}[v]^2 - v^2\right)\mathbf{H}\left[v^2\right]_x - \frac{1}{3}\mathbf{H}\big[v\mathbf{H}[v]\big]v_x\mathbf{H}[v]\right.\\ &\left.+\frac{1}{6}\left(v^3\mathbf{H}[v]_x + v_x\mathbf{H}[v]^3\right)\right\}dx. \end{aligned}\tag{68}$$

The convolution theorem (A.3), with $w = v$, implies that

$$\mathbf{H}\big[v\mathbf{H}[v]\big] = \frac{1}{2}\left(\mathbf{H}[v]^2 - v^2\right).$$

Using this identity in (68), simplifying the result, and writing $\mathbf{H}\partial_x = |\partial_x|$, we find that

$$\{\mathcal{H}_2, \mathcal{F}_4\}(v) = \frac{1}{12}\int\left\{\mathbf{H}[v]^2\,|\partial_x|\left[v^2\right] - v^2\,|\partial_x|\left[v^2\right] + \frac{4}{3}v^3\,|\partial_x|\,[v]\right\}dx. \tag{69}$$

Using (67) and (69) in (66), and simplifying the result, we get

$$\mathcal{K}_4(v) = \int_{-\infty}^{\infty} \left\{ \frac{1}{6} v^3 \left|\partial_x\right| [v] - \frac{1}{8} v^2 \left|\partial_x\right| \left[v^2\right] \right\} dx. \tag{70}$$

After neglecting terms of higher order than the quartic terms, setting the amplitude parameter ε equal to one, and using (64), (70) in (65), we see that the near-identity transformation $v \mapsto u$ defined by $\mathcal{F}$ transforms the Hamiltonian equations (13)–(15) into (59)–(60).

8. LINEARIZED STABILITY OF PERIODIC WAVES

The harmonic-wave solution (7)–(8) is the constant-frequency analog of the Stokes-wave solution for a weakly nonlinear dispersive waves. A significant difference, however, is that the dependence on the wave amplitude A appears as a coefficient of the dispersive term, rather than as a shift in the frequency.

A single positive-wavenumber harmonic is an exact solution of (4) because four-wave resonant interactions of the form $k+k-k$ do not generate any new spatial harmonics. On the other hand, resonant interactions involving more than one harmonic generate infinitely many new harmonics (for example, $3k = 2k + 2k - k$, $4k = 3k + 2k - k$, and so on).

8.1. Linearized stability

To study the linearized stability the solution (7)–(8), we normalize $A = k = \omega = 1$, without loss of generality, and write

$$\psi(x,t) = e^{i(x-t)} \left[1 + \varphi(x,t)\right]. \tag{71}$$

Using this expression in (4), and linearizing the result with respect to φ, we get

$$\varphi_t - i\varphi = e^{-ix} \mathbf{P} \left[e^{ix} \Big(\left|\partial_x\right| [m] - m + i\varphi_x - \varphi \Big) \right]_x$$

where $m = \varphi + \varphi^*$. We write this equation as

$$\varphi_t - i\varphi = \tilde{\partial_x} \tilde{\mathbf{P}} \Big[\left|\partial_x\right| [\varphi^*] - \varphi^* + \left|\partial_x\right| [\varphi] + i\varphi_x - 2\varphi \Big]$$

where

$$\tilde{\partial_x} = e^{-ix} \partial_x e^{ix} = \partial_x + i\mathbf{I}, \qquad \tilde{\mathbf{P}} = e^{-ix} \mathbf{P} e^{ix}.$$

The Fourier transform $\hat{\psi}(k,t)$ is supported in $0 < k < \infty$, so from (71) $\hat{\varphi}(k,t)$ is supported in $-1 < k < \infty$. It follows that $\tilde{\mathbf{P}}[\varphi] = \varphi$, and therefore

$$\varphi_t - i\varphi = \tilde{\partial_x} \tilde{\mathbf{P}} \Big[\left|\partial_x\right| [\varphi^*] - \varphi^* \Big] + (\partial_x + i) \left(\left|\partial_x\right| [\varphi] + i\varphi_x - 2\varphi \right).$$

To compute the action of $\tilde{\mathbf{P}}$ on φ^*, we decompose φ as

$$\varphi(x,t) = \xi(x,t) + \eta(x,t)$$

where

$$\xi(x,t) = \int_{-1}^{1} \hat{\varphi}(k,t)e^{ikx}\,dk, \qquad \eta(x,t) = \int_{1}^{\infty} \hat{\varphi}(k,t)e^{ikx}\,dk.$$

Then

$$\tilde{\partial_x}\tilde{\mathbf{P}}\Big[|\partial_x|\,[\varphi^*] - \varphi^*\Big] = (\partial_x + i\mathbf{I})\Big[|\partial_x|\,[\xi^*] - \xi^*\Big],$$

and

$$\varphi_t + 3\varphi_x - i\,|\partial_x|\,[\varphi] + i\varphi = |\partial_x|\,[\varphi]_x + i\varphi_{xx} + (\partial_x + i)\left(|\partial_x|\,[\xi^*] - \xi^*\right). \quad (72)$$

Projecting (72) onto Fourier components with $1 < k < \infty$, and using the fact that $|\partial_x|\,[\eta] = -i\eta_x$, we get

$$\eta_t + 2\eta_x + i\eta = 0.$$

Thus, perturbations with wavenumber $k > 1$ are independent of the other modes. They are stable, with velocity 2 and frequency 1.

Projecting (72) onto Fourier components with $-1 < k < 1$, we get

$$\xi_t + 3\xi_x - i\,|\partial_x|\,[\xi] + i\xi = |\partial_x|\,[\xi]_x + i\xi_{xx} + (\partial_x + i)\left(|\partial_x|\,[\xi^*] - \xi^*\right). \quad (73)$$

Thus, if $0 < k < 1$, the k and $(-k)$ modes are coupled through their interaction with the unperturbed wave.

To solve (73), we write

$$\xi(x,t) = \mu(x,t) + \nu^*(x,t)$$

where

$$\mu(x,t) = \int_{0}^{1} \hat{\varphi}(k,t)e^{ikx}\,dk, \qquad \nu(x,t) = \int_{0}^{1} \hat{\varphi}^*(-k)e^{ikx}\,dk.$$

Then $\mathbf{H}[\mu] = -i\mu$, $\mathbf{H}[\nu] = -i\nu$ and

$$|\partial_x|\,[\xi] = -i\mu_x + i\nu_x^*.$$

Using this equation in (73), projecting the result onto Fourier components with $0 < k < 1$ and $-1 < k < 0$, respectively, and simplifying the result, we get

$$\mu_t + 2\mu_x + i\mu + i\left(\partial_x^2 + 1\right)\nu = 0,$$
$$\nu_t + 4\nu_x - i\nu + 2i\nu_{xx} + i\left(\partial_x - i\right)^2\mu = 0.$$

We look for Fourier solutions of this system of the form

$$\mu(x,t) = Me^{ikx-i\omega t}, \qquad \nu(x) = Ne^{ikx-i\omega t}$$

where $0 < k < 1$. Then

$$\begin{bmatrix} -\omega + 2k + 1 & 1 - k^2 \\ -(1-k)^2 & -\omega - 1 + 4k - 2k^2 \end{bmatrix}\begin{bmatrix} M \\ N \end{bmatrix} = 0.$$

Setting the determinant of the matrix in this equation equal to zero, and introducing the phase speed $c = \omega/k$, we get

$$c^2 + 2(k-3)c + 6 - 2k - k^2 = 0.$$

The solutions are real, with

$$c = (3-k) \pm \sqrt{1 + 2(1-k)^2}. \tag{74}$$

It follows that the harmonic solutions (7)–(8) of (4) are linearly stable to small perturbations.

We remark that in this analysis, unlike the analogous analysis of dispersive waves using the NLS equation, the perturbations are not assumed to be of long wavelength relative to the wavelength of the carrier wave. Their frequency is, however, close to the frequency of the carrier wave.

8.2. Semi-classical solutions

To derive a semi-classical approximation for (4) it is convenient to proceed informally. The same results can be obtained by the explicit introduction of a small parameter.

We write

$$\psi(x,t) = a(x,t)e^{iS(x,t)} \tag{75}$$

where a, S are real-valued functions. Using (75) in (4), we get

$$iaS_t + a_t = e^{-iS}\mathbf{P}\left[e^{iS}\left\{a\left|\partial_x\right|\left[a^2\right] - a^3S_x + ia^2a_x\right\}\right]_x. \tag{76}$$

We suppose that the phase S varies much more rapidly than a, so that the spectrum of ψ is concentrated near S_x. Assuming that $S_x > 0$, we have for any slowly varying function b that

$$\mathbf{P}\left[be^{iS}\right] \sim be^{iS}.$$

We may therefore approximate (76) as

$$\begin{aligned} iaS_t + a_t &= iS_x \left\{a\,|\partial_x|\left[a^2\right] - a^3 S_x + ia^2 a_x\right\} \\ &\quad + \left\{a\,|\partial_x|\left[a^2\right] - a^3 S_x + ia^2 a_x\right\}_x . \end{aligned}$$

Expanding derivatives and equating real and imaginary parts in this equation we get

$$\begin{gathered} S_t + a^2 S_x^2 = S_x\,|\partial_x|\left[a^2\right] + a a_{xx} + 2a_x^2, \\ a_t + 4S_x a^2 a_x + a^3 S_{xx} = \left(a\,|\partial_x|\left[a^2\right]\right)_x . \end{gathered}$$

Introducing $n = a^2$ and $k = S_x$, we may write these equations as

$$k_t + \left(k^2 n\right)_x = \left(k\,|\partial_x|\,[n] + \frac{1}{2} n_{xx} + \frac{n_x^2}{4n}\right)_x , \tag{77}$$

$$n_t + \left(2kn^2\right)_x = 2n\,|\partial_x|\,[n]_x + n_x\,|\partial_x|\,[n] . \tag{78}$$

The terms on the right-hand side of (77)–(78) are small in the semi-classical limit. The leading order semi-classical equations are therefore

$$k_t + \left(k^2 n\right)_x = 0, \tag{79}$$

$$n_t + \left(2kn^2\right)_x = 0. \tag{80}$$

These equations form a hyperbolic system for (k, n):

$$\begin{bmatrix} k \\ n \end{bmatrix}_t + \begin{bmatrix} 2kn & k^2 \\ 2n^2 & 4kn \end{bmatrix} \begin{bmatrix} k \\ n \end{bmatrix}_x = 0. \tag{81}$$

The eigenvalues λ and eigenvectors R of the matrix in (81) are given by

$$\lambda = \gamma k n, \qquad R = \begin{bmatrix} k \\ (\gamma - 2)n \end{bmatrix}$$

where

$$\gamma = 3 \pm \sqrt{3}.$$

The hyperbolicity of the semi-classical equations means that periodic wave-trains are modulationally stable, and is consistent with the linearized stability of periodic waves. The characteristic velocities $\lambda = 3 \pm \sqrt{3}$ of (79)–(80)

at $kn = 1$ agree with the velocities in $c = 3 \pm \sqrt{3}$ at $k = 0$ in (74), both of which limits describe linearized, long-wave perturbations of the solution $\psi(x,t) = e^{ix-it}$.

Riemann invariants φ of (81) are given by

$$\varphi(k,n) = kn^{(\gamma-2)/2}.$$

It follows that $\{(k,n) : k > 0, n > 0\}$ is an invariant region for smooth solutions of (81), consistent with the assumption made in deriving the system that $k, n > 0$.

The characteristics of (81) are genuinely nonlinear for $kn \neq 0$, with

$$\nabla\lambda \cdot R = \beta kn, \qquad \beta = 9 \pm 5\sqrt{3}. \tag{82}$$

Thus, semi-classical solutions steepen until the higher-order dispersive terms on the right hand side of (77)–(78) become important.

The dominant long-wave dispersive terms in (77)–(78) are the ones proportional to $|\partial_x|\,[n]_x = \mathbf{H}[u]_{xx}$. Omitting the explicit introduction of a small parameter, we find that small-amplitude long-wave perturbations of k, n about $k = 1$, $n = 1$ are given by

$$k(x,t) = 1 + u(x,t) + \ldots, \qquad n(x,t) = 1 + (\gamma - 2)u(x,t) + \ldots$$

where $u(x,t)$ satisfies a Benjamin-Ono equation

$$u_t + \gamma u_x + \beta u u_x = \alpha \mathbf{H}[u]_{xx} \tag{83}$$

with

$$\alpha = 1 \pm \frac{2\sqrt{3}}{3}.$$

The nonlinear coefficient β agrees with the genuine-nonlinearity coefficient in (82), while the linear dispersive coefficient α agrees with the long-wave expansion of the linearized phase velocity (74), which is $c = \gamma - \alpha|k| + O(k^2)$ as $k \to 0$.

The soliton solutions of (83) are

$$u(x,t) = \frac{4\alpha}{\beta}\left[\frac{a}{(x-ct)^2 + a^2}\right]$$

where a is a large, positive parameter, and

$$c = \gamma + \frac{\alpha}{a}.$$

For the (+)-branch, with $\alpha > 0$, the solitons travel faster than periodic long waves, while for the (−)-branch, with $\alpha < 0$, the solitons travel slower than periodic long waves. The solitons have the same phase velocity as a periodic short wave with wavenumber close to $\kappa = 4 \pm 2\sqrt{3}$. This suggests that solitary waves corresponding to the BO-solitons may not exist in the full asymptotic equation due to the radiation of short waves, but we will not pursue that question further here.

9. NUMERICAL SOLUTIONS

We used a pseudo-spectral method to numerically integrate both the Burgers-Hilbert equation (1) and the asymptotic equation (4). We computed the spatial derivatives and the Hilbert transform spectrally, and carried out the time-integration by means of a fourth order Runge-Kutta scheme. The numerical solutions show that small-amplitude solutions of the Burgers-Hilbert equation are well-described by the corresponding solutions of the asymptotic equation. They also show that solutions of the asymptotic equation steepen and form a singularity.

Using 2^{12} points for $x \in [0, 2\pi]$, we could integrate the equations quickly in MATLAB up to the formation-time of the singularity, which will be discussed below. For both equations, energy was conserved to within a relative error of less than 10^{-7} for the duration of the integration; and, for the asymptotic equation, the momentum and action were conserved to within a relative error of less than 10^{-8}.

We performed several integrations of the Burgers-Hilbert equation with the same initial profile scaled to different amplitudes:

$$u(x, 0) = A\left\{\cos x + \frac{1}{2}\cos\left[2\left(x + 2\pi^2\right)\right]\right\} \tag{84}$$

where A is a real constant. This data has maximum negative slope

$$\epsilon = |\min u_x(x, 0)|$$

where $\epsilon = cA$ with $c \approx 1.225$.

A linear timescale T_h for (1) is given by the period $T_h = 2\pi$ of solutions of the linearized equation (17). A nonlinear timescale T_b is given by the shock formation time $T_b = 1/\epsilon$ for the inviscid Burgers equation. Thus,

$$2\pi\epsilon = \frac{T_h}{T_b}$$

is an appropriate parameter for measuring the effect of nonlinearity on solutions of the Burgers-Hilbert equation. If ϵ is small, which is the regime in which the asymptotic equation applies, then singularities form over the course of many oscillations; while if ϵ is large, they form quickly relative to the period of an oscillation.

For smaller amplitude data, with $2\pi\epsilon \leq 1.2$, the solutions evolve in a qualitatively similar fashion. Over shorter times, they approximately oscillate with period 2π between a profile and its Hilbert transform. Over longer times, the profile deforms and steepens until a singularity in the derivative u_x develops.

Figure 1 shows a numerical solution of (1) for the initial data (84) with $A = 10^{-2}$ and $\epsilon \approx 1.2 \times 10^{-2}$. In the top frame of figure 1, the solid line is the initial data and the dashed line is the solution at $t = \pi/2$. Since A is small, the dashed line is almost equal to the Hilbert transform of the solid line. In the bottom frame of figure 1, the solid line shows the solution $u(x,t)$ at $t = 2\pi\, n$ with $n = 10^4$, just prior to the development of the singularity, and the dashed line shows the solution at $t = 2\pi\, n + \pi/2$. The dashed curve is again almost equal to the Hilbert transform of the solid curve.

Several snapshots in time of $u(x,t)$ are shown in figure 2 where the times of each slice are integer multiples of the linearized oscillation period. The singularity develops ahead of a large peak in u. Although the this peak steepens in a similar way to what happens for the inviscid Burgers equation, the Burgers-Hilbert equation has an additional fast oscillation. When the solution is steep in one phase of the oscillation, it has a cusp or filament in the other phase.

As the singularity forms, the solution of the Burgers-Hilbert equation develops a structure on a smaller spatial scale than occurs in shock-formation for the Burgers equation. We show a magnification of the region around the singularity in figure 3. The solid lines show the solution at the times $T_s - 2\pi$, T_s and $T_s + 2\pi$, where the singularity-formation time T_s was determined numerically. The dashed lines show the solution at times $T_s - 1.5\pi$, $T_s + 0.5\pi$ and $T_s + 2.5\pi$. In both phases, the wave-profile forms a small dip ahead of its steepest part, followed by a corner, or 'knee', after which the profile becomes relatively flat. The integration can be continued to times slightly beyond the development of the singularity, presumably because de-aliasing in the pseudo-spectral method introduces a small amount of numerical dissipation and dispersion.

For larger amplitude data, with $2\pi\epsilon \geq 1.7$, the qualitative nature of the solution is dramatically different. The characteristic time of the nonlinear term, $T_b = 1/\epsilon < 3.7$, is then significantly less than the period 2π of the linear term. The solutions do not oscillate between one profile and its Hilbert transform; instead, they quickly develop shocks in much the same way as for Burgers equation, albeit modified by the linear Hilbert transform term. In this regime, the asymptotic equation no longer provides a good approximation for the evolution of u.

The transition between these two qualitatively different regimes is remarkably rapid. For example, when $2\pi\epsilon = 1.154$, the singularity forms at time $T_s \approx 77.9$ in the twelfth oscillation of period 2π; but when $2\pi\epsilon = 1.731$, the singularity forms at $T_s \approx 3.94$ in the first oscillation. Thus, once the data is small enough that a singularity does not form in the first few oscillations, the effect of the nonlinearity becomes much weaker as a result of the alternation between compression and expansion in each oscillation, leading to a greatly increased lifespan of smooth solutions.

Figure 4 shows a log-log plot of the numerically computed singularity-formation time T_s against $2\pi\epsilon$ for several integrations. The equation

$$\epsilon^2 T_s = 2.37 \tag{85}$$

provides an excellent fit to the numerical values for $2\pi\epsilon \leq 1$. The scaling $T_s \sim k\,\epsilon^{-2}$ as $\epsilon \to 0$ agrees with the cubically-nonlinear scaling used to derive the asymptotic equation (4) from equation (1), and the value $k \approx 2.37$ is obtained from a numerical integration of the asymptotic equation described below.

For large values of ϵ, we expect that (1) should behave like the inviscid Burgers equation and that

$$T_s \sim \epsilon^{-1} \qquad \text{as } \epsilon \to \infty.$$

This line is also plotted on figure 4; the agreement with the numerical values is excellent for $2\pi\epsilon \geq 2$.

In the transition regime, $1.2 < 2\pi\epsilon < 1.7$, the singularity-formation time T_s appears to increase in a series of steps as ϵ decreases. These results suggest that $T_s : (0,\infty) \to \mathbb{R}$ is a decreasing, lower-semicontinuous function of ϵ with jump discontinuities at a decreasing sequence $\{\epsilon_n : n \in \mathbb{N}\}$ of values of ϵ, where ϵ_n is the largest value of ϵ for which the singularity forms in the n^{th} oscillation. We would further expect that as $n \to \infty$, $\epsilon_n \to 0$ and

$$T_s\left(\epsilon_{n+1}\right) - T_s(\epsilon_n) \sim 2\pi.$$

Next, we describe a corresponding numerical solution of the asymptotic equation. We integrated (4) for $\psi(x,t)$ with initial data

$$\psi(x,0) = e^{ix} + \frac{1}{2}e^{2i(x+2\pi^2)}. \tag{86}$$

The corresponding asymptotic solution of the Burgers-Hilbert equation (1) with initial data (84) is

$$u(x,t) \sim \frac{1}{2} A\,\psi\left(x, \frac{1}{4}A^2 t\right) e^{-it} + \text{c.c.} \qquad \text{as } A \to 0.$$

The real and imaginary parts of $\psi(x,t)$ are shown in figure 5 just prior to the development of the singularity at $t = \tau_s$ where $\tau_s \approx 0.395$. Since u oscillates between the real and imaginary parts of ψ on the fast time scale, the curves in figure 5 should be compared to the corresponding curves in figure 1 (b). While $\Im(\psi)$ shows a sharper feature right at the point of the singularity, the two figures otherwise correspond extremely well to one another.

The asymptotic theory predicts that the singularity formation time T_s for the Burgers-Hilbert equation has the asymptotic behavior

$$T_s \sim 4\tau_s A^{-2} \qquad \text{as } A \to 0.$$

Setting $A = \epsilon/c$, and using the numerically computed value for τ_s, we get

$$T_s \approx 2.37\epsilon^{-2} \qquad \text{as } \epsilon \to 0.$$

which is the relation used to fit the Burgers-Hilbert data in equation (85). The mean of $\epsilon^2 T_s$ for the five values of ϵ from the Burgers-Hilbert data whose T_s lie near the asymptotic fit is given by

$$\overline{\epsilon^2 T_s} = 2.47.$$

This differs by 4% from the coefficient predicted by the asymptotic equation; moreover, the difference is less for the smaller values of ϵ. Thus, the solution of the asymptotic equation is in excellent quantitative agreement with the solutions of the Burgers-Hilbert equation.

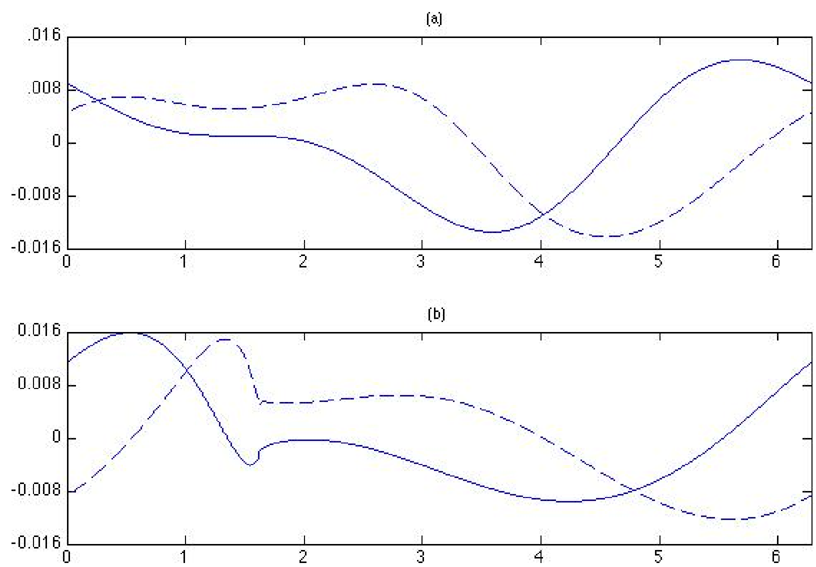

FIG. 1. A numerical solution of the Burgers-Hilbert equation (1) for the initial condition (84) with $A = 0.01$. (a) The solid line shows $u(x, 0)$. The dashed line shows the solution $u(x, \pi)$, one half-period after the initial condition. (b) The solution for the initial condition in (a) just prior to the formation of a singularity. The solid and dashed line show $u(x, T_s)$ for $T_s = 2\pi n$ for $n = 10^4$ and $n = 10^4 + 1/4$, respectively.

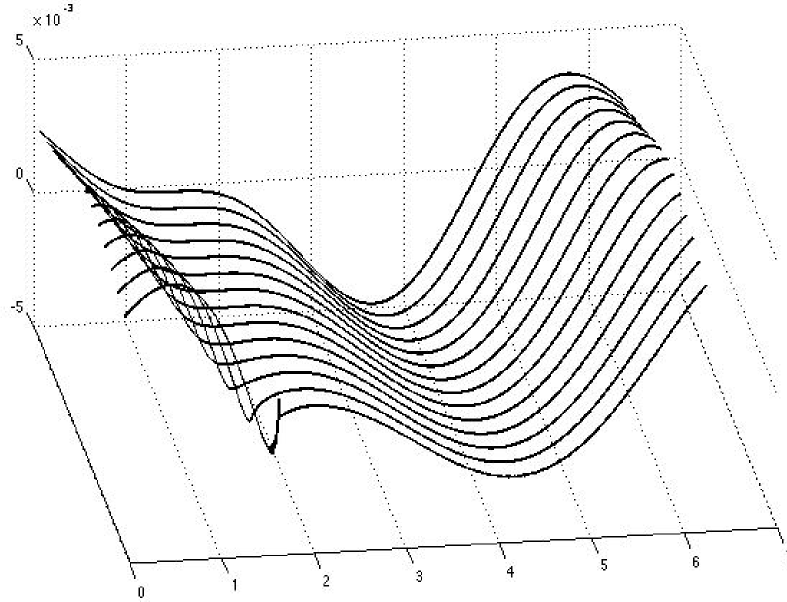

FIG. 2. Snapshots of $u(x,t)$ for the solution in figure 1. Time increases to the foreground from the initial condition to the development of the singularity, and each time slice is taken at an integer multiple of 2π in order to factor out the fast oscillation. The singularity develops as a steepening ahead of the maximum of $-u_x(x,0)$.

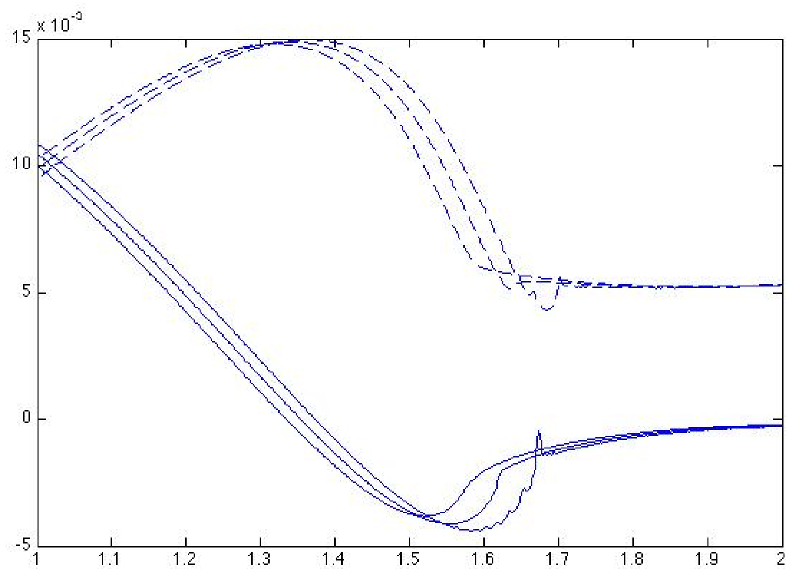

FIG. 3. Magnification of $u(x,t)$ near the singularity for the solution in figure 1. The solid curves show the solution one period before, at, and one period after the singularity. The dashed curves show the solution a quarter period before, a quarter period after, and one and a quarter periods after the formation of the singularity; these are approximately the Hilbert transforms of the solid curves.

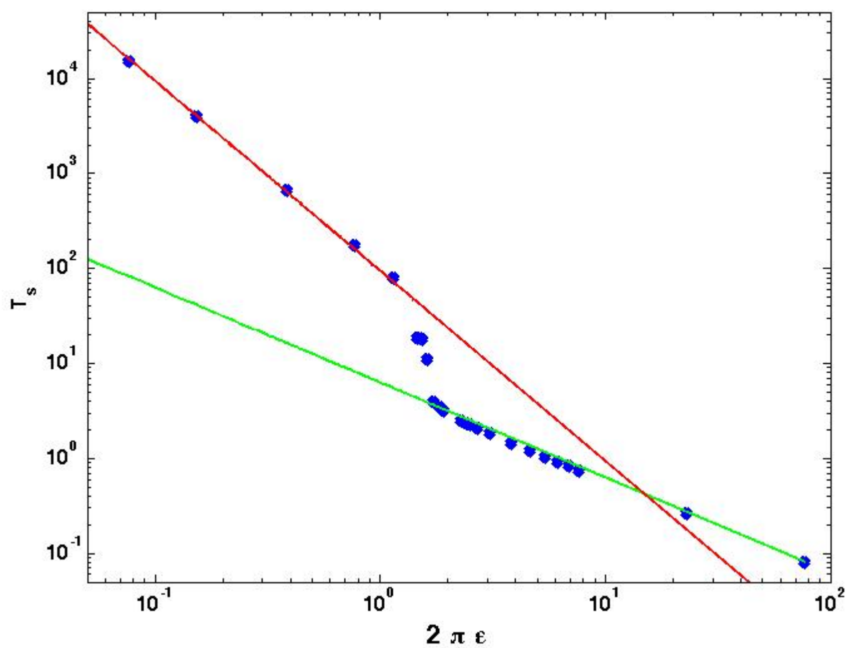

FIG. 4. Logarithm of the singularity formation time T_s for the Burgers-Hilbert equation versus the logarithm of $2\pi\epsilon$ for the numerical experiments (diamonds) along with the prediction from the asymptotic equation, $T_s = 2.37\,\epsilon^{-2}$ (steeper line) and the prediction from the Burgers equation $T_s = \epsilon^{-1}$ (shallower line).

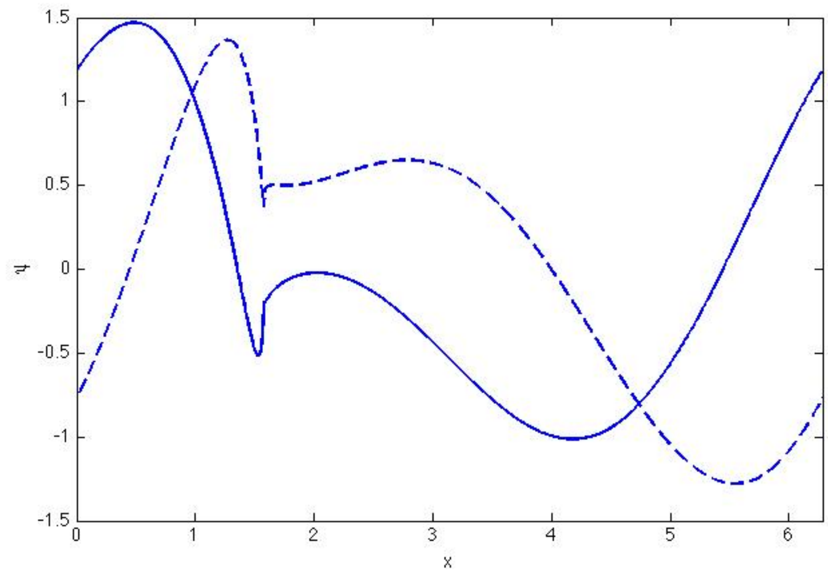

FIG. 5. The solution of the asymptotic equation (4) with initial data (86) at the onset of the singularity. The solid line is $\Re(\psi)$, the dashed line is $\Im(\psi)$. Compare with figure 1 (b).

APPENDIX: NOTATION

In this appendix we summarize some definitions and notation that are used throughout the paper. For definiteness, we consider square-integrable functions defined on $\mathbb{R}$. Similar considerations apply to periodic functions, in which case we project constant Fourier modes to zero.

We denote the Fourier transform of $f \in L^2(\mathbb{R})$ by $\hat{f} \in L^2(\mathbb{R})$, where

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx}\, dk.$$

We define the Hilbert transform $\mathbf{H} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, by

$$\widehat{\mathbf{H}[f]}(k) = -i(\operatorname{sgn} k)\hat{f}(k), \tag{A.1}$$

where

$$\operatorname{sgn} k = \begin{cases} +1 & \text{if } k > 0, \\ 0 & \text{if } k = 0, \\ -1 & \text{if } k < 0, \end{cases} \tag{A.2}$$

Equivalently,

$$\mathbf{H}[f] = \left(\text{p.v.}\frac{1}{\pi x}\right) * f.$$

The Hilbert transform is a skew-adjoint isometry on $L^2(\mathbb{R})$, and $\mathbf{H}^2 = -\mathbf{I}$.

If $u : \mathbb{R} \to \mathbb{R}$ is a real-valued L^2-function, then $u + i\mathbf{H}[u]$ is the boundary value on the real axis of a holomorphic function in the upper-half plane with uniformly bounded L^2-norms on lines with constant positive imaginary part. If F, G are holomorphic functions whose boundary values have real parts v, w, respectively, then a consideration of the holomorphic functions FG and F^3 implies that

$$\mathbf{H}\Big[vw - \mathbf{H}[v]\mathbf{H}[w]\Big] = v\mathbf{H}[w] + w\mathbf{H}[v], \tag{A.3}$$

$$v^2\mathbf{H}[v] - \frac{1}{3}\mathbf{H}[v]^3 = \mathbf{H}\left[\frac{1}{3}v^3 - v\mathbf{H}[v]^2\right], \tag{A.4}$$

under suitable assumptions on v, w. For example, it is sufficient that $v, w \in L^p$ for $p > 2$ in (A.3), and $v \in L^p$ for $p > 3$ in (A.4).

We define projection operators

$$\mathbf{P}, \mathbf{Q} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

onto positive and negative wavenumber components by

$$\mathbf{P}\left[\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}\,dk\right] = \int_{0}^{\infty} \hat{f}(k)e^{ikx}\,dk, \tag{A.5}$$

$$\mathbf{Q}\left[\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}\,dk\right] = \int_{-\infty}^{0} \hat{f}(k)e^{ikx}\,dk. \tag{A.6}$$

Then $\mathbf{P}$, $\mathbf{Q}$ are orthogonal self-adjoint projections on $L^2(\mathbb{R})$, and

$$\mathbf{P} = \frac{1}{2}\left(\mathbf{I} + i\mathbf{H}\right), \qquad \mathbf{Q} = \frac{1}{2}\left(\mathbf{I} - i\mathbf{H}\right). \tag{A.7}$$

Moreover,

$$\mathbf{P}[f]^* = \mathbf{Q}\left[f^*\right], \qquad \mathbf{Q}[f]^* = \mathbf{P}\left[f^*\right], \tag{A.8}$$

where the star denotes the complex conjugate, and

$$\int_{-\infty}^{\infty} \mathbf{P}[f](x)\,g(x)\,dx = \int_{-\infty}^{\infty} f(x)\,\mathbf{Q}[g](x)\,dx. \tag{A.9}$$

We denote the differentiation operator by ∂_x, and define

$$|\partial_x| = \mathbf{H}\partial_x, \tag{A.10}$$

so that $\widehat{|\partial_x|\,[f]}(k) = |k|\hat{f}(k)$.

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