NONLINEAR HIGH-FROUDE-NUMBER FREE-SURFACE PROBLEMS

by

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1. *Introduction*

Many authors have applied the concepts of slender-body theory to the problem of a body translating along a free surface. Several years ago there was a spate of papers in which the problem of calculating ship wave resistance was attacked in this way. See particularly Tuck (1965)(1), which lists many other references. These ideas have been applied to the problem of a planing boat by Tulin (1959)(2); an indication of the same approach had already appeared in the famous paper by Wagner (1932)(3). There has been a conspicuous lack of success in predicting wave resistance in this way, and so slender-body theory has fallen into some disrepute for ship hydrodynamics problems, although it has certainly not been adequately tested in the planing problem and perhaps not in the resistance problem. It is possible that valuable information may still be extracted from the application of slender-body theory to steady-motion problems of ship hydrodynamics.

One major feature distinguishes slender-body theories for ship problems from the usual slender-body theory of aerodynamics, viz., gravity (or Froude number) appears as an extra parameter, and a special assumption must be made about its order of magnitude. The basic parameter of slender-body theory is, of course, that quantity ϵ which specifies how slender the body is. We may take it to be the beam/length ratio, the draft/length ratio, or some other measure of slenderness, such as the maximum slope of the body-surface tangent plane with respect to the longitudinal axis. (We continue to avoid being specific on this matter. It is often very convenient to think of overall length, L, as being a quantity of order unity and beam or draft of order ϵ.) In free surface problems, one must investigate what reasonable assumptions can be made about the magnitude of g with respect to ϵ.

In the ship resistance problem, it has generally been assumed that** $g = O(1)$ as $\epsilon \to 0$, or, more precisely, that $1/F^2 = O(1)$, where $F = U/\sqrt{gL}$, the Froude number. On the other hand, in planing problems one is usually concerned with high Froude numbers, and so Wagner, Tulin, and others have assumed that gravity is negligible, which, as we shall see, amounts to assuming that $g = O(\epsilon^2)$, or $F = O(\epsilon^{-1})$. In all, it will appear that there are four interesting cases:

(1) $g = O(\epsilon^{-1})$. Gravity dominates in the free-surface conditions, i.e., the free surface condition is everywhere equivalent to a rigid wall condition.

(2) $g = O(1)$. Gravity dominates *near* the body, but ordinary gravity waves occur at large distances from the body. This is the case usually considered in the analysis of ship wave resistance.

(3) $g = O(\epsilon)$. Gravity waves occur near the body, but gravity effects vanish far away, leaving the far-field free surface as a simple pressure-relief surface (with no wave motion possible). This case is developed in the present paper.

(4) $g = O(\epsilon^2)$. The effects of gravity vanish everywhere. This is the usual planing case.

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**All statements involving O, o, and asymptotic expansions are based on the definitions in Erdélyi (1956)(4).

The simple interpretations given here for each case apply only to the lowest-order approximations, but in practice one is not likely to carry calculations beyond the first approximation. It should be noted that this list of four cases is complete, at least with respect to the lowest-order problems which result. This fact will be demonstrated later.

The distinction in cases (2) and (3) between near-field and far-field behavior suggests that a method of analysis should be followed which makes the differences very clear. Tuck has done this for case (2) by using the method of matched asymptotic expansions. In that method (Van Dyke (1964)(5)), one assumes the existence of different asymptotic expansions according to whether distance from the longitudinal axis is order ϵ or order 1. The first is supposed to be valid on the body but not at infinity, whereas the latter is valid at infinity but not on the body. Characteristically in such analyses, the two expansions must be found simultaneously, for there are not sufficient boundary conditions to determine either expansion alone. Step-by-step, the two are matched to each other as successive terms in each are found.

We shall set up the problem of case (3). We obtain very simply the general solution for the first term in the outer region. The first term in the inner solution cannot be found analytically, for it must come from the numerical solution of a boundary-value problem. This problem involves the satisfaction of the full nonlinear boundary conditions which are typical of free-surface problems; we show that it would be improper to try to linearize these conditions. However, the numerical solution needs to be found in two dimensions only. This appears to be a reasonable computer project to undertake; an outline of a proposed procedure for doing this is presented.

The matching of these two first terms requires that we know the behavior of the inner solution at infinity. However, we do not know it analytically anywhere. There are two ways of resolving this difficulty. Firstly, we could assume that there is an overlap in the domains of validity of inner and outer solutions and thus infer from the outer solution what is the behavior of the inner solution at infinity. It turns out that this is indeed a correct inference, but it is still an inference. Secondly, we can find an "intermediate-region" solution, and this solution must necessarily overlap the inner and outer solutions. This procedure provides a proof of the correctness of the first procedure. To carry it out here would lead us too far afield from our main concern; we limit ourselves to an indication of the method and its results in an appendix.

The establishment of case (3) not only completes our hierarchy of problems. It provides a mathematical formulation for the physical problems of steady ship or planing-surface motion in situations in which the speed is high but not so high that gravity can be ignored. The application to high-performance, small planing boats is rather obvious.

It may happen that application to problems of steady ship motion is reasonable too, although one can only speculate on this matter at present. Certainly, consideration of the usual values of Froude number leads to the contrary conclusion, for a fast ship operates at a Froude number less than 0.4, which is hardly large, i.e., of order $\epsilon^{-\frac{1}{2}}$, as required by the present theory. However, one should not jump to conclusions too quickly on the basis of asymptotic solutions, and the lack of success of the conventional slender-body theory in predicting ship resistance suggests that a basically different approach is needed. In applying asymptotic solutions to practical problems, one tries to use finite values for the small parameter which is supposed to be approaching zero. One never knows in practice how small the parameter must be for the expansion to be useful; one can only try the expansion and examine the results.

Let us now abandon all *a priori* notions about what constitutes a "reasonable" value of a small parameter and consider the physical implications of the different approaches. In case (2), a logical consequence of the assumptions is that gravity is strong enough to make the free surface act like a rigid wall near the ship. The analysis which follows (case (3)) leads to the conclusion that inertial and gravitational effects are comparable in the near field. The latter seems physically to be a much more acceptable result.

In the far field, case (2) predicts the existence of gravity waves, whereas case (3) does not. Here, case (2) seems to be more reasonable. But to some extent the anomaly in case (3) is an illusion which disappears on closer examination of the asymptotic solution. The result which will be proven for the far field shows that as $\epsilon \to 0$ the effects of gravity disappear. But the waves are still there; they are represented by higher-order terms in the outer expansion and it can be shown that these terms are controlled in the matching procedure primarily by the lowest-order inner-solution term -- which contains the wave effects. Since in any case our interest is primarily in finding the pressure distribution on the body, it is not unreasonable to demand the greatest possible accuracy of the inner solution, while we disregard the outer solution as far as possible.

It must be emphasized that this is just speculation. However, the application to planing problems is sufficiently attractive to warrant the work of solving the inner problem numerically, and it should not be much additional work to apply the completed computer program to conventional ships. Only the numerical results so obtained can settle the question.

2. *The problem for $g = O(\epsilon)$*

In this section we formulate the steady-motion translation problem for a ship under the assumption that $g = O(\epsilon)$. In order to display this assumption explicitly, we define a new constant,

$$G = g/\epsilon = O(1).$$

We assume that the incident flow is a uniform stream with velocity U in the positive x-direction. The body surface is specified by the equation:

$$z - h(x,y) = 0, \tag{1}$$

where

$$h(x,y) = O(\epsilon), \tag{2}$$

and $h(x,y)$ is defined only for $y = O(\epsilon)$. The free surface is specified by:

$$z - \zeta(x,y) = 0. \tag{3}$$

We assume that the generated waves are at most of the same order of magnitude as the body beam and draft, i.e.,

$$\zeta(x,y) = O(\epsilon). \tag{4}$$

The fluid velocity is represented as the gradient of a potential function, $\varphi(x,y,z)$, satisfying Laplace's equation:

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0. \tag{5}$$

The potential also satisfies four boundary conditions:

$$0 = \varphi_x h_x + \varphi_y h_y - \varphi_z \qquad \text{on} \quad z - h(x,y) = 0; \tag{6}$$

$$\tfrac{1}{2}U^2 = g\zeta + \tfrac{1}{2}(\varphi_x^2 + \varphi_y^2 + \varphi_z^2) \quad \text{on} \quad z - \zeta(x,y) = 0; \tag{7}$$

$$0 = \varphi_x \zeta_x + \varphi_y \zeta_y - \varphi_z \quad \text{on} \quad z - \zeta(x,y) = 0; \tag{8}$$

$$\varphi = Ux \text{ for both } x = -\infty \text{ and } y^2 + z^2 = \infty. \tag{9}$$

Equation (6) is the kinematic condition on the body boundary, i.e., it is equivalent to $\partial\varphi/\partial\nu = 0$ on the body, where $\partial\varphi/\partial\nu$ is the rate of change of φ normal to the body. Equations (7) and (8) are, respectively, the dynamic and kinematic conditions on the free surface. Equation (9) is a radiation condition, sufficiently strong to render the solution unique in the treatment presented here.

We express Equation (6) in a more useful form as follows. In any cross-section $x = x_0$ = constant, let $\partial\varphi/\partial n$ denote the rate of change of φ in the direction normal to the section curve, $z - h(x_0, y) = 0$ (positive for flow into the body). Then we have:

$$\frac{\partial\varphi}{\partial n} = \frac{-\varphi_y h_y + \varphi_z}{\sqrt{1 + h_y^2}} = \frac{\varphi_x h_x}{\sqrt{1 + h_y^2}}. \tag{6'}$$

This expression represents an apparent normal flow velocity across a cylinder with a shape identical to the body section at the chosen value of $x = x_0$.

In order to treat the inner and outer regions systematically, we introduce new inner-region variables:

$$X = x; \quad Y = y/\epsilon; \quad Z = z/\epsilon. \tag{10}$$

We note that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial X}; \quad \frac{\partial}{\partial y} = \epsilon^{-1} \frac{\partial}{\partial Y}; \quad \frac{\partial}{\partial z} = \epsilon^{-1} \frac{\partial}{\partial Z}. \tag{10'}$$

Since the function $h(x,y)$ is $O(\epsilon)$, we define also

$$h(x,y) = \epsilon\, H(X,Y). \tag{11}$$

The body surface is thus specified by

$$Z - H(X,Y) = 0. \tag{11'}$$

Finally, we introduce assumed asymptotic expansions for the dependent variables, φ and ζ, as follows:

$$\varphi(x,y,z) \sim \sum_{n=0}^{N} \varphi_n\, (x,y,z;\epsilon), \tag{12}$$

with $\varphi_{n+1} = o(\varphi_n)$ as $\epsilon \to 0$ for (fixed) $\sqrt{y^2+z^2} = O(1)$;

$$\varphi(x,y,z) \sim \sum_{n=0}^{N} \Phi_n\, (X,Y,Z;\epsilon), \tag{13}$$

with $\Phi_{n+1} = o(\Phi_n)$ as $\epsilon \to 0$ for (fixed) $\sqrt{Y^2+Z^2} = O(1)$;

$$\zeta(x,y) \sim \sum_{n=0}^{N} \zeta_n\, (x,y;\epsilon), \tag{14}$$

with $\zeta_{n+1} = o(\zeta_n)$ as $\epsilon \to 0$ for (fixed) $y = O(1)$;

$$\zeta(x,y) \sim \epsilon \sum_{n=0}^{N} Z_n(X,Y;\epsilon), \tag{15}$$

with $Z_{n+1} = o(Z_n)$ as $\epsilon \to 0$ for (fixed) $Y = O(1)$.

The sense of the symbol "$\sim$" is the same as in Erdélyi (1956)[4]. We must require that none of the functions in the asymptotic expansions vanish identically.

We now start to find the solution by substituting the far-field expansions, (12) and (14), into the differential equation and the relevant boundary conditions. Since we cannot use the body boundary condition for the far-field potential, we must find the most general solution which satisfies the appropriate boundary conditions and which has arbitrary form near the body. Then we shall have to turn to the near-field problem.

In the far field, if we let $\epsilon \to 0$, there is no body at all, and so obviously

$$\varphi_0(x,y,z;\epsilon) = Ux. \tag{16}$$

From the kinematic boundary condition, we obtain the result that

$$0 = U\zeta_{0_x} + o(1),$$

and so

$$\zeta_0(x,y;\epsilon) = o(1).$$

The next term, $\varphi_1(x,y,z;\epsilon)$, of the asymptotic series (12) satisfies:

$$\varphi_{1_{xx}} + \varphi_{1_{yy}} + \varphi_{1_{zz}} = 0;$$

$$0 = U\zeta_{0_x} - \varphi_{1_z} + \text{smaller terms, on } z = \zeta(x,y);$$

$$0 = \epsilon G\zeta_0 + U\varphi_{1_x} + \text{smaller terms, on } z = \zeta(x,y);$$

$$\varphi_1 \to 0 \text{ as } x \to -\infty \quad \text{and} \quad y^2+z^2 \to \infty.$$

The "smaller terms" include those terms which are necessarily of higher order than those kept. *A priori*, we cannot state anything about the relative orders of magnitude of the terms retained above. However, a brief study of the possibilities shows that only one combination* of free surface conditions is possible:

$$0 = U\zeta_{0_x} - \varphi_{1_z}; \tag{17}$$

$$0 = \varphi_{1_x}.$$

Since $\zeta_0 = o(1)$, we can apply these on $z = 0$ rather than on $z = \zeta(x,y)$ without incurring errors of consequence (at this stage of the solution). Then the last equation can be simplified to:

$$0 = \varphi_1. \tag{18}$$

The problem for φ_1 is now easily solved. We see that the solution satisfies Laplace's equation in the lower half-space. However, since $\varphi_1 = 0$

*Other combinations are either mutually exclusive or lead to forbidden conclusions, e.g., $\varphi_1 \equiv 0$.

on $z = 0$, it can be extended analytically into the upper half-space as an odd harmonic function of z. Furthermore, it must vanish upstream and at large distance from the x-axis but we cannot specify its behavior on the x-axis. The general solution* is:

$$\varphi_1(x, y, z;\epsilon) = \sum_{m=1}^{\infty} \frac{\sin m\theta}{r^m} \int_0^{\infty} \frac{\left[\sqrt{(x-\xi)^2 + r^2} + x - \xi\right]^m f_{1m}(\xi;\epsilon)\, d\xi}{\sqrt{(x-\xi)^2 + r^2}}, \tag{19}$$

where

$$y = r\cos\theta, \quad z = r\sin\theta,$$

and the functions $f_{1m}(x;\epsilon)$ are (for the moment) a set of m arbitrary functions. For future reference, we give here the approximation** to φ_1 valid for small r:

$$\varphi_1(x, y, z;\epsilon) = \sum_{m=1}^{\infty} \frac{2^m \sin m\theta}{r^m} \int_0^{x} (x-\xi)^{m-1} f_{1m}(\xi;\epsilon)\, d\xi \left[1 + O(r)\right]. \tag{20}$$

It may be noted that there is really no justification at this point for choosing zero as the lower limit of the integrals in (19). However, φ_1 will soon be matched to the inner solution, Φ_1, and, since there is no body where $x < 0$, Φ_1 disappears there. Actually, it can be shown that φ_1 as given above represents a uniformly valid approximation to φ as $r \to 0$, provided $x \le \delta < 0$, where δ is an arbitrary but fixed negative number.

Nothing more can be said about φ_1 until we attempt to match it to the inner solution, and so we proceed now to formulate the problem for Φ_0 and Φ_1. Here we have a differential equation subject to boundary conditions on the body and the free surface. However, we do not have a radiation condition or other condition at infinity.

First we re-express all conditions in terms of inner variables. Laplace's equation becomes:

$$\Phi_{0_{YY}} + \Phi_{0_{ZZ}} = 0; \tag{21}$$

$$\Phi_{1_{YY}} + \Phi_{1_{ZZ}} = -\epsilon^2 \Phi_{0_{XX}}; \text{ etc.} \tag{22}$$

In obvious analogy to (10') we define a new operator $\partial/\partial N = \epsilon\, \partial/\partial n$, and so the body boundary condition, (6'), becomes:

$$\frac{\partial\varphi}{\partial N} = \left[\frac{\partial\Phi_0}{\partial N} + \frac{\partial\Phi_1}{\partial N} + \dots\right]$$
$$= \epsilon^2 H_X \left\{\Phi_{0_X} + \Phi_{1_X} + \dots\right\} \Big/ \sqrt{1 + H_Y^2},$$

applied on $Z - H(X, Y) = 0$. Asymptotically, as $\epsilon \to 0$, we have:

$$\frac{\partial\Phi_0}{\partial N} = 0; \tag{23}$$

*See Section 9.3 of Ward (1955)[6].

**Ibid.

$$\frac{\partial \Phi_1}{\partial N} = \epsilon^2 H_X \Phi_{0_X} \Big/ \sqrt{1 + H_Y^2}; \tag{24}$$

etc.

It can easily be shown that $\partial \Phi_0 / \partial N = 0$ also on the free surface. Thus Φ_0 satisfies the Laplace equation in two dimensions (2D), with homogeneous Neumann conditions on the boundaries. Moreover, it must match at infinity with $\varphi_0 = Ux$, which has no gradient in the y-z plane. Accordingly, for any X, Φ_0 must be constant, i.e., Φ_0 is a function of X only. The matching process immediately prescribes that

$$\Phi_0 = UX. \tag{25}$$

The body boundary condition for Φ_1 now becomes:

$$\frac{\partial \Phi_1}{\partial N} = \frac{\epsilon^2 U H_X}{\sqrt{1 + H_Y^2}}, \tag{24'}$$

and Φ_1 satisfies the 2D Laplace equation:

$$\Phi_{1_{YY}} + \Phi_{1_{ZZ}} = 0. \tag{22'}$$

To leading order of magnitude, the free surface conditions are:

$$0 = \epsilon^2 GZ_0 + U \Phi_{1_X} + \frac{1}{2\epsilon^2}\left[\Phi_{1_Y}^{\ 2} + \Phi_{1_Z}^{\ 2}\right]; \tag{26}$$

$$0 = \epsilon^2 U Z_{0_X} + \Phi_{1_Y} Z_{0_Y} - \Phi_{1_Z}. \tag{27}$$

These are to be applied on the actual free surface. However, $Z_1 = o(Z_0)$, by definition, and so we can apply them on $Z = Z_0$, thereby incurring only higher-order errors. We have no basis for applying them on $Z = 0$.

From (24') we see now that $\Phi_1 = O(\epsilon^2)$. In (26) it is apparent that the quadratic terms are the same order as the linear term, $U\Phi_{1X}$, and also that $Z_0 = O(1)$. In the kinematic condition, Equation (27), all terms are the same order of magnitude. Thus Φ_1 satisfies the 2D Laplace equation, subject to the full, nonlinear, free-surface conditions, (26) and (27). The only simplification over the original nonlinear problem is that here we have a problem in two dimensions rather than in three. However, this is a significant simplification.

There seems to be no alternative to solving this nonlinear problem numerically, but fortunately it is of a type for which precedents exist. A later section presents a discussion of a method for attempting such a solution.

It should be noted specifically that $Z_0 = O(1)$, and so the free-surface disturbance is not infinitesimal in the stretched coordinates. However, in natural coordinates, we have from (15) that

$$\zeta(x, y) \sim \epsilon\, Z_0(X, Y; \epsilon),$$

which agrees with the assumption expressed in (4).

Normally, in order to carry out the matching procedure, we would find Φ_1 analytically, approximate it for large $R = \sqrt{Y^2 + Z^2}$, and match the result to the outer solution evaluated for small r. Since we know nothing about the analytical form of Φ_1 in any region, we cannot do this, and we must devise a different method for matching. One procedure, which actually leads to the correct result (to lowest order), is to assume that Φ_1 can be expanded for large R in a series of cylindrical harmonics. This is not really justified, because Φ_1 is harmonic beyond a certain radius, but only in a half-space. We could also use the outer solution in the form given by (20) to infer the large-R behavior of Φ_1, but this is also not justified unless we are sure that inner and outer expansions have an overlap in their domains of validity, although this again happens to give the correct result. The proper procedure in this problem is to find the "intermediate solution", which is really an asymptotic expansion which acts as an outer expansion to the inner expansion and as an inner expansion to the outer expansion. This intermediate expansion has an overlap with both of the other expansions, and so it can give a true picture of the large-R behavior of Φ_1. The results of the Appendix show that, for large R, Φ_1 has the form

$$\Phi_1 = \frac{A_{11}(X;\epsilon)\sin\theta}{R}\left[1 + \bar{O}(1/R)\right]. \tag{28}$$

(We could also indicate another error-term which is $o(A_{11})$ as $\epsilon \to 0$.)
Since $\Phi_1 = O(\epsilon^2)$, we must have

$$A_{11}(X;\epsilon) = O(\epsilon^2).$$

In terms of real (outer) coordinates, Φ_1 is (for large R):

$$\Phi_1 = \frac{\epsilon\, A_{11}(X;\epsilon)\sin\theta}{r}\left[1 + O\left(\frac{\epsilon}{r}\right)\right].$$

This must match $\varphi_1(x, y, x;\epsilon)$, evaluated for small r. (See (20).) Clearly,

$$\epsilon\, A_{11}(X;\epsilon) = 2\int_0^X f_{11}(\xi;\epsilon)\, d\xi.$$

The other terms in (20) have no counterparts in Φ_1 to match; they must be higher order in ϵ. We can see this in another way too. If a term in (20) of the form

$$\frac{2^m \sin m\theta}{r^m}\int_0^x (x - \xi)^{m-1} f_{1m}(\xi;\epsilon)\, d\xi$$

is to match a term in Φ_1, then the latter must have the form

$$\frac{A_{1m}(X;\epsilon)\sin m\theta}{R^m}$$

for large R, with $A_{1m} = O(\epsilon^2)$ again. But, since $R = r/\epsilon$, we have

$$\frac{A_{1m}(X;\epsilon)\sin m\theta}{R^m} = \frac{\epsilon^m A_{1m}(x;\epsilon)\sin m\theta}{r^m}$$

and so

$$\epsilon^{m} A_{1m}(x;\epsilon) = 2^{m} \int_0^x (x-\xi)^{m-1} f_{1m}(\xi;\epsilon)\, d\xi, \tag{29}$$

i.e.,

$$f_{1m}(x;\epsilon) = O(\epsilon^{2+m}).$$

This means that the terms containing f_{1m}, $m > 1$, do not really belong in φ_1, since they are of higher order than the leading term.

We now have the result that

$$\varphi_1(x,y,z;\epsilon) = \frac{\sin\theta}{r} \int_0^\infty \frac{\left[\sqrt{(x-\xi)+r^2} + (x-\xi)\right] f_{11}(\xi;\epsilon)\, d\xi}{\sqrt{(x-\xi)^2 + r^2}} \tag{30}$$

with

$$f_{11}(x;\epsilon) = \frac{\epsilon}{2} A_{11_X}(X;\epsilon) = O(\epsilon^3). \tag{30'}$$

Thus

$$\varphi_1(x,y,z;\epsilon) = O(\epsilon^3). \tag{31}$$

The most important result to come out of the matching process is the condition that $\Phi_1 \to 0$ as $R \to \infty$. This is the missing boundary condition on the inner solution term, and we can now formulate the problem for Φ_1 completely, viz., Φ_1 satisfies (22'), (24'), (26), (27), and

$$\Phi_1(X,Y,Z;\epsilon) = \frac{(2/\epsilon)\, F_{11}(X;\epsilon)\sin\theta}{R}\left[1 + O(1/R)\right] \tag{32}$$

as $R \to \infty$, where

$$F_{11}(x;\epsilon) = \int_0^x f_{11}(\xi;\epsilon)\, d\xi. \tag{33}$$

From Bernoulli's equation (properly stretched), we obtain for the pressure on the body:

$$P_0(X,Y,H(X,Y);\epsilon) = -\rho\left\{\epsilon^2 GH + U\Phi_{1_X} + \frac{1}{2\epsilon^2}\left[\Phi_{1_Y}^2 + \Phi_{1_Z}^2\right]\right\} \tag{34}$$

(This is really the first term in an inner asymptotic expansion for $p(x,y,z)$, as indicated by the subscript on P_0.)

There has been nothing in the formulation of this problem which implies that the body is symmetrical in y. Therefore, the fact that Φ_1 is symmetrical in Y for large R simply implies that the unsymmetrical part dies out more rapidly than $1/R$ as $R \to \infty$. It might be assumed that it does not appear here because it is higher order in ϵ, but this is not the case. In fact, the unsymmetrical part of Φ_1, if it exists at all, is $O(\epsilon^2)$. Nevertheless, it must be matched to φ_2, which is $O(\epsilon^4)$, which indicates that it is indeed higher order *in the far field* than the first approximation.

It is of some interest to consider the nature of the far-field solution, φ_1. With the definition (33), we can perform some simple manipulations on (30) to show that

$$\varphi_1(x,y,z;\epsilon) = \int_0^\infty \frac{z\ F_{11}(\xi;\epsilon)\ d\xi}{\left[(x-\xi)^2 + y^2 + z^2\right]^{3/2}}, \tag{35}$$

which is the potential of a line of vertical dipoles along the x-axis, of density $F_{11}(x;\epsilon)$. This can also be interpreted as a system of "vortex-pairs", which is equivalent to a system of horseshoe vortices compressed down to zero span. If $F_{11}(x;\epsilon)$ does not vanish aft of the body, there exists a vortex wake.

Finally, we note form (17) and (35) that we can write down explicitly the expression for the far-field free-surface disturbance:

$$\zeta_0(x,y;\epsilon) = \frac{1}{Uy^2}\int_0^\infty d\xi\ F_{11}(\xi;\epsilon)\left[1 + \frac{x-\xi}{\left[(x-\xi)^2 + y^2\right]^{1/2}}\right]. \tag{36}$$

3. *The variety of possible problems*

In the Introduction, it was stated that there are four - and only four - free-surface, slender-body problems of physical interest (for steady motion). These were arranged according to the order of magnitude of gravity or, what is equivalent, according to the order of magnitude of Froude number. Physically, the four problems were distinguished according to the occurrence or nonoccurrence of gravity waves in the first approximations for near-field and far-field solutions.

Setting out now to demonstrate these assertions, we find it convenient to dispose of another question at the same time, namely, the possibility of generating other problems of physical interest by a different selection of the stretching parameter. Whenever the method of matched asymptotic expansions is used, this question should be considered, for there is no absolutely dependable general procedure for deciding how to distort the coordinates in the inner problem. We shall show that no new physical consequences can be found from other stretching arrangements, at least within a large class of such distortions. To be specific, we shall limit our consideration to distortions in which the longitudinal scale is not altered and in which the two transverse scales are stretched by like amounts, in proportion to a power of ϵ. It should be emphasized that the statements and conclusions which follow apply only to the lowest-order nontrivial terms in the asymptotic expansions.

To express the full generality allowed above, we define a new gravity constant,

$$G = g\epsilon^{-\gamma} = O(1),$$

where γ is arbitrary, and we stretch the coordinates as follows:

$$X = x; \quad Y = y\epsilon^{-\beta}; \quad Z = z\epsilon^{-\beta},$$

β being greater than zero but otherwise arbitrary. Now we see what this generality does to the formulations of the outer and inner problems.

The outer-region behavior does not depend directly on the stretching (although absolute orders of magnitude do). Therefore we need only consider the effect of varying γ. It is easily seen that the outer solution satisfies the 3D Laplace equation and furthermore that it is necessarily linear, so that we have the two free-surface condition:

$$\text{A.} \quad 0 = \epsilon^{\gamma} G\zeta_0 + U\varphi_{1_x};$$

B. $0 = U\zeta_{0_x} - \varphi_{1_z}$;

both are applied on $z = 0$. The ζ_0-terms cannot be higher order than the φ_1-terms in *both* equations, since that implies that $\varphi_{1x} = \varphi_{1z} = 0$ on $z = 0$, i.e., $\varphi_1 \equiv 0$ everywhere. Furthermore, the ζ_0-term cannot be lower order in *either* equation, since that implies that $\zeta_0 \equiv 0$. Now we have three all-inclusive conditions, which we present in tabular form:

OUTER REGION

$\gamma < 0$	$\gamma = 0$	$\gamma > 0$
$\varphi_{1z} = 0$ on $z = 0$. (rigid-wall condition)	gravity waves	$\varphi_1 = 0$ on $z = 0$. (pressure-relief condition)

In the inner region, we distort the space variables as described, obtaining the 2D Laplace equation, the two free-surface conditions:

$$\text{A.} \quad 0 = \epsilon^{\gamma+\beta} Z_0 + U\,\Phi_{1_X} + \epsilon^{-2\beta}\left[\Phi_{1_Y}^{2} + \Phi_{1_Z}^{2}\right],$$

$$\text{B.} \quad 0 = \epsilon^{\beta}\,U\,Z_{0_X} + \epsilon^{-\beta}\,\Phi_{1_Y}\,Z_{0_Y} - \epsilon^{-\beta}\,\Phi_{1_Z},$$

and the body boundary condition:

$$\text{C.} \quad \frac{\partial \Phi_1}{\partial N} = \epsilon^{2\beta}\,UH_X \Big/ \sqrt{1 + H_Y^2},$$

where $H(X, Y) = \epsilon^{-\beta} h(x, y)$. Condition C implies that $\Phi_1 = O(\epsilon^{2\beta})$. We now find that there are again three possible conditions:

INNER REGION

$\gamma < \beta$	$\gamma = \beta$	$\gamma > \beta$
$\Phi_{1Z} = 0$ on $Z = O$. (rigid-wall condition)	gravity waves of finite amplitude	pressure-relief condition (finite-amplitude disturbance)

We see that the nature of the outer (or inner) solution depends only on whether γ is less than, equal to, or greater than 0 (or β). The actual value of β is of no consequence, except that it must be (strictly) greater than zero. Therefore we may as well take it to be unity. Then there are in all five cases to be considered for γ:

a.	b.	c.	d.	e.
$\gamma < 0$	$\gamma = 0$	$0 < \gamma < 1$	$\gamma = 1$	$\gamma > 1$

Cases a, b, d, and e cerrespond, respectively, to the cases (1) - (4) in the Introduction.

We discard c. as offering nothing new physically. It is really a strange hybrid case: the solution satisfies the rigid-wall condition in the near field and the simple pressure-relief condition (without gravity) in the far-field. In other words, the near-field solution corresponds to the limit of low Froude number, the far-field solution to the limit of high Froude number. The interesting effects of gravity are submerged somewhere between the two regions. In the language of the Appendix, we could assume that there would be an "intermediate problem", in which gravity would appear explicitly. In fact, it is clear that there must be more to the problem of case c. than simply finding inner and outer expansions, for the following reason. The inner solution satisfies $\Phi_{1_Z} = 0$ on $Z = 0$, and so it must have a large-R representation of the form

$$\sum_{m=1}^{\infty} \cos m\theta \left[\frac{A_m(X;\epsilon)}{R^m} + B_m(X;\epsilon)\, R^m\right] + A_0(X;\epsilon) + C(X;\epsilon)\log R.$$

The outer solution satisfies $\varphi_1 = 0$ on $z = 0$, and so it has a small-r representation of the form

$$\sum_{m=1}^{\infty} \sin m\theta \, \frac{a_m(x;\epsilon)}{r^m}.$$

Obviously, these cannot be matched.

The methods of the Appendix can be applied to provide a solution. It can be shown that in addition to the inner and outer expansions there is a third expansion to be found, which is valid between the other two. That is, it is an inner expansion with respect to the original outer expansion and an outer expansion with respect to the original inner expansion. It must be obtained in detail through two matching procedures. It satisfies the 2D Laplace equation and linear free-surface conditions. (However, it is not a conventional 2D gravity wave problem.) Physically, it cannot provide any information not already implied in cases b and d.

With respect to our basic set of four problems, arranged according to the order of magnitude of g, it was indicated in the Introduction that gravity generally has a greater effect in the inner region than in the outer. This point deserves some emphasis and elaboration. In Tuck's (1965)[1] analysis ($g = O(1)$), gravity seems to disappear altogether in the near field, for there he obtains a gravity-free boundary-value problem. However, what actually happens is that gravity *dominates* the near-field flow to such an extent that the first approximation is a perturbation about an infinite-g process. If we assume that $g = O(\epsilon^{-1})$, this behavior extends into the outer field as well.

If now we assume that gravity is weak, viz., $g = O(\epsilon)$, the already-small effect of gravity in Tuck's far-field problem disappears altogether. That is, the free surface becomes a simple pressure-relief surface. In the near field, the dominance of gravity is weakened to the point where gravity waves occur. With the severe constraint of gravity reduced, the disturbance to the free surface is much increased, and we have the finite-amplitude problem described in detail in the previous section. If we degrade gravity one degree further, say let $g = O(\epsilon^2)$, the outer problem is unchanged, since gravity already has no effect there. In addition, the effect of gravity drops out of the inner region.

4. *Application of the theory for $g = O(\epsilon)$*

Planing Surfaces. The equation describing the body surface was originally chosen in the form (1) specifically for the application to problems of planing surfaces. There seems to be little question that the theory as developed here *ought* to apply to such problems, provided the configurations studied do not violate the assumptions of slender-body theory. These assumptions are most likely to be of concern as they relate to the geometry of the bow and stern.

If the leading edge of the planing surface is perpendicular to the direction of travel, a violation of the assumptions does occur there. However, if the planing surface has just a small amount of deadrise, the leading edge of the wetted area will be appreciably swept-back in shape, and the geometry at the bow does not prevent application of slender-body theory. Fortunately, most planing boats do have some deadrise, for other reasons.

The stern causes more of a problem, at least in principle. A planing boat usually has a sharp trailing edge which produces an effect like that of an airfoil trailing edge, namely, it constrains the fluid to pass off the planing surface smoothly at a given angle. In linearized analyses of planing surfaces, one usually treats this sutuation by postulating a Kutta condition -- which is just as valid as in the airfoil problem.

In the numerical solution of our nonlinear inner problem, it is simple to guarantee the same result, i.e., that the fluid leaves the stern smoothly, provided the lifting surface ends abruptly. (Obviously, an ideal-fluid theory cannot predict a separation from a smoothly curving surface.) A proposed method for obtaining the numerical solution is described in the next section. Anticipating that discussion somewhat, we would use Equation (27) to obtain the free surface shape at $x = x_0 + \Delta x$ from the shape at $x = x_0$ (through a finite-difference approximation). Now, if $x = x_0$ denotes the trailing edge (assumed perpendicular to the x-axis), we still could use (27) to step the solution to the next section, $x = x_0 + \Delta x$, using the values of all quantities as calculated at $x = x_0$. The free surface will then automatically extend smoothly off the trailing edge. There is no problem in principle about this procedure even if the trailing edge is not perpendicular to the x-axis. Thus the kinematic free-surface condition provides the means for including in the theory a sharp free-surface breakway from a typical planing-boat stern.

On the other hand, the dynamic free-surface condition leads to fundamental difficulties. The pressure on the body is given by Equation (34). If there is not to be a discontinuity in pressure at the stern ($x = x_0$), then the pressure must approach zero as $x \rightarrow x_0$ on the hull. But there is no reason in general to predict that Equation (34) will act in such a convenient manner. It is characteristic of the first approximation in this theory (as in all slender-body theories) that a disturbance at a particular section can never have an effect upstream of that section, and so there is no mechanism by which the flow (as described by the theory) can adjust upstream to produce the smooth behavior expected at the trailing edge. Thus the theory will generally predict an abrupt change in pressure and in the other variables, notably Φ_{1x}, at the stern of a planing surface. This is clearly not consistent with the slender-body assumptions.

It would be easy to say simply that we should not attempt to apply slender-body theory to such problems, but experience in aerodynamics suggests that we may be more optimistic than that. The same failing occurs in many aerodynamic applications of slender-body theory; nevertheless, it is well-known that the resulting predictions are far from useless. The effects of the trailing edge are indeed manifested at upstream sections, and the pressure does adjust itself so that the boundary conditions are satisfied smoothly. Such effects, however, are significant over only a small percentage of the chord-length. The total force on the body is still predicted fairly accurately, the moment somewhat less accurately.

The application of slender-body theory to thin wings may be used to suggest further reasons for optimism in this matter. For a thin wing with no lateral curvature, sufficient conditions for satisfaction of the trailing-edge condition are that at the trailing edge (a) the wing have no longitudinal curvature and (b) the rate of change of span vanish. Under these conditions, the predicted pressure discontinuity (between upper and lower surfaces) *automatically* approaches zero at the trailing edge. Conventional planing hulls often satisfy such geometrical restrictions, and so one may expect similar propitious consequences. Unfortunately there does not seem to be any way of proving this.

Displacement Hulls. Much of the preceding discussion for planing surfaces can be carried over to displacement hulls, and so only the differences will be noted here.

Firstly, a conventional ship is likely not to have a sharply cut-off stern; so there is no problem in such cases with pressure discontinuity. On the other hand, a smoothly rounded stern may be surrounded by a separated flow, and this phenomenon cannot be predicted by the present method (or any other method in existence). The error in predictions may be of greater consequence than the error due to trailing-edge discontinuities.

Secondly, it appears that the speeds of displacement ships are far too low for the present theory to be applicable. Certainly, the usual values of Froude number for displacement ships lie below 0.4, contradicting the assumption herein that $F = O(\epsilon^{-\frac{1}{2}})$. Nevertheless, an *a priori* judgment about reasonable values of a small parameter in an asymptotic solution is a chimera. This point has been argued at length in the Introduction. All that remains now is to try the method to see how well it works out.

Even if the present method does not provide a realistic description of the flow around a conventional ship, there is a distinct aspect of the same problem in which it should still be useful. Slender-body analysis is characterized by the fact that the flow at any cross-section is assumed to be unaffected by phenomena at after sections. This suggests that the flow around the bow is largely independent of total ship length, and so it should be possible to predict the bow flow independently of the flow around the rest of the ship.

Since Froude number is generally based on overall ship length, we might suppose that an unconventional Froude number which relates to local conditions would be more appropriate for the analysis of the local flow around the bow. Formally, we can define a "running Froude number", similar to the Reynolds number often used in studying boundary layers, say,

$$F_x = U/\sqrt{gx}. \tag{37}$$

The next logical step would be to apply the high-Froude-number analysis to that part of the ship in which F_x is greater than some fixed number. This part of the ship must still be geometrically consistent with the slenderness assumptions, of course.

Calculating the flow around the bow of a conventional ship at conventional ship speed is an old problem on which practically no progress has ever been made. It is basically a nonlinear problem, and the usual linear ship theories can at best include some account of the important nonlinearities only in the form of singularities. Since the small-g slender-body theory presented herein actually involves a nonlinear near-field description, there is a reasonable chance that it can provide a more detailed description of the bow flow than has been possible heretofore.

It may also be noted that the present theory includes all cases for which $g = o(\epsilon)$. The effect of gravity enters into the boundary value problem only through Equation (26), the near-field dynamic free-surface condition. If Froude number is exceedingly large, the first term in (26) simply becomes very small, until it vanishes, and then we have the appropriate equation for the case $g = o(\epsilon)$. Thus, the use of the "running-Froude-number" argument does not invalidate the theory as developed for $g = O(\epsilon)$ even at the bow where, by (37), $F_x = \infty$.

The fact that slender-body theory treats the cross-sections in succession is also a limitation. In particular, it prevents one from obtaining a description of the accelerated flow just ahead of the bow. Of course, if the bow is blunt, the whole slender-body treatment fails. Tuck (1964)[7] has provided an analysis technique for treating the end singularity in an infinite fluid, but his technique has not been found applicable to the case of the surface ship. It seems quite possible that with the assumptions presented herein (concerning gravity) the Tuck procedure could in fact be applied to the ship bow problem.

5. *Formulation of the numerical problem*

The solution of the near-field problem has been reduced in the first approximation to finding a function, Φ_1, which satisfies the 2D Laplace equation, (24'), subject to the body boundary condition, (22'), the two free surface conditions, (26) and (27), and a condition at infinity, (32). The pressure on the body is then given by Equation (34). There seems to be no alternative to attempting a numerical solution of this problem.

Through the use of integral equations, each of the two-dimensional problems can be reduced to a one-dimensional problem. Fortunately, the computational procedure can be set down fairly easily. We start with Green's theorem in two dimensions for the potential at any point in the fluid:

$$\Phi_1(X, Y, Z) = \frac{1}{2\pi}\int\left[\Phi_{1_N} \log R_1 - \Phi_1 \frac{\partial}{\partial N}(\log R_1)\right] dS', \tag{38}$$

where $R_1^2 = (Y - Y')^2 + (Z - Z')^2$.

The integration must extend over the body, the free surface, and a closing surface at infinity, but Equation (32) shows that the last of these can be ignored. On the body, we know the normal derivative, from (24'), but not the value of the potential. If we know the value of the potential at any section, we can use Equation (26) in a finite difference calculation to find its value on the free surface at the section located a short distance aft. However, we do not have a direct way of finding the normal derivative on the free surface. The procedure then is to let the point (X, Y, Z) in Equation (38) approach the fluid boundary and solve the resulting integral equation for (a) Φ_1 on the body and (b) $\partial \Phi_1/\partial N$ on the free surface. The position of the free surface at the section is known from the solution at the previous section, through use of Equation (27). Equation (34) then gives the pressure on the body.

APPENDIX -- THE INTERMEDIATE PROBLEM

Previously we set up inner- and outer-region problems in which the transverse coordinates were, respectively, stretched by a factor $1/\epsilon$ and unstretched. Now we set up a new problem in which the amount of stretching is not completely specified. (Actually, we set up an infinite number of problems.) We define new coordinates as follows:

$$X = x; \quad Y = y\epsilon^{-\alpha}; \quad Z = z\epsilon^{-\alpha}; \quad 0 < \alpha < 1. \tag{39}$$

The stretching parameter α is not to be confused with the β used previously. Here we take $\alpha = \beta = 1$ always to correspond to the inner coordinates and $\alpha = 0$ to correspond to the outer coordinates; other values of α remain simultaneously under consideration, as they lead to other problems which are related to the outer and inner problems. For $0 < \alpha < 1$, we can apply neither the body boundary condition nor the radiation condition; the intermediate solution must allow arbitrary behavior at $R = 0$ and $R = \infty$, so that it can be matched to both inner and outer solutions. However, as we see presently, the intermediate problem is simpler than the problems for $\alpha = 0$ or $\alpha = 1$.

As before, we assume the existence of an asymptotic expansion for $\varphi(x, y, z)$, but there are now an infinity of expansions, depending on the value of α, and so we write α as an extra parameter in the terms of the expansion:

$$\varphi(x, y, z) \sim \sum_{n=0}^{N} \Phi_n \, (X, Y, Z; \epsilon; \alpha),$$

$\Phi_{n+1} = o(\Phi_n)$ for fixed $\sqrt{Y^2 + Z^2} = O(1)$, fixed α, as $\epsilon \to 0$. Similarly there is an expansion for ζ:

$$\zeta(x, y) \sim \sum_{n=0}^{N} \epsilon^{\alpha} Z_n \, (X, Y; \epsilon; \alpha),$$

$Z_{n+1} = o(Z_n)$ for fixed $Y = O(1)$, fixed α, as $\epsilon \to 0$. It must be understood that for *each* α the small-ϵ expansion is to be found and then re-ordered appropriately in terms of ϵ; this is what is meant by the expression "fixed α".

The formulation of the problem proceeds in a manner quite analogous to that of the inner problem previously, and so it will only be outlined here. First, it is rather obvious that

$$\Phi_0 \, (X, Y, Z; \epsilon; \alpha) = U X$$

for any α. Then we find that Φ_1 satisfies the 2D Laplace equation and the linear free-surface conditions:

$$0 = \Phi_1 \quad \text{on} \quad Z = 0;$$

$$0 = \epsilon^{2\alpha} U Z_{0_X} - \Phi_{1_Z} \quad \text{on} \quad Z = 0. \tag{40}$$

Thus Φ_1 satisfies the simpler aspects of each of the two limit solutions, that is, the simpler differential equation of the inner problem and the linear boundary conditions of the outer problem. The general solution is:

$$\Phi_1(X, Y, Z; \epsilon; \alpha) = \sum_{m=1}^{\infty} \sin m\theta \left\{ \frac{a_{1m}(X; \epsilon; \alpha)}{R^m} + b_{1m}(X; \epsilon; \alpha) R^m \right\}, \tag{41}$$

where $Y = R \cos \theta$, $Z = R \sin \theta$. Of course, nothing can be said about the coefficients a_{1m} and b_{1m} until the matching process is considered. There must be an overlap between the domain of validity of this solution (for some range of α) and that of the general outer solution, Equation (19). This means that in the small-r approximation to the latter we change variables according to (39); the result should be asymptotically the same as the intermediate solution, for α fixed, as $\epsilon \to 0$. In this way we obtain:

$$a_{1m}(X; \epsilon; \alpha) = 2^m \cdot \epsilon^{-m\alpha} \int_0^X (X - \xi) \, f_{1m}(\xi, \epsilon) \, d\xi; \tag{42}$$

$$b_{1m}(X; \epsilon; \alpha) = 0;$$

and thus we have matched the outer and intermediate solutions. The above formula can be compared with Equation (29).

In order to distinguish between independent variables of the inner and intermediate problems, let us temporarily denote them by subscripts 1 and α, respectively, i.e., $R_1 = r\epsilon^{-1}$ and $R_\alpha = r\epsilon^{-\alpha}$, etc. The intermediate problem as formulated above is appropriate for α arbitrarily close to unity, and so we can match the series (41) to the inner solution, $\Phi_1(X, Y, Z;\epsilon;1) \equiv \Phi_1(X_1, Y_1, Z_1;\epsilon;1)$. In the latter we let $(Y_1^2 + Z_1^2)^{\frac{1}{2}} = R_1 = R_\alpha \epsilon^{-1+\alpha}$; the matching principle requires that, to leading order,

$$\Phi_1(X_\alpha, Y_\alpha \epsilon^{-1+\alpha}, Z_\alpha \epsilon^{-1+\alpha};\epsilon;1) = \sum_{m=1}^{\infty} \frac{a_{1m}(X;\epsilon;\alpha) \sin m\theta}{R_\alpha^m}.$$

This is equivalent to specifying the behavior of $\Phi_1(X_1, Y_1, Z_1;\epsilon;1)$ for large R_1, that is, if we now resubstitute $R_1 = R_\alpha \epsilon^{-1+\alpha}$, we have, for large R_1,

$$\Phi_1(X_1, Y_1, Z_1;\epsilon;1) = \sum_{m=1}^{\infty} \frac{a_{1m}(X;\epsilon;\alpha) \sin m\theta}{R^m \; \epsilon^{m(1-\alpha)}}.$$

This is $O(\epsilon^2)$, and so $a_{1m}(X;\epsilon;\alpha) = O(\epsilon^{2+m-m\alpha})$. From (42), we see that $f_{1m}(X;\epsilon) = O(\epsilon^{2+m})$, which was stated but not quite proved earlier. This implies then that the only term which should appear in the outer solution is that for which $m = 1$.

In the intermediate problem, we now see that the leading term (and thus the only term) is

$$\Phi_1(X, Y, Z;\epsilon;\alpha) = \frac{a_{11}(X;\epsilon;\alpha) \sin \theta}{R}$$

$$= \frac{2\epsilon^{-\alpha}}{R} F_{11}(X;\epsilon) \sin \theta. \tag{43}$$

(See (33) and (42).) We can also obtain very simply from Equation (40) that

$$Z_0(X, Y;\epsilon;\alpha) = \frac{2\epsilon^{-3\alpha}}{U\; Y^2} \int_0^X d\xi \;\; F_{11}(\xi;\epsilon).$$

We have not really obtained any new results here, except to prove the validity of (43), which implies that

$$\Phi_1(X, Y, Z;\epsilon;1) \cong \frac{2\; F_{11}(X;\epsilon) \sin \theta}{\epsilon R}$$

for large R. Further contributions to Φ_1 at large R will be small in comparison either because of the R-dependence or because of the ϵ-dependence. In particular, we note that any antisymmetric part of Φ_1 must be $o(1/R)$ as $R \to \infty$ and/or $o(\epsilon^2)$ as $\epsilon \to 0$.

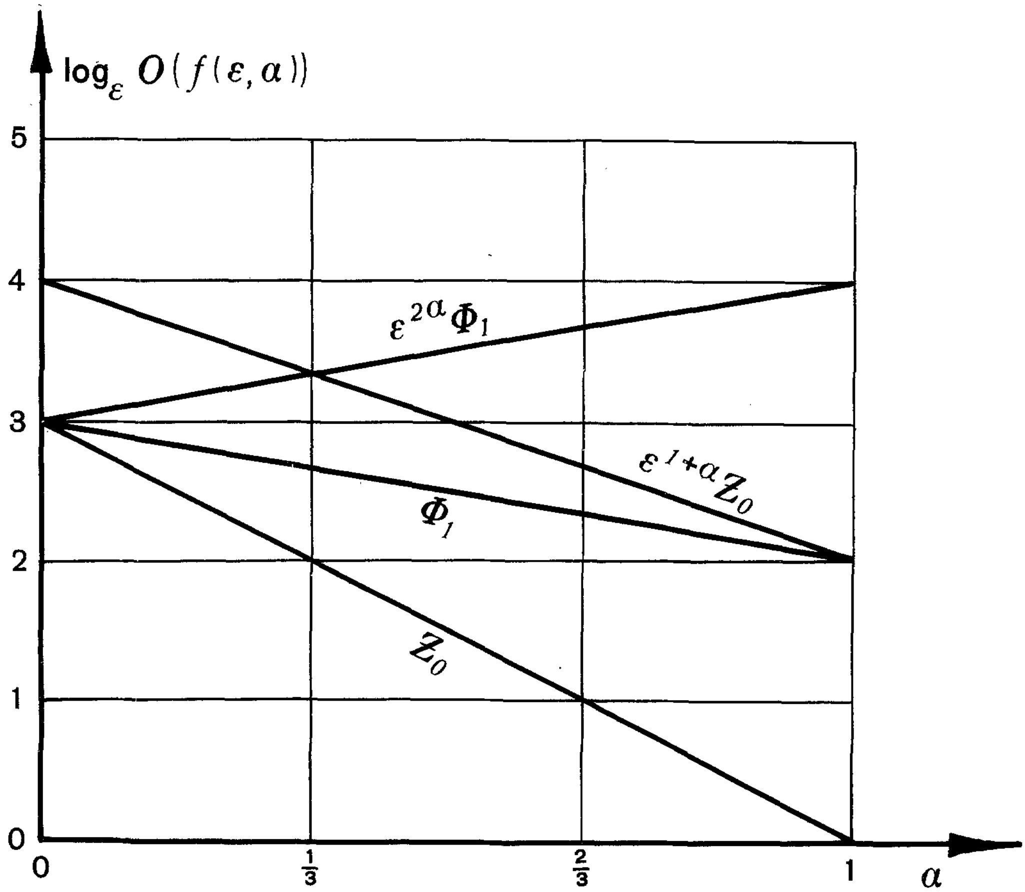
This completes the analysis of the lowest order nontrivial terms in the asymptotic expansions for $\varphi(x, y, z)$. The extension to higher-order terms is not likely to yield useful results, but we shall take a brief look at the next stages, for it throws some light on the structure of the solutions in the different regions.

We have found that $\Phi_1 = O(\epsilon^{3-\alpha})$ and $Z_0 = O(\epsilon^{3-3\alpha})$. If we use these facts and reconstruct the differential equation and boundary conditions to a higher order than previously, we obtain conditions on Φ_2 and Z_1, for $0 < \alpha < 1$.

First is the differential equation for Φ_2:

$$\Phi_{2_{YY}} + \Phi_{2_{ZZ}} = -\epsilon^{2\alpha} \Phi_{1_{XX}} .$$

The right hand side is $O(\epsilon^{3+\alpha})$, and so Φ_2 is no higher in order than this. However it could be of lower order, in which case the right hand side vanished. We can represent this situation to advantage by a simple figure



in which the abscissa is α and the ordinate is the exponent of ϵ in the order-of-magnitude statement. Thus, $\Phi_1 = O(\epsilon^{3-\alpha})$ is represented by the straight line, ordinate = $3 - \alpha$. We could call the ordinate $\log_\epsilon O(\Phi_1)$, in a symbolic sense. The representation for Φ_2 in this figure cannot lie above that for $\epsilon^{2\alpha}\Phi_1$.

Second is the dynamic free-surface condition, which is:

$$0 = \epsilon^{1+\alpha} G Z_0 + U \Phi_{2_X} \quad \text{on} \quad Z = 0.$$

The first term is $O(\epsilon^{4-2\alpha})$, which is also indicated in the figure. Since this term is completely known by the time we undertake the Φ_2-problem, it cannot be zero, and so it cannot be lower order than the second term. In other words, Φ_2 cannot be higher order than $\epsilon^{4-2\alpha}$.

From the figure, we see now that the intermediate problem for Φ_2 must be broken into two parts:

$0 < \alpha < 1/3$ | *$1/3 < \alpha < 1$*

$$\Phi_2 = O(\epsilon^{3+\alpha}) \qquad\qquad \Phi_2 = O(\epsilon^{4-2\alpha})$$

$$\Phi_{2_{YY}} + \Phi_{2_{ZZ}} = -\epsilon^{2\alpha}\,\Phi_{1_{XX}} \qquad\qquad \Phi_{2_{YY}} + \Phi_{2_{ZZ}} = 0$$

$$\Phi_{2_X} = 0 \quad \text{on} \quad Z = 0 \qquad\qquad \epsilon^{1+\alpha}\, G\, Z_0 + U\,\Phi_{2_X} = 0 \quad \text{on} \quad Z = 0.$$

In both cases we obtain the kinematic condition:

$$0 = \epsilon^{2\alpha}\, U\, Z_{1_X} - \Phi_{2_Z} \quad \text{on} \quad Z = 0.$$

This division is reasonable enough. The previous outer problem differs from the corresponding intermediate problem in that its solution satisfies the 3D Laplace equation, and this difference is having its effect in the present problem for $\alpha > 0$. On the other hand, the previous inner problem involved finite free-surface displacements, and these are now affecting the intermediate problem.

It is also obvious that the problems for $\alpha = 0$ and $\alpha = 1$ must be treated separately, for, by definition, $\Phi_2 = o(\Phi_1)$ in either case. Yet the intermediate problems formulated above show that Φ_2 approaches $O(\Phi_1)$ as $\alpha \to 0$ or 1. The outer problem can in fact be solved explicitly (if not uniquely); its solution is:

$$\begin{aligned}
\varphi_2(x, y, z;\epsilon) &= \Phi_2(X, Y, Z;\epsilon;0) \qquad (44)\\
&= -\frac{\epsilon G}{2U^2}\left\{\frac{\cos 2\theta}{R^2}\int_0^\infty d\xi\; F_{11}(\xi;\epsilon)\,\frac{\left[\sqrt{(X-\xi)^2 + R^2} + (X-\xi)\right]^2}{\sqrt{(X-\xi)^2 + R^2}} + \int_0^\infty \frac{F_{11}(\xi;\epsilon)\, d\xi}{\sqrt{(X-\xi)^2 + R^2}}\right\}\\
&\quad + \sum_{m=1}^\infty \frac{\sin m\theta}{R^m}\int_0^\infty d\xi\; f_{2m}(\xi;\epsilon)\,\frac{\left[\sqrt{(X-\xi)^2 + R^2} + (X-\xi)\right]^m}{\sqrt{(X-\xi)^2 + R^2}}
\end{aligned}$$

The first two terms are presumably known; they are both $O(\epsilon^4)$. The sum involves another set of unknown functions, $f_{2m}(X;\epsilon)$, to be found through an appropriate matching process. One might expect Φ_2, for $\alpha = 0$, to be $O(\epsilon^4)$, and this is indeed the case; certainly it cannot be $o(\epsilon^4)$.

To effect the matching properly, one does not attempt to match just Φ_2 in the different α-domains, of course, but rather the whole asymptotic solution as far as it is known. That is, we must write down $\Phi_0 + \Phi_1 + \Phi_2$ in each domain, reorder the terms with respect to ϵ, and match the resulting expressions in the overlap regions. The general three-term solutions in $0 < \alpha < 1/3$ and $1/3 < \alpha < 1$ are readily found, respectively:

$0 < \alpha < 1/3$

$$\begin{aligned}
\Phi_0 + \Phi_1 + \Phi_2 &= U X + \frac{2\epsilon^{-\alpha}}{R} F_{11}(X;\epsilon)\sin\theta \qquad (45)\\
&\quad - \epsilon^{\alpha} F_{11_{XX}}(X;\epsilon)\, R\log R\sin\theta\\
&\quad + \sum_{m=1}^\infty \sin m\theta\left\{\frac{a_{2m}(X;\epsilon;\alpha)}{R^m} + b_{2m}(X;\epsilon;\alpha)\, R^m\right\}.
\end{aligned}$$

1/3 < α < 1

$$\Phi_0 + \Phi_1 + \Phi_2 = U X + \frac{2\epsilon^{-\alpha}}{R} F_{11}(X;\epsilon) \sin\theta \tag{46}$$

$$- \frac{2\epsilon^{1-2\alpha}}{U^2} \frac{G \cos 2\theta}{R^2} \int_0^X (X - \xi) F_{11}(\xi;\epsilon)\, d\xi$$

$$+ \sum_{m=1}^{\infty} \sin m\theta \left\{ \frac{a_{2m}(X;\epsilon;\alpha)}{R^m} + b_{2m}(X;\epsilon;\alpha) R^m . \right\}$$

The three-term solution for the outer region will not be rewritten; it consists of the terms already given in (16), (30), and (44). The inner solution is not known explicitly. We might include here also the case that $\alpha = 1/3$, but this would add nothing to the results.

For α near to zero, we find that Φ_2 has a term proportional to R log R, and this term is $O(\epsilon^{3+\alpha})$. Since this term arose in the solution of the Poisson equation which must be satisfied in this domain, one may expect that it is related to the three-dimensional nature of the first outer solution. In fact, this is the case, for if one carries the small-r approximation given in Equation (20) further than just the one term, one finds that the next term exactly matches the term in question in Equation (45) above. In the usual way with these expansions, we can speak loosely of a logarithm as being O(1), and then we can go one step further in this same matching process, finding a further term in the outer solution for Φ_1 which matches the term containing b_{2m} in (45), thus determining this coefficient.

If we consider the term Φ_3 for a moment, we find that it satisfies a Poisson equation too for small α:

$$\Phi_{3_{YY}} + \Phi_{3_{ZZ}} = - \epsilon^{2\alpha} \Phi_{2_{XX}} = O(\epsilon^{3+3\alpha}), \quad 0 < \alpha < 1/5.$$

The order-of-magnitude statement alone indicates that at least the first term of Φ_3 will be controlled by Φ_1 in the outer region. However, the domain of α in which this occurs has been reduced; we must have $\alpha < 1/5$. We can expect Φ_n to be controlled by $\Phi_1(X, Y, Z;\epsilon;0)$ in a small range of α near $\alpha = 0$ *for every n*.

We can avoid all of this difficulty, as follows. The term Φ_1 in the intermediate solution is identical to the leading term of $\Phi_1(X, Y, Z;\epsilon;0)$ evaluated for $r = R\,\epsilon^{\alpha}$, as $\epsilon \to 0$. So we could use $\Phi_1(X, Y, Z;\epsilon;0)$ directly to generate the first term of the intermediate solution - to as close to $\alpha = 1$ as we like. In other words, the intermediate solution is simply the intermediate-range limit of the outer solution. (This is not true in many problems.) We have shown directly that this was true for Φ_1, and it can be seen that it is true for Φ_2 as well, for the established part of $\Phi_2(X, Y, Z;\epsilon;\alpha)$ given above in Equation (46) exactly matches the intermediate expansion of $\Phi_2(X, Y, Z;\epsilon;0)$ in the range $1/3 < \alpha < 1$ (the term containing cos 2θ). We note that the term containing R log R is dropped because in this range it is higher order in ϵ.

With respect to matching intermediate and inner solutions, not much can be said. The unknown functions, $f_{2m}(X;\epsilon)$, will be determined in this process, and they will almost all vanish, for they will be found to be higher order in ϵ. Clearly, the term containing f_{21} must match Φ_2 of the inner solution, since the $\sin\theta/R$ -behavior of Φ_1 is already accounted for in the first-order intermediate and outer solutions. On the other hand, the term containing f_{22} will match the antisymmetric behavior (if any) of the first inner solution. If the body is not symmetric in y, f_{22} will be $O(\epsilon^4)$, and this term must appear in the intermediate and outer solutions, Φ_2.

Thus, asymmetries arise in the far field first in the Φ_2 -term and are of order ϵ^4.

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