NONLINEAR INTEGRODIFFERENTIAL EQUATIONS IN A BANACH SPACE (*)

by ENZO MITIDIERI (in Udine) and IOAN I. VRABIE (in Iasi)(**)

SOMMARIO.- *Si prova un risultato di esistenza per una classe di equazioni integrodifferenziali del tipo*

$$[u'(t) + Au(t)] \cap \int_0^t k(t-s)F(s,u(s))ds \neq \emptyset,\ 0 \leq t \leq T$$

$$u(0) = u_0,$$

dove A è un operatore m-accretivo su uno spazio di Banach reale X con risolvente $(I + \lambda A)^{-1}$ compatto per ogni $\lambda > 0$, $k : [0,T] \to L(X)$ è un nucleo operatoriale ed $F : [0,T] \times D(A) \to 2^X$ è una applicazione multivoca soddisfacente ed una certa condizione di continuità.

SUMMARY.- *We prove an existence result for a class of integrodifferential equations of the form*

$$[u'(t) + Au(t)] \cap \int_0^t k(t-s)F(s,u(s))ds \neq \emptyset,\ 0 \leq t \leq T$$

$$u(0) = u_0,$$

where A is an m-accretive operator acting in a real Banach space X with $(I + \lambda A)^{-1}$ compact for each $\lambda > 0$, $k : [0,T] \to L(X)$ is a C^1 operator kernel and $F : [0,T] \times D(A) \to 2^X$ is a multivalued mapping satisfying a certain continuity condition.

(*) Pervenuto in Redazione il 18 maggio 1989.
Supported by M.P.I. (40% - Equazioni Differenziali).

(**) Indirizzi degli Autori: E. Mitidieri: Dipartimento di Matematica e Informatica, Università di Udine - 33100 Udine e Dipartimento di Scienze Matematiche - Università di Trieste - 34100 Trieste (Italy); I.I. Vrabie: Department of Mathematics - Polytechnic Institute of Iasi - 6600 Iasi (Romania).

1. Introduction.

In the present paper we prove an existence result for a class of integrodifferential equations of the form

$$(1.1)\qquad \begin{aligned} &[u'(t) + Au(t)] \cap \int_0^t k(t\text{-}s)F(s,u(s))ds \neq \emptyset, 0 \leq t \leq T \\ &u(0) = u_0, \end{aligned}$$

where A is an m-accretive operator acting in a real Banach space X with $(I+\lambda A)^{-1}$ compact for each $\lambda > 0, k : [0,T] \to L(X)$ is a C^1 operator kernel and $F : [0,T] \times D(A) \to 2^X$ is a multivalued mapping satisfying a certain continuity condition. We note that (1.1) represents the abreviated writing for the problem

$$(1.2)\qquad \begin{aligned} &u'(t) + Au(t) \ni \int_0^t k(t-s)f_u(s)ds, \qquad 0 \leq t \leq T \\ &u(0) = u_0 \\ &f_u \in L^2([0,T];X), f_u(s) \in F(s,u(s)) \text{ a.e. for } s \in (0,T). \end{aligned}$$

Problems of this kind have been intensively studied by many authors and we refer the reader to [1, 2, 3, 5, 8, 9, 10, 11, 12, 13] and to the references therein. The case when both A and $-F(s, \cdot)$ are the subdifferentials of some proper l.s.c. convex functions acting from a real Hilbert space H into $\bar{R}$ has been considered in [1, 3, 6]. In [8] both A and $-F(s, \cdot)$ are allowed to be multivalued but maximal monotone in H. The monotonicity assumption on $- F(s, \cdot)$ has been discarded in [12] but there $F : [0,T]\times X \to X$ is single-valued and continuous - a condition which does not allow F to be a partial differential operator. In [9, 10] no monotonicity condition on $- F(s, \cdot)$ is assumed, but the equation there considered are slightly different from (1.1) and F is single-valued.

Our aim here is to analyse (1.1) in the case when $-F(s, \cdot)$ is neither monotone nor single-valued.

It should be pointed out that many problems of great practical importance may be written in an abstract form (1.1). Here we discuss only an equation - which may be interpreted as a possible model describing the heat flow in a material with memory - in order to emphasize how the multivalued case could occur in a very natural fashion in some concrete situations. For other examples with F single-valued or not see [9, 10, 13].

The paper is divided into seven sections. In sections 2 and 3 we recall for easy reference some basic facts on m-accretive operators and on multivalued mappings respectively. The statement of our main result is contained in section 4, while in section 5 we focus our attention on its complete proof. Some results concerning the continuation of the solutions are included in section 6. Finally, in the last section 7 we analyse an example in order to illustrate how the abstract theory applies to some classes of nonlinear integro-partial differential equations.

2. Preliminairies on m-accretive operators.

Let X be a real Banach space let $x, y \in X$, and let us denote by $[x,y]_+$ the right directional derivative of the norm $\|\cdot\| : X \to R_+$ calculated at x in the direction y, i.e.

$$[x,y]_+ := \lim_{t \downarrow 0} \frac{1}{t} (\|x+ty\| - \|x\|).$$

An operator $A : D(A) \subset X \to 2^X$ is called *accretive* if

$$[x-\hat{x}, y-\hat{y}]_+ \geq 0$$

for each $x, \hat{x} \in D(A)$, $y \in Ax$ and $\hat{y} \in A\hat{x}$. An accretive operator is called *m-accretive* if $I+\lambda A$ is surjective for each $\lambda > 0$, where I is the identity on X.

Let us consider the Cauchy problem

$$(2.1) \qquad \begin{cases} u'(t) + Au(t) \ni f(t), & 0 \leq t \leq T \\ u(0) = u_0, \end{cases}$$

where $A : D(A) \subset X \to 2^X$ is m-accretive, $f \in L^1([0,T]; X)$ and $u_0 \in \overline{D(A)}$.

By a *strong solution* of (2.1) we mean a function $u \in W^{1,\infty}([0,T]; X)$ with $u(0) = u_0$, $u(t) \in D(A)$ a.e. for $t \in (0,T)$ and such that

$$f(t) - u'(t) \in Au(t) \text{ a.e. for } t \in (0,T).$$

By an *integral solution* of (2.1) we mean a continuous function $u : [0,T] \to \overline{D(A)}$ with $u(0) = u_0$ and satisfying

$$\|u(t)-x\| \leq \|u(s)-x\| + \int_s^t [u(\tau)-x, f(\tau)-y]_+ \, d\tau$$

for each $x \in D(A)$, $y \in Ax$ and $0 \leq s \leq t \leq T$.

It readily follows that each strong solution of (2.1) is an integral solution of the same problem, but the converse statement is not true.

We recall for easy references several results we need later.

THEOREM 2.1 (Kato). *Let X be reflexive and let $A : D(A) \subset X \to 2^X$ be m-accretive. Then for each $u_0 \in D(A)$ and each $f \in W^{1,1}([0,T]; X)$ there exists a unique strong solution $u : [0,T] \to \overline{D(A)}$ of (2.1) satisfying*

$$\|u'(t)\| = |Au(t) + f(t)| \leq |Au_0 + f(0)| + \int_0^t \|f'(s)\| \, ds \tag{2.2}$$

a.e. for $t \in (0,T)$, where $|Az+z| := \inf\{\|y+z\|; y \in Ax\}$, for each $x \in D(A)$ and $z \in X$.

For the proof of Theorem 2.1 see [4, Theorem 2.2, p. 131].

THEOREM 2.2 (Benilan). *Let $A : D(A) \subset X \to 2^X$ be m-accretive. Then for each $(u_0,f) \in \overline{D(A)} \times L^1([0,T]; X)$ there exists a unique integral solution $u = \mathcal{T}(u_0,f)$ of (2.1). In addition, for each (v_0,g) belonging to $\overline{D(A)} \times L^1([0,T]; X)$, the integral solution $v = \mathcal{T}(v_0,g)$ satisfies together with u*

$$\|u(t)-v(t)\| \leq \|u(s)-v(s)\| + \int_s^t \|f(\tau)-g(\tau)\| \, d\tau \tag{2.3}$$

for each $0 \leq s \leq t \leq T$.

See [4, Theorem 2.1, p. 124].

THEOREM 2.3 (Mitidieri-Vrabie). *Let X be a real Banach space whose dual is uniformly convex, let $A : D(A) \subset X \to 2^X$ be m-accretive with $(I + \lambda A)^{-1}$ compact for each $\lambda > 0$ and let u_0 be a fixed element in $D(A)$. Then the mapping $f \to \mathcal{T}(u_0,f)$ - the unique integral solution of (2.1) corresponding to (u_0,f) - is sequentially continuous from each bounded subset in $W^{1,1}([0,T]; X)$ endowed with the weak topology of $L^1([0,T]; X)$ into $C([0,T]; X)$ endowed with its strong topology. In particular, the mapping above is sequentially continuous from $W^{1,p}([0,T]; X)$ endowed with its own weak topology into $C([0,T]; X)$ endowed with its strong topology, for each $1 \leq p \leq \infty$.*

For the proof of Theorem 2.3 see [9] or [13, Corollary 2.4.2, p. 75].

LEMMA 2.1 (Mitidieri-Vrabie). *Let X be a real Banach space whose dual is uniformly convex, let $A : D(A) \subset X \to 2^X$ be m-accretive and let $u_0 \in D(A)$. Then for each $r > 0$ and each $T_0 \in (0,T]$ there exists a constant $C(T_0,r) > 0$ such that, for each $f \in W^{1,2}([0,T_0];X)$ verifying*

$$[\int_0^{T_0} (\|f(t)\|^2 + \|f'(t)\|^2)dt]^{1/2} \le r,$$

the unique strong solution u_f of (2.1) *corresponding to (u_0,f) satisfies*

$$|Au_f(t)| \le C(T_0,r) \text{ a.e. for } t \in (0,T).$$

In addition, we have

$$C(T_0,r) \le C(T_1,r)$$

for each $r > 0$ and $0 < T_0 \le T_1 \le T$.

See [9], or [13, Lemma 5.2.1, p. 273].

3. Preliminairies on multivalued mappings.

A mapping $F : [0,T] \to 2^X$ is called *weakly measurable* if $F^{-1}(C) := \{t \in [0,T]; F(t) \cap C \neq \emptyset\}$ is Lebesgue measurable whenever C is closed in X.

The next result is a specific form of a general selection theorem due to Kuratowski and Ryll-Nardzewski.

THEOREM 3.1. *Let X be a separable real Banach space and let $F : [0,T] \to 2^X$ be a nonempty and closed valued mapping which is weakly measurable. Then F has at least one strongly measurable selection, i.e. there exists at least one strongly measurable function $f : [0,T] \to X$ such that $f(t) \in F(t)$ a.e. for $t \in (0,T)$.*

See [13, Theorem 3.1.1, p. 117].

Let X, Y be two real Banach spaces and let U be a nonempty subset in Y. A nonempty and closed valued mapping $P : U \to 2^X$ is called *weakly upper semicontinuous* (*weakly u.s.c.*) at $u_0 \in U$ if for each weakly open subset D in X with $P(u_0) \subset D$ there exists a neighbourhood V of u_0 in the weak topology of Y such that $P(u) \subset D$ for each $u \in V$. A mapping $P : U \to 2^X$ is called *weakly u.s.c. on U* if it is weakly u.s.c. at each $u \in U$. A mapping $P : U \to 2^X$ is called *bounded* if its range is bounded in X.

THEOREM 3.2. *Let X, Y be two reflexive real Banach spaces, let U be a nonempty, bounded, closed and convex subset in Y and let $P : U \to 2^X$ be a nonempty, closed and convex valued mapping which is bounded. Then P is weakly u.s.c. on U if and only if its graph is weakly$\times$weakly sequentially closed in $U \times X$.*

See [13, Theorem 3.1.3, p. 121].

Then next result is a variant of the well-known Kakutani-Ky Fan's fixed point theorem.

THEOREM 3.3. *Let K be a nonempty, weakly compact convex subset in a reflexive real Banach space Y and let $P : K \to 2^K$ be a nonempty, closed and convex valued mapping whose graph is weakly$\times$weakly sequentially closed in $K \times K$. Then P has at least one fixed point in K, i.e. there exists at least one element $u \in K$ such that $u \in P(u)$.*

Theorem 3.3 is a direct consequence of Theorem 3.2 combined with [5, Corollary to Theorem 6.3, p. 75].

4. The main result.

We introduce first some concepts we need in the statement of our main result.

Let X be a reflexive and separable real Banach space and let $A : D(A) \subset X \to 2^X$ be an m-accretive operator. We denote by $W_A^{1,2}([0,T];X)$ the space of all functions $u \in W^{1,2}([0,T];X)$ such that $u(t) \in D(A)$ a.e. for $t \in (0,T)$ and for which there exists $v \in L^\infty([0,T];X)$ with $v(t) \in Au(t)$ a.e. for $t \in (0,T)$.

DEFINITION 4.1. A mapping $F : [0,T] \times D(A) \to 2^X$ is called *A-demiclosed* if

(i) for each $(t,x) \in [0,T] \times D(A)$, the set $F(t,x)$ is nonempty, closed and convex;

(ii) for each $u \in W_A^{1,2}([0,T];X)$ the mapping $t \to F(t,u(t))$ is weakly measurable from $[0,T]$ into X;

(iii) if (u_n) is a sequence in $W_A^{1,2}([0,T];X)$ and (v_n) is a sequence in $L^\infty([0,T];X)$ such that $v_n(t) \in Au_n(t)$ for each $n \in N$ and a.e. for $t \in (0,T)$, and

$$\lim u_n = u \text{ strongly in } C([0,T];X),\ u \in W_A^{1,2}([0,T];X),$$

$$\lim v_n = v \text{ weakly star in } L^\infty([0,T];X),$$

and if (f_n) is a sequence of measurable selections of $(F(\cdot, u_n(\cdot)))$ such that

$$\lim f_n = f \text{ weakly in } L^2([0,T]; X),$$

then $f(t) \in F(t,u(t))$ a.e. for $t \in (0,T)$.

DEFINITION 4.2. A mapping $F : [0,T] \times D(A) \to 2^X$ is called *A-dominated* if there exists a nondecreasing function $m : R_+ \to R_+$ such that

$$|F(t,x)|_S \leq m(|Ax|)$$

for each $x \in D(A)$ and a.e. for $t \in (0,T)$, where $|F(t,x)|_S := \sup\{\|f\| : f \in F(t,x)\}$ and $|Ax| := \inf\{\|y\|; y \in Ax\}$.

By a *strong solution* of (1.1) on $[0,T_0] \subset [0,T]$ we mean a function $u \in W^{1,\infty}([0,T_0]; X)$ - with $u(t) \in D(A)$ a.e. for $t \in (0,T)$ - for which there exists $f_u \in L^2([0,T_0]; X)$ with $f_u(t) \in F(t,u(t))$ a.e. for $t \in (0,T)$ and such that u is a strong solution of (2.1) on $[0,T_0]$ corresponding to $f \in W^{1,2}([0,T_0]; X)$ defined by

$$f(t) := \int_0^t k(t-s) f_u(s) ds$$

for each $t \in [0,T]$.

The hypotheses we need in the sequel are listed below.

(H_1) X is a separable real Banach space whose dual is uniformly convex.

(H_2) $A : D(A) \subset X \to 2^X$ is an m-accretive operator with $(I+\lambda A)^{-1}$ compact for each $\lambda > 0$.

(H_3) $F : [0,T] \times D(A) \to 2^X$ is an A-demiclosed and A-dominated mapping.

(H_4) $k : [0,T] \to L(X)$ - the space of all linear continuous operators from X into itself endowed with the operator norm - is of class C^1.

Now we may proceed to the statement of our main result.

THEOREM 4.1. *If* (H_1), (H_2), (H_3) *and* (H_4) *are satisfied then for each* $u_0 \in D(A)$ *there exists* $T_0 = T(u_0) \in (0,T]$ *such that the problem* (1.1) *has at least one strong solution defined on* $[0,T_0]$.

5. The proof of the main result.

The idea of the proof consists in showing that a suitably defined operator has at least one fixed point.

Let $T_0 \in (0,T]$ and $r > 0$ and let us define

$$K_r^{T_0} : = \{f \in W^{1,2}([0,T_0];X);\ \|f\|_{1,2} \leq r, f(0) = 0\}$$

where $\|\cdot\|_{1,2}$ is the usual norm on $W^{1,2}([0,T_0];X)$, i.e.

$$\|f\|_{1,2} : = [\int_0^{T_0} \|f(t)\|^2 + \|f'(t)\|^2)dt]^{1/2}$$

for each $f \in W^{1,2}([0,T_0];X)$.

Clearly $K_r^{T_0}$ is nonempty, closed, convex and bounded. In addition, since the dual of X is uniformly convex, X is reflexive - see [13, Corollary 1.4.1, p. 18] - and thus $W^{1,2}([0,T_0];X)$ is reflexive too. Hence $K_r^{T_0}$ is weakly compact in $W^{1,2}([0,T_0];X)$ - see [13, Corollary 1.2.4, p. 7].

Let $u_0 \in D(A)$ be arbitrary, but fixed. For each $f \in K_r^{T_0}$ let us denote by u_f the unique strong solution of the Cauchy problem

$$\text{(5.1)} \quad \begin{cases} u'_f(t) + Au_f(t) \ni f(t),\ 0 \leq t \leq T_0 \\ u_f(0) = u_0, \end{cases}$$

whose existence is ensured by Theorem 2.1. Now, let us define the mapping $P : D(P) \subset K_r^{T_0} \to 2^{W^{1,2}([0,T_0];X)}$ by

$$Pf : = \{k^*h_f\,;\, h_f \in L^2([0,T_0];X), h_f(t) \in F(t,u_f(t)) \text{ a.e. } t \in (0,T_0)\}$$

for each $f \in D(P)$, where

$$(k^*h_f)(t) : = \int_0^t k(t-s)h_f(s)ds, \text{ for } t \in [0,T_0],$$

and $D(P) = \{f \in K_r^{T_0}\,;\, (\exists)h_f \in L^2([0,T_0];X), h_f(t) \in F(t,u_f(t))$ a.e. for $t \in (0,T_0)\}$.

From (2.2) in Theorem 2.1 it follows that for each $f \in K_r^{T_0}$ we have $u_f \in W_A^{1,2}([0,T_0];X)$ and therefore, by (ii) in Definition 4.1, the mapping $t \to F(t,u_f(t))$ is weakly measurable from $[0,T_0]$ into X. Next (i) in Definition 4.1 and the remark above show that the set of all strongly measurable selections of $t \to F(t,u_f(t))$ is nonempty. See Theorem 3.1. Finally, from (2.2) and (H_3) - see Definition 4.2 - we conclude that each measurable selection of $t \to F(t,u_f(t))$ belongs to $L^\infty([0,T_0];X)$, and thus to

$L^2([0,T_0];X)$. Consequently $D(P) = K_r^{T_0}$ and P is nonempty, closed and convex valued. See (i) and (iii) in Definition 4.1.

At this point it is quite obvious that (1.1) has at least one strong solution u defined on $[0,T_0]$ if and only if P has at least one fixed point $f \in K_r^{T_0}$. Indeed $f \in K_r^{T_0}$ is a fixed point of P if and only if the strong solution u_f of (5.1) is actually a strong solution of (1.1).

Thus, our aim in that follows is to show that, for some suitably chosen $r > 0$ and $T_0 \in (0,T]$, the mapping P has at least one fixed point. The proof of this fact - based mainly on Theorem 3.3 - consists in two steps.

First step. We show that for some $r > 0$ and $T_0 \in (0,T]$, P maps $K_r^{T_0}$ into itself.

Second step. With $r > 0$ and $T_0 \in (0,T]$ as above, we prove that the graph of P is weakly$\times$weakly sequentially closed in $K_r^{T_0} \times K_r^{T_0}$.

First step. Since $k \in C^1([0,T]; L(X))$, there exists $k_0 > 0$ such that

$$\|k(t)\|_{L(X)} \le k_0, \ \|k'(t)\|_{L(X)} \le k_0 \text{ for each } t \in [0,T]. \tag{5.2}$$

Let $f \in K_r^{T_0}$ and let us denote by $\mathrm{Sel}(F(\cdot,u_f(\cdot)))$ the set of all strongly measurable selections of $t \to F(t,u_f(t))$. As we already have seen, this set is nonempty and included in $L^2([0,T_0];X)$. By Definition 4.2 and (5.2) we easily conclude that

$$\|(k*h_f)(t)\| \le (\|k\|_{L(X)} * \|h_f\|)(t) \le k_0 \int_0^t m(|Au_f(s)|)ds$$

for each $h_f \in \mathrm{Sel}(F(\cdot,u_f(\cdot)))$ and $t \in [0,T_0]$. From Lemma 2.1 we then have

$$\|(k*h_f)(t)\| \le k_0T_0m(C(T_0,r)) \tag{5.3}$$

for each $h_f \in \mathrm{Sel}(F(\cdot,u_f(\cdot)))$ and $t \in [0,T_0]$.

Similarly, we get

$$\|(k*h_f)'(t)\| = \|k(0)h_f(t) + (k'*h_f)(t)\| \le \|k(0)\|_{L(X)}\|h_f(t)\| +$$

$$+ (\|k'\|_{L(X)} * \|h_f\|)(t) \le k_0m(|Au_f(t)|) + k_0 \int_0^t m(|Au_f(s)|)ds$$

$$\le k_0m(C(T_0,r)) + k_0T_0m(C(T_0,r)),$$

for each $h_f \in \mathrm{Sel}(F(\cdot, u_f(\cdot)))$ and $t \in [0,T_0]$. A simple computational argument involving (5.3) and the last inequality yields

$$(5.4) \qquad \|k^*h_f\|_{1,2} \le k_0 m(C(T_0,r))(2T_0^2 + 2T_0 + 1)^{1/2}\, T_0^{1/2},$$

for each $h_f \in \mathrm{Sel}(F(\cdot,u_f(\cdot)))$.

Now, let us fix $r > 0$. From Lemma 2.1 combined with the fact that m is nondecreasing it follows that

$$\lim_{T_0 \downarrow 0} k_0 m(C(T_0, r))\,(2T_0^2 + 2T_0 + 1)^{1/2} T_0^{1/2} = 0$$

Consequently, for a sufficiently small $T_0 \in (0,T]$ we have

$$k_0 m(C(T_0,r))(2T_0^2 + 2T_0 + 1)^{1/2}\, T_0^{1/2} \le r.$$

But this inequality along with (5.4) shows that

$$\|k^*h_f\|_{1,2} \le r$$

for each $f \in K_r^{T_0}$ and $h_f \in \mathrm{Sel}(F(\cdot,u_f(\cdot)))$. Since $(k^*h_f)(0) = 0$, we conclude that for $r > 0$ and $T_0 \in (0,T]$ as above, P maps $K_r^{T_0}$ into itself thereby completing the first part of the proof.

Second step. Next we prove that for $r > 0$ and $T_0 \in (0,T]$ as above the graph of P is weakly$\times$weakly sequentially closed in $K_r^{T_0} \times K_r^{T_0}$. To this aim, let $((f_n, k^*h_n))$ be a sequence in graph (P) such that

$$(5.5) \qquad \lim f_n = f \text{ and } \lim k^*h_n = \tilde{h} \text{ weakly in } W^{1,2}([0,T_0]; X).$$

Let (u_n) be the sequence of strong solutions of (5.1) corresponding to (f_n). By Theorem 2.3 it follows that

$$\lim u_n = u \text{ strongly in } C([0,T_0]; X),$$

where u is the unique strong solution of (5.1) corresponding to f.

Now, let (v_n) be the sequence defined by

$$v_n(t) := -u'_n(t) + f_n(t) \text{ a.e. for } t \in (0,T_0),$$

for each $n \in N$. Since $v_n(t) \in Au_n(t)$ for each $n \in N$ and a.e. for $t \in (0,T_0)$ and (f_n) is bounded in $W^{1,2}([0,T_0];X)$, from Lemma 2.1 it follows that (v_n) is bounded in $L^\infty([0,T_0];X)$. Recalling that X is reflexive - being the predual of a uniformly convex space - and X^* is separable - being the predual of a separable Banach space X^* - we may assume with no loss of generality that

$$\lim v_n = v \text{ weakly star in } L^\infty([0,T_0];X).$$

At this point, let us define the operator $\mathcal{A} : D(\mathcal{A}) \subset L^2([0,T_0];X) \to 2^{L^2([0,T_0];X)}$ by

$$\mathcal{A}u := \{v \in L^2([0,T_0;X);\ v(t) \in Au(t) \text{ a.e. for } t \in (0,T_0)\}$$

for each $u \in D(\mathcal{A})$, where

$$D(\mathcal{A} = \{u \in L^2([0,T_0];X);\ u(t) \in D(A) \text{ a.e. for } t \in (0,T_0) \text{ and there exists } v \in L^2([0,T_0];X) \text{ with } v(t) \in Au(t) \text{ a.e. for } t \in (0,T_0)\}.$$

We may easily verify that $\mathcal{A}$ is m-accretive in $L^2([0,T_0];X)$. Recalling that the dual of $L^2([0,T_0];X)$ is $L^2([0,T_0];X^*)$ - see [7, Theorem 8.20.5, p. 607] - and X^* is uniformly convex, by [14, Theorem 4.2 and Remark 4.7, p. 365] it follows that $L^2([0,T_0];X)$ has a uniformly convex dual. Consequently, by [4, Proposition 3.5, p. 75], $\mathcal{A}$ is demiclosed, i.e. its graph is strongly$\times$weakly sequentially closed in $L^2([0,T_0];X)\times L^2([0,T_0];X)$. Hence $v \in \mathcal{A}u$, i.e. $v(t) \in Au(t)$ a.e. for $t \in (0,T_0)$. Now Lemma 2.1 comes into play and shows - via (H_3) and Definition 4.2 - that (h_n) is bounded in $L^\infty([0,T_0];X)$. Thus we may assume with no loss of generality - by extracting a subsequence if necessary - that

$$\lim h_n = h \quad \text{weakly star in } L^\infty([0,T_0];X).$$

Since $h_n(t) \in F(t,u_n(t))$ for each $n \in N$ and a.e. for $t \in (0,T_0)$ - see the definition on P - condition (iii) in Definition 4.1 implies that $h(t) \in F(t,u(t))$ a.e. for $t \in (0,T_0)$. Finally, inasmuch as the operator $g \to k*g$ is weakly-weakly continuous from $L^2([0,T_0];X)$ into $W^{1,2}([0,T_0];X)$ - being strongly-strongly continuous and linear - we have

$$\lim k*h_n = k*h \quad \text{weakly in } W^{1,2}([0,T_0];X).$$

But this remark along with (5.5) shows that the graph of P is weakly$\times$weakly sequentially closed in $K_r^{T_0} \times K_r^{T_0}$. Hence Theorem 3.3 applies and consequently P has at least one fixed point $f \in K_r^{T_0}$. Since u_f - unique strong solution of (5.1) corresponding to f - is actually a strong solution of (1.1) on $[0,T_0]$, the proof is complete.

6. Continuation of the strong solutions.

Concerning the continuation of the strong solutions of (1.1) we have the following results

THEOREM 6.1. *If* (H_1), (H_2), (H_3) *and* (H_4) *are satisfied, then a strong solution* $u : [0,T_0) \to \overline{D(A)}$ *of* (1.1) *is noncontinuable (as a strong solution) if and only if the mapping* $t \to |Au(t)|$ *is unbounded on* $[0,T_0]$ *in the* $L^\infty([0,T_0]; R)$*-norm.*

The proof of Theorem 6.1 is quite similar to that of [13, Theorem 5.2.2, p. 278] and therefore we do not give details.

THEOREM 6.2. *If* (H_1), (H_2), (H_3) *and* (H_4) *are satisfied and there exist* $c_i \in L^\infty_{loc}([0,T); R_+)$, $i = 1,2$, *such that*

$$|F(t,u)|_S \le c_1(t)|Au| + c_2(t)$$

for each $u \in D(A)$ *and a.e. for* $t \in (0,T)$, *then each noncontinuable strong solution of* (1.1) *is defined either on* $[0,T)$ *or on* $[0,T]$.

Since the proof of Theorem 6.2 follows exactly the same lines as those in the proof of [13, Corollary 5.2.1, p. 279], we omit it.

7. An application to heat conduction in materials with memory.

Our aim here is to illustrate the degree of applicability of Theorem 4.1 by an example of a nonlinear integro-partial differential equation which may be interpreted as a possible model describing the heat flow in a material with memory.

First we recall for easy references some basic facts about the generalized gradient in the sense of Clarke.

Let $g : R \to R$ be a given function which is Lipschitz on bounded subsets in R. The *generalized gradient* of g calculated at $u \in R$ is defined *by*

$$\tilde{\nabla} g(u) := \mathrm{conv}\{v \in R; (\exists)(u_u) \subset R, \lim u_n = u, \lim g'(u_n) = v\}$$

where conv B is the closed convex hull of the subset $B \subset R$.

Since g is Lipschitz on bounded subsets in R, g is almost everywhere differentiable on R and g' is bounded on bounded subsets in R. Therefore, for each $u \in R$, $\tilde{\nabla}g(u)$ is nonempty, closed, convex and bounded. In addition, we may easily verify that $\tilde{\nabla}g : R \to 2^R$ is upper semicontinuous and maps bounded subsets in R into bounded subsets in R.

Now, we may proceed to the statement of the problem to be studied in this section. Namely, let us consider the nonlinear integro-partial differential equation

$$
\begin{aligned}
&u_t - u_{xx} = \int_0^t a(t-s)(f(u))_{xx}ds \text{ a.e. for } (t,x) \in (0,T) \times (0,1) \\
&u_x(t,0) \in \beta(u(t,0)), -u_x(t,1) \in \beta(u(t,1)) \text{ a.e. for } t \in (0,T) \qquad (7.1)\\
&u(0,x) = u_0(x) \text{ a.e. for } x \in (0,1),
\end{aligned}
$$

where $a : [0,T] \to R$ is a C^1 kernel, $f : R \to R$ is a C^1 function with f Lipschitz on bounded subsets in R and $\beta : D(\beta) \subset R \to 2^R$ is an m-accretive operator with $0 \in D(\beta)$ and $0 \in \beta(0)$.

Since f is not a C^2 function, we define the solution of (7.1) as the solution of the relaxed problem below.

$$
\begin{aligned}
&u_t - u_{xx} = \int_0^t a(t-s)[\tilde{\nabla}f'(u)u_x^2 + f'(u)u_{xx}]ds \text{ a.e. for } (t,x) \in (0,T) \times (0,1) \\
&u_x(t,0) \in \beta(u(t,0)), -u_x(t,1) \in \beta(u(t,1)) \text{ a.e. for } t \in (0,T) \qquad (7.2)\\
&u(0,x) = u_0(x) \text{ a.e. for } x \in (0,1),
\end{aligned}
$$

where $\tilde{\nabla}f'$ is the generalized gradient of f'.

Using Theorem 4.1 we may prove

THEOREM 7.1. *Let $a : [0,T] \to R$ and $f : R \to R$ be C^1 functions with f' Lipschitz on bounded subsets in R and let $\beta : D(\beta) \subset R \to 2^R$ be an m-accretive operator with $0 \in D(\beta)$ and $0 \in \beta(0)$. Then, for each u_0 in $H^2([0,1])$ with $u'(0) \in \beta(u(0))$ and $-u'(1) \in \beta(u(1))$ there exists $T_0 \in (0,1)$ such that the problem (7.2) has at least one solution $u : [0,T_0] \to L^2([0,1])$ satisfying*

(7.3) $u(t) \in H^2([0,1]), u_x(t,0) \in \beta(u(t,0)), -u_x(t,1) \in \beta(u(t,1))$ *a.e.* $t \in (0,T_0)$;

(7.4) $t \to u_t$ *belongs to* $L^2([0,T_0]; L^2([0,1]))$, *and*

(7.5) $t \to \frac{1}{2}\int_0^1 u_x^2 dx + j(u(t,1)) - j(u(t,0))$ *belongs to* $AC([0,T_0]; R_+)$.

where $j : R \to R$ *is a proper, l.s.c. convex function whose subdifferential* ∂j *coincides with* β.

Proof. First, we rewrite (7.2) as an abstract integrodifferential equation of the form (1.1). Thus, let $X = L^2([0,1])$ and let us define the operator $A : D(A) \subset X \to 2^X$ by

$$Au := \{-u''\}$$

for each $u \in D(A) = \{u \in H^2([0,1]); u'(0) \in \beta(u(0)), -u_x(1) \in \beta(u(1))\}$.
Next, let us define $k : [0,T] \to L(X)$ by

$$k(t) := a(t)I$$

for each $t \in [0,T]$, where I is the identity on X, and $F : D(A) \to 2^X$ by

$$F(u) := \{pu'^2 + f'(u)u''; p \in L^\infty([0,1]), p(x) \in \tilde{\nabla} f'(u(x)) \text{ a.e. for } x \in (0,1)\}$$

for each $u \in D(A)$.

Clearly (7.2) may be rewritten in the form (1.1) with X, A, k and F defined as above. We note that where F does not depend on $t \in [0,T]$.

Since X is a separable real Hilbert space, (H_1) is satisfied. In addition, it is well-known that A is m-accretive and, for each $\lambda > 0$, the resolvent operator $(I+\lambda A)^{-1}$ is compact. Thus (H_2) holds. Inasmuch as (H) is obviously verified, in that follows we focus our attention in order to check (H_3). To this aim, let us observe that, by a well-known regularity result for elliptic equations, we have

(7.6) $$\|u\|_{H^2([0,1])} \leq C\|Au\|_{L^2([0,1])}$$

for each $u \in D(A)$, where $C > 0$ does not depend on u.

We recall that $\tilde{\nabla} f'$ is a nonempty, closed and convex valued upper semicontinuous mapping which maps bounded subsets in R into bounded subsets in R. Then, it readily follows that for each $u \in D(A)$ - which obviously is included in $C([0,1])$ - the set of all measurable selections of $x \to \tilde{\nabla} f'(u(x))$ is nonempty, closed, convex and bounded in $L^\infty([0,1])$.

Thus F satisfies (i) in Definition 4.1. Next, let $u \in W_A^{1,2}([0,T];X)$. For (7.6) we deduce that $u \in L^\infty([0,T];L^\infty([0,1]))$ and therefore, the same argument as above shows that F satisfies (ii) in Definition 4.1.

Now, let (u_n) be a sequence in $W_A^{1,2}([0,T];X)$ such that

$$\lim u_n = u \text{ strongly in } C([0,T];X),\ u \in W_A^{1,2}([0,T];X), \text{ and}$$

$$\lim Au_n = Au \text{ weakly star in } L^\infty([0,T];X).$$

Also by (7.6) it follows that (u_n) is bounded in $L^\infty([0,T]\times[0,1])$, as thus, we may assume with no loss of generality - by extracting a subsequence if necessary - that

$$\lim u_n = u \text{ a.e. for } (t,x) \in (0,T)\times(0,1). \tag{7.7}$$

Let (f_n) be a sequence of selections of $(F(u_n))$ such that

$$\lim f_n = \hat{f} \text{ weakly in } L^2([0,T];X).$$

Let (p_n) be such that

$$p_n(t,x) \in \tilde{\nabla} f'(u_n(t,x))$$

for each $n \in N$ and a.e. for $(t,x) \in (0,T)\times(0,1)$ and

$$f_n(t,x) = p_n(t,x)u_{nx}^2(t,x) + f'(u_n(t,x))u_{nxx}(t,x) \tag{7.8}$$

for each $n \in N$ and a.e. for $(t,x) \in (0,T)\times(0,1)$. Since (u_n) is bounded in $L^\infty([0,T]\times[0,1])$ and $\tilde{\nabla} f'$ maps bounded subsets in R into bounded subsets in R, we easily deduce that (p_n) is bounded in $L^\infty([0,T]\times[0,1])$. Then, we may assume with no loss of generality that

$$\lim p_n = p \text{ weakly star in } L^\infty([0,T]\times[0,1]). \tag{7.9}$$

Inasmuch as $\tilde{\nabla} f'$ is upper semicontinuous and has convex and closed values, by [13, Theorem 3.1.2, p. 120] and (7.7) it follows that

(7.10) $$p(t,x) \in \tilde{\nabla} f'(u(t,x)) \text{ a.e. for } (t,x) \in (0,T)\times(0,1).$$

Observing that

$$\int_0^T \int_0^1 |u_{nx} - u_x|^2\, dx \cdot dt \leq - \int_0^T \int_0^1 (u_{nxx} - u_{xx})(u_n - u) dx \cdot dt$$

for each $n \in N$, we deduce that

$$\lim u_{nx} = u_x \text{ strongly in } L^2([0,T]; L^2([0,1])).$$

From (7.7), (7.8), (7.9), (7.10) and the last remark we conclude that $\hat{f}(t) \in F(t,u(t))$ a.e. for $t \in (0,T)$, and thus (iii) in Defintion 4.1 is also satisfied.

Finally, we show that F is A-dominated. To this aim, let $r > 0$ and let $M_r := \{u \in D(A); \|Au\|_{L^2([0,1])} \leq r\}$. Since $H^2([0,1])$ is continuously embedded in $H^1([0,1])$ and the letter is continuously embedded in $C([0,1])$, from (7.6) it follows that

$$\|u\|_{C([0,1])} \leq C\|Au\|_{L^2([0,1])} \leq C\cdot r, \text{ and}$$

$$\|u'\|_{C([0,1])} \leq C\|Au\|_{L^2([0,1])} \leq C\cdot r$$

for each $r > 0$ and $u \in M_r$, where $C > 0$ does not depend on $r > 0$.

Now, let us define $m : R_+ \to R_+$ by

$$m(r) := \sup\{ |\tilde{\nabla} f'(u)|_S C^2 r^2 + |f'(u)| r; u \in [-C\cdot r, C\cdot r]\}$$

where $|\tilde{\nabla} f'(u)|_S = \sup\{|v|; v \in \tilde{\nabla} f'(u)\}$ for each $u \in R$.

Obviously m is nondecreasing and satisfies the condition in Definition 4.2. Then F is A-dominated. From Theorem 4.1 we conclude that for each $u_0 \in D(A)$ there exists $T_0 \in (0,T]$ such that the problem (7.2) has at least one strong solution defined on $[0,T_0]$. Since (7.3), (7.4) and (7.5) follows from [13, Theorem 1.9.3, p. 42] the proof is complete.

REFERENCES

[1] S. AIZICOVICI, *On a class of functional-differential equations*, Rend. Mat., *8* (1975), 685-706.

[2] S. AIZICOVICI, *Time-dependent Volterra integrodifferential equations*, J. Integral Equations, *10* (1985), Suppl.., p. 45-60.

[3] V. BARBU, *Integro-differential equations in Hilbert spaces*, An. St. Univ. Al. I. Cuza, XIX (1973), p. 365-383.

[4] V. BARBU, *Nonlinear semigroups and differential equations in Banach spaces*, Noordhoff, 1976.

[5] F.E. BROWDER, *Nonlinear operators and nonlinear equations of evolution*, Amer. Math. Soc. Proc. of Symposia in Pure Math., Vol. XVIII, Part. 2, 1976.

[6] M.G. CRANDALL, S.O. LONDEN and J.A. NOHEL, *An abstract nonlinear Volterra integrodifferential equation*, J. Math. Anal. Appl., *64* (1978), p. 701-735.

[7] R.E. EDWARDS, *Functional analysis*, Holt, Rinehart and Winston, 1965.

[8] G. GRIPENBERG, *On some integral and integro-differential equations in a Hilbert space*, An. Mat. pura appl., *118* (1978), p. 181-198.

[9] E. MITIDIERI, I.I. VRABIE, *Existence for nonlinear functional differential equations*, Hiroshima Math. J., *17* (1987), p. 627-649.

[10] E. MITIDIERI, I.I. VRABIE, *A class of strongly nonlinear functional differential equations*, Annali di Mat. Pura e Appl. (IV). Vol. CLI, (1988), p. 125-147.

[11] I.I. VRABIE, *The nonlinear version of Pazy's local existence theorem*, Israel J. Math., *32* (1979), p. 221-235.

[12] I.I. VRABIE, *Compactness methods for an abstract Volterra integrodifferential equation*, Nonlin. Anal. TMA, *5* (1981), p. 355- 371.

[13] I.I. VRABIE, *Compactness methods for nonlinear evolutions*, Pitman Monographs and Surveys in Pure and Applied Mathematics, Vol. 32, Longman Scientific & Technical, 1987.

[14] C. ZALINESCU, *On uniformly convex functions*, J. Math. Anal. Appl., *95* (1983), p. 344-374.