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Nonlinear Model Reduction by Moment Matching

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Contents

Abstract

Mathematical models are at the core of modern science and technology. An accurate description of behaviors, systems and processes often requires the use of complex models which are difficult to analyze and control. To facilitate analysis of and design for complex systems, model reduction theory and tools allow determining "simpler" models which preserve some of the features of the underlying complex description. A large variety of techniques, which can be distinguished depending on the features which are preserved in the reduction process, has been proposed to achieve this goal. One such a method is the moment matching approach.

This monograph focuses on the problem of model reduction by moment matching for nonlinear systems. The central idea of the method is the preservation, for a prescribed class of inputs and under some technical assumptions, of the steady-state output response of the system to be reduced. We present the moment matching approach from this vantage point, covering the problems of model reduction for nonlinear systems, nonlinear time-delay systems, data-driven model reduction for nonlinear systems and model reduction for "discontinuous" input signals. Throughout the monograph linear systems, with their simple structure and strong properties, are used as a paradigm to facilitate understanding of the theory and provide foundation of the terminology and notation. The text is enriched by several numerical examples, physically motivated examples and with connections to well-established notions and tools, such as the phasor transform.

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Introduction

The availability of mathematical models is essential for the analysis, control and design of modern technological devices. As the computational power has advanced, the complexity of these mathematical descriptions has increased. This has maintained the computational needs at the top or above the available possibilities. A solution to this problem is represented by the use of reduced order models, which are exploited in the prediction, analysis and control of a wide class of behaviors. For instance, reduced order models are used to simulate weather forecast models and design very large scale integrated circuits and networked dynamical systems. The model reduction problem can be informally formulated as the problem of finding a simplified description of a dynamical system in specific operating conditions, preserving at the same time specific properties, *e.g.* stability. For linear systems, the problem has been addressed from several perspectives which can be divided into two main groups: singular value decomposition (SVD) approximation methods and Krylov approximation methods. The theory of balanced realizations, the use of Hankel operators and of proper orthogonal decomposition (POD) belong to the first group, whereas the use of interpolation theory belongs to the latter.

The additional difficulties of the reduction of nonlinear systems carry the need to develop different or "enhanced" techniques. Several methods which extend balancing and proper orthogonal decomposition to nonlinear systems have been proposed. Reduction of special classes of nonlinear systems and local reduction (for instance around a limit cycle) represent another approach. Although many results and efforts have been made, at present there is no complete theory of model reduction for nonlinear systems or, at least, not as complete as the theory developed for linear systems.

In this chapter we briefly recall the main model reduction methods which have been presented in the literature. We then establish the objective of this monograph and summarize its contribution and content. The chapter continues with a section in which the notation used throughout the monograph is gathered and is concluded with some bibliographical remarks on the methods described in this introduction.

1.1 Main Methods of Model Reduction for Linear Systems

Since the order of a dynamical system is usually defined as the number of states that the system has, model reduction methods require the elimination of some state variables. If we want that the reduced order model preserves some sort of "likeness" to the system to be reduced, then the elimination of the states cannot be arbitrary. To render precise this problem formulation two questions need to be answered.

Q1. What are the characteristics and properties that the reduction method aims to preserve?

Q2. What is lost in the reduction process and how can we quantify/alleviate this loss?

Depending on how these two questions are answered we obtain a multitude of different reduction techniques. It is important to stress from the onset that there is no "perfect" or "best" reduction method. In fact, the problem of model reduction epitomizes typical engineering problems in which there exists a trade off between the accuracy or performance achieved and the cost required to achieve it. In the following we briefly recall the main ideas behind the most common model reduction methods.

1.1.1 Singular value decomposition methods

Balancing and balanced approximations

With the objective of economizing on the order of the system, we wonder which states should be eliminated. It seems reasonable that unobservable and uncontrollable states should be the first candidates in the elimination process since they do not contribute to the input-output behavior of the system. This implies that if our objective is to economize on the order of the systems, these modes should be eliminated by a sensible method. The information on the degree of controllability and observability of a state is given by the controllability Gramian and observability Gramian, respectively. In particular, a difficult to control state is a state which requires high control energy to be steered to zero. However, a problem arises when we have mixed situations, such as a state which may be difficult to control but easy to observe. To be able to rank all the states with respect to a common criterion, it is fundamental to introduce the concept of balancing. From a mathematical viewpoint, balancing methods consist in the simultaneous diagonalization, by means of a singular value decomposition, of the reachability and observability Gramians. In this way we can identify the states that are simultaneously the least controllable and least observable. Then the reduction simply consists in eliminating these states. Moreover, balanced truncation methods preserve stability and naturally provide an upper bound on the approximation error in terms of the $\mathcal{H}_\infty$-norm. This quantifies what is lost in the reduction process. Finally, note that since the Gramians are related to the solutions of Riccati equations, variations of the balanced truncation method can be obtained using variations of the Riccati equations. Among these variations we mention stochastic balancing, bounded real balancing and positive real balancing. All these methods share the same answer to question Q1, namely the characteristics on which we base the reduction are the observability and the controllability Gramians, however, they differ in the answer to question Q2, namely in the properties and the type of approximation error of the reduced order model.

Hankel-norm approximation

While balanced truncation provides a bound on the approximation error, the reduced order model obtained is not optimal with respect to any given norm. An alternative method, still based on a singular value decomposition, is represented by the optimal approximation in the Hankel-norm. With this method, an optimal reduced order model is sought with respect to a 2-induced norm of the Hankel operator of the system. Similarly to balancing, the Hankel-norm approximation yields a stable reduced order model and an upper bound on the $\mathcal{H}_\infty$ norm which depends on the neglected Hankel singular values. However, the main characteristic of the method is that the model obtained is optimal with respect to an optimality criterion, *i.e.* with respect to the Hankel-norm. Note that the optimal model in the Hankel-norm is not optimal in the $\mathcal{H}_\infty$ norm. As a consequence, with respect to this last norm, balancing may offers a better approximation.

Proper orthogonal decomposition

Proper orthogonal decomposition is a method which is widely applied in practice since it does not necessarily require a high order model to begin with. In the proper orthogonal decomposition method a cloud of state measurements is obtained at several instants of time. These measurements are collected in data matrices, known as time-snapshots, which are then decomposed along orthonormal directions in a linear fashion. A reduced order model is obtained truncating the number of orthonormal directions used with respect to some optimality criterion (often a 2-induced norm). Proper orthogonal decomposition is strictly linked to other singular value decomposition methods, such as balancing, however, POD has the important advantage with respect to other SVD methods of operating on data clouds instead of on the matrices of the systems. As a consequence the method can be attempted also on systems which are not described by linear differential equations.

1.1.2 Model reduction using Krylov methods

Model reduction using Krylov methods, also known as moment matching methods, or interpolation methods, belongs to a different category of reduction techniques with respect to the SVD methods. The interpolation theory relies on the notion of moment. Note that a linear differential system which is observable and controllable is fully described by its transfer function. Given a set of complex interpolation points (which have to be selected with respect to some criterion), we determine the coefficients of the Laurent series expansion of the transfer function at these interpolation points. These coefficients are called moments. The moment matching method consists in determining a lower order model which has a transfer function that, at the same interpolation points, possesses the same coefficients of the Laurent expansion (up to a certain order). In other words, in moment matching a reduced order model is such that its transfer function (and derivatives of this) takes the same values of the transfer function (and derivatives of this) of the system to be reduced at the same interpolation points. This is graphically represented in Fig. 1.1 in which the magnitude (top) and phase (bottom) of the transfer function of a reduced order model (dashed/red line) matches the respective quantities of a given system (solid/blue line) at 30 rad/s.

The advantage of moment matching over the SVD methods is that the numerical implementation is much more efficient. Since only matrix-vector multiplications are used, *i.e.* no matrix factorizations or inversions are needed, the number of operations required to compute a reduced order model of order ν given a system of order $n >> \nu$ is $\mathcal{O}(\nu n^2)$. This is to be compared with a $\mathcal{O}(n^3)$ computational complexity of balancing and Hankel-norm approximations. On the other hand, among the drawbacks of moment matching methods there are the difficulty in preserving important properties of the original system, such as stability, and the lack of a bound on the estimation error.

Note, finally, that model reduction of linear systems is an active area of research in various domains of engineering and mathematics, and many variations and improvements have been proposed for all of these methods. For instance, mixed singular value decomposition and

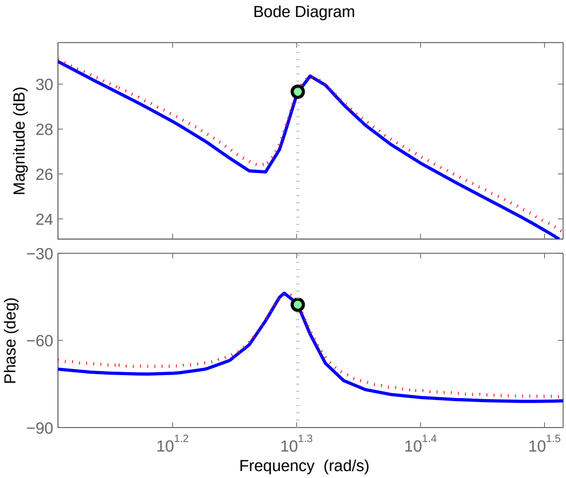

Figure 1.1: Diagrammatic illustration of the interpolation approach. Magnitude (top graph) and phase (bottom graph) plots of a given system (solid/blue line) and of a reduced order model (dashed/red line). The green circle represents the interpolation point.

Krylov methods are capable of yielding reduced order models that simultaneously maintain some of the properties of the system to be reduced and are determined in a computationally efficient manner.

1.2 Contents of the Monograph

The goal of this monograph is to present, in a uniform and complete way, moment matching techniques for nonlinear systems. The focus is on the so-called "steady-state" notion of moment. The moment is defined using the steady-state output response of the system interconnected with an interpolating signal generator. While the theory and the techniques are developed from a pure nonlinear perspective, throughout the monograph we point out several connections with the interpolation theory and the classical "interpolation-based" notion of moment. This justifies the terminology used and improves the understanding of the nonlinear theory. The chapters are enriched with examples and conclude with bibliographical notes. The monograph comprises:

Chapter 2 begins with a very general formulation of the problem that we call "model reduction by moment matching". The problem is formulated from a "general-system perspective" and not from a linear system point of view. We then specialize the problem to nonlinear differential systems and we introduce the notions of steady-state and of moment for this class of systems, clarifying the nature of the relation between these two objects. We also relate the introduced notions with classical interpolation theory. We then present families of nonlinear reduced order models and we study the possibility of achieving specific properties, such as assigning prescribed zero dynamics. We specialize these results to linear systems, proposing families of linear reduced order models which preserve specific properties (properties which are usually difficult to maintain in the interpolation-based approach). We conclude the chapter with a selection of additional topics regarding systems in special form.

Chapter 3 deals with the problem of model reduction for nonlinear time-delay systems. The center manifold theory for time-delay systems is used to extend the definition of moment to nonlinear time-delay systems and a family of systems achieving moment matching for nonlinear time-delay systems is given. The possibility of interpolating multiple moments increasing the number of delays but maintaining the number of equations is investigated and the problem of obtaining a reduced order model of an unstable system is discussed. Similarly to the previous chapter, the results are also specialized to linear time-delay systems. Several examples illustrate the theory.

Chapter 4 presents a theoretical framework and a collection of techniques to obtain reduced order models by moment matching from input/output data for nonlinear, possibly time-delay, systems. We begin providing algorithms for the determination of an approximation of the moment which converges asymptotically to the actual moment of the nonlinear system. The computational complexity is discussed and the results are also specialized to linear systems. Several examples illustrate the theory.

Chapter 5 investigates the limitations of the characterization of moment based on a signal generator described by differential equations. With the final aim of solving the model reduction problem for a class of input signals generated by exogenous systems which do not have an implicit (differential) form, a time-varying parametrization of the steady-state of the system is used to extend, exploiting an integral matrix equation, the definition of moment to this class of input signals. The equivalence of the new definition and the classical interpolation-based notion of moment is proved under specific conditions. Special attention is given to periodic signals due to the wide range of practical applications where these are encountered. Reduced order models matching the steady-state response to explicit signal generators are given for linear systems and several connections with the classical reduced order models are drawn.

1.3 Notation

Standard notation has been adopted, most of which is defined in this section and used throughout the remainder of the monograph. When additional notation (not included in this section) is introduced, this is defined in the relevant parts of the monograph.

The symbol $\mathbb{R}_{\geq 0}$ ($\mathbb{R}_{>0}$) denotes the set of non-negative (positive) real numbers; $\mathbb{C}_{<0}$ denotes the set of complex numbers with strictly negative real part; $\mathbb{C}_0$ denotes the set of complex numbers with zero real part and $\mathbb{D}_{<1}$ the set of complex numbers with modulo less than one.

The symbol I denotes the identity matrix and $\sigma(A)$ denotes the spectrum of the matrix $A \in \mathbb{R}^{n\times n}$. The symbol $\otimes$ indicates the Kronecker product and $||A||$ indicates the induced Euclidean matrix norm. Given a list of n elements a_i, $\mathrm{diag}(a_i)$ indicates a diagonal matrix with diagonal elements equal to the a_i's. The vectorization of a matrix $A \in \mathbb{R}^{n\times m}$, denoted by $\mathrm{vec}(A)$, is the $nm \times 1$ vector obtained by stacking the columns of the matrix A one on top of the other, namely $\mathrm{vec}(A) = [a_1^\top, a_2^\top, \dots, a_m^\top]^\top$, where $a_i \in \mathbb{R}^n$ are the columns of A and the superscript $\top$ denotes the transposition operator. The superscript $*$ indicates the conjugate transpose operator. The symbol $\mathrm{adj}(A)$ denotes the adjugate (known also as classical adjoint or adjunct) of A, namely the transpose of its cofactor matrix.

The symbol $\Re[z]$ indicates the real part of the complex number z, $\Im[z]$ denotes its imaginary part and ι denotes the imaginary unit. The symbol ϵ_k indicates a vector with the k-th element equal to 1 and with all the other elements equal to 0. Given a function f, $\overline{F}$ represents its phasor at ω, whereas $<f(t)>$ indicates its time average.

Given a set of delays $\{\tau_j\}$, the symbol $\mathfrak{R}^n_T = \mathfrak{R}^n_T([-T,0],\mathbb{R}^n)$, with $T = \max_j\{\tau_j\}$, indicates the set of continuous functions mapping the interval $[-T,0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. The subscripts "τ_j" and "χ_j" denote the translation operator, *e.g.* $x_{\tau_j}(t) = x(t-\tau_j)$.

Let $\bar{s} \in \mathbb{C}$ and $A(s) \in \mathbb{C}^{n\times n}$. Then $\bar{s} \notin \sigma(A(s))$ means that $\det(\bar{s}I - A(\bar{s})) \neq 0$. $\sigma(A(s)) \subset \mathbb{C}_{<0}$ means that for all $\bar{s}$ such that $\det(\bar{s}I - A(\bar{s})) = 0$, $\bar{s} \in \mathbb{C}_{<0}$.

The symbol $\mathcal{L}(f(t))$ denotes the Laplace transform of the function f (provided that f is Laplace transformable) and $\mathcal{L}^{-1}\{F(s)\}$ denotes the inverse Laplace transform of $F(s)$ (provided it exists). With some abuse of notation, $\sigma(\mathcal{L}(f(t)))$ denotes the set of poles of $\mathcal{L}(f(t))$. The symbol $\delta_0(t)$ denotes the Dirac δ-function.

Given two functions, $f : Y \to Z$ and $g : X \to Y$, with $f \circ g : X \to Z$ we denote the composite function $(f \circ g)(x) = f(g(x))$ which maps all $x \in X$ to $f(g(x)) \in Z$. $L_f h(x)$ denotes the Lie derivative of the smooth function h along the smooth vector field f, *i.e.* $L_f h(\cdot) = \frac{\partial h}{\partial x} f(x)$. Given a function $y : \mathbb{R} \to \mathbb{R}$ the symbol $y^{(k)}$ denotes the k-th order time derivative of y (provided it exists). Given a scalar function $V : \mathbb{R}^n \to \mathbb{R} : x \mapsto V(x)$, the symbols V_x and V_{xx} denote, respectively, the gradient and the Hessian matrix of the function V, provided they exist.

1.4 Bibliographical Notes

To report all the developments and results on model reduction and to give credit to all the researchers who have contributed to the field would be a titanic effort (if at all possible) considering the enormous research activity which has contributed to this field. The following references should not be considered at all exhaustive but should be seen as a possible starting point for the interested reader.

1.4.1 Model reduction for linear systems

Several survey papers have been written on the topic of model reduction for linear systems. Here we list a few examples known to the authors. For a survey paper on balanced truncation see, *e.g.* Gugercin and Antoulas [2004]. For survey papers on model reduction based on Krylov subspaces see, *e.g.* Bai [2002] and Freund [2003]. Other survey papers on model reduction of linear systems are, for instance, Fortuna et al. [1992], Antoulas et al. [2001] and Baur et al. [2014]. For further detail and an extensive list of references on the problem of model reduction for linear systems see the monograph Antoulas [2005].

Balanced approximations and Hankel-norm approximations

Balanced truncation has been originally introduced by Moore [1981], which recognizes that the idea is closely related to the "principal axis realization" proposed by Mullis and Roberts [1976]. Almost immediately it has been shown that the method possesses the property of preserving the stability of the system, see Pernebo and Silverman [1982], and provides a computable error bound, see Enns [1984] and Glover [1984]. Modifications of the standard method to achieve preservation of passivity have been proposed in *e.g.* Phillips et al. [2003], Saraswat et al. [2005], Yan et al. [2007] and Reis and Stykel [2010]. An efficient and numerically robust implementation of balanced truncation is the square-root method, see Laub et al. [1987] and Tombs and Postlethwaite [1987], which is based on the Cholesky factorizations of the Gramians. The Schur method proposed by Safonov and Chiang [1989] enhances some of the robustness properties of the square-root method. The balancing-free square-root method proposed in Varga [1992] combines the square-root method and the Schur method. Another class of methods based on the Gramians is the family of Cross-Gramian methods given in Fernando and Nicholson [1983, 1984], Aldhaheri [1991], Antoulas et al. [2001], Sorensen and Antoulas [2002] and Baur and Benner [2008], which have properties similar to balanced truncation (preservation of stability and computable error bound). Numerical efficient implementations of the balanced truncation methods have been proposed in Rabiei and Pedram [1999], Van Dooren [2000], Benner et al. [2000, 2003], Benner [2004], Gugercin and Li [2005] and Baur and Benner [2008]. Other numerically efficient methods related to balanced truncation are the singular perturbation approximation, see Liu and Anderson [1986], Varga [1992] and Benner et al. [2000], frequency weighted balanced truncation, see Enns [1984], Gawronski and Juang [1990] and Gugercin and Antoulas [2004], fractional balanced reduction, see Meyer [1990], and balanced stochastic truncation, see Benner et al. [2001]. The numerical stability of the balanced truncation methods often relies upon the stability of the system. Generalizations of the method to unstable systems have been proposed in Zhou et al. [1999]. Extensions to time-varying systems have been given in Lall and Beck [2003], Sandberg and Rantzer [2004] and Sandberg [2006]. Several approximated versions of the balanced truncation method have been presented. Willcox and Peraire [2002] have proposed a method which they interpreted as frequency-domain POD, and that later has been called Poor Man's Truncated Balanced Reduction method in Phillips and Silveira [2005]. The dominant subspace projection method is another heuristic balanced-free method which approximates, in a certain sense, balanced truncation. see Penzl [2006] (see also Li and White [2001] for another version). Finally, some results on model reduction for linear systems based on the notion of Hankel operators are given in Adamjan et al. [1971], Glover [1984], Safonov et al. [1990], Kavranoğlu and Bettayeb [1993] and Benner et al. [2004].

Krylov methods

The origin of this approach can be traced back to the related problems of Nevanlinna-Pick interpolation and of partial realization of covariance sequences, see Georgiou [1983], Kimura [1983, 1986], Antoulas et al. [1990], Byrnes et al. [1995], Georgiou [1999] and Byrnes et al. [2001]. An early result based on Krylov methods is the asymptotic waveform evaluation method proposed in Pillage and Rohrer [1990]. This method computes the moments explicitly and, consequently, is numerically unstable and inefficient. The problem of numerical instability has been tackled in several works, starting with the Lanczos Padé method proposed in Gallivan et al. [1994] and Feldmann and Freund [1995], and the "passive reduced-order interconnect macromodeling algorithm" presented in Odabasioglu et al. [1998]. Techniques to double the number of interpolated points have been firstly proposed in Grimme [1997] and are referred to as dual rational Arnoldi and Lanczos methods. In general Krylov methods do not preserve stability and passivity. Stability of the reduced order model can be enforced using the restarting techniques given in Grimme et al. [1995] or the restarted dual Arnoldi method proposed by Jaimoukha and Kasenally [1997]. Other techniques to preserve these and other structural properties are presented in, *e.g.* Bai and Freund [2001], Freund [2004], Li and Bai [2005], Beattie and Gugercin [2008], Polyuga and Van der Schaft [2010, 2011, 2012] and Gugercin et al. [2012]. An important problem for Krylov methods is the selection of the interpolation points. Early results on this aspect are given in Chiprout and Nakhla [1995], where the complex frequency hopping, which is based on a binary search, is proposed. Another approach using a binary search is given in Achar and Nakhla [2001]. Recent results on the problem of selecting the interpolation points are given in Chu et al. [2006] and Gugercin et al. [2008]. In this last paper the iterative rational Krylov algorithm (IRKA) is prospposed, which is becoming increasingly popular. While stability is not guaranteed in all instances, the method is numerical effective and solves first-order necessary conditions of optimality with respect to the $\mathcal{H}_2$ norm. Several modifications of this method have been proposed in Gugercin et al. [2008], Van Dooren et al. [2008] and Bunse-Gerstner et al. [2010] for MIMO systems, and in Flagg et al. [2013] for the $\mathcal{H}_\infty$ case. Another adaptive algorithm, more efficient than IRKA, but less precise, has been presented in Druskin and Simoncini [2011]. An algorithm less efficient than IRKA, but that allows to select the order of the reduced order model adaptively is given in Panzer et al. [2013a]. A data-driven Krylov approach has been presented under the name of Loewner framework in Mayo and Antoulas [2007]. A drawback of the Krylov methods is the lack of an error bound. This problem is addressed in several works in which results for systems in special form are obtained, see *e.g.* Grimme [1997], Bai et al. [1999], Bechtold et al. [2004], Panzer et al. [2013b] and Konkel et al. [2014].

1.4.2 Model reduction for nonlinear systems

From the '90s, considerable research effort has been dedicated to the problem of model reduction for nonlinear systems. The problem of model reduction for special classes of systems, such as differential-algebraic systems, bilinear systems and mechanical/Hamiltonian systems has been studied in Al-Baiyat et al. [1994], Lall et al. [2003], Soberg et al. [2007] and Fujimoto [2008]. Several results rely on approximating the nonlinearity with a polynomial, see *e.g.* Chen [1999], Phillips [2000, 2003], Rewienski and White [2003] and Benner [2004], or the ability of transforming the system into a quadratic bilinear form, see *e.g.* Gu [2009, 2011], Benner and Breiten [2015] and Antoulas et al. [2016]. The first breakthrough on balancing for nonlinear systems has been made in Scherpen [1993]. This paper originated subsequent results (sometimes referred to as energy-based methods, see Scherpen and Gray [2000]) which exploit balancing, see Scherpen and Van der Schaft [1994] and Gray and Mesko [1997], or the notion of Hankel operator, see Gray and Scherpen [2001], Scherpen and Gray [2002] and Fujimoto and Scherpen [2005, 2010]. Techniques based on the reduction around a limit cycle or a manifold have been presented in Verriest and Gray [1998] and Gray and Verriest [2006]. Model reduction methods based on proper orthogonal decomposition have been developed for linear and nonlinear systems, see *e.g.* Kunisch and Volkwein [1999], Willcox and Peraire [2002], Hinze and Volkwein [2005], Grepl et al. [2007], Kunisch and Volkwein [2008] and Astrid et al. [2008]. Finally, some computational aspects have been investigated in Lall et al. [2002], Willcox and Peraire [2002], Gray and Verriest [2006] and Fujimoto and Tsubakino [2008].

Moment matching for nonlinear systems

A fundamental preliminary result for the development of model reduction by moment matching for nonlinear systems has been to recognize

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