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Nonlinear Models in $2 + \epsilon$ Dimensions

by

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Abstract

The general nonlinear scalar model is studied at asymptotically low temperature near two dimensions. The low temperature expansion is renormalized and effective algorithms are derived for calculation to all orders in the renormalized expansion. The renormalization group coefficients are calculated in the two loop approximation and topological properties of the renormalization group equations are investigated. Special attention is paid to the infrared instabilities of the fixed points, since they provide the continuum limits of the model.

The model consists of a scalar field ϕ on Euclidean $2 + \epsilon$ space whose values $\phi(x)$ lie in a finite dimensional differentiable manifold M. The action is

$$S(\phi) = \Lambda^{\epsilon} \int dx \;\; \frac{1}{2} T^{-1} g_{ij}(\phi(x)) \;\; \partial_\mu \phi^i(x) \;\; \partial_\mu \phi^j(x)$$

where Λ^{-1} is the short distance cutoff and $T^{-1} g_{ij}$ is a (positive definite) Riemannian metric on M, called the metric coupling.

The standard nonlinear models are the special cases in which M is

a homogeneous space (the quotient G/H of a Lie group G by a compact subgroup H) and g_{ij} is some G- invariant Riemannian metric on M. G acts as a global internal symmetry group.

The renormalization of the model is divided into two parts: showing that the action retains its form under renormalization and showing that renormalization respects the action of the diffeomorphisms (i.e. the reparametrizations or transformations) of M. The techniques used are the standard power counting arguments combined with generalizations of the BRS transformation and the method of quadratic identities.

The second part of the renormalization is crucial for renormalizing the standard models, since it implies the renormalization of internal symmetry. It is carried out to the point of identifying the finite dimensional cohomology spaces containing possible obstructions to the renormalization of the transformation laws, and of noting the absence of obstructions when M has finite fundamental group and nonabelian semisimple isometry group.

The renormalization group equation for the metric coupling is

$$\Lambda^{-1} \frac{\partial}{\partial \Lambda^{-1}} g_{ij} = - \beta_{ij}(g)$$

$$\beta_{ij}(T^{-1}g) = - \epsilon T^{-1} g_{ij} + R_{ij} + \frac{1}{2} T R_{ipqr} R_{jpqr} + O(T^2) .$$

R_{ipqr} is the curvature tensor and $R_{ij} = R_{ipjp}$ the Ricci tensor of the metric g_{ij}. The β- function $\beta_{ij}(g)$ is a vector field on the infinite dimensional space of Riemannian metrics on M.

Two results on global properties of β are obtained. When M is a homogeneous space G/H, the β- function is shown to be a gradient on the finite dimensional space of G- invariant metric couplings on M. And, when M is a two dimensional compact manifold, the β- function is shown to be a gradient on the infinite dimensional space of metrics on M. The rest of the results are concerned with fixed points. The fixed points are shown to correspond to the metrics satisfying a generalized Einstein equation:

$$R_{ij} - a\, g_{ij} = \nabla_i v_j + \nabla_j v_i , \quad a = \pm 1 \text{ or } 0$$

for v^i some vector field on M. Known solutions to these equations are discussed and some of their general properties described. In particular, it is shown that infrared instability occurs in at most a finite number of directions in the infinite dimensional space of metric couplings.

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Typographical Note

This thesis was prepared on a PDP-11 computer using the NEQN and NROFF typesetting programs of the UNIX operating system. Limitations of the printer required that Greek letters and special symbols be constructed as combinations of other characters. The composite symbols and their meanings are:

α	alpha	$\int$	integral
β	beta	∂	partial derivative
γ	gamma	Σ	sum
Γ	GAMMA	∞	infinity
δ	delta	∇	del, covariant derivative
ϵ	epsilon	$\vec{\nabla}$	gradient
λ	lambda	Δ	laplacian
Λ	LAMBDA	$\perp$	perpendicular
μ	mu	$\oplus$	direct sum
π	pi	$\otimes$	tensor product
Π	Pi, product	Ψ	PSI
ϕ	phi	ρ	rho
Φ	PHI	σ	sigma
ψ	psi	τ	tau

Contents

I. The Low Temperature Expansion

1. Introduction

This is the first part of a study of the general nonlinear scalar model at aymptotically low temperature near two dimensions. It treats the renormalization of the low temperature expansion. The second part is an investigation of the topological properties of the renormalization group equations near zero temperature. A partial summary of both parts is to be found in [1].

The model consists of a scalar field ϕ on Euclidean $2+\epsilon$ space whose values $\phi(x)$ lie in a finite dimensional differentiable manifold M. The distribution of the fields is

$$\prod_x d\phi(x) \quad \exp[-S(\phi)] \;, \tag{1.1}$$

where $\prod_x d\phi(x)$ is the a priori measure on the fields and

$$S(\phi) = \Lambda^{\epsilon} \int dx \;\; \frac{1}{2} T^{-1} g_{ij}(\phi(x)) \;\; \partial_\mu \phi^i(x) \;\; \partial_\mu \phi^j(x) \tag{1.2}$$

is the action. Λ^{-1} is the short distance cutoff.

The parameters of the model are: (1) $T^{-1} g_{ij}$, a (positive definite) Riemannian metric on M, called the metric coupling; and (2) $d\phi(x)$, a volume element on M (independent of x), called the a priori volume element, which is taken to be some natural volume element, such as the metric volume element, associated with g_{ij}.

parameters: parameters describing nonrenormalizable vertices are ignorable. Power counting determines that the bare parameter λ can be written as a function of a renormalized parameter λ^r and the ratio of scales $\mu^{-1}\Lambda$ so that $Z(\Lambda, \lambda)$, when expanded in λ^r, has a sensible limit order by order in λ^r as $\Lambda \to \infty$. To lowest order, λ is λ^r scaled by appropriate powers of $\mu^{-1}\Lambda$ so that the renormalized distribution of fields is, at lowest order, independent of Λ. At higher order, λ consists of cutoff dependent counterterms (containing powers of $\log \mu^{-1}\Lambda$) needed to cancel the primitive divergences in the Feynman diagrams of the perturbative expansion.

By power counting, the primitive divergences depend only on the short distance properties of the model. Therefore the perturbation theory can be made cutoff independent by means of counterterms which are independent of the infrared regularization.

The space of renormalized parameters λ^r must be large enough to contain all counterterms permitted by power counting, because the distinction between renormalized parameter and counterterm is arbitrary, up to cutoff independent reapportionments between the two.

The continuum limit of the perturbation theory, which depends on μ and λ^r, is defined by (6.1.3). Renormalization group equations follow from the equivalence of cutoff and continuum theories at distances much larger than the cutoff. $Z(\Lambda, \lambda)$ is independent of μ, so differentiating the expression on the left in (6.1.3) with respect to μ, holding Λ and λ fixed, gives the renormalization group equation appropriate to renormalize perturbatively. The internal O(N) symmetry equates all constants, so it is only necessary to investigate fluctuations about any one of them. The O(N) symmetry of the action, and its approximate scale invariance near two dimensions, give the result that the distribution of fields retains the form (1.1-2) under renormalization. The renormalized distribution depends on a renormalized temperature (and a renormalized field). The renormalization group acts on the fields and on the one parameter space of the temperature T.

A first order perturbative calculation of the renormalization group coefficients finds an infrared unstable fixed point at a temperature of order ϵ. The smallness of ϵ justifies the use of perturbative technique to find the fixed point. From the point of view of Wilson[7], the unstable manifold of the renormalization group action at the fixed point describes a Euclidean quantum field theory or, equivalently, the universal scaling limit of a nearly critical extended statistical system. In two dimensions the fixed point is at zero temperature and the model is asymptotically free.

Brezin, Zinn-Justin and Le Guillou[8-10] systematized Polyakov's results on the O(N)- model in the language of perturbative quantum field theory. (See also [11].) The double expansion in T and ϵ is found to be a renormalizable perturbation series, so that standard perturbative field theory algorithms can be applied to the calculation of renormalization group coefficients to all orders.

The standard perturbative version of the model is used. Small fluctuations of the nonlinear field $\phi(x)$ are represented as linear fields $\sigma^i(x)$ by means of coordinates about some point on the (N-1)-sphere M. Because of the homogeneity of M, all such points are equivalent. An O(N-1) subgroup of the internal symmetries acts by linear transformations on σ, the rest by nonlinear transformations. A special choice of coordinates is made in order to simplify the form of the nonlinear symmetry.

The distribution (1.1-2), rewritten in terms of the linear fields, describes an (N-1)- component massless scalar field governed by an action consisting of the integral over space of an infinite power series in the linear field times a product of two of its derivatives:

$$\tilde{S}(\sigma) = \Lambda^{\epsilon} \int dx \; \frac{1}{2} T^{-1} \tilde{g}_{ij}(\sigma(x)) \; \partial_{\mu}\sigma^i(x) \; \partial_{\mu}\sigma^j(x) \tag{1.3}$$

$$\tilde{g}_{ij}(v) = \delta_{ij} + (1 - v_k v_k)^{-1} \; v_i v_j \; . \tag{1.4}$$

The expansion in T becomes a sum of Feynman diagrams.

Power counting determines that the renormalized perturbative action remains the integral of a power series in the linear field times two of its derivatives. The nonlinear symmetries of the bare action give rise to quadratic identities on the renormalized action.[12] The most general solution of these identities consistent with power counting is exactly the bare action, up to a renormalization of the temperature and a multiplicative renormalization of the field.

The equivalence of bare and renormalized descriptions of the model implies renormalization group equations for the temperature and field, whose coefficients can be determined at each order in T and ϵ from the ultraviolet divergences of a finite number of Feynman diagrams. In [12], the coefficients are calculated in the two loop approximation.

The aim of the present work is to extend the results of Polyakov to the general nonlinear model, using an elaboration of the methods of perturbative field theory. Part I is concerned with the renormalization of the double expansion in T and ϵ.

The treatment of renormalization divides into two conceptually distinct tasks. The first task is to show that after renormalization the distribution of the fields retains the form (1.1-2). Linear fields are introduced to represent the fluctuations around the constant fields $\phi(x) = m$, using coordinates on M near m. In the absence of homogeneity all constants must be included. For each constant m, the distribution of linear fields is governed by an action of the form (1.3). Vertices are provided by the Taylor series expansion at m of the metric coupling in coordinates around m. Power counting determines that each distribution of linear fields retains its form under renormalization. The problem is to show that the renormalized vertices and linear fields associated with the various constants can be made to fit together as the Taylor expansions of a single renormalized metric coupling for a single nonlinear field. This is accomplished by expressing the conditions for compatibility of the vertices as an invariance of the collection of distributions of linear fields under simultaneous change of m and ϵ. Resulting quadratic identities on the renormalized distributions of linear fields are solved to find a renormalized distribution of nonlinear fields of the form (1.1-2). The result is that, under any renormalization scheme, the continuum limit $\Lambda \to \infty$ can be taken, order by order in T and ϵ, when the bare metric and field of (1.1-2) are expressed as cutoff dependent functions of a renormalized metric and a renormalized field. It follows that the renormalization group acts on the fields and on the metric couplings. Different renormalization schemes give rise to equivalent renormalization group actions.

Effective algorithms are derived for performing manifestly covariant calculations to all orders in T and ϵ; and the coefficients of the renormalization group equations are calculated in the two loop approximation. For the metric coupling, the result is

$$\Lambda^{-1} \frac{\partial}{\partial \Lambda^{-1}} g_{ij} = - \beta_{ij}(g) \tag{1.5}$$

$$\beta_{ij}(T^{-1}g) = - \epsilon T^{-1} g_{ij} + R_{ij} + \frac{1}{2} T R_{ipqr} R_j{}^{pqr} + O(T^2) . \tag{1.6}$$

R_{ipqr} is the curvature tensor and $R_{ij} = R_{ipjp}$ the Ricci tensor of the metric g_{ij}. The field $\phi(x)$ is renormalized within the space of order parameters: the nonnegative unit measures $\Phi(x)$ on M. The renormalization group equation is linear in the order parameter:

$$\Lambda^{-1} \frac{\partial}{\partial \Lambda^{-1}} \Phi(x) = \gamma(g)^{*} \Phi(x) \tag{1.7}$$

$$\gamma(T^{-1}g) = -\frac{1}{2} T \nabla_i \nabla_i + O(T^3) \tag{1.8}$$

where ∇_i is the covariant derivative for the metric g_{ij}. These equations are studied in Part II.

The second task in the study of renormalization is to investigate the effects on the transformation properties of the model. The diffeomorphisms of M (i.e., the transformations or reparametrizations of M) act on the fields and parameters of the model as a group of equivalence transformations. The diffeomorphisms which leave the parameters unchanged are global internal symmetries. The question is whether it is possible, given an arbitrary renormalization scheme, to find finite corrections which make the renormalization preserve the structure of the equivalence transformations. This seems crucial to the interpretation of the renormalized model. In particular, the preservation of internal symmetry is needed for the renormalizability of the standard models.

The investigation of the renormalizability of the transformation laws is not carried to completion here, but stops with identification of the finite dimensional cohomology spaces which contain the possible obstructions. The next step, which is an examination of the extent to which the action of the renormalization group removes the obstructions, is not taken; nor is any interpretation offered for the pathologies associated with the obstructions to renormalizability of the transformation laws.

The construction of the renormalized model and the calculation of the renormalization group coefficients require no conditions on the global properties of the manifold M. The discussion of the renormalization of the transformation laws, on the other hand, is limited here to the cases in which M is either a compact manifold or a noncompact homogeneous space. In the latter case, additional qualifying assumptions are made when convenient.

The organization of Part I is as follows. Section 2 describes basic structural features of the nonlinear models: the parameters of the models; the transformation properties; the structure of the standard models; and the definition of correlation functions, the order parameter and the generating functions. Section 3 sketches the construction of the regularized low temperature expansion. The technical details are given in sections 4 and 5. Section 4 is a treatment of systems of coordinates on the manifold M. Section 5 discusses the representation of the small fluctuations by linear fields, describes the distributions of linear fields, and derives invariance properties. Section 6 treats the renormalization. It discusses the renormalization group in general, constructs the renormalized nonlinear model, and begins the investigation of the renormalization of the transformation laws. Section 7 summarizes rules for calculation, including special rules adapted to the standard models; and presents the results of several calculations, notably the two loop calculation of the renormalization group coefficients. Material specific to the standard models is given in sections 2.3, 4.8, 5.8, 6.5, and 7.2.

The essential ingredients of Part I are manipulations of formal power series. Analytic niceties are either suppressed or ignored. Tensor analysis is done using index notation, which is regarded from the point of view of [13]. The indices $\{i,j,k, \cdots ,p,q,r, \cdots\}$ are used for tangent vectors on M. The summation convention is used throughout. [14] is a reference for basic facts and notation of differential geometry.

2. Structure of the Nonlinear Models

2.1. The distribution of fields

The form of the distribution of fields (1.1-2) is determined by Euclidean invariance, by the scalar character of the field ϕ, by the requirement that all interactions be short range and order inducing, and by certain assumptions of regularity.

The a priori measure $\prod_x d\phi(x)$ is, by itself, the most general Euclidean invariant distribution of fields in which the values of the field at different points in space are completely independent. It is the first term in an expansion in the range of interaction (having range zero). The full distribution of fields can be written as the a priori measure times the exponential of (minus) an action. Because only short range interactions are admitted, the action must be the integral over space of a local expression: a sum of products of spatial derivatives of the field.

A derivative $\partial_\mu\phi^i(x)$ of the field, in the μ direction at x, is a tangent vector to the manifold M at the point $\phi(x)$. Because ϕ is assumed to be scalar, only products of even numbers of derivatives can occur in the integrand of the action; and the spatial indices must be contracted in pairs with the Euclidean metric. The result, for each point x, is a partially symmetrized tensor at $\phi(x)$ in M. This tensor must be contracted with a dual tensor in order to obtain a real number which can be integrated over space. The dual tensor must in general depend on $\phi(x)$; but, by Euclidean invariance, it cannot depend explicitly on x . Thus the coupling associated with each term in the action is a tensor field on M.

A term in the action containing no derivatives of the field takes the form

$$\Lambda^{2+\epsilon} \int dx\ h(\phi(x)) \qquad (2.1.1)$$

where h is a real valued function on M (a tensor field of rank zero). h is the generalization of a constant external field. It can always be absorbed into the a priori volume element:

$$d\phi(x) \rightarrow d\phi(x)\ \exp[\ h(\phi(x))\ ]\ . \qquad (2.1.2)$$

Moreover, the ratio of two volume elements, being a positive function on M, can always be written as the exponential of a function h. The range zero portion of the distribution of fields is parametrized equivalently either by a priori volume elements or by constant external fields.

The action (1.2) is the most general possibility containing the product of two derivatives of the field. The two derivatives must appear in the form $\partial_\mu \phi^i(x)\ \partial_\mu \phi^j(x)$, which is a symmetric two-tensor at $\phi(x)$. It must be contracted against a symmetric quadratic form on tangent vectors at $\phi(x)$. The quadratic form should be nonnegative in order that the action be order inducing. A field of nonegative symmetric quadratic forms is a (possibly degenerate) Riemannian metric on M.

Contributions to the action containing a product of more than two derivatives of the field have naive length dimension $\geq 2 + O(\epsilon)$. Since true scaling behavior consists of naive scaling behavior plus corrections of order T, these contributions are suppressed under renormalization at low temperatures and small ϵ. In the language of perturbative field theory, they are nonrenormalizable. In the language of statistical mechanics, they are irrelevant.

The regularity assumptions are: (1) that the manifold in which the field takes values is smooth (infinitely differentiable); (2) that the a priori volume element is smooth and nowhere vanishing; and (3) that the metric coupling is smooth and nowhere degenerate.

The temperature T in the coupling $T^{-1}g_{ij}$ is not a separate parameter. Multiplying T by a positive constant c while multiplying g_{ij} by c^{-1} leaves the coupling unchanged. The temperature is written separately only to make the expansion parameter visible and appears only in the combination $(T\, g^{-1})^{ij}$. Except when an explicit expansion parameter is needed, the temperature will be absorbed into the metric, the coupling written simply g_{ij}.

The parameters of the general nonlinear model are the a priori volume element $d\phi(x)$ and the metric coupling g_{ij}. Two a priori volume elements are equivalent if their ratio is a constant, because the corresponding a priori measures differ only in their normalizations.

It is convenient to select a particular a priori volume element $d_g\phi(x)$ for each metric coupling g_{ij}, and to parametrize the model by metric couplings and constant external fields h_o. Two constant external fields are equivalent if they differ by a constant function on M. The distribution of fields becomes

$$\prod_x d_g\phi(x) \quad \exp[\, - S(\phi) + H_o(\phi)\,] \qquad (2.1.3)$$

$$H_o(\phi) \;=\; \Lambda^{2+\epsilon} \int dx\; h_o(\phi(x)) \;. \qquad (2.1.4)$$

The obvious choice of $d_g\phi(x)$ is the metric volume element associated with g, but it will be useful to allow for a more general choice.

2.2. The manifold M and its diffeomorphism group

The manifold M is taken to be finite dimensional, connected and smooth. If M is not connected, then, because fluctuations between different connected components are negligible at low temperature, the model decomposes into a collection of independent nonlinear models, each based on one of the connected components. Therefore M might as well be assumed connected.

In the construction of the renormalized low temperature expansion, which sees only small fluctuations of the nonlinear field, only the local properties of M are significant. Global conditions on M, such as compactness or completeness, are not needed. It is expected, however, that, for the model to be sensible (both perturbatively and nonperturbatively), some global conditions are required. In particular, the global properties of M seem to be relevant to the existence of the infinite volume limit of the low temperature expansion, order by order in T. (Compare [15].) Compactness should certainly be enough to give a sensible model. Of the noncompact manifolds, certain homogeneous spaces, at least, should have sensible low temperature expansions. The discussion of the renormalization of the transformation properties of the model is limited to these cases because they are technically accessible.

The diffeomorphisms of M (i.e., the transformations or reparametrizations) are the smooth maps of M to itself which have a smooth inverses. They form a group $\underline{D}$. The infinitesmal diffeomorphisms are the vector fields on M. A diffeomorphism Ψ acts on the fields by acting simultaneously on their values everywhere in space, taking the field $\phi(x)$ to the field $\Psi \circ \phi(x)$. It carries the distribution of fields to a transformed distribution. It also transforms volume elements and metrics on M, taking the metric g_{ij} and volume element $d\phi(x)$ at $\phi(x)$ to the metric $\Psi_* g_{ij}$ and volume element $\Psi_* d\phi(x)$ at $\Psi \circ \phi(x)$.

The transformed distribution of fields retains the form (1.1-2), the metric coupling and the a priori volume element replaced by the transformed metric and the transformed a priori volume element. It follows immediately that any diffeomorphism of M which leaves the metric and volume element unchanged acts as a global symmetry of the model.

When the model is parametrized by a constant external field h_o (with respect to a special choice $d_g\phi(x)$ of a priori volume element for each metric coupling), the transformed distribution of fields corresponds to the transformed metric $\Psi_* g_{ij}$ and the transformed external field $\Psi_* h_o$ if and only if the choice of a volume element for each metric is natural; that is,

$$\Psi_* d_g\phi(x) = d_{\Psi_* g}\phi(x) \tag{2.2.1}$$

for all diffeomorphisms Ψ. This certainly holds when $d_g\phi(x)$ is the metric volume element for g. Henceforth $d_g\phi(x)$ is assumed natural in g.

The transformed distribution of fields is entirely equivalent in its observable properties to the original. The manifold M is not itself directly accessible to observation because there is no means of singling out a distinguished parametrization of the values of the field by points in M. The only observables are the spectral properties of the Euclidean motions (and of the internal symmetries), and are not affected by the diffeomorphisms of M.

The group of diffeomorphisms of M acts on the space of parameters of the model as a group of equivalence transformations. The space of parameters is, after selection of a natural volume element for each metric coupling, the infinite dimensional space $\tilde{\underline{R}}$ of Riemannian metrics on M together with the space of real valued functions on M (modulo the constants). The true models are the equivalence classes under the action of the diffeomorphism group. The space $\underline{R}$ of equivalence classes of metrics is an infinite dimensional manifold except at the metrics with isometries, where there are singularities.[16] The true parameter space is a vector bundle (with singularities) over $\underline{R}$, whose fibers are equivalent to the space of real valued functions on M (modulo constants).

A covariant renormalization scheme is one in which the bare and renormalized parameters (and fields) transform in identical fashion, which is to say that the renormalization and diffeomorphism groups commute. A renormalization scheme is manifestly covariant if it is natural in the metric coupling and the a priori volume element (or external field). The manifestly covariant schemes developed below are natural in the metric alone.

2.3. Standard models

The standard models are characterized by the property that the transformations of M leaving the metric coupling and a priori volume element invariant act transitively on M. That is, for any pair of points in M there is a symmetry transformation taking one point to the other.

The isometries of a Riemannian manifold always form a finite dimensional Lie group. The condition that the a priori volume element also be preserved determines a closed subgroup. Therefore the internal symmetry group is a finite dimensional Lie group G. The symmetries leaving fixed a single point m_o in M form a closed subgroup H of G, called the isotropy (or "little") group at m_o. The map $\pi: G \to M$ which takes the transformation Ψ in G to the point $\Psi(m_o)$ in M identifies M with the quotient G/H, the space of right H- cosets in G. Varying the base point m_o amounts to conjugating H by an element in G. The action of G on M is therefore determined by the conjugation class in G of the compact subgroup H.

Since H leaves m_o fixed, it acts by linear transformations on the tangent vectors to M at m_o. This is called the isotropy action of H. Since H is a group of isometries of a Riemannian metric on M, and since the exponential map for such a metric identifies the tangent space at m_o with a neighborhood of m_o in H- invariant fashion, and since M is connected, it follows that any element in H which acts as the identity on tangent vectors at m_o is in fact the identity transformation of M. H is therefore identified with a closed subgroup of an orthogonal group and must be compact.

The Lie algebra of G is written $\underline{g}$, that of H, $\underline{h}$. H is a subgroup of G, so $[\underline{h}, \underline{h}]$ is contained in $\underline{h}$ (where $[\cdot,\cdot]$ is the Lie bracket in $\underline{g}$). Since H is compact, there exists a linear subspace $\underline{m}$ complementary to $\underline{h}$ in $\underline{g}$:

$$\underline{g} = \underline{m} \oplus \underline{h} \tag{2.3.1}$$

such that the adjoint action of H on $\underline{g}$ (by conjugation) takes $\underline{m}$ to itself. On the Lie algebra level, $[\underline{h}, \underline{m}]$ is contained in $\underline{m}$. The letters {a,b,c ...} are used for indices taking values in $\underline{h}$ and the letters {i,j,k ... p,q,r ...} for indices taking values in $\underline{m}$. The nonzero structure constants of $\underline{g}$ are C^c_{ab}, $C^j_{ai} = -C^j_{ia}$, C^k_{ij}, and C^a_{ij}, where the structure constants are given, for example, by $[v, w]^k = v^i w^j C^k_{ij}$.

The O(N)- model has $G = O(N)$, $H = O(N-1)$, $M = S^{N-1}$, and $\underline{m} = R^{N-1}$. The adjoint action of $H = O(N-1)$ on $\underline{m} = R^{N-1}$ is the defining representation. The chiral SU(N)- model has $G = SU(N)\times SU(N)$, $H = SU(N)$ (the diagonal subgroup), $M = SU(N)$, and $\underline{m} = su(N)$. The adjoint action of H on $\underline{m}$ is conjugation.

The tangent space to M at m_o can be identified with the subspace $\underline{m}$ in $\underline{g}$ by means of the derivative $d_e\pi$ of the quotient map π: $G \to M$ at the identity e in G, because $d_e\pi(\underline{h}) = 0$. An infinitesmal transformation v in $\underline{m}$ moves the point m_o infinitesmally in M along the corresponding tangent vector $d_e\pi(v)$. The isotropy action of H on tangent vectors at m_o is the same as its adjoint action on $\underline{m}$.

Since G acts transitively on M, all G - invariant tensor fields on M, including the metric and volume element, are determined by their values at m_o, and are in one to one correspondence, via $d_e\pi$, with the H- invariant tensors on $\underline{m}$. In particular, the invariant metric is an H- invariant inner product on $\underline{m}$ and the invariant volume element an H- invariant volume element on $\underline{m}$. There is only one volume element on $\underline{m}$ (up to multiplication by a positive real number): $|e_1 \wedge e_2 \wedge \cdots|$ in a basis $\{e_i\}$ for $\underline{m}$. So the metric coupling is the only free parameter in a standard model.

Because H is compact, it leaves the volume element on $\underline{m}$ invariant. Any nondegenerate inner product on $\underline{m}$ can be averaged over H to make an H- invariant inner product. Therefore, for any G/H, H compact, there exists a standard model.

Any positive multiple of a G- invariant metric is also G- invariant, so the temperature is always a free parameter of a standard model. Whether there are more parameters depends on the isotropy action of H. If $(g_1)_{ij}$, and $(g_2)_{ij}$ are two H- invariant inner products on $\underline{m}$, then $(g_1^{-1}\ g_2)^i_j$ is an H- invariant symmetric linear transformation on $\underline{m}$, whose eigenspaces reduce the representation of H on $\underline{m}$. If H acts irreducibly, then $g_1^{-1}\ g_2$ is always a multiple of the identity and the temperature is the only free parameter. Wolf[17] has classified all such isotropy irreducible homogeneous spaces.

More generally, the isotropy representation of H contains k inequivalent irreducible representations in multiplicities $n_1, n_2, \cdots n_k$. The space $\bar{\underline{R}}_G$ of G- invariant metrics on M is a product of k factors. The i-th factor is the space of positive definite symmetric forms on a real vector space of dimension n_i. It is a noncompact smooth manifold of dimension $\frac{1}{2} n_1 (n_1 + 1)$. If some of these metrics have isometry groups G' larger than G, then the G'-invariant metrics form a submanifold of the space of all G- invariant metrics.

Inside the infinite dimensional manifold $\underset{\sim}{\tilde{R}}$ of all metrics on M is the infinite dimensional manifold $\underset{\sim}{\tilde{R}}_{[G]}$ of all metrics which have (sub-) groups of isometries equivalent to G under conjugation by diffeomorphisms of M. The diffeomorphisms of M act as equivalence transformations on $\underset{\sim}{\tilde{R}}_{[G]}$. Restricting to the finite dimensional manifold $\underset{\sim}{\tilde{R}}_G$ of G- invariant metrics eliminates most but not necessarily all of these equivalence transformations. For the standard models, preservation of the transformation laws under renormalization includes the preservation both of the internal symmetries and of the residual equivalence transformations.

A general characterization of the residual equivalence transformations will not be given. The problem is of a cohomological character. For example, the space of infinitesmal equivalences at a given G-invariant metric g_{ij} is the first cohomology group of the Lie algebra $\underset{\sim}{g}$ with coefficients in the full Lie algebra of infinitesmal isometries of g_{ij}.

There is always, however, a natural Lie group of equivalence transformations on $\underset{\sim}{\tilde{R}}_G$, namely the group $\underset{\sim}{D}_G$ of diffeomorphisms of M which commute with all transformations in G. Its Lie algebra is the space of all G- invariant vector fields on M, which is identical to the space of H- invariant vectors in $\underline{m}$, equipped with the with Lie bracket C_{ij}^{k}.

When G is semisimple, the G- invariant vector fields exhaust the infinitesmal equivalence transformations by diffeomorphisms of M, because all of the first cohomology groups of $\underline{g}$ are trivial. The space of metric couplings is then a finite dimensional noncompact manifold $\underline{\tilde{R}}_G$ on which a finite dimensional Lie group $\underline{D}_G$ of diffeomorphisms of M acts as a group of equivalence transformations. The space $\underline{R}_G$ of true parameters of the model is the quotient $\underline{\tilde{R}}_G/\underline{D}_G$. $\underline{R}_G$ is a manifold except at metrics with isometry groups larger than G (more precisely, larger than the generic isometry group of the G- invariant metrics).

An example is the model $M = G = G/\{e\}$ in which M is itself a Lie group, H is trivial and the metric coupling is assumed left, but not necessarily right, G- invariant. The space $\underline{\tilde{R}}_G$ of G- invariant metrics is the space of positive definite symmetric forms on $\underline{m} = \underline{g}$. It has dimension $\frac{1}{2}\, n\, (n + 1)$ where n is the dimension of G. All of $\underline{m}$ is H- invariant, so the residual transformations of M (those commuting with left multiplication) form the Lie group G (acting by right multiplication). The space $\underline{R}_G$ of equivalence classes of G- invariant metrics has dimension $\frac{1}{2}\, n\, (n + 1) - n = \frac{1}{2}\, n\, (n - 1)$ wherever it is nondegenerate.

It will sometimes be convenient to assume that G/H is unimodular, meaning that all transformations in $\underline{D}_G$ preserve the G- invariant volume element on M. For the infinitesmal transformations in $\underline{D}_G$, this is equivalent to the condition $c^j_{ij} = 0$ on the structure constants. If G is a unimodular Lie group (i.e. the left invariant volume element on G is also right invariant), then G/H is automatically unimodular. The compact and the semisimple Lie groups are all unimodular.

2.4. Correlation functions, the partition function and the free energy

The correlation function $< \phi(x_1)\ \phi(x_2) \cdots \phi(x_k) >$ of the nonlinear model is, for each k-tuplet $(x_1, \cdots, x_k)$ of (distinct) points in space, a nonnegative unit measure on M^k. It is the probability distribution induced from (1.1) on the values of the field at the points $x_1, \cdots, x_k$. Equivalently, it is the average over distribution (1.1) of the point measure in M^k located at $(\phi(x_1), \cdots, \phi(x_k))$.

A real valued function F on M^k is integrated against $< \phi(x_1)\ \phi(x_2) \cdots \phi(x_k) >$ according to

$$(F, <\phi(x_1) \cdots \phi(x_k)>) =$$

$$Z(0)^{-1} \int \prod_x d\phi(x) \quad \exp[-S(\phi)] \quad F(\phi(x_1), \cdots, \phi(x_k)) \qquad (2.4.1)$$

where $Z(0)$ normalizes the distribution of fields. If F is nonnegative then the integral is also, so the correlation function is a nonnegative measure on M^k. If $F = 1$ then its integral is also 1, so the correlation function is a unit measure.

In particular, the order parameter $< \phi(x) >$ is a nonnegative unit measure (a probability measure) on M.

In the presence of an external source the distribution of fields is

$$\int \prod_x d\phi(x) \quad \exp [- S(\phi) + H(\phi)] \qquad (2.4.2)$$

where the source term is

$$H(\phi) = \Lambda^{2+\epsilon} \int dx \quad h(x) \; (\; \phi(x) \;) \; . \qquad (2.4.3)$$

h is a space dependent external field. Its value h(x) at each point x is a real valued function on M. The partition function

$$Z = \int \prod_x d\phi(x) \quad \exp [- S(\phi) + H(\phi)] \; . \qquad (2.4.4)$$

depends on the external field and on the metric coupling.

Adding an external field to the action is equivalent to making the a priori volume element $d\phi(x)$ vary with x. In the infinite volume limit it is feasible to make a distinction between global (or thermodynamic) and local parameters: the a priori volume element, remaining constant in space, is the thermodynamic parameter; the external field, compactly supported or at least tempered in space, the local parameter. But renormalization depends only on short distance effects (and will be carried out at finite volume) so does not see the distinction. The form of the renormalization of the range zero part of the distribution of fields is more transparent in the language of external fields (in fact, it is linear in the external field), so it is convenient to fix an a priori volume element $d\phi(x) = d_g\phi(x)$ for each coupling g_{ij} and to consider the spatially varying external field h to be the global and local parameters combined.

The correlation functions are derivatives of the partition function with respect to the external field:

$$< \phi(x_1) \cdots \phi(x_k) > =$$

$$Z(0)^{-1} \frac{\delta}{\delta h(x_1)} \cdots \frac{\delta}{\delta h(x_k)} Z_{/h=0} . \qquad (2.4.5)$$

The dual to the space of functions on M is the space of measures on M; thus a derivative with respect to $h(x)$ is a measure. The derivatives are always unit measures because the partition function changes trivially when a constant function on M is added to $h(x)$.

The free energy Γ is the Legendre transform of $\log Z$:

$$\Gamma = \sup_h [- \log Z + \Lambda^{2+\epsilon} \int dx \, (h(x), \Phi(x))] . \qquad (2.4.6)$$

Γ is a function of the metric coupling and of the (spatially dependent) local order parameter Φ, which at each x is a nonnegative unit measure $\Phi(x)$ on M. The pairing between $h(x)$ and $\Phi(x)$ in (2.4.6) is the integral of a real valued function on M against a measure on M. The Legendre transform of Γ is, in turn, $\log Z$.

In the low temperature expansion, the supremum over external fields in (2.4.6) is achieved by evaluating

$$- \log Z + \Lambda^{2+\epsilon} \int dx \, (h(x), \Phi(x)) \qquad (2.4.7)$$

at its critical point as a function of Φ; that is, by inverting

$$\Phi(x) = Z^{-1} \frac{\delta}{\delta h(x)} Z \qquad (2.4.8)$$

to express $h(x)$ as a function of $\Phi(x)$, and then substituting for $h(x)$ in (2.4.7).

The partition function Z, or the free energy Γ, remains unchanged when the metric and the external field, or the order parameter, are transformed by the same diffeomorphism of M. They are functions, therefore, of the equivalence classes under the action of the diffeomorphism group of M. The content of the model is summed up in the dependence of the partition function or the free energy on the global and local parameters. The true space of parameters is therefore the space of equivalence classes.

The equivalence classes of metric couplings and spatially dependent external fields (modulo trivial external fields) make up a vector bundle (with singularities) over the equivalence classes $\underline{R}$ of metrics. The equivalence classes of metrics and local order parameters form a conical sub-bundle of the dual bundle.

2.5. The order parameter

The essential property of the order parameter is its averageability. The renormalization group acts by averaging the variables of the model over small regions in space (and by an overall rescaling of distances). Points in a manifold M can only be averaged if M is embedded in a space in which convex combination makes sense (for example, a vector space), and then the average of points in M will in general not remain in M. There are many embeddings of a given manifold M in a finite dimensional vector space, but none which is natural. Any such embedding involves arbitrary choices obscuring the character of the nonlinear model, which depends only on the intrinsic structure of the abstract manifold M. The only natural embedding is the one which places M inside the space of all unit measures on M itself, sending each point in M to the corresponding point measure. The order parameter then varies over all possible averages of point measures, which is to say over all the probability measures on M.

In a standard model (M the homogeneous space G/H) this picture can be considerably simplified. The internal symmetry group G acts on M, so acts by linear transformations on the real valued functions on M. Let V be a finite dimensional subrepresentation which separates points in M; that is, which, along with the products among its members, generates all the functions on M. Without loss of information, the values $h(x)$ of the external field can be assumed to lie in V. Each point m in M can be identified with a distinct point in the dual space V^*: the linear functional which assigns to each function h in V its value $h(m)$ at m. M is thus embedded in V^*, and the order parameter takes its values there. The correlation functions have their values in tensor products of V^*. In the $O(N)$- model, such a subrepresentation is given by the N linear coordinate functions on R^N, restricted to the unit sphere M.

When more than one G- invariant metric coupling exists, it is necessary to use a reducible subrepresentation V of functions on M as external fields in order to ensure, by appropriate choice of G- invariant inner product on V^*, that, for any G- invariant metric on M, the embedding of M in V^* can be made an isometry. An isometric embedding is desirable because it allows the model to be written as a free field subjected to constraints. This formulation of the model suffers from possible redundancy in the parametrization of G- invariant metrics on M by inner products on V^*.

The general manifold M possesses no distinguished generating subspace of functions, so all functions must be allowed as possible values of the external field, and all probability measures as possible values of the order parameter.

An asymptotically small action of the renormalization group on the model has the effect of smearing the field $\phi(x)$ whose values are in M to produce a field $\Phi(x)$ whose values are in the probability measures on M, close by the point measures. The renormalized distribution of fields is a distribution of the $\Phi(x)$. The renormalized action, as a function on this convex space of fields, has, presumably, a degenerate set of minima which is identical to the manifold M. (This is true of the minima of the free energy at low temperature in the mean field theory, which also requires for its formulation an averageable order parameter.) The renormalized nonlinear fields correspond to those Φ whose values $\Phi(x)$ lie in this copy of M. The fluctuations transverse to the space of renormalized nonlinear fields are integrated out of the renormalized distribution of the $\Phi(x)$ without any loss of information associated with distances much larger than the cutoff Λ^{-1}. The issue becomes the form of the resulting renormalized distribution of nonlinear fields and the effect of the renormalization procedure on the action of the diffeomorphism group of M. These issues are addressed, in somewhat different language, in the discussion of the renormalization of the low temperature expansion.

3. The Regularized Low Temperature Expansion

3.1. Linear fields

In the low temperature expansion only asymptotically small fluctuations about the constant fields have any significance. In order to apply the standard techniques of perturbative field theory, some linear representation for the fluctuations is needed. That is, a neighborhood of the constant $\phi(x) = m$ in the space of nonlinear fields must be replaced by a neighborhood of zero in some linear space of fields. This is most conveniently done by choosing coodinates on a neighborhood of m in M. Then points in M near m are represented by vectors in the linear space of coordinates. The value $\phi(x)$ of the nonlinear field is represented by the vector $\sigma^i(x)$ which is $\phi(x)$ in coordinates around m. σ is the linear field.

The advantages in defining the linear field by means of coordinates are that (1) manifest Euclidean invariance is maintained (when the coordinates are independent of x), and (2) power counting is simplified by the fact that the nonlinear field is local and of zeroth order in the linear field. These two conditions exactly characterize the definition of the linear field by means of coordinates.

Without loss of information, as far as the low temperature expansion is concerned, the distribution of nonlinear fields (2.4.2) is reexpressed in terms of the linear fields which represent the small fluctuations. The a priori measure $\prod_x d\phi(x)$ becomes

$$\tilde{d}\sigma = \prod_x d\sigma(x) \quad \exp[\tilde{J}(m,\sigma)] \tag{3.1.1}$$

where $d\sigma(x)$ is the a priori volume element on M at the constant m (which is independent of $\sigma^i(x)$), and

$$\tilde{J}(m,\sigma) = \Lambda^{2+\epsilon} \int dx \quad \tilde{j}(m,\sigma(x)) \tag{3.1.2}$$

is the logarithmic jacobian of the map from the linear field σ to the nonlinear field ϕ. $\tilde{j}(m,\sigma(x))$ is the logarithmic jacobian of the coordinate map from $\sigma^i(x)$ to $\phi(x)$ (with respect to the appropriate volume elements).

The action becomes

$$\tilde{S}(m,\sigma) = \Lambda^{\epsilon} \int dx \quad \frac{1}{2} T^{-1} \tilde{g}_{ij}(m,\sigma(x)) \quad \partial_\mu \sigma^i(x) \; \partial_\mu \sigma^j(x) \tag{3.1.3}$$

where $\tilde{g}_{ij}(m,\sigma(x))$ is the metric g_{ij} at the point $\sigma^i(x)$ in coordinates around m.

The external source becomes

$$\tilde{H}(m,\sigma) = \Lambda^{2+\epsilon} \int dx \quad \tilde{h}(x) \; (m,\sigma(x)). \tag{3.1.4}$$

where $\tilde{h}(x)\ (m,\sigma(x))$ is the external field $h(x)$ evaluated at $\sigma^i(x)$ in coordinates around m.

The distribution of the linear fields is

$$\tilde{d}\sigma \quad \exp[- \tilde{A}(m,\sigma)]$$

$$\tilde{A} = \tilde{S} - \tilde{H} . \tag{3.1.5}$$

The low temperature expansion for the fluctuations around $\phi(x) = m$ is calculated using standard Feynman diagram technique on the functional integral

$$\int \tilde{d}\sigma \quad \exp[- \tilde{A}] . \tag{3.1.6}$$

The propagator and vertices come from expanding $\tilde{S}$, $\tilde{H}$ and $\tilde{J}$ in powers of σ. Since coordinates are used to provide the linear fields, this amounts to expanding the metric $\tilde{g}_{ij}(m,v)$, the functions $\tilde{h}(x)$ (m,v) and the logarithmic jacobian $\tilde{j}(m,v)$ in powers of the coordinate v^i.

To achieve a manifestly covariant low temperature expansion, the coordinates should be such that the Taylor series coefficients of $\tilde{g}_{ij}$, $\tilde{h}(x)$, and $\tilde{j}$ are themselves manifestly covariant. That is, the linear space of coordinates around m should be the tangent space $T_m M$ to M at m; and the Taylor series coefficients should be tensors formed from the metric, the curvature and its covariant derivatives, and the external field and its covariant derivatives.

One set of coordinates having this property are the geodesic normal coordinates defined with respect to the metric coupling g_{ij}.

For the standard models it is convenient to use coordinates which are defined with reference only to the internal symmetry group, not to any particular invariant metric coupling (when more than one exists). Geodesic normal coordinates defined with respect to the canonical connection on M are of this type.

In section 4, effective algorithms are derived for calculating manifestly covariant Taylor series expansions in normal coordinates to arbitrary order.

The value of the functional integral (3.1.6) is at least formally independent of the choice of coordinates used to define it, so the construction of the general low temperature expansion should not require a particular choice of coordinates, even a manifestly covariant choice.

3.2. Infrared regularization

At T near zero, the distribution of linear fields (3.1.5) approaches asymptotically the gaussian distribution

$$\prod_x d\sigma(x) \quad \exp[\,-\,\tilde{S}_0(m,\sigma)]$$

$$\tilde{S}_0(m,\sigma) \;=\; \Lambda^{\epsilon} \int dx \;\; \tfrac{1}{2}\, T^{-1} \;\; \tilde{g}_{ij}(m,0) \;\; \partial_\mu \sigma^i(x)\; \partial_\mu \sigma^j(x) \;. \quad (3.2.1)$$

This determines the propagator for σ to be that of massless scalar field (a spin wave). But the massless propagator is infrared divergent in two dimensions:

$$\int_0 d^2k \; \frac{e^{ikx}}{k^2} = \infty \; . \qquad (3.2.2)$$

Therefore some kind of infrared regularization must be introduced.

The renormalization of the model sees only short distance effects, so the form of the infrared regularization cannot be essential. However, certain features would be especially attractive. The infrared regularization ought to be applicable to the full nonlinear model (2.4.2) and not merely to the sum over small fluctuations, in the hope that the low temperature expansion is that of a (nonperturbatively) sensible theory (having a good infinite volume limit). There should also be at least a plausible scenario for removal of the regularization, order by order in T, leaving behind a well-behaved set of correlation functions.[cf. 15] Finally, the infrared regularization should be specified without reference to a particular choice of coordinates (even a manifestly covariant one).

By these criteria, the simplest forms of infrared regularization are unsatisfactory. A direct low momentum cutoff for the linear field, or the addition of a regulator mass

$$\int dx \; \frac{1}{2} T^{-1} \; a^2 \; \tilde{g}_{ij}(m,0) \; \sigma^i(x) \; \sigma^j(x) \; , \qquad (3.2.3)$$

to the action are not applicable to the nonlinear field and depend on a choice of coordinates for their specification, changing form under change of coordinates.

One form of regularization which does respect the nonlinearity of the model is modifying the distribution (2.4.2) of nonlinear fields by including in the ac ion a constant external field $-\frac{1}{2} T^{-1} h_o(\phi(x))$. When T is asymptotically small the nonlinear field fluctuates only around the point m_o in M which minimizes h_o. The fluctuations are described by the distribution of linear fields (3.1.5). The external source $\tilde{H}$ includes the $O(T^{-1})$ constant external field. The leading, quadratic, term in the exponent of (3.1.5) provides a massive propagator for linear field, the masses being the eigenvalues of the Hessian of h_o at m_o. This is the form of regularization which was used in [9,10].

Infrared regularization by means of a constant external field has three principal disadvantages. First, it requires choice of a function on M and, in general, there is no choice which is natural in the metric $T^{-1} g_{ij}$, is of order T^{-1} and can be guaranteed to have a non-degenerate minimum. Second, it complicates calculation by separating the external field into two pieces, placing the quadratic part of one piece in the propagator as a mass term. Third, it singles out one point m_o in M, which is not in keeping with the fact that, in two dimensions, the presence of spin waves in the model prevents the spontaneous emergence of order. (See section 6.4 below.) The external field, being of negative length dimension, is a soft operator. Therefore, even in models in which order is imposed by a nontrivial external field, the short distances properties are those of the disordered state which sees all of the points in M. In a standard model all points in M are equivalent, so no harm is done by selecting one of them. But, in the general nonlinear model, the global structure of M is obscured by the constant external field. This is signalled by the persistence of infrared divergences in the correlation functions after h_o is sent to zero.

On the other hand, the constant external field infrared regularization is entirely suitable for separating infrared divergences from ultraviolet divergences in order to perform the renormalization. It is simply necessary to use a different external field to renormalize the distributions of linear fields about each of the constants.

An alternative to the constant external field is the finite volume form of infrared regularization. The system is placed in a box, which for simplicity is taken to be square, of side L. Periodic boundary conditions are imposed in order to mimic the anticipated disordering effect of the spin waves.

Finite volume regularization presents two complications. First, there is a loss of global spatial symmetry. The standard expectation is that this returns in the infinite volume limit. In any case, renormalization involves only the short distance property of local Euclidean invariance, which is not affected by the finiteness of the volume. The second complication is that, in the evaluation of Feynman diagrams, momenta are summed over a discrete set of values. This has no consequence for renormalization. All of the primitive divergences of the Feynman diagrams are either quadratic or logarithmic. The quadratic divergences can be extracted using only operator theoretic properties of the propagator, without need for an eigen" ' expansion. Logarithmic divergences are calculated by approximating the finite volume propagator with the continuum massless propagator k^{-2}. Corrections, which are $O(L^{-1}k^{-1})$, do not contribute to the primitive logarithmic divergences.

It seems to be possible, with periodic boundary conditions, to collect terms at each order in T so that, in the infinite volume limit, the bare propagator is, in effect, the subtracted (infrared finite) massless propagator. (Compare [15].) If this is so, then the discreteness of momentum space need never complicate actual calculation of infinite volume correlation functions. This would justify the use of periodic boundary conditions for the low temperature expansion.

Under periodic boundary conditions, the distribution of fields (2.4.2) is dominated at low temperature by all of the absolute minima of the action (1.2). These are the constant fields, forming a copy of M inside the space of all nonlinear fields. Fluctuations around all of the constants participate in the low temperature expansion of the partition function Z:

$$Z = \int_M Z(m) \tag{3.2.4}$$

where $Z(m)$ is the sum over fluctuations around the constant $\phi(x) = m$.

Each sum $Z(m)$ is calculated using a linear field $\sigma^i(x)$, defined by means of coordinates around m, to represent the fluctuations around the constant $\phi(x) = m$. Such a choice of coordinates around each point in M is called a system of coordinates for M. Metric and canonical normal coordinates are examples. Systems of coordinates in general, and these two in particular, are discussed in section 4.

If each $Z(m)$ were calculated by the functional integral (3.1.6), then the constant nonlinear fields would be overcounted, since they occur as constant fluctuations around all nearby constant fields. The degeneracy of the minimum of the action is reflected in the fact that there are zero modes in the inverse propagator for the linear field at each constant: namely, the constant linear fields $\sigma^i(x) = v^i$.

In each sum over fluctuations a gauge condition $\tilde{P}^i(m,\sigma) = 0$ is needed to avoid the overcounting and eliminate the zero modes. A gauge function of the form

$$\tilde{P}^i(m,\sigma) = L^{-(2+\epsilon)} \int dx \;\; \tilde{p}^i(m,\sigma(x)). \qquad (3.2.5)$$

maintains manifest Euclidean invariance and simplifies the power counting. $\tilde{p}^i(m,v)$ is, for each m, a vector valued function on the coordinate space at m. The simplest of gauge functions is

$$\tilde{p}^i(m,\sigma(x)) = \sigma^i(x) \; . \qquad (3.2.6)$$

The gauge condition is imposed by including a delta-function

$\delta(P(m,\sigma))$ in the distribution of linear fields (3.1.5). A Fadeev-Popov determinant must also be included to correctly reproduce the a priori measure. The gauge function has a finite number of components, so the determinant is of a finite dimensional matrix $\tilde{F}^i_j(m,\sigma)$:

$$\tilde{F}^i_j(m,\sigma) = L^{-(2+\epsilon)} \int dx \ \tilde{f}^i_j(m,\sigma(x)). \qquad (3.2.7)$$

A multiplier γ_i is used to enforce the gauge condition and a finite set of anticommuting ghost variables give the Fadeev-Popov determinant. The distribution of linear fields and auxiliary variables describing the fluctuations around $\phi(x) = m$ is

$$d\gamma \ dc^* \sim dc \ \tilde{d}\sigma \ \exp[- \tilde{A}(m,\sigma,\gamma,c,c^*)] \qquad (3.2.8)$$

$$\tilde{A}(m,\sigma,\gamma,c,c^*) = \tilde{S}(m,\sigma) - \tilde{H}(m,\sigma)$$

$$- i \gamma_i \tilde{P}^i(m,\sigma) - c^j \tilde{F}^i_j(m,\sigma) c^*_i . \qquad (3.2.9)$$

The contribution to the partition function of the fluctuations around m is

$$Z(m) = \int d\gamma \ dc^* \sim dc \ \tilde{d}\sigma \ \exp[- \tilde{A}(m,\sigma,\gamma,c,c^*)] . \qquad (3.2.10)$$

$\tilde{S}$ is given in (3.1.3), $\tilde{H}$ in (3.1.4) and $\tilde{d}\sigma$ in (3.1.1-2). $Z(m)$

is a volume element on M at m, which is integrated in (3.2.4) to give the partition function Z. In section 5 these constructions are described in more detail and a formula for $\tilde{F}_j^1(m,\sigma)$ is derived.

Both the constant external field and the finite volume forms of infrared regularization are used below. The simplicity of (3.1.5) in comparison with (3.2.8-9) is an advantage of the former. The arguments for renormalizability are correspondingly simpler. Below, when the relationship between the two arguments is sufficiently clear, only the simpler of the two is made explicit.

3.3. Ultraviolet regularization

Ultraviolet regularization is needed to tame the short distance divergences occurring in the Feynman diagrams which give the low temperature expansion of the functional integrals (3.1.6) or (3.2.10). The ultraviolet regularization should be applicable beyond the low temperature expansion. Among available forms of regularization, only the lattice has this property.

The action (1.2) makes no sense on the lattice. A nonlinear analogue of the finite difference operator must be found to take the place of the continuum spatial derivative ∂_μ. One possible lattice action is

$$S(\phi) = \sum_{(x,y)} \frac{1}{2} D^2(\phi(x), \phi(y)) \qquad (3.3.1)$$

where the sum ranges over unordered nearest neighbor pairs on a periodic, cubical lattice; and where $D^2(m_1,m_2)$ is the distance squared between the two points m_1 and m_2 in M, calculated with respect to the Riemannian metric coupling g_{ij}. In place of D^2 might be used any function $K(m_1,m_2)$ which is minimized when its two arguments are identical and whose second derivatives at the minima are the values of the metric coupling:

$$-\frac{1}{2}\frac{\partial^2 K}{\partial m_1^i \partial m_2^j}(m_1,m_2)_{/m_1=m_2} = g_{ij}(m_1) . \qquad (3.3.2)$$

If $v_{1,2}$ are $m_{1,2}$ in coordinates around m, then $K(m_1,m_2)$ is

$$\tilde{K}(m,v_1,v_2) = \tilde{g}_{ij}(m,<v>)\,(\delta v)^i\,(\delta v)^j + O((\delta v)^3) \qquad (3.3.3)$$

where $<v> = \frac{1}{2}(v_1 + v_2)$ and $\delta v = v_2 - v_1$. The terms of order $(\delta v)^3$ depend on the choice of K.

The lattice action in coordinates around m is

$$\tilde{S}(r,\sigma) = \Lambda^{-2} \sum_{(x,\mu)} \frac{1}{2}\tilde{g}_{ij}(m,<\sigma>_\mu)\ \partial_\mu\sigma^i(x)\ \partial_\mu\sigma^j(x)$$

$$+ \text{ terms containing more than two derivatives.} \qquad (3.3.4)$$

The sum is over points x and directions μ in the lattice; Λ^{-1} is the lattice spacing; $<\sigma>_\mu = \frac{1}{2}(\sigma(x) + \sigma(x+\mu))$; and ∂_μ is the finite difference operator in the μ direction:

$$\partial_\mu \sigma^i(x) = \Lambda [\sigma^i(x+\mu) - \sigma^i(x)] . \qquad (3.3.5)$$

At asymptotically low temperature the terms containing more than two derivatives are irrelevant to the continuum limit. Therefore the arbitrariness in the lattice action is of no consequence.

The propagator of the finite volume lattice σ field is the usual massless lattice propagator, the zero mode eliminated by the gauge condition. In Feynman diagrams, the momenta are summed over a periodic finite set $\{k_\mu\}$:

$$k_\mu = (2\pi L^{-1})\, n_\mu$$

$$n_\mu = -\tfrac{1}{2}\Lambda L \; \cdots \; -2, -1, 0, 1, 2, \cdots \; \tfrac{1}{2}\Lambda L$$

$$\sum_\mu |k_\mu| \neq 0 . \qquad (3.3.6)$$

To define the value of a diagram in noninteger dimensions, the ϵ dependence must be isolated. This is done by proper time parametrization of the propagators, exactly as in the case of continuous unbounded momenta:

$$G_o(k) = [\sum_\mu 2\Lambda^2 (1 - \cos \Lambda^{-1} k_\mu)]^{-1}$$

$$= \int_0^\infty dt \prod_\mu \exp(- t [2\Lambda^2 (1 - \cos \Lambda^{-1} k_\mu)]) . \qquad (3.3.7)$$

Summing over the loop momenta leaves an explicitly ϵ - dependent integral over the proper times, which can be evaluated at noninteger dimensions. There does not seem to be known a nonperturbative extension of the lattice regularization to noninteger dimensions.

Among the forms of short distance regularization which are applicable only in the low temperature expansion, the most attractive are those which do not depend upon the choice of coordinates on M. Dimensional regularization is one such. It is carried out by calculating the partition function Z order by order in T for a model based on a torus (periodic box) of side L and dimension $2 + \epsilon$. Again, the discreteness of the values of the momenta due to finite volume does not impede isolation of the ϵ dependence of the Feynman diagrams. The diagrams are evaluated at ϵ sufficiently negative for the result to be well defined and then analytically continued to $\epsilon = 0$.

Regularization using a cutoff in momentum space is also feasible, but depends for its specification on the choice of linear fields.

4. Systems of Coordinates

4.1. Introduction

A system of coordinates is a set of coordinates around each point m in the manifold M. In this section a technical apparatus is developed for describing systems of coordinates in general. This apparatus is then used to find recursive procedures for calculating Taylor series coefficients of a Riemannian metric and other tensor fields in special systems of coordinates: Riemannian geodesic normal coordinates and canonical geodesic normal coordinates in particular. Equivalent procedures for calculating the Taylor series coefficients of a metric in Riemannian normal coordinates were derived by more direct arguments by Cartan[18]. The Taylor series coefficients provide manifestly covariant vertices for the Feynman diagrams of the low temperature expansion of the nonlinear model. The general technical apparatus is used in succeeding sections in the description of the Fadeev-Popov determinant and in the renormalization of the low temperature expansion.

A natural linear coordinate space for a neighborhood of a point m in M is the tangent space T_mM to M at m. With these as coordinate spaces, a system of coordinates is a collection of coordinate maps E_m: $T_mM \to M$ identifying, for each m, a neighborhood of zero in T_mM with a neighborhood of m in M. The coordinate maps are assumed to fit smoothly together to give a single map E: $TM \to M$ from the tangent bundle TM to M. In perturbation theory only formal power series expansions have significance, so domains of definition in the tangent spaces T_mM are not made explicit, here and where relevant below.

4.2. The compatibility operator D_i

A system of coordinates E: $TM \to M$ determines, for m' sufficiently close to m, a transition function $E^{-1}_{m'} \circ E_m$: $T_mM \to T_{m'}M$. The infinitesimal version of the transition functions is a first order differential operator D_i acting on real valued functions $\bar{h}$ on TM:

$$w^i D_i \tilde{h}\ (m,v) = \frac{d}{dt/t=0}\ \tilde{h}(m(t),\ E^{-1}_{m(t)} \circ E_m(v))\ , \qquad (4.2.1)$$

where w and v are tangent vectors in T_mM, and $m(t)$ is a curve in M with $m = m(0)$ and $w = \frac{d}{dt/t=0}\, m(t)$.

Written in coordinates $\{m^i\}$ on M, D_i takes the form

$$D_i = \frac{\partial}{\partial m^i} - \bar{Q}^j_i(m,v)\, \frac{\partial}{\partial v^j}\ . \qquad (4.2.2)$$

D_i can be regarded as defining a locally flat, incomplete, nonlinear connection in the tangent bundle; it is the ordinary derivative on TM followed by horizontal projection. The transition functions are the path independent parallel transport functions of this flat nonlinear connection.

D_i has two defining properties. First, acting on functions on M, D_i is identical to the ordinary derivative d_i. That is, if

$\tilde{h}(m,v) = h(m)$, then

$$D_i \tilde{h}\,(m,v) = d_i h\,(m)\ . \tag{4.2.3}$$

Second, D_i satisfies the integrability condition

$$D^2_{ij} = 0. \tag{4.2.4}$$

where D^2_{ij} is defined by

$$v^i w^j\, D^2_{ij} = [v^i D_i,\, w^j D_j] - [v,\, w]^i D_i \tag{4.2.5}$$

for v,w vector fields on M and $[v,\, w]$ their Lie bracket. With respect to coordinates $\{m^i\}$ on M,

$$D^2_{ij} = [D_i,\, D_j]\ . \tag{4.2.6}$$

D^2_{ij} is the curvature of the nonlinear connection (represented as an operator on functions); $D^2_{ij} = 0$ expresses the local flatness.

A function $\tilde{h}(m,v)$ on TM can represent, via the various coordinate maps E_m, many distinct functions h_m on neighborhoods in M:

$$h_m(m') = \tilde{h}(m,\, E_m^{-1} m')\ . \tag{4.2.7}$$

D_1 is called the compatibility operator because it measures the extent to which these functions depend on m. $\tilde{h}$ is the expression in coordinates of a single function h on M,

$$\tilde{h}(m,v) = h(E_m(v)) , \qquad (4.2.8)$$

if and only if $D_1\tilde{h} = 0$.

Any diffeomorphism $\Psi: M \to M$ near the identity gives rise to a transformed system of coordinates $\Psi \circ E$. The transition functions and the compatibility operator do not change. Therefore, the compatibility operator can at most determine the system of coordinates up to such transformations by diffeomorphisms of M. In fact, integration of D_1 as a one form on M recreates the transition functions, and the transition functions clearly determine the system of coordinates exactly up to diffeomorphisms of M. Moreover, any first order operator satisfying (4.2.3-4) is the compatibility operator for some system of coordinates.

The additional data which are needed to specify completely the system of coordinates are the coordinate origins $o(m) = E_m^{-1}(m)$. A real valued function h on M is represented in coordinates E by the unique solution of the compatibility equation $D_1\tilde{h} = 0$ with initial conditions $\tilde{h}(m,o(m)) = h(m)$. The obvious choice of origin in T_mM is zero. But the ambiguity in the system of coordinates associated with a given compatibility operator will be important in the discussion of renormalization.

A tensor valued function $\tilde{t}^{j\cdots}_{i\cdots}$ on TM is a function whose value at (m,v) is a tensor at m. The compatibility operator extends to act on these functions. For each m, $\tilde{t}$ is regarded as a tensor field on the tangent space T_mM. The transition functions are used to differentiate with respect to m:

$$w^k D_k \tilde{t}^{i\cdots}_{j\cdots} = \frac{d}{dt/t=0} (E^{-1}_{m(t)} \circ E_m)^* \tilde{t}^{i\cdots}_{j\cdots} , \qquad (4.2.9)$$

where $m = m(0)$ and $w = \frac{d}{dt/t=0} m(t)$.

A tensor valued function $\tilde{t}^{i\cdots}_{j\cdots}(m,v)$ is the expression in coordinates of a single tensor field $t^{i\cdots}_{j\cdots}(m)$ on M,

$$\tilde{t}(m,v) = E^*_m t \ (v) , \qquad (4.2.10)$$

if and only if $\tilde{t}$ satisfies the compatibility equation $D_i \tilde{t} = 0$ with initital conditions

$$\tilde{t}^{i\cdots}_{j\cdots}(m,o(m)) = (d_{o(m)} E_m)^q_j \cdots (d_{o(m)} D^{-1}_m)^i_p \cdots t^{i\cdots}_{j\cdots}(m) \ . \ (4.2.11)$$

The extended compatibility operator continues to satisfy the integrability condition $D^2_{ij} = 0$.

4.3. The linear connection

It is useful to combine a system of coordinates with a linear connection in the tangent bundle TM. The linear connection serves two distinct purposes. The first is to provide a covariant derivative ∇_i, allowing D_i to be written as ∇_i plus an operator which acts independently at each point m. This is a technical convenience which presupposes no special relationship between the system of coordinates and the linear connection. The second purpose is to define geodesic normal coordinates for the linear connection. Two types of connection are of special interest: the torsion free Levi-Civita connection when M is Riemannian, and the canonical connection when M is homogeneous.[14]

A linear connection in TM determines a set of path dependent linear parallel transport functions between tangent spaces T_mM and $T_{m'}M$. As in (4.2.1), the infinitesmal version of the parallel transport functions is a first order operator, the covariant derivative ∇_i, acting on real valued and tensor valued functions on TM. In coordinates $\{m^i\}$, writing ∂_k for $\frac{\partial}{\partial v^k}$,

$$\nabla_i \tilde{h}\,(m,v) = \left(\frac{\partial}{\partial m^i} - \Gamma_{ji}^{\ \ k}(m)\, v^j\, \partial_k \right) \tilde{h}\,(m,v) \qquad (4.3.1)$$

$$\nabla_i \tilde{w}^j\,(m,v) = \left(\frac{\partial}{\partial m^i} - \Gamma_{pi}^{\ \ k}(m)\, v^p\, \partial_k \right) \tilde{w}^j\,(m,v)$$

$$+ \Gamma_{ki}^{\ \ j}(m)\, \tilde{w}^k\,(m,v) \qquad (4.3.2)$$

where Γ_{ji}^{k} is the Christoffel symbol for the linear connection. If $\tilde{t}^{i\ldots}_{j\ldots}(m,v)$ does not depend on v then $\nabla_k \tilde{t}^{i\ldots}_{j\ldots}$ is the ordinary covariant derivative.

Any tensor valued function $\tilde{t}$ on TM has a formal Taylor series expansion:

$$\tilde{t}^{i\ldots}_{j\ldots}(m,v) =$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} v^{k_1}\cdots v^{k_n} \;\partial_{k_1}\cdots\partial_{k_n} \tilde{t}^{i\ldots}_{j\ldots}(m,v)_{/v=0} \,, \qquad (4.3.3)$$

where the coefficients $\partial_{k_1}\cdots\partial_{k_n} \tilde{t}^{i\ldots}_{j\ldots}(m,0)$ are tensor fields on M. From (4.3.2) it follows immediately that

$$\nabla_i v^j = 0 \qquad (4.3.4)$$

and

$$[\nabla_i \,,\, \partial_j] = 0 \,. \qquad (4.3.5)$$

Thus the Taylor series coefficients of $\nabla_i \tilde{t}$ are given by the covariant derivatives of the coefficients of $\tilde{t}$.

∇^2_{ij}, defined as in (4.2.5), is

$$\nabla^2_{ij}\tilde{h}\;(m,v) \;=\; -\,R^{k}{}_{pij}(m)\; v^p\; \partial_k \tilde{h}\;(m,v) \qquad (4.3.6)$$

$$\nabla^2_{ij}\tilde{w}^k = -R^n{}_{pij}\, v^p\, \partial_n\, \tilde{w}^k + R^k{}_{pij}\, \tilde{w}^p \, . \qquad (4.3.7)$$

$R^k{}_{pij}$ is the curvature tensor of the linear connection:

$$u^i v^j w^p R^k{}_{pij} = [u^i \nabla_i ,\, v^j \nabla_j]\, w^k - [u,\, v]^q\, \nabla_q\, w^k \qquad (4.3.8)$$

for u, v, w vector fields on M.

Define a matrix valued fuction $Q^j_i(m,v)$ on TM by

$$D_i = \nabla_i - Q^j_i(m,v)\, \partial_j \, . \qquad (4.3.9)$$

Equivalently,

$$\frac{d}{dt}\Big/_{t=0}\; E^{-1}_{m(t)} \circ E_m\,(v) = (w,\, -\, Q(m,v)\, w\,) \, . \qquad (4.3.10)$$

The expression on the left in (4.3.10) is a vector tangent to TM at (m,v). On the right is the same tangent vector decomposed into horizontal and vertical parts with respect to the linear connection in TM.

Both the linear and the nonlinear parallel transport functions preserve the Lie brackets of vector fields on the tangent spaces $T_m M$. It follows that D_i is given on tensor valued functions by

$$D_i \tilde{w}^j = \nabla_i \tilde{w}^j - Q^k_i\, \partial_k \tilde{w}^j + \tilde{w}^k\, \partial_k Q^j_i \qquad (4.3.11)$$

$$D_i \tilde{t}^{k\cdots}_{j\cdots} = \nabla_i \tilde{t}^{k\cdots}_{j\cdots} - [Q_i, \tilde{t}]^{k\cdots}_{j\cdots}, \qquad (4.3.12)$$

where Q_i at m is the vector field $Q^j_i \partial_j$ on $T_m M$ and $[Q_i, \tilde{t}]$ at m is the Lie bracket of this vector field with the tensor field $\tilde{t}(m,v)$ on $T_m M$.

The integrability condition $D^2_{ij} = 0$ becomes, substituting (4.3.9) in (4.2.5),

$$Q^p_i \partial_p Q^k_j - Q^p_j \partial_p Q^k_i - \nabla_i Q^k_j + \nabla_j Q^k_i =$$

$$T^p_{ij} Q^k_p + R^k{}_{pij} v^p . \qquad (4.3.13)$$

T^k_{ij} *is the torsion tensor of the linear connection:*

$$v^i w^j T^k_{ij} = v^i \nabla_i w^k - w^j \nabla_j v^k - [v, w]^k \qquad (4.3.14)$$

for v, w vector fields on M.

The derivative of E_m at the origin $o(m) = E^{-1}_m(m)$ is given by

$$(d_{o(m)} E^{-1}_m)^i_j = Q^i_j (m, o(m)) + \nabla_j o^i (m). \qquad (4.3.15)$$

Q depends on both the system of coordinates and the linear connection, but the particular combination on the right in (4.3.15) depends only on the system of coordinates.

4.4. Normal coordinates

Given a linear connection on TM, normal coordinates around the point m are defined by

$$E_m(v) = p_{(m,v)}(1) \tag{4.4.1}$$

where $p_{(m,v)}(t)$ is the geodesic leaving m with initial velocity v. By construction, the origin $o(m)$ is at zero.

The velocity field is covariant constant along a geodesic, so, in the language of (4.3.10),

$$\frac{d}{dt}\Big/_{t=0} \quad E^{-1}_{p_{(m,v)}(t)} \circ E_m(v) = (v, -v) . \tag{4.4.2}$$

That is,

$$v^i Q^j_i(m,v) = v^j . \tag{4.4.3}$$

Also, since $p_{(m,v)}(t)$ has velocity v at $t = 0$,

$$Q(m,0) = 1 . \tag{4.4.4}$$

Conditions (4.4.3–4) determine Q uniquely. The contraction of v^i with both sides of equation (4.3.13) gives a matrix equation

$$\delta Q + Q^2 - Q = Q T + R . \qquad (4.4.5)$$

The first order operator $\delta = - v^i \nabla_i$ is

$$\delta = v^i (\partial_i - \nabla_i) \qquad (4.4.6)$$

The matrix valued functions $T(m,v)$ and $R(m,v)$ are

$$T^i_j(m,v) = v^k \, T^i_{kj}(m) \qquad (4.4.7)$$

$$R^i_j(m,v) = v^k v^l \, R^i{}_{klj}(m) . \qquad (4.4.8)$$

Equation (4.4.5) has a unique solution $Q^i_j(m,v)$ satisfying the initial condition (4.4.4).

The Taylor series in normal coordinates of a real valued function h on M is the expansion of $\tilde{h}(m,v) = h(E_m(v))$ in powers of v. The compatibility condition $D_i \tilde{h} = 0$ implies $0 = - v^k D_k \tilde{h} = \delta\tilde{h}\delta\,\tilde{h} = 0$. With initial condition $\tilde{h}(m,0) = h(m)$, the formal power series solution is

$$\tilde{h}(m,v) = \sum_{n=0}^{\infty} \frac{1}{n!} v^{k_1} \cdots v^{k_n} \nabla_{k_1} \cdots \nabla_{k_n} h(m) . \qquad (4.4.9)$$

The Taylor series coefficients of $\tilde{h}$ are the symmetrized covariant derivatives of h.

4.5. The vielbein $V^i_j(m,v)$

A vector field w^i on M is represented in normal coordinates by the vector valued function $\tilde{w}^i(m,v)$ which solves, from (4.3.11),

$$0 = -v^i D_i \tilde{w} = (\partial - 1 + Q)\tilde{w}\ , \qquad (4.5.1)$$

$$\tilde{w}(m,0) = w(m)\ . \qquad (4.5.2)$$

The vector valued function $\bar{w}^i(m,v)$ whose Taylor series coefficients are the symmetrized covariant derivatives of w is the solution of $\partial\bar{w} = 0$ with initial condition (4.5.2). The two vector valued functions $\tilde{w}$ and $\bar{w}$ are related by a linear transformation: $\bar{w} = V\tilde{w}$, where the matrix valued function $V^i_j(m,v)$ is the solution of

$$\partial V = V(Q - 1) \qquad (4.5.3)$$

$$V(m,0) = 1\ . \qquad (4.5.4)$$

∂ is applied to both sides of (4.5.3) and (4.4.5) is used to obtain

$$\partial^2 V + (\partial V)(1 - T) - V(T + R) = 0\ . \qquad (4.5.5)$$

(4.5.5) has a unique solution satisfying initial condition (4.5.4).

It is possible to calculate the Taylor series coefficients of Q recursively using (4.4.4-5), but, because the equation is nonlinear in Q, this is an inefficient method. (4.5.4-5) is linear in V, so is more suited for practical calculation. Q is given by rewriting (4.5.3):

$$Q = 1 + V^{-1} \partial V . \qquad (4.5.6)$$

The tensor valued function $\bar{t}^{i\ldots}_{j\ldots}(m,v)$ which represents in normal coordinates the tensor field $t^{i\ldots}_{j\ldots}$ on M is found by a direct extension of (4.5.1-3). First, the tensor valued function $\bar{t}(m,v)$ which solves $\partial\bar{t}\ (m,v) = 0$, $\bar{t}(m,0) = t(m)$ is found. Its Taylor series coefficients are the symmetrized covariant derivatives of t:

$$\bar{t}^{i\ldots}_{j\ldots}(m,v) = \sum_{n=0}^{\infty} \frac{1}{n!} v^{k_1} \ldots v^{k_n} \nabla_{k_1} \ldots \nabla_{k_n} t^{i\ldots}_{j\ldots}(m) . \qquad (4.5.7)$$

Then

$$\bar{t}^{i\ldots}_{j\ldots} = V^{i}_{p}\ldots \ \bar{t}^{p\ldots}_{q\ldots} \ V^{q}_{j}\ldots . \qquad (4.5.8)$$

The Taylor series expansion in normal coordinates of any tensor field on M is thus obtained immediately, once the Taylor series expansion of $V^{i}_{j}(m,v)$ is known.

4.6. Metrics and volume elements in coordinates

In a coordinate system E, a metric g_{ij} is represented by the tensor valued function $\tilde{g}_{ij}(m,v) = E_m^* g_{ij}(v)$. It satisfies the compatibility condition

$$D_i \tilde{g}_{jk} = 0 \tag{4.6.1}$$

and the initial condition

$$\tilde{g}_{ij}(m, o(m)) = (d_{o(m)} E_m)_i^p \; g_{pq}(m) \; (d_{o(m)} E_m)_j^q \, . \tag{4.6.2}$$

When the origin is at zero, the initial condition is, by (4.3.15),

$$\tilde{g}_{ij}(m,0) = (Q^{-1})_i^p(m,0) \; g_{pq}(m) \; (Q^{-1})_j^q(m,0) \, . \tag{4.6.3}$$

When the coordinates are normal with respect to a linear connection in TM, $\tilde{g}$ is given by (4.5.7-8):

$$\tilde{g}_{ij}(m,v) = V_i^p(m,v) \; \bar{g}_{pq}(m,v) \; V_j^q(m,v) \tag{4.6.4}$$

$$\bar{g}_{pq}(m,v) = \sum_{n=0}^{\infty} \frac{1}{n!} \, v^{k_1} \cdots v^{k_n} \, \nabla_{k_1} \cdots \nabla_{k_n} g_{pq}(m) \, . \tag{4.6.5}$$

In particular, when the metric g_{ij} is covariant constant,

$$\tilde{g}_{ij}(m,v) = V^p_i(m,v) \; g_{pq}(m) \; V^q_j(m,v) \; . \qquad (4.6.6)$$

A volume element dm on M is represented in coordinates by the tensor valued function $\tilde{d}v\,(m,v) = E^*_m dm\,(v)$, which satisfies the compatibility condition $D_i \tilde{d}v = 0$, with initial condition

$$\tilde{d}v\,(m,o(m)) = dm \;\det d_{o(m)} E_m \; . \qquad (4.6.7)$$

The ratio between $\tilde{d}v$ and dm is a positive real valued function $\exp \tilde{j}(m,v)$ on TM:

$$\tilde{d}v = dm \exp \tilde{j}(m,v) . \qquad (4.6.8)$$

$\tilde{j}(m,v)$ is the logarithmic jacobian of the coordinate map E_m at (m,v) with respect to the volume element dm at m and at $E_m(v)$.

$\tilde{d}v$ consists of a volume element on $T_m M$ for each m, so integration against $\tilde{d}v$ turns a real valued function $\tilde{h}$ on TM into a real valued function $\int \tilde{d}v \, \tilde{h}$ on M. The compatibility condition $D_i \tilde{d}v = 0$ implies the integration by parts formula

$$\int \tilde{d}v \;\; D_i \tilde{h} = d_i \int \tilde{d}v \;\; \tilde{h}. \qquad (4.6.9)$$

In perturbation theory the integrations are asymptotic expansions, so conditions on the support of $\tilde{h}$ are unnecessary.

When the coordinates are normal with respect to a linear connection, $\bar{d}v$ is given by (4.5.7-8):

$$\tilde{d}v = \bar{d}v \det V(m,v) \qquad (4.6.10)$$

$$\bar{d}v\,(m,v) = \sum_{n=0}^{\infty} \frac{1}{n!}\, v^{k_1} \cdots v^{k_n}\, \nabla_{k_1} \cdots \nabla_{k_n} dm \,. \qquad (4.6.11)$$

In particular, when the volume element is covariant constant,

$$\tilde{d}v = dm \det V(m,v) \,. \qquad (4.6.12)$$

The logarithmic jacobian $\tilde{j}(m,v)$ of the coordinate map E_m at (m,v) is then $\operatorname{tr} \log V(m,v)$.

4.7. Calculation of Taylor series: torsion free normal coordinates

The system of coordinates is assumed to be normal with respect to a torsion free linear connection in TM. An example of such a connection is the Levi-Civita connection of a Riemannian metric on M.

The matrix valued function $V^i_j(m,v)$ is expanded in in powers of v with nonstandard coefficients:

$$V = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\, V^{(n)} \qquad (4.7.1)$$

$V^{(n)}$ is homogeneous in v of degree n and $V^{(0)} = 1$. Equation

(4.5.5) gives $V^{(1)} = 0$ and, for $n > 1$, the recursion relation

$$V^{(n)} = 2\nabla V^{(n-1)} - \nabla^2 V^{(n-2)} + V^{(n-2)} R \qquad (4.7.2)$$

where, in this context, $\nabla = v^i \nabla_i$. The matrix valued function R is defined in (4.4.8). The first few terms in the expansion are

$$V = 1 + \frac{1}{3!} R + \frac{1}{4!} (2 \nabla R) + \frac{1}{5!} (3 \nabla^2 R + R^2) \qquad (4.7.3)$$

$$+ \frac{1}{6!} (4 \nabla^3 R + 3 \nabla (R^2) - [R, \nabla R]) + \cdots$$

$$\log V = \frac{1}{6} R + \frac{1}{12} \nabla R + (\frac{1}{40} \nabla^2 R - \frac{1}{180} R^2) \qquad (4.7.4)$$

$$+ \frac{1}{6!} (4 \nabla^3 R - 2 \nabla (R^2) - [R, \nabla R]) + \cdots .$$

Note that $\mathrm{tr}\, R = - v^i v^j R_{ij}$.

(4.5.6) gives

$$Q = 1 + \frac{1}{3} R + \frac{1}{12} \nabla R + (\frac{1}{60} \nabla^2 R - \frac{1}{45} R^2)$$

$$+ \frac{1}{6!} (2 \nabla^3 R - 6 \nabla (R^2)) + \cdots . \qquad (4.7.5)$$

If g_{ij} is a covariant constant metric on M, then the linear connection, being torsion free, must be the Levi-Civita connection for g_{ij}.

The curvature matrix is then symmetric: $R = g^{-1} R^{*} g$. From (4.6.6),

$$g^{-1} \tilde{g} = 1 + \frac{1}{3} R + \frac{1}{6} \nabla R + \left(\frac{1}{20} \nabla^2 R + \frac{2}{45} R^2 \right)$$

$$+ \frac{1}{90} \left(\nabla^3 R + 2 \nabla (R^2) \right) + \cdots . \qquad (4.7.6)$$

More explicitly:

$$\tilde{g}_{ij}(m,v) = g_{ij}(m) + \frac{1}{3} v^k v^l R_{iklj}(m) + \frac{1}{6} v^k v^l v^n \nabla_k R_{ilnj}(m)$$

$$+ v^k v^l v^n v^p \left(\frac{1}{20} \nabla_k \nabla_l R_{inpj} + \frac{2}{45} R_{iklq} R^q{}_{npj} \right) (m)$$

$$+ \cdots . \qquad (4.7.7)$$

4.8. Homogeneous spaces

In this subsection, M is a homogeneous space G/H and E is a G- invariant system of coordinates. That is,

$$E_m = \Psi^{-1} \circ E_{\Psi(m)} \circ \Psi_* \qquad (4.8.1)$$

for all m in M and all Ψ in G. If t is a G- invariant tensor field on M, i.e. $\Psi_* t = t$ for all Ψ in G, then its representation in coordinates $\tilde{t}(m,v) = E^*_m t\ (v)$ is a G- invariant tensor valued function on TM:

$$\tilde{t}(\Psi(m), \Psi_* v) = \Psi_* \, t(m,v) . \qquad (4.8.2)$$

The canonical connection in TM is a natural linear connection defined using the group theoretic structure of the quotient G/H (see section 2.3 and [14]). It can be defined by giving the operator ∇_i on tensor valued functions on TM. ∇_i is defined at the H- invariant base point m_o:

$$w^i \nabla_i \tilde{t} \, (m_o, v) = \frac{d}{dt/t=0} \; e_*^{-tw} \, \tilde{t}(e^{tw} m_o, e_*^{tw} v) , \qquad (4.8.3)$$

where w and v are vectors in $\underline{m} = T_{m_o} M$. ∇_i at m_o respects the action of H on $T_{m_o} M$, so it extends to a G- invariant operator on all of TM.

It follows immediately from (4.8.3) that G- invariant tensor valued functions on TM are annihilated by ∇_i. In particular, the torsion and curvature of the canonical connection are G- invariant and therefore covariant constant. Since G- invariant tensor valued functions are completely determined by their values on $\underline{m} = T_{m_o} M$, they need only be studied there.

The matrix valued function $Q^i_j(m,v)$, defined with respect to the canonical connection and the system of coordinates E, is G- invariant, so $\nabla_i Q^k_j = 0$. (4.3.13) becomes

$$Q_i^p \partial_p Q_j^k - Q_j^p \partial_p Q_i^k = T_{ij}^p Q_p^k + R^k{}_{pij} v^p , \qquad (4.8.4)$$

or,

$$[Q_i, Q_j] = T_{ij}^k Q_k + R^k{}_{pij} v^p \partial_k . \qquad (4.8.5)$$

The torsion and curvature (at m_o) are (see section 2.3 and [14]):

$$T_{ij}^k = - C_{ij}^k \qquad (4.8.6)$$

$$R^l{}_{ijk} = - C_{jk}^a C_{ai}^l . \qquad (4.8.7)$$

The vector fields $- Q_i$ and $- R^k{}_{pij} v^p \partial_k$ on $\underline{m}$ generate the ideal $\underline{m} + [\underline{m}, \underline{m}]$ in $\underline{g}$. The Q_i are nonlinear vector fields; the rest are linear. Note that unimodularity of G/H is expressed in the condition

$$C_{ik}^k = 0 \qquad (4.8.8)$$

($C_{ak}^k = 0$ follows automatically from the compactness of H.)

Canonical normal coordinates are described by the matrix valued function $V_j^i(v)$ on $\underline{m}$ satisfying (4.5.4-5). In this context, $\partial = v^i \partial_i$. $R(v)$ and $T(v)$ are matrix valued functions on $\underline{m}$ defined by (4.4.7-8). Explicitly,

$$T(v)\, w = -\, [v, w]_{\underline{m}} \tag{4.8.9}$$

$$R(v)\, w = [v, [v, w]_{\underline{h}}] \tag{4.8.10}$$

where $[v, w]_{\underline{m}}$ is the component of the Lie bracket lying in $\underline{m}$ and $[v, w]_{\underline{h}}$ is the component in $\underline{h}$. Note that $\partial T = T$ and $\partial R = 2\, R$.

Recursion relations for the Taylor series coefficients of $V^i_j(v)$ are obtained by writing (4.5.5) in terms of the expansion (4.7.1):

$$V^{(0)} = 1, \quad V^{(1)} = T \tag{4.8.11}$$

and, for $n > 1$,

$$V^{(n)} = V^{(n-1)}\, T + V^{(n-2)}\, R\,. \tag{4.8.12}$$

The first few terms are:

$$V = 1 + \frac{1}{2} T + \frac{1}{3!} (T^2 + R) + \frac{1}{4!} (T^3 + R\, T + T\, R) + \cdots \tag{4.8.13}$$

$$\log V = \frac{1}{2} T + \frac{1}{24} (T^2 + 4\, R) + O(v^4) \tag{4.8.14}$$

In the special cases in which $[R, T] = 0$, (4.5.4-5) can be solved exactly:

$$V = e^{T/2} f(R + \frac{1}{4} T^2) \qquad (4.8.18)$$

where

$$f(u) = u^{-1/2} \sinh(u^{1/2}) \qquad (4.8.19)$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} u^n . \qquad (4.8.20)$$

The Taylor series of Q is calculated using (4.5.6) and (4.8.13):

$$Q = 1 + \frac{1}{2} T + \frac{1}{12} (T^2 + 4 R)$$

$$+ \frac{1}{24} (R T - T R) + \cdots . \qquad (4.8.15)$$

A G- invariant metric on M is determined by an H- invariant inner product g_{ij} on $\underline{m}$. Infinitesmal H- invariance is

$$0 = C^k_{ai} g_{kj} + g_{ik} C^k_{aj} . \qquad (4.8.16)$$

The representation of the metric in canonical normal coordinates is given by (4.6.6) and (4.8.13):

$$\tilde{g} = V^* g V$$

$$= g + \frac{1}{2}(T^* g + g T) + \frac{1}{4} T^* g T$$

$$+ \frac{1}{6}((T^2 + R)^* g + g(T^2 + R)) + \cdots . \qquad (4.8.17)$$

5. Linear Fields

5.1. Introduction

The space of fields $\phi(x)$ taking values in the manifold M is itself a nonlinear manifold. For calculation in the low temperature expansion, a linear representation is needed for the fields near each constant $\phi(x) = m$. Natural linear fields at m are the tangents to the manifold of nonlinear fields at the constant m, i.e., the fields σ taking values $\sigma^i(x)$ in the tangent space to M at m. The linear representation is a collection of maps $\tilde{E}_m$ from linear to nonlinear fields, defined near the zero linear field, taking the linear field σ at m to the nonlinear field $\phi = \tilde{E}_m(\sigma)$ which it represents.

It is convenient to use linear representations which respect the spatial symmetry of the model, in order that the symmetry remain manifest in the low temperature expansion. It simplifies power counting to use representions which are local and zeroth order in the linear field. Any representation satisfying these two criteria is determined by a system of coordinates E on M:

$$\tilde{E}_m(\sigma)\ (x) = E_m\ (\sigma(x)) \ . \tag{5.1.1}$$

5.2. The compatibility operator $\tilde{D}_i$

Each nonlinear field is represented by many linear fields, associated with different constants. A compatibility condition determines when a real valued function of the linear fields represents a single function of the nonlinear fields. The compatibility operator $\tilde{D}_i$ is defined to act on real valued functions $\tilde{G}(m,\sigma)$ of the linear fields by

$$w^i \tilde{D}_i \tilde{G}\,(m,\sigma) = \frac{d}{dt/t=0}\ \tilde{G}(\ m(t),\ \tilde{E}_{m(t)}^{-1} \circ \tilde{E}_m(\sigma)\ )\ , \qquad (5.2.1)$$

where $m = m(0)$ and $w = \frac{d}{dt/t=0}\, m(t)$. $\tilde{G}(m,\sigma)$ is the representation of a single function $G(phi)$ of the nonlinear fields,

$$\tilde{G}(m,\sigma) = G(\tilde{E}_m(\sigma))\ , \qquad (5.2.2)$$

if and only if

$$\tilde{D}_i \tilde{G} = 0\ . \qquad (5.2.3)$$

The linear fields form a vector bundle over the constants. The compatibility operator is the infinitesmal expression of a flat nonlinear connection in this bundle whose path independent parallel transport functions are the transition functions $\tilde{E}_{m'}^{-1} \circ \tilde{E}_m$. The flatness of the nonlinear connection is expressed in the integrability condition

$$\tilde{D}^2_{ij} = 0 \; . \tag{5.2.4}$$

$\tilde{D}^2_{ij}$ is defined as in (4.2.5).

5.3. Linear connections

It is convenient to have use of a linear connection in the bundle of linear fields. It provides a first order operator $\tilde{\nabla}_i$ which acts on real valued functions of the linear fields, as the infinitesmal form of linear parallel transport of linear fields along paths in the constants M.

In calculation of the low temperature expansion it is convenient to use linear connections which respect the spatial symmetry and which are local and zeroth order in the linear field, thus which are determined by linear connections in TM. With respect to coordinates $\{m^i\}$ on M,

$$\tilde{\nabla}_i = \frac{\partial}{\partial m^i} - \Gamma_{ji}^{\;k}(m)\, \sigma^j(x)\, \frac{\partial}{\partial \sigma^k(x)} \; . \tag{5.3.1}$$

$\Gamma_{ji}^{\;k}$ is the Christoffel symbol of the linear connection on M in coordinates $\{m^i\}$. The summation convention applies to the index x as well as to the ordinary indices.

Given a linear connection in the bundle of linear fields the compatibility operator can be written in the form

$$\tilde{D}_i = \tilde{\nabla}_i - \tilde{Q}_i(m,\sigma)\, \frac{\partial}{\partial \sigma} \tag{5.3.2}$$

where $\tilde{Q}_i(m,\sigma)\frac{\partial}{\partial\sigma}$ is a vector field on the space of linear fields at m. When the linear representation is based on a system of coordinates and $\tilde{\nabla}_i$ is determined by a linear connection on M, then

$$\tilde{D}_i = \tilde{\nabla}_i - Q_i^j(m,\sigma(x))\frac{\partial}{\partial\sigma^j(x)} . \qquad (5.3.3)$$

$Q_j^i(m,\sigma)$ is the matrix valued function on TM, defined by (4.3.9), which describes the system of coordinates with respect to the linear connection in TM.

A linear connection in the bundle of linear fields is not essential to the representation of nonlinear by linear fields. It merely provides, as in (5.3.2-3) a convenient separation of the compatibility operator into a linear covariant derivative plus an operator which acts independently on each space of linear fields.

5.4. Extensions to tensor valued functions

In extending these operators to tensor valued functions of the linear fields, two types of functions must be distinguished.

Those whose values are tensors in the linear fields themselves are treated exactly as were tensor valued functions in section 4.2. A vector valued function of this type is of the form $\tilde{W}^i(x)(m,\sigma)$, an example being $\sigma^i(x)$ itself. The extension of $\tilde{\nabla}_i$ to such functions satisfies

$$\tilde{\nabla}_i \, \sigma^j(x) = 0 \tag{5.4.1}$$

$$[\tilde{\nabla}_i, \frac{\delta}{\delta\sigma^j(x)}] = 0 \; . \tag{5.4.2}$$

The second kind of tensor valued function takes its value at (m,σ) in a tensor space of $T_m M$. It is of the form $\tilde{T}^{i\cdots}_{j\cdots}(m,\sigma)$. In order to extend the operators $\tilde{\nabla}_i$ and $\tilde{D}_i$ to these functions, an auxiliary linear connection in TM is needed to transport the tensors of TM.

This auxiliary linear connection in TM is in principle distinct from the linear connection in the bundle of linear fields which gives $\tilde{\nabla}_i$, even when the latter derives from a single linear connection in TM. It is also distinct from a linear connection used to define normal coordinates.

The operator $\tilde{\nabla}_i$ extends to:

$$\tilde{\nabla}_i \tilde{W}^j \,(m,\sigma) = \left(\frac{\delta}{\delta m^i} - \Gamma^{\;p}_{qi}(m)\, \sigma^q(x) \frac{\delta}{\delta\sigma^p(x)} \right) \tilde{W}^j \,(m,\sigma)$$

$$+ \hat{\Gamma}^{\;j}_{ki}(m)\, \tilde{W}^k(m,\sigma) \; . \tag{5.4.3}$$

$\Gamma^{\;p}_{qi}$ is the Christoffel symbol for the linear connection determining $\tilde{\nabla}_i$, and $\hat{\Gamma}^{\;j}_{ki}$ is the Christoffel symbol for the auxiliary linear connection.

The compatibility operator $\tilde{D}_i$ extends to:

$$\tilde{D}_i \tilde{W}^j = \tilde{\nabla}_i \tilde{W}^j - \tilde{Q}_i(m,\sigma) \frac{\partial}{\partial \sigma} \tilde{W}^j . \tag{5.4.4}$$

If $\tilde{T}(m,\sigma)$ is a tensor valued function which depends only on m, i.e. a tensor field on M, then $\tilde{D}_i \tilde{T}$ and $\tilde{\nabla}_i \tilde{T}$ both equal $\hat{\nabla}_i \tilde{T}$, the ordinary covariant derivative with respect to the auxiliary linear connection. The extended operator satisfies

$$\tilde{D}^2_{ij} \tilde{W}^k (m,\sigma) = \hat{R}^k{}_{pij}(m) \tilde{W}^p(m,\sigma) . \tag{5.4.5}$$

$\hat{R}^k{}_{pij}$ is the curvature of the auxiliary linear connection.

5.5. The action, source and a priori measure

In terms of linear fields at the constant m, the action and external source are

$$\tilde{S}(m,\sigma) = S(\tilde{E}_m(\sigma)) \tag{5.5.1}$$

$$\tilde{H}(m,\sigma) = H(\tilde{E}_m(\sigma)) , \tag{5.5.2}$$

satisfying the compatibility conditions

$$\tilde{D}_i \tilde{S} = \tilde{D}_i \tilde{H} = 0 . \tag{5.5.3}$$

In a linear representation based on coordinates, $\tilde{S}$ and $\tilde{H}$ are

$$\tilde{S}(m,\sigma) = \int dx \ \frac{1}{2} T^{-1} \ \tilde{g}_{ij}(m,\sigma(x)) \ \partial_\mu \sigma^i(x) \ \partial_\mu \sigma^j(x) \tag{5.5.4}$$

$$\tilde{H}(m,\sigma) = \Lambda^{2+\epsilon} \int dx \ \tilde{h}(x)(m,\sigma(x)) \ , \tag{5.5.5}$$

where $\tilde{g}_{ij}$ and $\tilde{h}(x)$ are the metric and external field in coordinates around m. The compatibility condition $\tilde{D}_i \tilde{H} = 0$ is equivalent to $D_i \tilde{h}(x) = 0$, and $\tilde{D}_i \tilde{S} = 0$ is equivalent to $D_i \tilde{g}_{jk} = 0$.

The a priori measure $d\phi$ is represented in terms of the linear fields at m by

$$\tilde{d}\sigma \ (m,\sigma) = \tilde{E}^*_m d\phi \ (\sigma) \ . \tag{5.5.6}$$

It satisfies the compatibiliy condition

$$\tilde{D}_i \tilde{d}\sigma = 0 \ . \tag{5.5.7}$$

It can be written

$$\tilde{d}\sigma = d\sigma \ \exp \tilde{J}(m,\sigma) \tag{5.5.8}$$

where $d\sigma$ is the measure $d\phi$ at the constant $\phi = m$, and $\tilde{J}(m,\sigma)$ is the logarithmic jacobian at σ of the linear representation $\tilde{E}_m$.

A spatially invariant a priori measure $d\phi$ takes the form $\prod_x d\phi(x)$, where $d\phi(x)$ is the a priori volume element on M. When the linear representation is by means of a system of coordinates, then

$$d\sigma = \prod_x d\sigma(x) \tag{5.5.9}$$

$$\tilde{J}(m,\sigma) = \Lambda^{2+\epsilon} \int dx \ \tilde{j}(m,\sigma(x)) \ . \tag{5.5.10}$$

$d\sigma(x)$ is the a priori volume element $d\phi(x)$ at $\phi(x) = m$. $\tilde{j}(m,\sigma(x))$ is the logarithmic jacobian of the coordinate map E_m at $\sigma^i(x)$.

$\tilde{d}\sigma$ consists, for each m, of a volume element on the linear fields at m. Integration against $\tilde{d}\sigma$ turns a real valued function $\tilde{G}(m,\sigma)$ of the linear fields into a real valued function $\int \tilde{d}\sigma \ \tilde{G}$ on M. The compatibility condition $\tilde{D}_i \tilde{d}\sigma = 0$ implies the integration by parts formula

$$\int \tilde{d}\sigma \ \ \tilde{D}_i \tilde{G} = d_i \int \tilde{d}\sigma \ \ \tilde{G} \ . \tag{5.5.11}$$

$\tilde{d}\sigma$ also integrates tensor valued functions of the linear fields, producing tensor fields on M. The integration by parts formula is

$$\int \tilde{d}\sigma \ \ \tilde{D}_i \ \tilde{T}^{j\cdots}_{k\cdots} = \hat{\nabla}_i \int \tilde{d}\sigma \ \ \tilde{T}^{j\cdots}_{k\cdots} \ . \tag{5.5.12}$$

5.6. The gauge condition

This section and the next are concerned with technical aspects of the degenerate perturbation theory associated with finite volume infrared regularization of the low temperature expansion.

The minima of the action consist of the constant fields $\phi(x) = m$. The low temperature expansion of the integral over nonlinear fields is calculated by integrating over the linear fields at each constant. To prevent overcounting, the integral over the linear fields $\sigma^i(x)$ at the constant $\phi(x) = m$ must avoid the constant linear field $\sigma^i(x) = v^i \neq 0$ which represents $\phi(x) = m'$. Integration over the nonlinear fields is replaced by constrained integration over the linear fields:

$$\int d\phi \; G(\phi) \;=$$

$$\int_m \int \tilde{d}\sigma \;\; \delta(\tilde{P}(m,\sigma)) \;\; \det \tilde{F}(m,\sigma) \;\; G(\tilde{E}_m(\sigma)) \qquad (5.6.1)$$

where $G(\phi)$ is any real valued function of the nonlinear fields, $\tilde{P}^i(m,\sigma)$ is a vector valued gauge function and $\tilde{F}^i_j(m,\sigma)$ is a matrix valued function to be determined. $\det \tilde{F}$ is a Fadeev-Popov determinant which compensates for the distorting effect of the gauge condition. The δ- function in (5.6.1) is the natural point measure at the zero in T_mM with values in the volume elements at m. A more explict notation would be $d_o m \; \delta(\tilde{P})$ where $d_o m$ is an arbitrary volume element at m and $\delta(\tilde{P})$

is the standard δ- function on T_mM defined with respect to $d_o m$. The product $d_o m\,\delta(\tilde{P})$ does not depend on the choice of $d_o m$. The integration $\int_m$ over m in (5.6.1) is integration against the volume element left by $\delta(\tilde{P})$.

The gauge condition is enforced by means of a multiplier γ_j in $T^*_m M$:

$$\delta(\tilde{P}) = \int d\gamma \;\; \exp(i\gamma_j \tilde{P}^j) \; . \tag{5.6.2}$$

$d\gamma$ is the natural volume element on $T^*_m M$ whose value is a volume element on M. That is, for f a real valued function on $T^*_m M$, $\int d\gamma\, f(\gamma)$ is a volume element on M at m.

To ensure that the gauge condition $\tilde{P}^i(m,\sigma) = 0$ selects from the collection of all linear fields near zero a faithful copy of the space of all nonlinear fields near the constants, $\tilde{P}^i$ must satisfy the nondegeneracy condition:

$$\det \tilde{D}_j \tilde{P}^i \;(m,0) \neq 0 \; . \tag{5.6.3}$$

The gauge function ought to respect the spatial symmetry of the model and, to simplify the power counting, should be local and zeroth order in the linear field:

$$\tilde{P}^j(m,\sigma) = L^{-(2+\epsilon)} \int dx \;\; \tilde{p}^j(m,\sigma(x)) \; . \tag{5.6.4}$$

A useful choice is

$$\tilde{P}^i(m,\sigma) = L^{-(2+\epsilon)} \int dx \ \sigma^i(x) . \qquad (5.6.5)$$

5.7. The Fadeev-Popov determinant

The strategy for finding $\tilde{F}^i_j(m,\sigma)$ is to fix an arbitrary constant m and to represent the linear fields at nearby constants m' by the linear fields at m, using the nonlinear parallel transport functions $\tilde{E}^{-1}_{m'} \circ \tilde{E}_m$. The auxiliary linear connection is used to transport vectors in TM. For convenience of exposition, coordinates $\{m^i\}$ are used on M. Only the first order in $(m' - m)$ is of interest, so path dependence of the auxiliary parallel transport does not matter.

The vector valued gauge function $\tilde{P}^i(m',\sigma)$ becomes, for each m', a function on the linear fields at m with values in $T_m M$. The a priori measures $\tilde{d}\sigma\ (m',\sigma)$, because of their compatibility, are all represented by $\tilde{d}\sigma\ (m,\sigma)$. For ϕ and m' both near m, equation (5.6.1) becomes

$$\int d\phi \ G(\phi) =$$

$$\int_{m'} \int \tilde{d}\sigma\ (m,\sigma) \ \delta(\tilde{P}(m',\sigma)) \ \det \tilde{F}(m',\sigma) \ G(\tilde{E}_m \sigma) . \qquad (5.7.1)$$

Since $\tilde{d}\sigma\ (m,\sigma)$ represents $d\phi$ in terms of the linear fields at m,

$$\int d\phi \; G(\phi) = \int \tilde{d}\sigma \; (m,\sigma) \quad G(\tilde{E}_m \sigma) \; . \qquad (5.7.2)$$

Therefore the Fadeev-Popov determinant is determined by the condition

$$1 = \int_{m'} \delta(\tilde{P}(m',\sigma)) \quad \det \tilde{F}(m',\sigma) \; . \qquad (5.7.2)$$

Since m is arbitrary and since σ participates in the integral over fluctuations only if $\tilde{P}^i(m,\sigma) = 0$, $\tilde{F}^i_j(m,\sigma)$ need only satisfy (5.7.2) on the gauge slice $\tilde{P}^i(m,\sigma) = 0$. To first order in $(m' - m)$,

$$\tilde{P}^i(m',\sigma) = \tilde{P}^i \; (m,\sigma) + (m' - m)^j \; \tilde{D}_j \tilde{P}^i \; (m,\sigma) \; . \qquad (5.7.4)$$

So $\det \tilde{F}(m,\sigma)$ is determined, on the gauge slice, by

$$1 = \int_{m'} \delta(\; (m' - m)^j \, \tilde{D}_j \tilde{P}^i \; (m,\sigma) \;) \quad \det \tilde{F}(m,\sigma) \; . \qquad (5.7.5)$$

Therefore the correct Fadeev-Popov determinant is provided by

$$\tilde{F}^i_j(m,\sigma) = - \tilde{D}_j \tilde{P}^i \; (m,\sigma) \; . \qquad (5.7.6)$$

When the linear representation is given by a system of coordinates, when $\tilde{Q}$ is determined by a linear connection in TM, when the auxiliary connection is the same linear connection, and when the gauge function is (5.6.5); then, using (5.3.3), (5.4.1) and (5.4.4) to calculate (5.7.6),

$$\tilde{F}^i_j(m,\sigma) = L^{-(2+\epsilon)}\int dx \;\; Q^i_j(m,\sigma(x)) \; . \qquad (5.7.7)$$

$\tilde{F}(m,\sigma)$ is well defined by (5.7.6) even off the gauge slices $\tilde{P}^i(m,\sigma) = 0$, but its definition depends on the choice of auxiliary linear connection. On the gauge slices, however, the definition depends only on the linear representation of the fields, because when the auxiliary condition varies, the change in $\tilde{D}_i\tilde{P}^j$, being linear in $\tilde{P}$, vanishes wherever $\tilde{P}$ does.

The Fadeev-Popov determinant can be represented, for each m, as an integral over a finite set of anticommuting ghosts variables:

$$\det \tilde{F}^i_j(m,\sigma) = \int dc^* \wedge dc \;\; c^j \tilde{F}^i_j c^*_i \; . \qquad (5.7.8)$$

The ghost c is in $T_m M$, c^* in $T^*_m M$. A function of c is an element of the Grassmann algebra $\Lambda^*(T^*_m M)$; a function of c^* is an element of the Grassmann algebra $\Lambda^*(T_m M)$. A monomial in the ghosts containing c r times and c^* s times is said to have bidegree (r,s) and ghost number $r - s$.

The volume element $dc^* \wedge dc$ integrates a function of the ghosts to the trace of its component of highest bidegree. Explicitly, in terms of a basis $\{c^*_i\}$ for $T^*_m M$ and the dual basis $\{c^i\}$ for $T_m M$, temporarily abandoning the summation convention,

$$dc^* \frown dc = (dc^*_1 \frown dc^1)\ (dc^*_2 \frown dc^2)\ \cdots\ , \tag{5.7.9}$$

$$0 = \int dc^*_i \frown dc^i\ (\ c^*_i \text{ or } c^i \text{ or } 1\ ) \tag{5.7.10}$$

$$1 = \int dc^*_i \frown dc^i\ (c^i\ c^*_i)\ . \tag{5.7.11}$$

The volume element $dc^* \frown dc$ is natural. It does not involve an arbitrary choice of volume element on $T_m M$ or on $T^*_m M$.

5.8. Redundancy equations and BRS invariance

When infrared regularization is provided by a constant external field, the distribution of linear fields at m is

$$\tilde{d}\sigma\ \ \exp[\ -\ \tilde{S}(m,\sigma) + \tilde{H}(m,\sigma)]\ . \tag{5.8.1}$$

The fact that this represents, for all m, the same distribution of nonlinear fields is expressed in the compatibility equations:

$$\tilde{D}_i \tilde{d}\sigma = 0 \tag{5.8.2}$$

$$\tilde{D}_i \tilde{S} = 0 \tag{5.8.3}$$

$$\tilde{D}_i \tilde{H} = 0 \; . \tag{5.8.4}$$

The compatibility equations state that the vertices contained in the Taylor series expansions of $\tilde{J}$, $\tilde{S}$ and $\tilde{F}$ at m determine those contained in the expansions for any m' infinitesmally close to m. $\tilde{D}_i$ satisfies the integrability condition

$$D^2_{ij} = 0 \; . \tag{5.8.5}$$

When finite volume infrared regularization is used, the distribution of linear fields and auxiliary variables is

$$d\mathcal{Y} \;\; dc^{*}\text{-}dc \;\; \tilde{d}\sigma \quad \exp \left[- \tilde{A}(m,\sigma,\mathcal{Y},c,c^{*}) \right] \tag{5.8.6}$$

$$\tilde{A}(m,\sigma,\mathcal{Y},c,c^{*}) = \tilde{S}(m,\sigma) - \tilde{H}(m,\sigma)$$

$$- \; i \; \mathcal{Y}_j \; P^j(m,\sigma) \; - \; c^j \; \tilde{F}^i_j(m,\sigma) \; c^{*}_i \tag{5.8.7}$$

$$\tilde{F}^i_j(m,\sigma) = - \tilde{D}_j P^i \; (m,\sigma) \; . \tag{5.8.8}$$

An extension of (5.8.2-4) is sought which includes the multiplier and ghost contributions to (5.8.7). After Becchi, Rouet and Stora[19], it is expected that there is an equation of the form $s(\tilde{A}) = 0$, where s is a first order operator which increases ghost number by one, which satisfies $s^2 = 0$, and which includes, in some sense, a term $c^i \tilde{D}_i$. Formally, s is to be a vector field of ghost number one on the supermanifold described by the variables (m, σ, y, c, c^*).

$\tilde{D}_i$ is extended to act on the multiplier and ghosts by means of the auxiliary linear connection in TM. In coordinates $\{m^i\}$ this amounts to adding to expression (5.3.2) for $\tilde{D}_i$ a term

$$\hat{\Gamma}_{ki}^{j} \;(y_j \frac{\partial}{\partial y_k} + c_j^* \frac{\partial}{\partial c_k^*} - c^k \frac{\partial}{\partial c^j}) . \tag{5.8.9}$$

The extended $\tilde{D}_i$ satisfies

$$\tilde{D}_i c^j = \tilde{D}_i c_j^* = \tilde{D}_i y_j = 0 \tag{5.8.10}$$

$$[\tilde{D}_i , \frac{\partial}{\partial y_j}] = [\tilde{D}_i , \frac{\partial}{\partial c_j^*}] = [\tilde{D}_i , \frac{\partial}{\partial c^j}] = 0 \tag{5.8.11}$$

$$\tilde{D}_{ij}^2 = \hat{R}^p{}_{qij} \;(c_p^* \frac{\partial}{\partial c_q^*} + y_p \frac{\partial}{\partial y_q} - c^q \frac{\partial}{\partial c^p}) . \tag{5.8.12}$$

$c^i \tilde{D}_i$ now makes sense, and

$$(c^i \tilde{D}_i)^2 = \frac{1}{2} c^i c^j \;(\tilde{D}_{ij}^2 - \hat{T}_{ij}{}^k \tilde{D}_k) . \tag{5.8.13}$$

The BRS operator is defined to be

$$s = c^i \tilde{D}_i - i\gamma_j \frac{\partial}{\partial c_j^*}$$

$$+ \frac{1}{2} \hat{T}^i_{jk} c^j c^k \frac{\partial}{\partial c^i} + \frac{1}{2} \hat{R}^i{}_{jkl} c_i^* c^k c^l \frac{\partial}{\partial(i\gamma_j)} \qquad (5.8.14)$$

where $\hat{R}$ and $\hat{T}$ are the curvature and torsion of the auxiliary linear connection.

It follows immediately from the Bianchi identities

$$0 = \underset{ijk}{\mathrm{Alt}} \left(\hat{R}^p{}_{ijk} + \hat{T}^q_{ij} \hat{T}^p_{kq} - \nabla_i \hat{T}^p_{jk} \right) \qquad (5.8.15)$$

$$0 = \underset{ijk}{\mathrm{Alt}} \left(\nabla_i \hat{R}^p{}_{qjk} + \hat{T}^r_{ij} \hat{R}^p{}_{qrk} \right) \qquad (5.8.16)$$

that

$$s^2 = 0 \, . \qquad (5.8.17)$$

Moreover,

$$s(\tilde{S}) = c^i \tilde{D}_i \tilde{S} = 0 \qquad (5.8.18)$$

$$s(\tilde{H}) = c^i \tilde{D}_i \tilde{H} = 0 \qquad (5.8.19)$$

and

$$\tilde{A} = \tilde{S} - \tilde{H} + s(c_j^* \tilde{P}^j(m,\sigma)) , \qquad (5.8.20)$$

so

$$s(\tilde{A}) = 0 . \qquad (5.8.21)$$

Also,

$$s(\tilde{d}\sigma) = 0 \qquad (5.8.22)$$

$$s(dc^* \sim dc) = 0 \qquad (5.8.23)$$

$$s(d\gamma) = 0 . \qquad (5.8.24)$$

Finally, an integration by parts formula which will be used below is

$$\int \tilde{d}\sigma \, d\gamma \;\; s(\bar{G}) = c^1 \hat{v}_1 \int \tilde{d}\sigma \, d\gamma \;\; \tilde{G} . \qquad (5.8.25)$$

5.9. Standard models

In this section, which is a continuation of section 4.8, M is a homogeneous space G/H, and all structures are assumed G- invariant, except the external source $\tilde{H}$. Because of the G- invariance, the distribution of fields need only be examined at a single point m_o in M.

The linear connection in TM which determines ∇_i and also $\tilde{\nabla}_i$ is the canonical connection. The auxiliary linear connection determining $\hat{\nabla}_i$ might be chosen the same, but, more generally,

$$\hat{\nabla}_i = \nabla_i - \hat{\Gamma}_{ki}^{\ j} v^k \frac{\partial}{\partial v^j} . \qquad (5.9.1)$$

$\hat{\Gamma}_{ki}^{\ j}$ is an H- invariant tensor at m_o. On G- invariant functions of the linear fields, $\tilde{D}_i$ is the vector field

$$- Q_i^j(m,\sigma(x)) \frac{\partial}{\partial \sigma^j(x)} . \qquad (5.9.2)$$

The compatibility equations (5.8.2-3) become the nonlinear symmetries

$$[\tilde{Q}_i, \tilde{d}\sigma] = 0 \qquad (5.9.3)$$

$$[\tilde{Q}_i, \tilde{S}] = 0 . \qquad (5.9.4)$$

The integrability condition (5.8.5) becomes, using (4.8.5), the commutation relation

$$[\tilde{Q}_i, \tilde{Q}_j] = T_{ij}^k \tilde{Q}_k + R^p{}_{qij} \sigma^q(x) \frac{\partial}{\partial \sigma^p(x)} . \qquad (5.9.5)$$

$-\tilde{Q}_i$ is extended to the auxiliary variables by adding to it the term (5.8.9). The commutation relation for the extended operator is, using (5.8.12),

$$[\tilde{Q}_i, \tilde{Q}_j] = T^k_{ij}\tilde{Q}_k + R^p{}_{qij}\sigma^q(x)\frac{\delta}{\delta\sigma^p(x)}$$

$$+ \hat{R}^p{}_{qij}\left(c^*_p \frac{\partial}{\partial c^*_q} + \gamma_p \frac{\partial}{\partial \gamma_q} - c^q \frac{\partial}{\partial c^p} \right) . \qquad (5.9.6)$$

The BRS operator is

$$s = c^i\tilde{Q}_i - i\gamma_j \frac{\partial}{\partial c^*_j}$$

$$+ \frac{1}{2}\hat{T}^i_{jk}\, c^j c^k \frac{\partial}{\partial c^i} + \frac{1}{2}\hat{R}^i{}_{jkl}\, c^*_i c^k c^l \frac{\partial}{\partial (i\gamma_j)} . \qquad (5.9.7)$$

All of (5.8.17-18,20,22-24) continue to hold. (5.8.19,21) become

$$s(\tilde{H}) = - c^i\, [\tilde{Q}_i, \tilde{H}] \qquad (5.9.8)$$

$$s(\tilde{A}) = s(\tilde{H}) . \qquad (5.9.9)$$

Note that the canonical connection on a Lie group has $\hat{R}^p{}_{ijk} = 0$. s (5.9.7) is then the original BRS transformation.[19]

6. Renormalization

6.1. Generalities

From the point of view of of Wilson[7], renormalization means eliminating from the distribution of fields of a model all fluctuations on scales smaller than a cutoff distance Λ^{-1}, leaving an effective distribution for the remaining degrees of freedom. The effective distribution has the same properties as the original at distances much larger than Λ^{-1}.

The most general distribution of fields, including all possible short range interactions, is characterized by a point λ in an infinite dimensional space of parameters. Appropriate powers of Λ are used to make λ dimensionless. Each effective distribution is characterized by an effective parameter $\lambda(\Lambda)$. λ is considered here to include a characterization of the local sources conjugate to the fields of the model, so that renormalization of the field is implied by renormalization of λ.

The invariance of the long distance properties of the model under simultaneous change of the cutoff Λ and the parameter λ is expressed in a differential equation for the partition function:

$$[\Lambda \frac{\partial}{\partial\Lambda} + \beta(\lambda) \frac{\partial}{\partial\lambda}] Z(\Lambda,\lambda) = 0 . \tag{6.1.1}$$

$\beta = \beta(\lambda) \frac{\partial}{\partial\lambda}$ is is a vector field on the space of parameters, called the β- function. The renormalization group equation (6.1.1) states that increasing the cutoff length from Λ^{-1} to $e^t\Lambda^{-1}$ while flowing along the vector field $-\beta$ for a "time" t has no net effect on the long distance properties of the model. The effective parameter $\lambda(\Lambda)$ corresponding to the cutoff distance Λ^{-1} satisfies the ordinary differential equation

$$\Lambda^{-1}\frac{\partial}{\partial\Lambda^{-1}}\lambda = -\beta(\lambda) . \qquad (6.1.2)$$

The vector field $-\beta$ is the infinitesmal generator of the renormalization group. The time t which indexes the action of the renormalization group is the logarithmic change of the cutoff length Λ^{-1}.

Flowing along $-\beta$ in parameter space has the effect on long distance properties only of decreasing all dimensionless characteristic lengths. That is, if $r(\lambda)$ is a dimensionless length in the model, so that $\Lambda^{-1}r(\lambda)$ is, for example, some correlation length, then it follows from the renormalization group equation (6.1.1) that when λ flows to $e^{-t\beta}(\lambda)$ and Λ is replaced by $e^{-t}\Lambda$ the length $\Lambda^{-1}r(\lambda)$ remains unchanged. Therefore, the dimensionless length obeys $r(e^{-t\beta}(\lambda)) = e^{-t}r(\lambda)$.

The model shows critical behavior at values of λ where some characteristic length μ^{-1} goes to infinity. The collection of such values of the parameter forms the critical surface. The divergence of the dimensionless length $\mu^{-1}\Lambda$ near the critical surface allows a scaling or continuum limit to be defined, in which μ^{-1} serves as the fundamental unit of length, measured against which the cutoff length disappears.

Critical behavior is associated with instability in the renormalization group action (so called infrared instability). The renormalization group leaves infinite lengths infinite, but sends finite dimensionless lengths towards zero. Therefore two nearly identical values of the parameter, one critical and the other only near critical, go to entirely different fates under the renormalization group. Such behavior characterizes instability.

The thermodynamic, or infinite distance, properties associated with a value λ of the parameter are determined by the ultimate fate of λ under the renormalization group. The abrupt change in this fate at the critical surface indicates that the critical behavior is associated with a phase transition.

The fact that infrared instability in the renormalization group implies critical behavior is most easily seen in the case of a fixed point with nontrivial unstable manifold. (The unstable manifold of a fixed point consists of the points in parameter space driven to the fixed point by the renormalization group as $t \rightarrow -\infty$. The stable manifold consists of the points driven to the fixed point as $t \rightarrow +\infty$.)

A parameter λ near the stable manifold is driven by the renormalization group into the vicinity of the fixed point and then away along a trajectory which conveges towards the unstable manifold. Parameters near the fixed point are almost left fixed by the renormalization group, so the trajectory spends a long time there. As λ approaches the stable manifold, the trajectory approaches a limit which consists of a path lying in the stable manifold teminating at the fixed point followed by a path in the unstable manifold leaving the fixed point. If λ^r is some point on the outgoing part of the limiting trajectory, then, as λ approaches the stable manifold, the time it takes for the trajectory to reach a neighborhood of λ^r grows without bound. Assuming some nonzero dimensionless length associated with λ^r, the corresponding length associated with λ diverges at the stable manifold of the fixed point. The stable manifold is therefore a critical surface.

The scaling or continuum limit of the model is characterized by a space of renormalized parameters λ^r. The renormalized partition function Z^r is defined as a function of the renormalized parameter λ^r and of the macroscopic length scale μ^{-1} by

$$Z^r(\mu,\lambda^r) = \lim_{\Lambda\to\infty} Z(\Lambda, \lambda(\mu^{-1}\Lambda, \lambda^r)) , \qquad (6.1.3)$$

in which the bare parameter λ is given as a function of λ^r and the ratio of scales by inverting

$$\lambda^r = (\mu^{-1}\Lambda)^{-\beta} (\lambda) . \qquad (6.1.4)$$

The expression on the right in (6.1.4) describes the point in parameter space reached by flowing from λ along $-\beta$ for a time $\log \mu^{-1}\Lambda$.

By the renormalization group equation (6.1.1), $Z(\Lambda, \lambda)$ is independent of Λ, in its long distance properties, when λ is given by (6.1.4). Therefore the renormalized partition function as a function of the renormalized parameter describes a continuum model. It follows from (6.1.1-4) that the renormalized partition function satisfies the renormalization group equation

$$[\mu \frac{\partial}{\partial\mu} + \beta(\lambda^r) \frac{\partial}{\partial\lambda^r}] Z^r(\mu, \lambda^r) = 0 . \qquad (6.1.5)$$

There remains the problem of describing the appropriate space of definition of the renormalized parameter. The space of renormalized parameters, as they are given in (6.1.4), is the same as the space of bare parameters. But the renormalization group is, strictly speaking, a semigroup, since there is no way to undo the elimination of degrees of freedom. The definition of the continuum model (6.1.3) requires that the renormalization group be run backwards an infinite amount of time. Pathological short distance behavior is to be feared unless the entire backwards trajectory can be exhibited inside the space of bare parameters. Therefore the renormalized parameter should lie in an unstable manifold associated with an infrared instability of the renormalization group.

Equivalently, the continuum model should be defined by making the bare parameter approach a critical surface within the space of bare parameters, rather than by allowing the bare parameter to follow an arbitrary trajectory of the renormalization group backwards into unknown territory.

Suppose $\lambda(s)$ a curve in parameter space such that $\lambda(0)$ lies on the critical surface, $r(s)$ being a dimensionless length associated with $\lambda(s)$ which diverges at $s = 0$. The continuum limit is defined by sending $s \to 0$ with the cutoff $\Lambda(s) = \mu\, r(s)$. By the renormalization group equation (6.1.1), the same continuum limit is obtained from $\lambda'(s) = e^{-t\beta}(\lambda(s))$, $\Lambda'(s) = e^{-t}\Lambda(s)$, for any $t << r(s)$. Letting $t \to \infty$, as $s \to 0$, brings $\lambda(s)$ to the unstable manifold. Therefore, the directions of infrared stability in parameter space are irrelevant to the continuum limit.

Recapitulating: the continuum or scaling limits of the model are described by the unstable manifolds associated with infrared instability of the renormalization group.

In perturbative field theory, the renormalization traces a reversed course. First, the renormalized partition function is shown to be well defined in the continuum limit as a function of the renormalized parameter, when the bare parameter is made to depend appropriately on the renormalized parameter and the ratio of scales. Renormalization group equations then follow from the existence of the continuum limit.

The perturbative field theory is given order by order in an expansion about a free field theory, that is, about a gaussian distribution of fields. Power counting limits the space of possibly relevant parameters: parameters describing nonrenormalizable vertices are ignorable. Power counting determines that the bare parameter λ can be written as a function of a renormalized parameter λ^r and the ratio of scales $\mu^{-1}\Lambda$ so that $Z(\Lambda, \lambda)$, when expanded in λ^r, has a sensible limit order by order in λ^r as $\Lambda \to \infty$. To lowest order, λ is λ^r scaled by appropriate powers of $\mu^{-1}\Lambda$ so that the renormalized distribution of fields is, at lowest order, independent of Λ. At higher order, λ consists of cutoff dependent counterterms (containing powers of $\log \mu^{-1}\Lambda$) needed to cancel the primitive divergences in the Feynman diagrams of the perturbative expansion.

By power counting, the primitive divergences depend only on the short distance properties of the model. Therefore the perturbation theory can be made cutoff independent by means of counterterms which are independent of the infrared regularization.

The space of renormalized parameters λ^r must be large enough to contain all counterterms permitted by power counting, because the distinction between renormalized parameter and counterterm is arbitrary, up to cutoff independent reapportionments between the two.

The continuum limit of the perturbation theory, which depends on μ and λ^r, is defined by (6.1.3). Renormalization group equations follow from the equivalence of cutoff and continuum theories at distances much larger than the cutoff. $Z(\Lambda, \lambda)$ is independent of μ, so differentiating the expression on the left in (6.1.3) with respect to μ, holding Λ and λ fixed, gives the renormalization group equation

$$[\,\mu \frac{\partial}{\partial\mu} + \beta^r(\lambda^r) \frac{\partial}{\partial\lambda^r}\,]\; z^r(\mu,\lambda^r) = 0\,, \tag{6.1.6}$$

where

$$\beta(\lambda^r) = \mu \frac{\partial}{\partial\mu}\Big/_{\lambda,\Lambda}\; \lambda^r\,. \tag{6.1.7}$$

More precisely, for $t = \log \mu^{-1}\Lambda$, let Π_t be the map from renormalized to bare parameter which provides the continuum limit:

$$z^r(\mu,\lambda^r) = \lim_{\Lambda\to\infty} z(\,\Lambda,\, \Pi_t^{-1}(\lambda^r)\,)\,. \tag{6.1.8}$$

Then

$$\beta^r(\lambda^r) = -\frac{\partial}{\partial t}\,(\,\Pi_t^{-1}\,) \circ \Pi_t\,. \tag{6.1.9}$$

$$= (\,\Pi_t^{-1}\,)_*\, \frac{\partial}{\partial t}\, \Pi_t\,. \tag{6.1.10}$$

Being dimensionless, the renormalization group coefficient β^r can depend on μ and Λ only in the combination $\mu^{-1}\Lambda$. But, at each order in λ^r, any term in β^r which diverges when $\mu^{-1}\Lambda\to\infty$ can be isolated as a separate invariance of the model and discarded without affecting the validity of the renormalization group equations. Therefore the renormalization group coefficients are independent of μ and Λ in the limit $\mu^{-1}\Lambda \to \infty$.

For properties associated with distances much greater than Λ^{-1}, the bare partition function is governed by a renormalization group equation of the form (6.1.1), the β- function being

$$\beta(\lambda) = \Lambda \frac{\partial}{\partial \Lambda}\bigg|_{\mu, \lambda^{r}} \lambda , \qquad (6.1.11)$$

or, more precisely,

$$\beta(\lambda) = \left(\frac{\partial}{\partial t} \Pi_t \right) \circ \Pi_t^{-1} . \qquad (6.1.12)$$

The above constructions depend on expansions which have a chance of making sense only when both λ and λ^r are small. But λ is given as a power series in λ^r with divergent coefficients. As the ratio of scales increases, λ^r and λ must be confined to smaller and smaller values. The perturbative renormalization group equations are at best asymptotic expansions of the nonperturbative equations. They are useful because the topological structure of a vector field such as the renormalization group generator, in the region of small values of the parameter, is exhibited in its asymptotic expansion.

The topological properties of the renormalization group determine _a posteriori_ the length scales at which the perturbative analysis is appropriate. When perturbation theory shows infrared instability in the region of small values of the parameter, the perturbative analysis is appropriate at short distances: it establishes the existence of the continuum limit and exhibits the short distance ($\ll \mu^{-1}$) scaling properties. When perturbation theory shows infrared stability, the perturbative analysis can be used to find long distance properties ($\gg \mu^{-1}$), but cannot pick out the renormalization group trajectory which leads back to an infrared unstable fixed point, or even guarantee that such a trajectory exists.

6.2. Power counting for the nonlinear models

The general program outlined in the previous section is now to be adapted to the nonlinear models. Two complications arise: (1) the degeneracy of the gaussian models at asymptotically small values of the parameter, and (2) the existence of a group of equivalence transformations on the parameters.

This section describes the constraints on perturbative renormalization determined by power counting. The arguments are applicable to models on Euclidean $2 + \epsilon$ space for asymptotically small ϵ, but for convenience only the case $\epsilon = 0$ is treated explicitly. For $\epsilon \neq 0$, the essential point is that the significant cutoff dependence of the Feynman diagrams consists of powers of $\log \mu^{-1}\Lambda$ and that, in the double expansion in T and ϵ, the power of $\log \mu^{-1}\Lambda$ occurring in each term of the expansion is controlled by the combined power of T and ϵ multiplying the term, which is therefore the appropriate number by which to order the expansion.

The parameters of the nonlinear model are the metric coupling and the (spatially dependent) external field. The perturbative expansion is in powers of the temperature. The model is to be renormalized by expressing the bare metric and external field as a renormalized metric and external field (scaled acccording to naive dimension) plus counterterms, so that the partition function (2.4.4), expanded in the renormalized temperature, is a cutoff independent function of the renormalized parameters. Renormalization of the external field is equivalent to renormalization of its dual, the order parameter.

The apparatus of perturbative renormalization cannot, however, deal directly with the nonlinear model. The theorems which support power counting arguments require a Feynman diagram expansion, which in turn is derived from a functional integral over linear fields. Therefore the perturbative renormalization must take place in the collection of distributions (3.1.6) or (3.2.8-9) of linear fields. It is necessary first to renormalize the individual distributions of linear fields, and then, as a separate matter, to show that the collection of renormalized distributions of linear fields is equivalent to a single renormalized nonlinear model.

It might seem that infrared regularization by means of a constant external field $-\frac{1}{2}T^{-1}h_o$ avoids the second issue by eliminating all but one distribution of linear fields from consideration. However, h_o is a soft operator whose effect is negligible at the short distances which are of concern in the renormalization. h_o should be considered a device by which one distribution of linear fields at a time is singled out for renormalization. Each distribution is renormalized in the presence of an appropriate external field. The issue of the compatibility of the resulting collection of renormalized distributions remains.

The parameters of the distribution of linear fields at m, (3.1.5) or (3.2.8-9), are the Taylor series coefficients of $\tilde{g}_{ij}$ and $\tilde{h}(x)$, and, in (3.2.8-9), also the coefficients of $\tilde{p}^i$ and $\tilde{f}^i_j$. The Taylor series coefficients of $\tilde{g}$, $\tilde{h}$, $\tilde{j}$, $\tilde{p}$ and $\tilde{f}$ are the couplings associated with the vertices of the Feynman diagram expansion. The logarithmic jacobian $\tilde{j}$ of the coordinate map is not an independent parameter, because the a priori volume element from which it derives is fixed in association with the metric coupling. The variability of $\tilde{j}$ is absorbed into that of $\tilde{h}$.

The distribution of linear fields at m is renormalized by expressing the bare vertices as (naively rescaled) renormalized vertices plus counterterms, so that the functional integral (3.1.6) or (3.2.10) at m is a cutoff independent function of the renormalized couplings. The renormalized vertices take the most general form prescribed by power counting for the counterterms.

Of the variables in (3.2.8-9), only the field $\sigma^i(x)$ has short distance fluctuations. Power counting reveals that the primitively divergent diagrams involving σ contain arbitrary numbers of the dimensionless vertices from the expansion of $\tilde{S}(m,\sigma)$. These are the vertices containing two derivatives of $\sigma^i(x)$. The rest of the vertices of (3.2.8-9) contain no derivatives of σ. They have length dimension -2. By power counting, at most one of them can occur in a primitively divergent Feynman diagram.

The diagrams containing only vertices from $\tilde{S}$ are quadratically divergent. The primitive divergences of such diagrams consist of integrals over space of two kinds of local expression in $\sigma^i(x)$: polynomials in $\sigma^i(x)$ multiplied by Λ^2; and polynomials in $\sigma^i(x)$ multiplied by $\partial_\mu \sigma^i(x)\, \partial_\mu \sigma^j(x)$. These have exactly the form of the vertices occurring in the expansions of $\tilde{S}$ and $\tilde{H}$. Since any number of vertices from $\tilde{S}$ can be present in these diagrams, the coefficients of the primitive divergences are polynomials in the Taylor series coefficients of the metric $\tilde{g}_{ij}$. The order of the polynomials can grow without bound as the number of loops increases.

The remaining primitively divergent diagrams contain exactly one vertex not from the expansion of $\tilde{S}$, and are logarithmically divergent. A primitive divergence of such a diagram is the integral over x of a polynomial in $\sigma^i(x)$ multiplied by a coefficient from the expansion of one of $\tilde{j}(m,\sigma(x))$, $\tilde{h}(m,\sigma(x))$, $i\gamma_j \tilde{p}^j(m,\sigma(x))$, or $c^j \tilde{f}_j^i(m,\sigma(x)) c_i^*$. These primitive divergences have exactly the form of $\tilde{H}$, $i\gamma_j \tilde{P}^j$, and $c^j \tilde{F}_j^i c_i^*$. The Taylor series coefficients of the metric occur nonlinearly, while those of $\tilde{j}$, $\tilde{h}$, $\tilde{p}$, and $\tilde{f}$ occur at most linearly.

The power counting argument appropriate to (3.1.5) is identical to the above, simply omitting mention of $\tilde{p}$ and $\tilde{f}$.

The renormalized distribution of linear fields at m therefore takes the same form as the bare one:

$$\tilde{d}^r\sigma \quad \exp[- \tilde{A}^r(m,\sigma)] \tag{6.2.1}$$

or

$$d\mathcal{Y} \quad dc^*\text{-}dc \quad \tilde{d}^r\sigma \quad \exp[- \tilde{A}^r(m,\sigma,\mathcal{Y},c,c^*)] \ , \tag{6.2.2}$$

where

$$\tilde{d}^r\sigma = \prod_x d^r\sigma(x) \quad \exp[- \tilde{J}^r(m,\sigma)] \tag{6.2.3}$$

$$\tilde{J}^r(m,\sigma) = \mu^{2+\epsilon} \int dx \quad \tilde{j}^r(m,\sigma(x)) \tag{6.2.4}$$

$$\tilde{A}^r(m,\sigma) = \tilde{S}^r(m,\sigma) - \tilde{H}^r(m,\sigma) \tag{6.2.5}$$

$$\tilde{A}^r(m,\sigma,\mathcal{Y},c,c^*) = \tilde{S}^r(m,\sigma) - \tilde{H}^r(m,\sigma) \tag{6.2.6}$$

$$- i \, \mathcal{Y}_i \, (\tilde{P}^r)^i(m,\sigma) - c^j \, (\tilde{F}^r)^i_j(m,\sigma) \, c^*_i$$

$$\tilde{S}^r(m,\sigma) = \mu^{\epsilon} \int dx \quad \frac{1}{2} T^{-1} \quad \tilde{g}^r_{ij}(m,\sigma(x)) \quad \partial_\mu\sigma^i(x) \, \partial_\mu\sigma^j(x) \tag{6.2.7}$$

$$\tilde{H}^r(m,\sigma) = \mu^{2+\epsilon} \int dx \quad \tilde{h}^r(x) \quad (m,\sigma(x)) \tag{6.2.8}$$

$$(\tilde{P}^r)^i(m,\sigma) = L^{-(2+\epsilon)} \int dx \ (\tilde{p}^r)^i(m,\sigma(x)) \tag{6.2.9}$$

$$(\tilde{F}^r)^i_j(m,\sigma) = L^{-(2+\epsilon)} \int dx \ (\tilde{f}^r)^i_j(m,\sigma(x)). \tag{6.2.10}$$

The renormalized parameters at m consist of the coefficients of the Taylor series expansions (in powers of v) of $\tilde{g}^r_{ij}(m,v)$, $\tilde{h}^r(x)\ (m,v)$, $(\tilde{p}^r)^i$, and $(\tilde{f}^r)^i_j(m,v)$. $\tilde{j}^r$ is not an independent renormalized parameter but is determined by the fixed relationship between $\tilde{d}^r\sigma$ and $\tilde{g}^r$. Any discrepancy between the counterterms for the $\tilde{j}$ vertices and $\tilde{j}^r$ becomes an inhomogeneous counterterm for $\tilde{h}$, that is, a counterterm of the form of an external field, but not linear in $\tilde{h}$.

The bare parameters for each m are expressed in terms of the ratio of scales $\mu^{-1}\Lambda$ and the renormalized parameters at m, so as to give a well defined renormalized partition function

$$Z^r(m) = \lim_{\Lambda\to\infty} Z(m), \tag{6.2.11}$$

which depends only on μ and the renormalized parameters at m.

For each constant m, $\tilde{g}_{ij}$ at m depends nonlinearly on $\tilde{g}^r_{ij}$ at m. $\tilde{h}$, $\tilde{p}$, and $\tilde{f}$ at m depend nonlinearly on $\tilde{g}^r_{ij}$ at m and at most linearly on $\tilde{h}^r$, $\tilde{p}^r$, and $\tilde{f}^r$ at m (respectively).

Power counting alone puts no restrictions on the renormalized vertices. Each term of each of the formal power series $\tilde{g}^r_{ij}$, $\tilde{h}^r$, $\tilde{p}^r$, and $\tilde{f}^r$ at m is an independent parameter.

6.2. Renormalization of the compatibility conditions

The power counting arguments apply independently at each constant m. But the bare parameters describing the various distributions of linear fields not independent. For example, the vertices contained in $\tilde{S}(m,\sigma)$ and $\tilde{H}(m,\sigma)$ for a given m determine those for any m′ infinitesmally close, because any small fluctuation around the constant $\phi(x) = m$ is also a small fluctuation around $\phi(x) = m'$. To consistently describe a renormalized nonlinear model, the renormalized perturbative parameters must contain an equivalent redundancy. That is, the collection of renormalized distributions of linear fields (6.2.1) or (6.2.2) must be the expression in some "renormalized" system of coordinates of a renormalized distribution of nonlinear fields of the form (3.1.5) or (3.2.8–9). If this were not the case then the space of parameters for the model would have grown enormously: from metrics and external fields to independent Taylor series of metrics and external fields at each point in M.

The redundancy in $\tilde{S}$, in $\tilde{H}$ and in the a priori measure $\tilde{d\sigma}$ is expressed as an invariance under a first order differential operator $\tilde{D}_1$, described in section 5.2. When infrared regularization is provided by a constant external field, the renormalization of the nonlinear structure of the model follows from the renormalization of the invariance of the collection of bare distributions of linear fields under $\tilde{D}_1$.

For finite volume infrared regularization, the collection of bare distributions of linear fields (3.2.8-9) is invariant under a single anticommuting transformation of the BRS type, described in section 5.8, which connects distributions at different values of m. The renormalization of the nonlinear structure follows from the renormalization of the BRS invariance.

The line of argument is an elaboration of that of [8-10]. An effective action is defined for each distribution of linear fields as the sum of one particle irreducible Feynman diagrams. The invariance properties of the collection of distributions of linear fields are used to obtain quadratic identiities on the collection of effective actions. At lowest order these identitities state the original invariance properties. The primitively divergent pieces of the effective actions satisfy the same quadratic identities. Therefore the renormalized distributions of linear fields can also be made to satisfy the identities. The quadratic identities are solved to obtain the result that the renormalized distribution of linear fileds is the expression of a renormalized distribution of nonlinear fields in terms of a renormalized system of coordinates, and, for finite volume infrared regularization, of a renormalized gauge function. The argument is presented in parallel for both forms of infrared regularizaiton.

The sum of connected Feynman diagrams for the distribution (3.1.5) or (3.2.8-9) is generated by

$$\tilde{W}(m,\mu) = \log \int \tilde{d}\sigma \ \exp[- \tilde{A}(m,\sigma) + (\mu,\text{sigma})] \tag{6.3.1}$$

or

$$\tilde{W}(m,\mu,p,c,c^*) = \log \int dY \, \tilde{d}\sigma \tag{6.3.2}$$

$$\exp[- \tilde{A}(m,\sigma,Y,c,c^*) + (\mu,\sigma) + iY_j p^j] \ .$$

$\mu_i(x)$ is a local source conjugate to $\sigma^i(x)$:

$$(\mu,\sigma) = \Lambda^{2+\epsilon} \int dx \ \mu_i(x) \ \sigma^i(x) \ . \tag{6.3.3}$$

The auxiliary variables of (3.2.8-9) can be kept as parameters for purposes of renormalization, because integration over them is finite dimensional and cannot introduce additional divergences. They global objects coupled only to the large distance fluctuations of the local field. But it is convenient to introduce a conjugate variable p^j for the multiplier Y_j and to integrate out Y_j, in order to remove the zero models from the integral over σ. It is not convenient to integrate out the ghost variables, because of the trilinear and quadrilinear terms in the BRS operator (5.8.14).

The effective action Γ, comprising the sum of one particle irreducible diagrams, is given by the Legendre transform of $\tilde{W}$:

$$\tilde{\Gamma}(\sigma) + \tilde{W}(\mu) = (\mu,\sigma) \qquad (6.3.4)$$

or

$$\tilde{\Gamma}(\sigma,\gamma) + \tilde{W}(\mu,p) = (\mu,\sigma) + i\gamma_j p^j \qquad (6.3.5)$$

where μ and p in (6.3.4-5) are given by inverting

$$\sigma = \frac{\partial}{\partial\mu}\tilde{W} \qquad (6.3.6)$$

$$i\gamma_j = \frac{\partial}{\partial p_j}\tilde{W} . \qquad (6.3.7)$$

In (6.3.5) the dependence on m, c, c^* is suppressed. Note that, for finite volume regularization, $\tilde{W}$ and $\tilde{\Gamma}$ transform as logarithms of volume elements on M (see section 5.6).

The generating functions are calculated in a perturbative expansion about the gaussian distribution

$$\prod_x d\sigma(x) \quad \exp \{ - \Lambda^{\epsilon} \int dx \;\; \frac{1}{2} T^{-1} \qquad (6.3.8)$$

$$[\tilde{g}_{ij}(m,0)\, \partial_\mu\sigma^i\, \partial_\mu\sigma^j + \Lambda^2\, \tilde{h}_{o,ij}(m,0)\, \sigma^i\, \sigma^j] \}$$

or

$$d\gamma \prod_x d\sigma(x) \quad \exp\{ -\Lambda^{\epsilon} \int dx \frac{1}{2} T^{-1} \tilde{g}_{ij}(m,0)\, \partial_\mu \sigma^i \partial_\mu \sigma^j \tag{6.3.9}$$

$$+ \; i\gamma_j L^{-2-\epsilon} \int dx \, (\tilde{p}^j(m,0) + \tilde{p}^j_{,k}(m,0)\, \sigma^k(x)\,)\,,$$

where $\tilde{h}_{o,ij}(m,0)$ is the hessian of the constant external field and $\tilde{p}^j_{,k}(m,0)$ is the first derivative of the gauge function $\tilde{p}^j(m,\sigma(x))$ at $\sigma^i(x) = 0$.

The consequences of the redundancy equations are derived first for constant external field regularization. As in (5.3.3), the compatibility operator is written in terms of a linear connection in TM as

$$\tilde{D}_i = \tilde{\nabla}_i - Q_i^j(m,\sigma(x)) \frac{\delta}{\delta\sigma^j(x)} \,. \tag{6.3.10}$$

The integration by parts formula (5.5.11) yields

$$\tilde{\nabla}_i(e^{\tilde{W}}) = \int \tilde{D}_i \{ \tilde{d}\sigma \exp[-\tilde{S} + \tilde{H} + (\mu,\sigma)] \} \tag{6.3.11}$$

$$= -\int \tilde{d}\sigma \quad \{ \exp[-\tilde{S} + \tilde{H} + (\mu,\sigma)] \tag{6.3.12}$$

$$\int dx \; Q_i^j(m,\sigma(x))\, \mu^j(x)$$

or

$$0 = \tilde{\nabla}_i \tilde{W} - \mu_j(x) \, (Q_i^j, -\frac{\delta}{\delta \tilde{h}(x)}) \tilde{W} \,. \tag{6.3.13}$$

The pairing $(Q_i^j, -\frac{\delta}{\delta\tilde{h}(x)})$ is defined as follows. The expression on the right in (6.3.12) requires integration of power series in $\sigma^i(x)$. Such integrals are generated by differentiation with respect to the Taylor series coefficients of $\tilde{h}(x)$ occurring in $\tilde{H}$. If

$$\tilde{h}(m,v) = \sum_{n=0} \frac{1}{n!} v^{k_1} \cdots v^{k_n} \tilde{h}(x)_{k_1 \cdots k_n} \tag{6.3.14}$$

and

$$Q_i^j(m,v) = \sum_{n=0} \frac{1}{n!} v^{k_1} \cdots v^{k_n} Q^j_{i,k_1 \cdots k_n} , \tag{6.3.15}$$

then define

$$(Q_i^j, \frac{\delta}{\delta\tilde{h}(x)}) = \sum_{n=0} Q^j_{i,k_1 \cdots k_n} \frac{\delta}{\delta\tilde{h}(x)_{k_1 \cdots k_n}} . \tag{6.3.16}$$

Pairings of $\frac{\delta}{\delta\tilde{h}(x)}$ with other functions of $\sigma^i(x)$ are defined similarly.

The Legendre transform of (6.3.13) is the quadratic identity

$$\tilde{D}_i^e \hat{\Gamma} = 0 \tag{6.3.17}$$

where

$$\tilde{D}_i^e = \tilde{\nabla}_i - (Q_i^j, -\frac{\delta\Gamma}{\delta h(x)})\frac{\delta}{\delta\sigma^j(x)} . \qquad (6.3.18)$$

At lowest order $-\frac{\delta\Gamma}{\delta h(x)} = 1$ so (6.3.17-18) is then the original redundancy equation.

The standard argument by induction in the order of perturbation theory gives the quadratic identity on the renormalized distribution of fields (6.2.1):

$$\tilde{D}_i^r \tilde{A}^r = 0 \qquad (6.3.19)$$

$$\tilde{D}_i^r = \tilde{\nabla}_i - (Q^r)_i^j(\sigma(x))\frac{\delta}{\delta\sigma^j(x)} \qquad (6.3.20)$$

$$(Q^r)_i^j(\sigma(x)) = (Q_i^j, -\frac{\delta\tilde{\Gamma}^r}{\delta h(x)}) . \qquad (6.3.21)$$

Power counting and Euclidean invariance give that $(Q^r)_i^j(\sigma(x))$ is a power series in $\sigma^i(x)$, containing no derivatives of $\sigma^i(x)$, with coefficients independent of x. Therefore (6.3.19) implies the separate identities

$$\tilde{D}_i^r \tilde{S}^r = \tilde{D}_i^r \tilde{H}^r = 0 . \qquad (6.3.22)$$

The operator $(\tilde{D}^r)_{ij}^2$, defined as in (4.2.5), satisfies, by (6.3.22),

$$(\tilde{D}^r)_{ij}^2 \tilde{H}^r = 0 . \qquad (6.3.23)$$

$(\tilde{D}^r)^2_{ij}$ does not differentiate with respect to m, because $\tilde{\nabla}^2_{ij}$ does not; that is, $(\tilde{D}^r)^2_{ij}$ acts independently on each space of linear fields. But for each m individually, $\tilde{H}^r(m,\sigma)$ can be chosen arbitrarily. Therefore (6.3.23) implies the renormalized integrability condition

$$(\tilde{D}^r)^2_{ij} = 0 . \tag{6.3.24}$$

Now define

$$D^r_i = \nabla_i - (Q^r)^j_i(m,v) \frac{\partial}{\partial v^j} \tag{6.3.25}$$

as a first order operator on functions on TM. (6.3.19-24) imply

$$(D^r)^2_{ij} = 0 \tag{6.3.26}$$

$$D^r_i \tilde{h}^r(x) = 0 \tag{6.3.27}$$

$$D^r_i \tilde{g}^r_{ij} = 0 . \tag{6.3.28}$$

It follows immediately, using the results of section 4.2, that D^r_i is the compatibility operator for a renormalized system of coordinates E^r on M, that $\tilde{g}^r_{ij}$ is the expression in that system of coordinates of a metric g^r_{ij} on M:

$$\tilde{g}^r_{ij}(m,v) = (E^r_m)^* g_{ij}\ (v)\ , \tag{6.3.29}$$

and that $\tilde{h}^r(x)$ is the exprssion in the renormalized coordinates of a function $h^r(x)$:

$$\tilde{h}^r(x)\ (m,v) = h^r(x)(E^r_m(x))\ . \tag{6.3.30}$$

The renormalized distribution of nonlinar fields, represented in the renormalized coordinates by the collection of renormalized distributions of linear fields, is

$$\prod_x d_{g^r}\phi(x) \quad \exp\,[\, - S^r(\phi) + H^r(\phi)\ ] \tag{6.3.31}$$

$$S^r(\phi) = \mu^{\epsilon} \int dx \ \frac{1}{2}\, g^r_{ij}(\phi(x)) \quad \partial_\mu \phi^i(x) \quad \partial_\mu \phi^j(x) \tag{6.3.32}$$

$$H^r(\phi) = \mu^{2+\epsilon} \int dx \ \ h(x)^r\ (\ \phi(x)\ )\ . \tag{6.3.33}$$

As discussed in section 4.2, the system of coordinates E^r is determined by D^r_i only up to transformations by diffeomorphisms Ψ of M (near the identity):

$$E^r \rightarrow \Psi \circ E^r\ . \tag{6.3.34}$$

Therefore the renormalized metric g^r and the renormalized external field $h^r(x)$ are only determined up to the transformations

$$g^r,\ h^r(x) \rightarrow \Psi_* g^r,\ h(x)\circ\Psi^{-1}\ . \qquad (6.3.35)$$

The relationship between $\tilde{h}(x)$ and $\tilde{h}^r(x)$ is, by power counting, a linear one:

$$\tilde{h}(x) = \tilde{Z}_h(m)\ \tilde{h}^r(x) + \tilde{h}_1(m)\ . \qquad (6.3.36)$$

For each m, $\tilde{Z}_h(m)$ is, to any finite order, a differential operator on functions on $T_m M$ and $\tilde{h}_1(m)$ is a real valued function on $T_m M$. Both depend on the cutoff and, nonlinearly, on $\tilde{g}^r$. The renormalized compatibility conditions imply

$$h(x) = Z_h\ h^r(x) + h_1 \qquad (6.3.37)$$

$$\tilde{Z}_h(m) = E_m^*\ Z_h\ (E_m^r)_* \qquad (6.3.38)$$

$$\tilde{h}_1(m) = E_m^*\ h_1 \qquad (6.3.39)$$

Z_h is, to any finite order, a differential operator on real valued functions on M and h_1 is a real valued function on M.

The inhomogeneous term h_1 in (6.3.34) takes the form

$$h_1 = h_{1,b} + (\mu^{-1}\Lambda)^{-2-\epsilon} h_{1,r} . \qquad (6.3.40)$$

$h_{1,r}$ is a finite contribution to the renormalized external field. $h_{1,b}$ contains the counterterms for whatever quadratic divergences appear, given the choices $d_g\phi(x)$ and $d_{g^r}\phi(x)$ of a priori volume elements. The dependence of $h_{1,b}$ on the cutoff is in the form of powers of $\log \mu^{-1}\Lambda$. h_1 is trivial for the standard models because there is no nonconstant invariant function on a homogeneous space M.

To lowest order,

$$T^{-1}g_{ij} = (\mu^{-1}\Lambda)^{-\epsilon} \; [\; (T^{-1}g)^r_{ij} + O(T^o) \;] \qquad (6.3.41)$$

and

$$\underline{Z}_h = (\mu^{-1}\Lambda)^{-2-\epsilon} \; [\; 1 + O(T^o) \;] \; .$$

The counterterms of order T^o in (6.3.41-2) are nonlinear in $T^{-1}g^r$ and depend on the cutoff through powers of $\log \mu^{-1}\Lambda$.

Loop counting constrains in the usual fashion the powers of T, $\log \mu^{-1}\Lambda$, and ϵ occurring in the counterterms. As the number of loops increases, more of the vertices play a role in the primitive divergences, so the primitive divergences come to depend on more and more derivatives (in M) of the parameters.

For finite volume regularization, the argument proceeds in essentially the same fashion. The integration by parts formula (5.8.25) and induction on the order of the perturbative expansion give the quadratic identity on the renormalized vertices

$$s^r(\tilde{A}^r) = 0 \tag{6.3.43}$$

where s^r is identical to s (5.8.14) except that $\tilde{D}_i^r$ (6.3.20) takes the place of $\tilde{D}_i$. There is also a linear identity

$$\frac{\delta}{\delta(i\gamma_j)} \tilde{A}^r = -(\tilde{P}^r)^j \tag{6.3.44}$$

$$(\tilde{P}^r)^j = (\tilde{P}^j, \frac{\delta\tilde{A}^r}{\delta h})$$

$$= L^{-2-\epsilon} \int dx \quad (\tilde{p}^j, \frac{\delta\tilde{A}^r}{\delta h(x)}) . \tag{6.3.45}$$

$(s^r)^2$ acts independently at each m, and $(s^r)^2(\tilde{A}^r) = 0$, so

$$(s^r)^2 = 0 . \tag{6.3.46}$$

From (6.3.43-46) it follows that

$$\tilde{A}^r = \tilde{S}^r - \tilde{H}^r + s^r(c_j^* (\tilde{P}^r)^j) \tag{6.3.47}$$

with

$$\tilde{D}_i^r \tilde{S}^r = \tilde{D}_i^r \tilde{H}^r = (\tilde{D}^r)^2_{ij} = 0 . \qquad (6.3.48)$$

Therefore the renormalized distribution of linear fields (6.2.2) is the expression of a renormalized distribution of nonlinear fields (6.3.31-33) in terms of a renormalized system of coordinates E^r and a renormalized gauge funciton $\tilde{P}^r$. From this point the argument is exactly as in the case of constant external field infrared regularization.

6.4. Renormalization group equations

The equivalence between bare and renormalized descriptions of the distribution of nonlinear fields is used to derive renormalization group equations. The renormalized partition function is given by

$$Z^r(\mu, g^r_{ij}, h^r) = \lim_{\Lambda \to \infty} Z(\Lambda, g_{ij}, h) . \qquad (6.4.1)$$

where the full dependence of Z and Z^r on parameters has been made explicit. The limit is taken with the bare parameters functions of $t = \log \mu^{-1}\Lambda$ and of the renormalized parameters:

$$g = g(t, g^r) \qquad (6.4.2)$$

$$h = Z_h(t, g^r)\, h^r + h_1(t, g^r) . \qquad (6.4.3)$$

The freedom to change origins in the coordinate spaces defining the linear fields gives the freedom to insert an arbitrary diffeomorphism of M in the transformation from renormalized to bare parameters.

The renormalization group equation for the bare partition function is

$$[\, \Lambda \frac{\partial}{\partial \Lambda} + \beta(g) \frac{\partial}{\partial g} + (\, \tilde{\gamma}(g)\, h(x) + \delta(g)\,) \frac{\partial}{\partial h(x)}\,]\, Z = 0 \,. \qquad (6.4.4)$$

$$\beta(g) = \frac{\partial}{\partial t}\Big/_{g^r}\, g \qquad (6.4.5)$$

$$\tilde{\gamma}(g) = (\frac{\partial}{\partial t}\Big/_{g^r}\, Z_h)\, Z_h^{-1} \qquad (6.4.6)$$

$$\delta(g) = \frac{\partial}{\partial t}\Big/_{g^r}\, h_1 - \tilde{\gamma}\, h_1 \,. \qquad (6.4.7)$$

The β- function, $\beta = \beta(g) \frac{\partial}{\partial g}$, is a vector field on the space of Riemannian metrics on M. $\tilde{\gamma}(g)$ is, for each metric coupling g, a linear operator on real valued functions on M. $\delta(g)$ is, for each metric g, a real valued function on M. The combination $\tilde{\gamma}(g)\, h(x) + \delta(g)$ is an inhomogeneous linear vector field on the space of external fields, which depends on the metric coupling. This pair of vector fields, on metric couplings and external fields, is (the negative of) the generator of the renormalization group. It is the object which, in the general discussion of the previous section, was called the β-function.

The inhomogeneous term in the renormalization group equation for the external field is an inconvenience which can be eliminated by an appropriate choice of bare and renormalized a priori volume elements. To see this, write $h_1 = h_{1,r} + h_{1,b}$, as in (6.3.40), and express $h_{1,b}$ as a function of $t = \log \mu^{-1}\Lambda$ and g. Then

$$\delta(g) = [\, \frac{\partial}{\partial t}\Big/_g + \beta(g) \frac{\partial}{\partial g}\Big/_t - \tilde{\gamma}(g) \,]\, h_{1,b} \qquad (6.4.8)$$

The arguement now goes by induction in the order of perturbation theory. The renormalization group coefficients are independent of the cutoff. Therefore, at each order, the most divergent part $h^{div}_{1,b}$ of $h_{1,b}(t,g)$ satisfies

$$[\, \beta(g) \frac{\partial}{\partial g}\Big/_t - \tilde{\gamma}(g) \,]\, h^{div}_{1,b} = 0 . \qquad (6.4.9)$$

But the lowest order contribution to the operator in brackets in (6.4.9) is the naive scaling value -2 of $\tilde{\gamma}$, implying that $h^{div}_{1,b} = 0$. Therefore $h_{1,b}$ must be independent of t. As a function of g alone, it can be absorbed into the bare a priori volume element, thereby eliminating the inhomogeneous coefficient from the renormalization group equation for the bare external field. An exactly parallel argument shows that $h_{1,r}$ can be absorbed into the renormalized a priori volume element, eliminating the inhomogeneous coefficient from the renormalization group equation for the renormalized external field.

Thus the renormalization group equation for the external field identifies a unique choice of volume element, given perturbatively order by order in T (and $\hbar$), with respect to which the external field is renormalized homogeneously. Homogeneous renormalization of the external field is the signal that the a priori volume element is chosen so that setting $h = 0$ in fact means the absence of an effective external field. With respect to any other choice of volume element, $h = 0$ leads to nonspontaneous ordering of the model. In the standard models this issue does not arise, because the distinguished, or neutral, volume element is fixed completely by the internal symmetry.

The distinguished bare volume element depends on the form of ultaviolet regularization. To lowest order, it is the metric volume element for g_{ij}. In dimensional regularization there are no quadratic divergences, so the distinguished volume element remains the metric volume element to all orders. On the lattice, however, a one loop calculation gives the distinguished bare volume element

$$d_{T^{-1}g}\phi(x) = d^{o}_{T^{-1}g}\phi(x) \quad \exp[\, \frac{T}{48} R + O(T^2)\,] \qquad (6.4.10)$$

where $d^{o}_{g}\phi(x)$ is the metric volume element and R is the scalar curvature of g_{ij}.

Henceforth it is assumed that the a priori volume elements are fixed at their respective neutral values. The renormalization group equations are then

$$[\,\Lambda \frac{\partial}{\partial\Lambda} + \beta(g)\frac{\partial}{\partial g} + \tilde{\gamma}(g)\; h(x)\frac{\partial}{\partial h(x)}\,]\; Z = 0\,. \tag{6.4.11}$$

$$[\,\Lambda \frac{\partial}{\partial\Lambda} + \beta(g)\frac{\partial}{\partial g} - \gamma(g)^{*}\;\Phi(x)\frac{\partial}{\partial\Phi(x)}\,]\; \Gamma = 0\,. \tag{6.4.12}$$

$$[\,\mu \frac{\partial}{\partial\mu} + \beta^{r}(g^{r})\frac{\partial}{\partial g^{r}} + \tilde{\gamma}^{r}(g^{r})\; h^{r}(x)\frac{\partial}{\partial h^{r}(x)}\,]\; Z^{r} = 0\,. \tag{6.4.13}$$

$$[\,\mu \frac{\partial}{\partial\mu} + \beta^{r}(g^{r})\frac{\partial}{\partial g^{r}} + \gamma^{r}(g^{r})^{*}\;\Phi^{r}(x)\frac{\partial}{\partial\Phi^{r}(x)}\,]\; \Gamma^{r} = 0\,. \tag{6.4.14}$$

The renormalized order parameter $\Phi^{r}(x)$ is the conjugate variable to the renormalized external field, and is related to the bare order parameter by

$$\Phi^{r}(x) = (\mu^{-1}\Lambda)^{2+\epsilon}\; Z_{h}^{*}\;\Phi(x)\,, \tag{6.4.15}$$

so $\Phi^{r} = \Phi + O(T)$. The renormalized free energy $\Gamma^{r}(\mu,g,\Phi)$ is the Legendre transform of $\log Z^{r}$, as in (2.4.7-8).

The coefficients of the renormalization group equations are given by

$$(\beta,\ \tilde{\gamma}\, h) = (\frac{\partial}{\partial t}\Pi_{t}) \circ \Pi_{t}^{-1} \tag{6.4.16}$$

$$(\beta^{r},\ \tilde{\gamma}^{r}\, h^{r}) = (\Pi_{t})_{*} \circ \frac{\partial}{\partial t}\Pi_{t}\,, \tag{6.4.17}$$

where

$$\Pi_t\ (g^r, h^r) = (g, h)\ . \qquad (6.4.18)$$

More concretely,

$$\beta(g) = \frac{\partial}{\partial t}\Big/_{g^r}\ g \qquad (6.4.19)$$

$$\tilde{\gamma}(g) = \left(\frac{\partial}{\partial t}\Big/_{g^r}\ Z_h \right) \circ Z_h^{-1} \qquad (6.4.20)$$

$$\beta^r(g^r) = \left(\frac{\partial g}{\partial g^r}\right)^{-1}\ \beta(g) \qquad (6.4.21)$$

$$\tilde{\gamma}^r(g^r) = Z_h^{-1} \circ \left(\frac{\partial}{\partial t} - \beta^r(g^r)\,\frac{\partial}{\partial g^r} \right)\ Z_h \qquad (6.4.22)$$

$$\gamma = (2 + \epsilon) + \tilde{\gamma} \qquad (6.4.23)$$

$$\gamma^r = (2 + \epsilon) + \tilde{\gamma}^r \qquad (6.4.24)$$

$\gamma(g)$ and $\gamma^r(g^r)$ are linear operators on real valued functions on M. $\gamma(g)^*$ and $\gamma^r(g^r)^*$ are the adjoint operators on measures on M. The former annihilate constant functions on M; their adjoints generate semigroups which preserve probability.

The effective parameters satisfy the renormalization group equations

$$\Lambda^{-1} \frac{\partial}{\partial \Lambda^{-1}} g = - \beta(g) \tag{6.2.25}$$

$$\Lambda^{-1} \frac{\partial}{\partial \Lambda^{-1}} h(x) = - \tilde{\gamma}(g)\ h(x) \tag{6.2.26}$$

$$\Lambda^{-1} \frac{\partial}{\partial \Lambda^{-1}} \Phi(x) = \gamma(g)^{*}\ \Phi(x) \tag{6.2.27}$$

$$\mu^{-1} \frac{\partial}{\partial \mu^{-1}} g^{r} = - \beta^{r}(g^{r}) \tag{6.4.28}$$

$$\mu^{-1} \frac{\partial}{\partial \mu^{-1}} h^{r}(x) = - \tilde{\gamma}^{r}(g^{r})\ h^{r}(x) \tag{6.4.29}$$

$$\mu^{-1} \frac{\partial}{\partial \mu^{-1}} \Phi^{r}(x) = \gamma^{r}(g^{r})^{*}\ \Phi^{r}(x) \tag{6.4.30}$$

Recall that flowing along $(-\beta, -\tilde{\gamma}\,h)$ is equivalent to decreasing all dimensionless lengths: dimensionful lengths remain constant while the unit of length Λ^{-1} increases.

The ambiguity in the relationship between bare and renormalized parameters, due to the freedom to choose arbitrarily the origins of the coordinate spaces, implies that the renormalization group coefficients are only defined up to infinitesmal diffeomorphisms:

$$\beta(g),\ \tilde{\gamma}(g) \rightarrow \beta(g) - [v(g), g],\ \ \tilde{\gamma}(g) - v(g) \tag{6.4.31}$$

where $v(g)$ is a vector field on M, and $[v, g]$ is the Lie bracket, (with singularities) over the equivalence classes $\underline{R}$ of metrics. The equivalence classes of metrics and local order parameters form a conical sub-bundle of the dual bundle.

2.5. The order parameter

The essential property of the order parameter is its averageability. The renormalization group acts by averaging the variables of the model over small regions in space (and by an overall rescaling of distances). Points in a manifold M can only be averaged if M is embedded in a space in which convex combination makes sense (for example, a vector space), and then the average of points in M will in general not remain in M. There are many embeddings of a given manifold M in a finite dimensional vector space, but none which is natural. Any such embedding involves arbitrary choices obscuring the character of the nonlinear model, which depends only on the intrinsic structure of the abstract manifold M. The only natural embedding is the one which places M inside the space of all unit measures on M itself, sending each point in M to the corresponding point measure. The order parameter then varies over all possible averages of point measures, which is to say over all the probability measures on M.

In a standard model (M the homogeneous space G/H) this picture can be considerably simplified. The internal symmetry group G acts on M, so acts by linear transformations on the real valued functions on M. Let V be a finite dimensional subrepresentation which separates infrared regularization. Only the infinitesmal diffeomorphisms are treated, although it should be possible to extend the arguments to deal with the full group of diffeomorphisms. The discussion is carried only to the point of identification of the finite dimensional cohomology spaces containing the potential obstructions to renormalizability.

In the case M compact, the problem divides into two stages. The first question is whether models which are equivalent under an infinitesmal diffeomorphism of M go to renormalized models which are also equivalent under some diffeomorphism of M. A negative answer clearly indicates some sort of pathology in the renormalization of the model. Given an affirmative answer to the first question, the second one arises: is it possible, by some finite modification of the renormalization procedure, to make the renormalization covariant? A covariant renormalization is one for models which are equivalent under a diffeomorphism of M are renormalized to models which are equivalent under the same diffeomorphism of M. That is, a covariant renormalization is one which commutes with the diffeomorphism group of M.

The renormalization of the equivalence relations is described by means of a set of quadratic identities on the renormalized action. Suppose v^i to be an infinitesmal diffeomorphism of M, i.e. a vector field on M. Let $\tilde{v}^i$ be its expression in coordinates around m. Then, for $\tilde{W}$ defined by (6.3.1),

$$\left([v, g] \frac{\partial}{\partial g} + [v, h(x)] \frac{\partial}{\partial h(x)} \right) e^{\tilde{W}} \qquad (6.5.1)$$

$$= \int e^{(\mu,\sigma)} \; [\, \tilde{v}^i(\sigma(x)) \frac{\partial}{\partial\sigma^i(x)} (\tilde{d}\sigma \; e^{-\tilde{A}}) \,]$$

or

$$\{ \; [v, g] \frac{\partial}{\partial g} \; + \; [v, h(x)] \frac{\partial}{\partial h(x)} \; \} \; \tilde{W} \tag{6.5.2}$$

$$= - (\tilde{v}^i, \frac{\partial \tilde{W}}{\partial \tilde{h}(x)}) \, \mu_i(x) \; .$$

The Legendre transform of (6.5.2) implies the transformation law for the renormalized action:

$$\{ \; [v, g] \frac{\partial}{\partial g} \; + \; [v, h(x)] \frac{\partial}{\partial h(x)} \; \} \; \tilde{A}^r \tag{6.5.3}$$

$$= (\tilde{v}^r)^i(m,\sigma(x)) \frac{\partial}{\partial\sigma^i(x)} \; \tilde{A}^r$$

where

$$(\tilde{v}^r)^i(m,\sigma(x)) \; = \; (\tilde{v}^i, - \frac{\partial \tilde{A}^r}{\partial \tilde{h}(x)} \; . \tag{6.5.4}$$

(6.5.3) is equivalent to

$$[v,g] \frac{\partial \tilde{g}^r}{\partial g} \; = \; [\tilde{v}^r(m), \tilde{g}^r] \tag{6.5.5}$$

and

$$[v,\ g]\ \frac{\delta \tilde{h}^r}{\delta g}\ +\ [v,\ h(x)]\ \frac{\delta \tilde{h}^r(x)}{\delta h(x)} \tag{6.5.6}$$

$$=\ [\tilde{v}^r(m)\ -\ \tilde{u}^r(m),\ \tilde{h}^r]$$

where $\tilde{v}^r(m) = (\tilde{v}^r)^i(m,v)\ \frac{\delta}{\delta v^i}$ and $\tilde{u}^r(m)$ is an arbitrary power series infinitesmal isometry of $\tilde{g}^r$ at m, which can freely be subtracted from $\tilde{v}^r(m)$ because it has no effect on (6.5.3).

(6.5.4) is rewritten, using (6.3.29),

$$[v,\ g]\ \frac{\delta g^r}{\delta g}\ =\ [v^r(m),\ g^r] \tag{6.5.7}$$

where

$$v^r(m)\ =\ (E^r_m)_*\ (\ \tilde{v}^r(m)\ -\ [v,\ g]\ \frac{\delta}{\delta g}\ E^r_m\ )\ . \tag{6.5.8}$$

The vector field v has been renormalized to a collection of power series vector fields $v^r(m)$ at the points m in M, defined up to addition of power series infinitesmal isometries $u^r(m)$ of g^r at m.

The issue now is whether there is some choice of $u^r(m)$ which makes the $v^r(m)$ the power series expansions of a single vector field v^r on M. Differentiating (6.5.7) with respect to m gives

$$[d_i v^r(m),\ g^r]\ =\ 0\ , \tag{6.5.9}$$

so $d_1 v^r(m)$ is a closed one form on M with values in the power series infinitesmal isometries of g^r.

If there exists a field $u^r(m)$ of power series infinitesmal isometries such that $d_1 v^r(m) = d_1 u^r(m)$, then $v^r = v^r(m) - u^r(m)$ is independent of m, so is a well-defined vector field on M. The obstruction, if any exists, to finding $u^r(m)$ is a cohomology class in the first cohomology of M with coefficients in the power series infinitesmal isometries of g^r.

If there is no obstruction, then (6.5.7), and (6.5.6) rewritten using (6.3.30), become

$$[v, g] \frac{\delta g^r}{\delta g} = [v^r, g^r] \qquad (6.5.10)$$

$$[v, g] \frac{\partial h^r}{\delta g} + [v, h] \frac{\partial h^r}{\delta h} = [v^r, h^r] . \qquad (6.5.11)$$

(6.5.10–11) state that bare parameters equivalent under an infinitesmal diffeomorphism v go to renormalized parameters equivalent by an infinitesmal diffeomorphism v^r. Therefore the obstructions to renormalization of the equivalence relations lie in the first cohomology of M with coefficients in the power series infinitesmal isometries of the metric g^r.

When the fundamental group $\pi_1(M)$ of M is finite, i.e. when the simply connected covering space of M is compact, then this cohomology space is zero. $u^r(m)$ is constructed on the covering space by integration of $d_1 v^r(m)$. It is then projected down to M by averaging over the fundamental group.

It will be noted in Part II that all known infrared unstable fixed points of the renormalization group equations have $\pi_1(M)$ finite, but that there are infrared stable fixed points for which the cohomology space of possible obstructions to renormalizability of the equivalence relations is nontrivial.

Assuming that the equivalence relations are renormalizable (for example, if $\pi_1(M)$ is finite), there remains the issue of the renormalizability of the group theoretic structure of the equivalence transformations. The transformation properties are renormalized if, for all infinitesmal diffeomorphisms v,

$$[v, g] \frac{\delta g^r}{\delta g} = [v, g^r] \qquad (6.5.12)$$

$$[v, g] \frac{\delta h^r}{\delta g} + [v, h] \frac{\delta h^r}{\delta h} = [v, h^r] \; . \qquad (6.5.13)$$

This is discussed in terms of the renormalization group equations. From (6.5.10-11), the renormalization group coefficients satisfy

$$[v, \beta] \qquad [v, g] \frac{\delta \beta}{\delta g} = [T(v), g] \qquad (6.5.14)$$

$$[v, \tilde{\gamma}] - [v, g] \frac{\delta \tilde{\gamma}}{\delta g} = T(v) \qquad (6.5.15)$$

where T is a linear transformation, depending on g, from vector fields to vector fields. The renormalization is covariant if $T = 0$.

As discussed in section 6.3, the renormalization group coefficients are only defined up to modification of the form

$$(\beta(g), \tilde{Y}(g)) \rightarrow (\beta(g) - [w(g), g], \tilde{Y}(g) - w(g)) , \qquad (6.5.16)$$

where, for each metric g, $w(g)$ is a vector field on M. The problem is to find $w(g)$ so that the modification (6.5.16) eliminates T in (6.5.14-15).

β lies in the tangent space $T_g\tilde{\underline{R}}$ to the space of metrics at g. The vertical subspace $V_g\tilde{\underline{R}}$ consists of tangents k_{ij} of the form $k = [w, g]$. $T_g\tilde{\underline{R}}$ splits, in $\underline{D}$- invariant fashion, into the vertical subspace and a horizontal subspace $H_g\tilde{\underline{R}}$ orthogonal in the inner product on $T_g\tilde{\underline{R}}$:

$$(k, k) = \int d_g^o m \; k_{ij} \; k_{ij} \qquad (6.5.17)$$

where $d_g^o m$ is the metric volume element for g.

Standard elliptic operator theory for *compact manifolds* gives a smooth decomposition of β into horizontal and vertical parts.[16] The freedom to modify β as in (6.5.16) is used to discard the vertical part. Now, since the horizontal subspaces go into each other under diffeomorphisms of M, the expression on the left in (6.5.14) must be a horizontal tangent vector. But the expression on the right is obviously a vertical tangent vector. So both are zero. Thus, by a suitable g-dependent transformation of M, the renormalization of the metric coupling can always be made *covariant*. It also follows that, for all v, $T(v)$ *is* an infinitesmal isometry of g.

The remaining problem is to make covariant the renormalization of the external field; that is, to find $w(g)$ such that, for all v,

$$v \cdot w = T(v) \tag{6.5.18}$$

where $v \cdot w$ is defined by

$$v \cdot w = [v, w] - [v, g] \frac{\delta w}{\delta g} \, . \tag{6.5.19}$$

From (6.5.15) follows the cocycle condition

$$v_1 \cdot T(v_2) - v_2 \cdot T(v_1) - T([v_1, v_2]) = 0 \, . \tag{6.5.20}$$

If $T(v) = 0$ for all infinitesmal isometries v of g, then (6.5.20) becomes the integrability condition for (6.5.18). So the problem reduces to solving (6.5.18) for v restricted to lie in $\underline{i}$, the Lie algebra of infinitesmal isometries of g. For v in $\underline{i}$,

$$v \cdot w = [v, w] \, , \tag{6.5.21}$$

so the obstruction, if it exists, is a cohomology class in the first cohomology of the Lie algebra $\underline{i}$ with coefficients in its adjoint representation.

Since M is compact, $\underline{i}$ is the Lie algebra of a compact group: the direct sum of an abelian Lie algebra $\underline{a}$ and a semi-simple Lie algebra $\underline{k}$. A semi-simple Lie algebra has no nontrivial first cohomology groups, so the cohomologically nontrivial T are all linear maps from $\underline{a}$ to $\underline{i}$. The cocycle condition states that, for v in $\underline{a}$, $T(v)$ must commute with all of $\underline{i}$. Therefore the first cohomology space, which is the space of possible obstructions to covariant renormalization of the external field, is exactly $a \otimes a^*$.

The case M a homogeneous space G/H must be handled somewhat differently. The compact homogeneous spaces are subsumed in the previous case. So the techniques are directed at the noncompact homogeneous spaces. It does not seem feasible to renormalize the metric everywhere on M, because of the lack of control at infinity. In particular, the elliptic operator theory used to obtain the covariance of the β- function is not available. An alternative is the usual treatment of the standard models: fluctuations are examined only at one typical point m_o in M. The arguments are sketched.

The first problem is to show that the homogeneous space structure is preserved under renormalization. In place of the compatibility conditions are the nonlinear symmetries described in section 5.9. These give rise to quadratic identities on the renormalized distribution of linear fields. The solution of the identities is a distribution of fields invariant under a deformation of the nonlinear representation of $\underline{g}$ on the $\sigma^i(x)$.

The question becomes whether there is a nonlinear transformation of the field $\sigma^i(x)$ which undoes the deformation of the representation of $\underline{g}$. The deformation is described by a one- cocycle on $\underline{g}$ with values in the power series vector fields on $T_{m_o}M$. It is removed if it is cohomologically trivial.

Note that $\underline{h}$ acts linearly on $T_{m_o}M$, so the vector fields on $T_{m_o}M$ which represent $\underline{h}$ vanish at the origin. The deformation of $\underline{g}$ need not preserve this property, so gives a linear map T from $\underline{h}$ to $\underline{m}$. The cocycle condition on the deformation becomes a cocycle condition on T, with respect to the isotropy representation of $\underline{h}$ on $\underline{m}$. It can be shown that if T can be eliminated then the entire deformation can be removed. Therefore the possible obstructions are cohomology classes in the first cohomology of $\underline{h}$ with coefficients in $\underline{m}$. It follows from the fact that $\underline{h}$ is the Lie algebra of a compact group H that this cohomology space is $\underline{a}^* \otimes \underline{m}_o$, where $\underline{a}$ is the abelian factor in $\underline{h}$ and $\underline{m}_o$ is the subspace of $\underline{h}$- invariant vectors in $\underline{m}$, i.e. the $\underline{g}$- invariant vector fields on M.

Assuming that the symmetry is preserved under renormalization, it remains to renormalize the residual equivalence transformations $\underline{D}_G$ (see section 2.3). This part of the argument proceeds as in the compact case. The first step is to attempt to renormalize the equivalence relations. The second is to use the geometry of the space $\tilde{\underline{R}}_G$ of G-invariant metrics to make β covariant under the equivalence transformations, and then to attempt to make the external field renormalization also covariant.

The analogue of the local isometry obstruction to renormalization of the equivalence relations is an obstruction in the first cohomology class of $\underline{m}_o$, the Lie algebra of $\underline{L}_G$, with coefficients in the space of all residual infinitesmal equivalences of G- invariant metrics (see section 2.3).

The obstruction to covariant renormalization of the external field turns out to be a cohomology class in the first cohomology of $\underline{m}_{oo}$ with coefficients in its adjoint representation, where $\underline{m}_{oo}$ is the subalgebra of $\underline{m}$ leaving the metric g invariant. Therefore the obstruction is possible if and only if $\underline{m}_{oo}$ has a nontrivial abelian factor.

7. Calculation

7.1. Rules for calculation

This section summarizes effective procedures for manifestly covariant calculation to all orders in the renormalized low temperature expansion, using the results and constructions of the previous sections. The procedures are discussed in less than the full generality those results and constructions allow; the more general procedures are left implicit. Infrared regularization is assumed to be by a constant external field or by finite volume, but details are given mainly for the latter. Metric normal coordinates (section 4.7) are used for the general model and canonical normal coordinates (section 4.8) for the standard models. Details are given mainly for dimensional regularization with renormalization by minimal subtraction.

The object is to calculate the renormalization group coefficients and the renormalized partition function Z^r (6.4.1) order by order in T as a function of the renormalized metric coupling $T^{-1}g^r_{ij}$ and the renormalized external field $h^r(x)$. The functional integral (2.4.4) for the partition function is rewritten in terms of the linear field $\sigma^i(x)$, as in (3.1.6) for constant external field regularization or as in (3.2.10) for finite volume regularization. Both require choice of a system of coordinates. Finite volume regularization also requires choice of a gauge function and an auxiliary linear connection in TM (see section 5.4).

The bare metric coupling is written as the naively rescaled renormalized metric coupling plus counterterms:

$$\Lambda^{\epsilon} T^{-1} g_{ij} = \mu^{\epsilon} T^{-1} (g_{ij}^{(k)} + O(T^{k+1})) \qquad (7.1.1)$$

$$g_{ij}^{(0)} = g_{ij}^{r} . \qquad (7.1.2)$$

Similarly, the bare external field is written as the naively rescaled renormalized external field plus counterterms:

$$\Lambda^{2+\epsilon} h(x) = \mu^{2+\epsilon} [h^{(k)}(x) + O(T^{k+1})] \qquad (7.1.3)$$

$$h^{(0)}(x) = h^{r}(x) . \qquad (7.1.4)$$

The central problem is to find the counterterms as functions of g^r and $h^r(x)$.

The a priori volume element is, to lowest order, the metric volume element. Both bare and renormalized a priori volume elements are to be adjusted, if necessary, at each order in T to ensure that the external fields are renormalized homogeneously (see section 6.4). Dimensional regularization eliminates all quadratic divergences, i.e. divergences proportional to $\Lambda^{2+\epsilon}$ vanish in the continuum limit for $\epsilon < -2$. So the a priori volume element is not an issue; in fact it plays no role in the calculations. The external field is renormalized homogeneously as long as no finite inhomogeneous counterterms are added to the external field. Minimal subtraction, in particular, allows no finite counterterms at all.

For the general model the coordinates are taken to be metric normal coordinates for the renormalized metric. For the standard models canonical normal coordinates are used. The gauge function is, for simplicity of calculation, (3.2.5,6). The auxiliary linear connection is the Levi-Civita connection for the renormalized metric in the general case and the canonical connection (section 4.8) in the standard case.

The coordinates, gauge function and auxiliary connection are held fixed as the low temperature expansion is renormalized. As a consequence, the Fadeev-Popov matrix $\tilde{f}^i_j(m,\sigma(x))$ (section 5.7) also stays fixed. By (5.7.6), (5.3.3) and (5.4.2),

$$\tilde{f}^i_j(m,\sigma(x)) = (Q^r)^i_j(m,\sigma(x)) \qquad (7.1.5)$$

where Q^r is the matrix valued function defined in (4.3.9) and given in normal coordinates by (4.5.6), (4.7.5) and (4.8.15).

As each order in perturbation theory for the functional integral (3.1.6) or (3.2.10) over the linear field is renormalized, the system of coordinates and, possibly, the gauge function also undergoes renormalization (see section 6.4). But the renormalized compatibility conditions (section 6.4) guarantee that the collection of counterterms for (3.1.6) or (3.2.10) can be reduced to counterterms for the functional integral

(2.4.4) over the nonlinear field, by using the renormalized system of coordinates. A new set of counterterms for (3.1.6) or (3.2.10) can then be constructed by returning to the original system of coordinates and gauge function.

The couplings for the vertices of the Feynman diagram representation of the functional integral (3.1.6) or (3.2.10) are provided by the Taylor series expansions in coordinates of the renormalized metric plus counterterms (7.1.1) and of the renormalized external field plus counterterms (7.1.3), by the expansion of the logarithmic jacobian $\tilde{j}(m,v)$ (absent for dimensional regularization), and by the expansion of $\tilde{f}^i_j(m,v)$ (in finite volume). Formulas (4.5.6), (4.6.4), (4.4.9) and (4.6.10–11) give these expansions in terms of the expansion of one quantity, the vielbein $V^i_j(m,v)$, which is calculated recursively using (4.7.2) or (4.8.12).

The gauge function (3.2.5–6) produces an especially simple propagator for $\sigma^i(x)$. From (6.3.9), the propagator is

$$G^{ij}(x,y) = T\,((g^r)^{-1})^{ij}\, G_o(x,y) \qquad (7.1.6)$$

where G_o is the finite volume (real space) propagator for a scalar field in which the constant fields have been projected out. The volume element $\prod_x d\sigma(x)$ restricted to the zero modes is the metric volume element, so integration over the multiplier γ using the metric volume element for $d\gamma$ leaves the propagator (7.1.6) and gives $Z^r(m)$ equal to the metric volume element dm times the Feynman diagram expansion generated by the action $\hat{\Lambda}$.

Since the techniques of calculation are all manifestly covariant, it is only necessary to calculate at a single point m in an arbitrary n- dimensional manifold M. The results of the calculation, which are expressions in the curvature tensor and other covariant objects, are immediately transferable to all points in any n- dimensional manifold. (It seems to be the case that with normal coordinates the dimension n never appears explicitly in calculation, in particular never appears explicitly in the renormalization group coefficients.)

The calculation procedure is now described recursively. Z^r is assumed known to order T^k. To order T^0,

$$Z^r = \int dm \ \exp [\mu^{2+\epsilon} \int dx \ h^r(x) \ (m)] \ . \tag{7.1.7}$$

To calculate Z^r to $O(T^k)$ requires knowledge of $g_{ij}^{(k)}$, $h^{(k)}(x)$ and the expansion of $V_j^i(m,v)$ to $O(v^{2k})$. In order to calculate Z^r to $O(T^{k+1})$ using (3.1.6) or (3.2.10) it is necessary to find V to $O(v^{2k+2})$, which is easily done using (4.7.2) or (4.8.12), and to find the $O(T^{k+1})$ counterterms

$$g_{ij}^{[k]} = g_{ij}^{(k+1)} - g_{ij}^{(k)} \tag{7.1.8}$$

$$h^{[k]}(x) = h^{(k+1)}(x) - h^{(k)}(x) \ . \tag{7.1.9}$$

The *first* step is to calculate counterterms for the divergences in the Feynman diagrams generated by the action (3.1.5) or (3.2.9) for the linear field σ. The metric and external field in coordinates are written as the rescaled renormalized metric and external field plus counterterms:

$$\Lambda^{\epsilon} T^{-1} \tilde{g}_{ij} = \mu^{\epsilon} T^{-1} [\tilde{g}_{ij}^{(k)} + O(T^{k+1})] \qquad (7.1.10)$$

$$\Lambda^{2+\epsilon} \tilde{h}(x) = \mu^{2+\epsilon} [\tilde{h}^{(k)}(x) + O(T^{k+1})] \qquad (7.1.11)$$

where $\tilde{g}^{(k)}$ and $\tilde{h}^{(k)}(x)$ are the expressions in coordinates of $g^{(k)}$ and $h^{(k)}(x)$.

Superficially it appears as if counterterms are needed at $O(T^{k+1})$ for all of the vertices given by the expansions of $\tilde{S}(m,\sigma)$ and $\tilde{H}(m,\sigma)$ in powers of σ. But the renormalized compatibility conditions (6.3.20–22) imply that only enough counterterms need be calculated to determine $g^{[k+1]}$ and $h^{[k+1]}(x)$. For this it is enough to calculate counterterms for the two point vertex

$$\int dx \;\; \tilde{g}^{r}_{ij}(m,0)\, \partial_\mu \sigma^i(x)\, \partial_\mu \sigma^j(x) \qquad (7.1.12)$$

and the zero point vertex

$$\int dx \;\; \tilde{h}^{r}(x)\,(m,0) \;. \qquad (7.1.13)$$

The counterterm for (7.1.12) is minus the part of the primitive divergence in the one particle irreducible two point function of $\sigma^i(x)$ which is quadratic in the external momentum. By the induction assumption the divergences are $O(T^{k+1})$ relative to the lpwest order contribution. With dimensional regularization there are no quadratic divergences, so on dimensional grounds the only divergence in the two point function must be quadratic in the external momentum. The diagrams which provide the primitive divergence have two external legs and contain only vertices from the expansion of $\tilde{S}$ (section 6.3), so the coefficient of the divergence is a symmetric tensor at m formed from the Taylor series coefficients of the metric coupling at m, which in turn are formed from the metric, its curvature tensor and the covariant derivatives of the curvature, all at m. The corresponding counterterm for $\tilde{g}^r_{ij}(m,0)$ is written $\tilde{g}^{<k>}_{ij}(m,0)$

The counterterm for (7.1.12) is minus the primitive divergence in the one particle irreducible zero point function. Again, by the induction assumption this is $O(T^{k+1})$. On dimensional grounds the primitive divergence is proportional either to a Taylor series coefficient of the external field at m or to $\Lambda^{2+\epsilon}$. The latter does not occur with dimensional regularization.

The diagrams which provide the part of the primitive divergence proportional to the external field have no external legs and one vertex from the expansion of $\tilde{H}$; the rest of the vertices are from the expansion of $\tilde{S}$. The coefficient of this part of the primitive divergence is a real number formed by contracting a covariant derivative of $h^r(x)$ at m with the metric coupling, its curvature tensor and covariant derivatives of the curvature, all at m.

The diagrams which give the quadratic primitive divergences have no external legs and vertices all from the expansion of $\tilde{S}$ except for at most one from the expansion of the logarithmic jacobian $\tilde{J}$. The coefficient of this divergence is a real number formed from the metric, curvature and covariant derivatives of the curvature at m.

The counterterm for $\tilde{h}^r(x)\ (m,0)$ is written $\tilde{h}^{<k>}(x)\ (m,0)$. By the above discussion,

$$\tilde{h}^{<k>}(x)\ (m,0) = [\underline{\tilde{z}}^{<k>}\ \tilde{h}^r(x)]\ (m,0) + \tilde{h}_1^{<k>}(m)\ . \qquad (7.1.14)$$

$\underline{\tilde{z}}^{<k>}$ is a differential operator in $\sigma^i(x)$ (independent of x) with constant coefficients formed from the metric, its curvature tensor and the covariant derivatives of the curvature at m. $\tilde{h}_1^{<k>}(m)$ is a real number formed from the metric, curvature and covariant derivatives of curvature at m. With dimensional regularization it is absent.

By (4.4.9) and (4.6.3), the counterterms $\tilde{g}_{ij}^{<k>}(m,0)$ and $\tilde{h}^{<k>}(x)\ (m,0)$ for the linear fields determine the counterterms $g_{ij}^{[k]}(m)$ and $h^{[k]}(x)\ (m)$ for the nonlinear fields:

$$h^{[k]}(x)\ (m) = \tilde{h}^{<k>}(x)\ (m,0) \qquad (7.1.15)$$

and

$$g_{ij}^{(k+1)}(m) = Q_i^p(m,0)\ [\tilde{g}_{pq}^{(k)}(m,0) + \tilde{g}_{pq}^{<k>}(m,0)]\ Q_j^q(m,0)\ , \qquad (7.1.16)$$

where, by the renormalized compatibility condition (6.3.21),

$$Q_i^p(m,0) = (Q^r)_i^p(m,0) + \tilde{\underline{z}}^{<k>}(Q^r)_i^p\ (m,0)\ . \qquad (7.1.17)$$

So

$$g_{ij}^{[k]}(m) = \tilde{g}_{ij}^{<k>}(m,0) + \tilde{\underline{z}}^{<k>}(Q^r)_i^p(m,0)\ g_{pj}^r(m)$$

$$+ g_{ip}^r(m)\ \tilde{\underline{z}}^{<k>}(Q^r)_j^p\ (m,0)\ . \qquad (7.1.18)$$

Since the counterterms for the external field are linear in the renormalized external field they can be written

$$h^{(k)}(x) = Z_h^{(k)}\ h^r(x) \qquad (7.1.19)$$

$$Z_h^{(k+1)} = Z_h^{(k)} + Z_h^{[k]} \qquad (7.1.20)$$

$$Z_h^{(0)} = 1 \qquad (7.1.21)$$

Note that (7.1.19) differs from (6.3.37) in that here Z_h does not contain the rescaling factor $(\mu^{-1}\Lambda)^{-2-\epsilon}$. $Z_h^{(k)}$ is a linear differential operator, natural in the renormalized metric, of order at most 2k. Homogeneous renormalization, if necessary by adjustment of the a priori volume element, has been assumed. By (7.1.14-15), $Z_h^{[k]}$ is given by

$$Z_h^{[k]}\ h^r(x)\ (m) = \underline{\tilde{z}}^{<k>}\ \tilde{h}^r(x)\ (m,0)\ . \qquad (7.1.22)$$

So, since the Taylor series coefficients of a function in normal coordinates are the covariant derivatives, $Z_h^{[k]}$ at m is $\underline{\tilde{z}}^{<k>}$ at m with covariant derivatives substituted for partial derivatives.

Once the counterterms $g^{[k]}$ and $h^{[k]}(x)$ are known, the renormalization group coefficients can be calculated to order T^{k+1} (relative to the lowest order contributions). For cutoff forms of ultraviolet regularization the formulas were given in section 6.4. They are presented here for dimensional regularization with minimal subtraction. The renormalization group coefficients give the change in g^r and $h^r(x)$ needed to keep $\mu^{\epsilon}\ g^{(k)}$ and $\mu^{2+\epsilon}\ h^{(k)}(x)$ fixed when making an infinitesimal logarithmic variation of μ:

$$\beta^{(k)} = -\left(\frac{\partial g^{(k)}}{\partial g}\right)^{-1} \epsilon\ g^{(k)} \qquad (7.1.23)$$

$$\tilde{\gamma}^{(k)} = -(2+\epsilon) - (Z_h^{(k)})^{-1}\ \beta \frac{\partial}{\partial g}\ Z_h^{(k)}\ . \qquad (7.1.24)$$

The r- superscripts have been suppressed because the renormalization of the bare parameters is of no interest.

The finiteness of the renormalization group coefficients at $\epsilon = 0$ implies that they are determined by the the simple poles in ϵ of the counterterms:

$$\beta^{(k)} = -\epsilon g + g \frac{\partial}{\partial g} g^{(k,1)} - g^{(k,1)} \qquad (7.1.25)$$

$$\tilde{\gamma}^{(k)} = -(2+\epsilon) + g \frac{\partial}{\partial g} Z_h^{(k,1)} . \qquad (7.1.26)$$

The residues of $g^{(k)}$, $Z_h^{(k)}$, $g^{[k]}$ and $Z_h^{[k]}$ are written $g^{(k,1)}$, $Z_h^{(k,1)}$, $g^{[k,1]}$ and $Z_h^{[k,1]}$.

Once the renormalization group coefficients are known to $O(T^{k+1})$, the full counterterms at $O(T^{k+1})$ are determined by (7.1.23-24). (See, for example, [20].) Therefore it is only necessary to find the primitive divergences which are simple poles in ϵ in order to find the full set of counterterms. With these counterterms, the Feynman diagram expansion of (3.1.6) or (3.2.10) gives the partition function Z^r finite to order T^{k+1}.

7.2. Renormalization group coefficients

This section summarizes the application of the procedures of 7.1 to the calculation of the renormalization group coefficients in the two loop approximation, using dimensional regularization and minimal subtraction. Metric normal coordinates define the linear fields. To eliminate factors of 2π, T is replaced by $2\pi T$. The r- superscripts are suppressed.

The primitive divergences in one loop diagrams give

$$T^{-1}\tilde{g}_{ij}^{<0>}(m,0) = -\frac{1}{3\pi} R_{ij}(m) \tag{7.2.1}$$

$$\tilde{h}^{<0>}(x)\ (m,0) = \frac{1}{2\pi} T\ (g^{-1})^{ij}\nabla_i\nabla_j h(x)\ (m)\ . \tag{7.2.2}$$

The operator $\underline{\tilde{z}}^{<0>}$ is therefore

$$\underline{\tilde{z}}^{<0>} = \frac{1}{2\pi} T\ (g^{-1})^{ij}\ \partial_i\partial_j \tag{7.2.3}$$

and

$$z_h^{(0)} = 1 + \frac{1}{2\pi} T\ (g^{-1})^{ij}\nabla_i\nabla_j h(x)\ (m)\ . \tag{7.2.4}$$

Using the expansion (4.7.5) for Q^i_j,

$$\underline{\tilde{z}}^{<0>}(Q)^i_j(m,0) = -\frac{1}{3\pi} T\ (g^{-1})^{ip}\ R^j_p(m)\ . \tag{7.2.5}$$

So, by (7.1.18),

$$T^{-1}g_{ij}^{[0]} = -\frac{1}{\pi} R_{ij}\ . \tag{7.2.6}$$

By (7.1.25-26), the one loop renormalization group coefficients are

$$\beta_{ij}^{(1)}(T^{-1}g) = -\epsilon T^{-1} g_{ij} + R_{ij} \tag{7.2.7}$$

$$\gamma^{(1)}(T^{-1}g) = -(2+\epsilon) - \frac{1}{2} T v_i v_i \; . \tag{7.2.8}$$

In the next order it is easily seen that there are no simple poles in the counterterms for (7.1.13), so

$$Z_h^{[1,1]} = 0 \tag{7.2.9}$$

and

$$g^{[1,1]}(m) = \tilde{g}^{<1,1>}(m,0) \tag{7.2.10}$$

where $\tilde{g}^{<1,1>}(m,0)$ is the residue of $\tilde{g}^{<1>}(m,0)$ at $\epsilon = 0$.

Calculation of the simple poles in the two loop Feynman diagrams for the two point function gives

$$T^{-1}\tilde{g}_{ij}^{<1,1>}(m,0) = -\frac{1}{6} T^2 (R_{ipqr} + R_{iqpr}) R_{jpqr} (m) \tag{7.2.11}$$

where contraction is with g_{ij}. The first Bianchi identity, (5.8.15) with $T = 0$, implies

$$R_{iqpr} R_{jpqr} = \frac{1}{2} R_{ipqr} R_{ipqr} . \tag{7.2.12}$$

Therefore

$$T^{-1} \tilde{g}_{ij}^{<1,1>}(m,0) = -\frac{1}{4} T^2 R_{ipqr} R_{jpqr} (m) . \tag{7.2.13}$$

By (7.2.10),

$$T^{-1} g_{ij}^{[1,1]} = -\frac{1}{4} T^2 R_{ipqr} R_{jpqr} (m) . \tag{7.2.14}$$

From (7.2.9), (7.2.14) and (7.1.25-26),

$$\beta_{ij}^{(2)} = \beta_{ij}^{(1)} + \frac{1}{2} T R_{ipqr} R_{jpqr} (m) . \tag{7.2.15}$$

$$\gamma^{(2)} = \gamma^{(1)} . \tag{7.2.16}$$

The two loop results are therefore (1.6,8).

Algebraic equations for the two loop β- function are obtained from (1.6) by expressing the metric curvature $R^p{}_{ijk}$ in terms of the metric g_{ij} and the canonical curvature and torsion, here written $\hat{R}$ and $\hat{T}$, and given in (4.8.6-7).

$$R^p{}_{ijk} = \hat{R}^p{}_{ijk} + \hat{T}^q_{jk} \Gamma^p_{qi} + \Gamma^p_{qj} \Gamma^q_{ik} - \Gamma^p_{qk} \Gamma^q_{ij} \tag{7.2.17}$$

where

$$\Gamma^{i}_{jk} = \frac{1}{2}\,\hat{T}^{i}_{jk} + \frac{1}{2}\,(g^{-1})^{ip}\,(g_{jq}\,\hat{T}^{q}_{kp} + g_{kq}\,\hat{T}^{q}_{jp})\ . \qquad (7.2.18)$$

When M is a locally symmetric space G/H, these equations reduce to

$$\beta_{ij}(T^{-1}g) = (-\epsilon + \frac{a}{2}\,T + c_1\,T^2)\,T^{-1}\,g_{ij} \qquad (7.2.19)$$

$$c_1 = \frac{n^2 + 4\,(\dim A)\,(\dim H - n)}{8\,n\,(\dim H - \dim A)}\ . \qquad (7.2.20)$$

a is +1 or -1 depending on whether G is compact or noncompact; n is the dimension of M; and A is the abelian factor in H. For $M = S^n = SO(n+1)/SO(n)$ this reproduces the result of [10].

II. The Renormalization Group Equation

1. Introduction

This is the second part of a study of the general nonlinear scalar model near two dimensions. In the first part, the model was described and its low temperature expansion renormalized. This part is an investigation of the topological properties of the renormalization group equations near zero temperature.

The fields $\phi(x)$ of the model are functions from $2+\epsilon$ dimensional Euclidean space to a finite dimensional differentiable manifold M. The coupling of the model is given by a (positive definite) Riemannian metric g_{ij} on M. The action is the energy integral

$$S(\phi) = \int dx \ \tfrac{1}{2}T^{-1} g_{ij} \ \partial_\mu \phi^i(x) \ \partial_\mu \phi^j(x) \ . \qquad (1.1)$$

The renormalization group is a one parameter group of transformations of the space $\underline{\tilde{R}}$ of Riemannian metrics on M, which describes the change in the effective coupling as the scale of distance in Euclidean space is increased. Finding its orbit picture is the crucial first step towards understanding the model. The large distance properties of the model are determined by the the long time properties of the renormalization group, the simplest of which are seen at attractive fixed points. Critical behavior (and the possibility of defining a continuum limit) is associated with instability in the long time properties, the simplest form of which is seen near fixed points with nontrivial unstable manifolds.

The infinitesmal form of the renormalization group is the renormalization group equation

$$\frac{d}{dt} g_{ij} = - \beta_{ij}(g) \tag{1.2}$$

where β is a vector field on $\underline{\tilde{R}}$, called the β- function. The tangent vector $\beta(g)$ to the space of metrics at g is the symmetric tensor field $\beta_{ij}(g)$ on M.

$\beta(T^{-1}g)$ is calculated by techniques of perturbative quantum field theory as an asymptotic expansion in the positive real number T, called the temperature. Effective algorithms for calculation were derived in Part I. To the second nontrivial order, the result is

$$\beta_{ij}(T^{-1}g) = - \epsilon T^{-1} g_{ij} + R_{ij} + \frac{1}{2} T R^2_{ij} + O(T^2) \tag{1.3}$$

$R^p{}_{ijk}$ is the curvature tensor of the metric g_{ij}, $R_{ij} = R^k{}_{ikj}$ is the Ricci tensor, and R^2_{ij} stands for $R_{ipqr}R_{jpqr}$.

Results on the properties of the β- function can be interpreted when $\epsilon = 0$ and also in the fictitious regime $\epsilon \simeq 0$. The latter is used as an approximation for the case $\epsilon = 1$, which cannot be studied directly by perturbative techniques. The interesting properties of β are those which depend smoothly on ϵ and whose quantitative features can be expanded as asymptotic series in ϵ.

β is, to all orders in T, a natural function of the metric, in the sense that

$$\beta(\Psi_* g) = \Psi_* \beta(g) \tag{1.4}$$

for all diffeomorphisms Ψ of M. More concretely, β is, to every finite order, a polynomial in the curvature tensor and its covariant derivatives. It follows that the renormalization group preserves symmetry.

Of special interest are the models in which M is a homogeneous space G/H, H compact, and g_{ij} lies in the space $\underline{\tilde{R}}_G$ of G-invariant metrics on M. Since it preserves symmetry, the renormalization group acts on $\underline{R}_G$.

The renormalization group commutes with the diffeomorphism group $\underline{D}$ acting on $\underline{\tilde{R}}$, so it acts on the space $\underline{R} = \underline{\tilde{R}}/\underline{D}$ of equivalence classes. In fact, only the equivalence classes are meaningful in the physical interpretation, so β is defined only up to replacements

$$\beta(g) \rightarrow \beta(g) - [v(g), g] , \tag{1.5}$$

where $v(g)$ is, for each metric g, a vector field on M defined up to infinitesmal isometries (Killing fields) of g, and satisfying

$$v(\Psi_* g) = \Psi_* v(g) \tag{1.6}$$

(up to Killing fields) for all diffeomorphisms Ψ.

When M is a homogeneous space G/H, a natural analogue of $\underline{D}$ is $\underline{D}_G$, the group of diffeomorphisms of M which commute with G. $\underline{D}_G$ acts on $\tilde{\underline{R}}_G$, the equivalence classes being $\underline{R}_G$. The vector fields $v(g)$ occurring in (1.5-6) are taken to be G- invariant, i.e. in the Lie algebra of $\underline{D}_G$. It is possible, at least when G is not semi-simple, that there exist equivalence relations between G- invariant metrics that are due to diffeomorphisms of M which are not in $\underline{D}$. This possibility is systematically ignored. In situations where it is realized, the treatment here will be incomplete.

The result (1.3) was calculated using dimensional regularization with minimal subtraction. There are other ways to calculate, none preferred. The β- functions calculated by different techniques are related by conjugations under transformations of $\tilde{\underline{R}}$ which commute with $\underline{D}$, in the absence of the obstructions discussed in section 6.5 of Part I. The properties of the renormalization group which are of interest are the invariants under these transformations.

The manifold M is assumed to be compact and/or homogeneous. General statements below are implicitly qualified by this assumption. If M is a noncompact homogeneous space, then it is assumed also to be unimodular, meaning that all invariant vector fields on M preserve any invariant volume element. Unimodularity must be assumed because almost all of the description of β presented here requires for its justification integration by parts on M. When M is compact, the integral of a function f over M with respect to the metric volume element is written $\int f$. The integration by parts formula states that $\int \nabla_i v^i = 0$ for any vector field v^i on M, where ∇_i is the covariant derivative associated with the metric. When M is homogeneous, any invariant function f on M must be constant. $\int f$ will mean simply the value of f at any point. Unimodularity is exactly the condition needed to give the integration by parts formula $\int \nabla_i v^i = 0$ for all invariant vector fields.

The space $\underline{R}$ is described in [1]. A reference for basic differential geometry is [2]. Tensor analysis is done here in index notation. For an explanation, see [3]. The aim of this presentation is to survey the accessible general features of the problem as an aid to further exploration. The issue of the existence of the renormalization group flow is not dealt with. Details are for the most part left to the reader.

The organization of Part II is as follows. Section 2 derives the fixed point equations, describes the known solutions and begins the discussion of topological properties near fixed points in terms of the linearization of β. Section 3 examines the special case M a homogeneous space; section 4 the special case M a two dimensional manifold. In both cases the β-function is shown to be a gradient. Section 5 continues the discussion of the fixed points, based on Bochner estimates for the linearization of the β- function.

2. Fixed Points (I)

2.1. The fixed point equation

The renormalization group equation is, following (1.4-5),

$$\frac{d}{dt} g = - \beta(g) + [v(g), g] \tag{2.1.1}$$

where $v(g)$ is an arbitrary vector field on M (satisfying (1.5)). The fixed points of the renormalization group are (the equivalence classes of) the metrics g_{ij} at which β is tangent to the orbit of the diffeomorphism group:

$$0 = - \beta(g) + [v(g), g] \tag{2.1.2}$$

for some vector field $v(g)$ on M. The perturbative expansion (1.3) for $\beta(T^{-1}g)$ can only be used reliably to locate fixed points at $T \simeq 0$.

A simple way to exhibit the small T structure of the β- function is to parametrize the general metric on M in the form $T^{-1}g_{ij}$ where g_{ij} is a metric of some fixed total volume (or, in the noncompact homogeneous case, of some fixed invariant volume element). The renormalization group equation (2.1.1) becomes

$$\frac{d}{dt} T = - \epsilon T + < \frac{1}{n} R > T^2 + < \frac{1}{2n} R_{ij}^2 > T^3 + O(T^4) \tag{2.1.3}$$

$$\frac{d}{dt} g_{ij} = - T \, (R_{ij} - < \frac{1}{n} R > g_{ij} - [v_o(g), g]_{ij}) \tag{2.1.4}$$

$$- T^2 \, (\frac{1}{2} R^2_{ij} - < \frac{1}{2n} R^2_{kk} > g_{ij} - [v_1(g), g]_{ij}) \; + \; O(T^3)$$

where $R = R_{kk}$ is the scalar curvature, R^2_{ij} stands for $R_{iklm}R_{jklm}$, n is the dimension of M, $v(T^{-1}g)$ is expanded as $v_o(g) + T\, v_1(g) + O(T^2)$, and $< f >$ is defined as $\int f \,/ \int 1$, the integrals taken with respect to the metric volume element for g_{ij}.

Of interest are the metrics left fixed by the renormalization group when $\leftarrow = 0$, and also those left fixed when $\leftarrow \simeq 0$ which depend smoothly on $\leftarrow$ and approach $\leftarrow = 0$ fixed points as $\leftarrow \rightarrow 0$. By (2.13), the $\leftarrow = 0$ fixed points are at $T = 0$. The metrics $T^{-1}g$ at $T = 0$ form a part of the boundary of the space of all nondegenerate metrics. Every metric at $T = 0$ is a fixed point, but not all are limits of renormalization group trajectories. At issue is the behavior of the renormalization group flow near the $T = 0$ metrics.

If $\frac{dT}{dt}$ were $O(T)$, then the orbits of the renormalization group would approach the $T = 0$ surface transversally at each g_{ij}, so that each point on the $T = 0$ surface would be a true fixed point. But when $\frac{dT}{dt} = O(T^2)$, as is the case in (2.1.3), the situation is quite different. This can be seen heuristically in an analogous equation in two variables:

$$\frac{d}{dt} (T,x) = (aT^2 + O(T^3), \, f(x)T + O(T^2)) \; . \tag{2.1.5}$$

The integral curves (T,x) of (2.1.5) are given for small T by

$$\log (T\, T_o^{-1}) = a \int_{x_o}^{x} \frac{dy}{f(y)} . \qquad (2.1.6)$$

As $T \to 0$ (or becomes $\gg T_o$) x is driven either to infinity or to a zero of f.

The true fixed points at $T = 0$ are the metrics at which $\frac{d}{dt} g_{ij} = O(T^2)$. They are the solutions of the nonlinear differential equation

$$R_{ij} - \langle \frac{1}{n} R \rangle g_{ij} - [v_o, g]_{ij} = 0 \qquad (2.1.7)$$

for some vector field v_o. Explicitly excluded from consideration here are fixed points "at ∞," the analogues of $T = 0$, $x = \infty$ in (2.1.5). (But see the example in section 3.3 below.) A fixed point at ∞ is one which lies on the boundary of the $T = 0$ metrics.

A solution of the fixed point equation (2.1.7) which has $\langle R \rangle \neq 0$ can be seen to survive as a fixed point whatever the higher order corrections to the β- function. But if $\langle R \rangle = 0$ then $\frac{dT}{dt} = O(T^3)$ and the existence of a fixed point depends on the possibility of eliminating the $O(T^2)$ contribution to $\frac{d}{dt} g_{ij}$ in (2.1.4) by perturbing the solution of (2.1.7) by an amount $o(1)$, so that the $O(T)$ term in (2.1.4) cancels the $O(T^2)$ term. This will be possible unless the $\langle R \rangle = 0$ solution of the fixed point equation lies in a manifold of solutions, so that perturbing along the manifold has no effect on the $O(T)$ term in (2.1.4). In this case, the $O(T^2)$ contribution in (2.1.4) projects onto the manifold of $\langle R \rangle = 0$ solutions to (2.1.7). The vanishing of the projection is an auxiliary nonlinear equation which must be satisfied by the fixed point metric.

A certain amount is to be learned about this problem by examining the linearization of the β- function at an $\langle R \rangle = 0$ solution. In particular, it is learned that the manifold of solutions is finite dimensional, so that the auxiliary fixed point equation is finite. The linearization for $\langle R \rangle = 0$ solutions is discussed in section 2.4 and again in 5.2-3.

The $\langle R \rangle = 0$ solutions face no more conditions beyond the auxiliary fixed point equations. If $\frac{dT}{dt}$ were $O(T^4)$ then $\langle R_{kk}^2 \rangle$ would vanish, implying flatness: $R_{ijkl} = 0$. Flat metrics are uninteresting from the point of view of perturbative renormalization and are henceforth ignored.

The topological properties of the renormalization group flow near a fixed point are examined in sections 2.4 and 5 using the linearization of the β- function. The behavior in the T- direction, however, can be seen immediately in (2.1.3). Near a solution of the fixed point equation (2.1.7) the behavior of the temperature is described qualitatively by

$$\frac{d}{dt} T = \begin{cases} -\epsilon T + \langle \frac{1}{n} R \rangle T^2 & \langle R \rangle \neq 0 \\ -\epsilon T + \langle \frac{1}{2n} R_{kk}^2 \rangle T^3 & \langle R \rangle = 0 \; . \end{cases} \qquad (2.1.8)$$

When $\epsilon = 0$, $\frac{dT}{dt}$ vanishes at $T = 0$ to first or second order. This is called aysmptotic freedom. It is ultraviolet asymptotic freedom when $\langle R \rangle \geq 0$ because at short distances ($t \to -\infty$) the effective temperature (slowly) approaches zero. The $T = 0$ models are free (gaussian) field theories. When $\langle R \rangle = 0$ the approach to freedom is especially slow. The fixed points with $\langle R \rangle < 0$ are asymptotically free in the infrared, the effective temperature approaching zero at long distances.

The $\epsilon \neq 0$ fixed points are exhibited also in (2.1.8). Besides the $T = 0$ fixed points (whose infrared stability or instability is determined by the sign of ϵ) there are fixed points: (1) at $T = O(\epsilon)$, unstable in the T- direction, when $\langle R \rangle$ and ϵ are positive; (2) at $T = O(|\epsilon|)$, stable in the T- direction, when $\langle R \rangle$ and ϵ are negative; and (3) at $T = O(\epsilon^{1/2})$, unstable in the T- direction, when ϵ is positive and $\langle R \rangle = 0$. For all three types of fixed point, the existence of a nontrivial manifold of solutions of the fixed point equation (2.1.7) gives rise to auxiliary fixed point equations, because the projection onto the solution set of the higher order contributions to the β- function are now $O(\epsilon^k)$ instead of $O(T^k)$, T asymptotically small, and have nontrivial effect as long as any degeneracy in the solution set remains.

2.2. Solutions of the fixed point equation

In this subsection are collected some general results on properties of the solutions of the fixed point equation (2.1.7) and general descriptions of the known examples. First, the fixed point equation is rewritten so that the solutions are normalized by curvature instead of volume:

$$R_{ij} - a\, g_{ij} = \nabla_i v_j + \nabla_j v_i \,, \quad a = \pm 1, \text{ or } 0 \,. \tag{2.2.1}$$

When $a \neq 0$, the resulting normalization is $< \frac{1}{n} R > = a$. When $a = 0$, $<R> = 0$ provides no normalization, so the additional assumption $< \frac{1}{2n} R_{kk}^2 > = 1$ is made to normalize the metric.

The solutions to

$$R_{ij} - a\, g_{ij} = 0 \tag{2.2.2}$$

are called Einstein metrics. They have long been of interest in geometry and general relativity. (See, for example, [4-10].) The solutions to

$$R_{ij} - a\, g_{ij} = \nabla_i v_j + \nabla_j v_i \neq 0 \tag{2.2.3}$$

might be called quasi-Einstein metrics. v^i is determined only up to addition of infinitesmal isometriea, or Killing fields, which are the

w^i satisfying $\nabla_i w_j + \nabla_j w_i = 0$. v^i can be fixed by requiring $\int v^i w^i = 0$ for all Killing fields w^i. It then follows immediately that v^i is invariant under the isometry group of g_{ij}.

A vector field w^i is a Killing field if and only if, for all vector fields u^i, including w^i itself,

$$0 = \frac{1}{2} \int (\nabla_i u_j + \nabla_j u_i)(\nabla_i w_j + \nabla_j w_i)$$

$$= \int u^i (- \nabla_j \nabla_j w_i - R_{ij} w^j - \nabla_i \nabla_j w_j) .$$

But $\nabla_j w_j = 0$, so the Killing fields are exactly the solutions to

$$0 = - \nabla_j \nabla_j w_i - R_{ij} w^j \qquad (2.2.4)$$

which also satisfy $0 = \nabla_i w^i$.

The next five propositions are concerned with the properties of a quasi-Einstein metric, i.e. a metric g_{ij} together with a vector field v^i satisfying (2.2.3).

Proposition 2.2.1.

v^i satisfies (2.2.4) but $\nabla_i v^i = \frac{1}{2}(R - n) \neq 0$.

Proposition 2.2.2.

The scalar curvature R is not constant.

Proposition 2.2.3.

M is not homogeneous.

<u>Proposition 2.2.4.</u>

The constant a can only be $+1$ and the scalar curvature satisfies $R \geq 0$.[10]

<u>Proposition 2.2.5.</u>

If $R \geq n - 2$ then the first betti number vanishes ($H^1(M) = 0$).

A theorem of Myers[11] gives the fundamental result on Einstein metrics with $a = 1$:

<u>Theorem 2.2.6.</u>

If $R_{ij} - g_{ij} = 0$ then M is compact.

Since the universal covering space of M has the same local geometry as M, it also is compact. Therefore the fundamental group $\pi_1(M)$ is finite.

Einstein metrics with $a = 0$ (Ricci-flat metrics) which have $b_1 \neq 0$ are locally the product of a b_1 dimensional flat manifold and an $n - b_1$ dimensional Ricci-flat manifold.[12] The metrics on the factors can be scaled independently without loss of Ricci-flatness, so there is always a degenerate solution set of the fixed point equation. When the two loop term in the β-function is taken into consideration, the ratio of the scales of the two factors is seen to diverge under the renormalization group. This is an example in which the auxiliary fixed point equation has no solution. Therefore only the Ricci-flat metrics with $b_1 = 0$ are of interest. From equation (2.2.4) it is apparent that on such manifolds all Killing fields are covariant constant, so give also harmonic one forms, so, since $b_1 = 0$, must be identically zero.

The theorem of Cheeger-Gromell[13] implies that, for $a = 0$ Einstein manifolds, $\pi_1(M)$ is finite. Aleksevskii and Kimel-Feld[14] have shown that the only homogeneous $a = 0$ Einstein manifolds are the flat ones.

Einstein metrics with $a = -1$ have no infinitesmal isometries, because (2.2.4) cannot be solved. (Note that for $M = G/H$ noncompact homogeneous this is the statement that there are no G- invariant Killing fields.)

In summary, the types of fixed point metric are, with references to the known examples:

<u>The</u> (-) <u>type</u> <u>Einstein</u> <u>metrics</u>: $R_{ij} + g_{ij} = 0$.

These are noncompact homogeneous spaces or compact manifolds without Killing fields. The corresponding fixed points are stable in the T- direction (asymptotically free at long distances).

The known noncompact homogeneous examples are the symmetric spaces of noncompact type (see [15]). The known compact examples are: (1) the compact locally symmetric spaces of noncompact type (which are noncompact symmetric spaces modulo discrete groups of isometries), and (2) the

Kahler-Einstein metrics of Yau[16,17]. The latter exist on all Kahler manifolds with negative first Chern class.

<u>The</u> (<u>0</u>) <u>type</u> <u>Einstein</u> (<u>Ricci-flat</u>) <u>metrics</u>: $R_{ij} = 0$.

The constraint $b_1 = 0$ is imposed. M is compact without Killing fields and is not homogeneous. The corresponding fixed points are unstable in the T- direction (asymptotically free at short distances, with an especially slow approach to the gaussian limit). If there is a degenerate solution set, an auxiliary fixed point equation on the solution set must be solved.

The only known examples are the Kahler-Einstein metrics of Yau[16,17] on Kahler manifolds with vanishing first Chern class.

<u>The</u> (+) <u>type</u> <u>Einstein</u> <u>metrics</u>: $R_{ij}-g_{ij} = 0$.

These are compact, with $\pi_1(M)$ finite. The corresponding fixed points are unstable in the T- direction (asymptotically free at short distances).

Any compact homogeneous space G/H for which the isotropy action of H is irreducible has, up to a scale, only one G- invariant symmetric tensor field, so is necessarily an Einstein manifold. These spaces have been classified by Wolf[18]. All but the noncompact symmetric spaces are of (+) type.

Many other homogeneous examples are known[19-22]. The first nonhomogeneous example was found by Page[23]. A number of others are now known[24].

<u>The quasi-Einstein metrics</u>: $R_{ij} - g_{ij} = v_i v_j + v_j v_i \neq 0$

These are not homogeneous. $R \geq 0$ and is not constant. The corresponding fixed points are unstable in the T- direction (asymptotically free at short distances).

No examples are known. (But see section 5.5 below.)

2.3. Linearization of the β- function at a fixed point

The aim here is to calculate enough derivatives of the β- function to establish, in the generic case, the existence of a true fixed point and the topological character of the renormalization group action nearby. In less favorable cases, the aim is to identify threats to the existence of the fixed point and the additional information needed to complete the portrait of the renormalization group action.

For the Einstein metrics, the renormalization group equations (2.1.3-4) are a suitable starting point. The metric coupling is written $T^{-1}(g_{ij} + k_{ij})$ where g_{ij} is an Einstein metric $(R_{ij} - a\, g_{ij} = 0)$ and k_{ij} is a small perturbation (leaving the volume fixed). The standard formula for the derivative of the Ricci tensor with respect to the metric gives[25-27]

$$\frac{dT}{dt} = -\epsilon T + a T^2 + b T^3 + O(T^4, T^3 k, T^2 k^2) \qquad (2,4,1)$$

$$\frac{d}{dt} k_{ij} = - T \left[\frac{1}{2} \Delta_\beta(k)_{ij} - \frac{1}{2} D_g \delta^*(k)_{ij} - D_g(dv_o(k)) \right]$$

$$- T^2 \left[\frac{1}{2} R^2_{ij} - b\, g_{ij} - D_g(v_1)_{ij} \right]$$

$$+ O(T^3, T^2 k, T k^2) \qquad (2.4.2)$$

where

$$\Delta_\beta(k)_{ij} = - \nabla_k \nabla_k k_{ij} - 2\, L(k)_{ij} \qquad (2.4.3)$$

$$L(k)_{ij} = R_{ikjl}\, k_{kl} \qquad (2.4.4)$$

$$(\delta^* k)_i = - \nabla^j (k_{ij} - \frac{1}{2} k_{kk}\, g_{ij}) \qquad (2.4.5)$$

$$D_g(v)_{ij} = [v, g]_{ij} = \nabla_i v_j + \nabla_j v_i \qquad (2.4.6)$$

$$dv_o(k)^i = \frac{d}{ds}\Big/_{s=0} v^i_o(g + s\, k)\ . \qquad (2.4.7)$$

dv_o should be chosen to be a convenient first order differential operator from symmetric tensor fields to vector fields, natural in g. v_1 is a vector field on M, also natural in g. The obvious choice for $dv_o(k)$ is $-\frac{1}{2}\delta^*(k)$, giving

$$\frac{d}{dt} k_{ij} = - \frac{1}{2} T\, \Delta_\beta(k) - T^2 S_{ij} + O(T^3, \cdots) \qquad (2.4.8)$$

$$S_{ij} = \frac{1}{2} R^2_{ij} - b\, g_{ij} - D_g(v_1)_{ij} \; . \qquad (2.4.9)$$

Δ_β is an elliptic operator with positive symbol. Therefore its spectrum is discrete and, of its eigenvalues, only a finite number have real part nonpositive. By construction, if k is of the form $D_g(w)$, then

$$\frac{1}{2} [\Delta_\beta(k) - D_g\delta^*(k)]_{ij} = [w, R_{ij} - a\, g_{ij}] = 0 \; . \qquad (2.4.10)$$

Therefore Δ_β maps Range(D_g) to Range(D_g). The k- directions in Range(D_g) are tangent to the orbit of $\underline{D}$, so are immaterial. (See [1].) Also, $k_{ij} = g_{ij}$ is an eigenvalue of Δ_β which is discarded because it represents a change of volume; it is a variation of g_{ij} in the T- direction.

The space of symmetric two tensors k_{ij} has a natural inner product

$$(k, k) = \int k_{ij}\, k_{ij} \qquad (2.4.11)$$

with respect to which Δ_β is essentially self-adjoint. Therefore all of its eigenvalues are real. Also, the orthogonal complement Range(D_g)$^\perp$ to Range(D_g) is taken to itself by Δ_β. The space of significant k- directions is the complement to g_{ij} in Range(D_g)$^\perp$. If Δ_β has no zero eigenvalues on the space of significant k-

directions, then, by the inverse function theorem, the Einstein metric corresponds to a true fixed point, which at most changes position slightly in response to higher order corrections. A neighborhood of the fixed point can be represented as the product of a stable manifold and a finite dimensional unstable manifold.

If Δ_β does annihilate some significant k_{ij}, then a more complicated situation is possible. The nature of the complications depends on the terms of order Tk^2 in the β- function. The k_{ij} on which Δ_β is zero might be tangents to a nontrivial manifold of solutions to the fixed point equation, in which case the $O(T^2)$ corrections are capable of eliminating the fixed point entirely (or of sending it to ∞). When $\epsilon = 0$, this is possible only for $a = 0$, but when $\epsilon \neq 0$, it is possible for any of the values of a. (Although, when $\epsilon \neq 0$, $a \neq 0$, the fixed point is only eliminated by effects at $O(\epsilon)$.)

It is also possible that the null k- directions are not tangents to curves of actual solutions of the fixed point equations. The terms in (2.4.8) of higher than first order in k might not be eliminable by a perturbation of the metric.

Answers to the following questions are to be sought in the linearization of the β- function.

(1) Are there significant null vectors for Δ_β?

(2) How many significant negative eigenvalues does Δ_β have? That is, how many additional directions of instability are there, beyond possibly the T- direction?

(3) What are the values of the topological invariants of the fixed point: the significant eigenvalues of Δ_β and the invariants of (2.4.1)?

The rest of this section lays groundwork for studying these questions. Some more detailed information is assembled in section 6.

The following gives a standard decomposition of the space of symmetric tensor fields.

Proposition 2.4.6.

The space S^2 of symmetric two tensor fields splits into

$$S^2 = (\mathrm{Range}(D_g) + H_C) \oplus H_{TT} . \qquad (2.4.12)$$

where

$$H_C = \{ f\, g_{ij} : f \text{ a function on } M \} \qquad (2.4.13)$$

$$H_{TT} = \{ k_{ij} : k_{ii} = 0,\ \nabla_j k_{ij} = 0 \} . \qquad (2.4.14)$$

HTT is clearly orthogonal to both $\mathrm{Range}(D_g)$ and H_C. Use of the Einstein condition yields

$$\Delta_\beta(f\, g_{ij}) = [(- \nabla_k \nabla_k - 2a) f]\, g_{ij} . \qquad (2.4.15)$$

Therefore Δ_β takes each of $\mathrm{Range}(D_g)$, H_C and H_{TT} to itself, and can be studied independently on each of them.

It remains to check whether $\mathrm{Range}(D_g)$ and H_C can intersect.

Proposition 2.4.7.

Given the Einstein condition, the intersection of $\mathrm{Range}(D_g)$ and H_C consists of the variations of the metric by infinitesimal conformal transformations

$$\{ \nabla_i \nabla_j f = \frac{-a}{n-1} f\, g_{ij} \;:\; (-\nabla_k \nabla_k - \frac{n}{n-1} a)\, f = 0 \} \;.(2.4.16)$$

In particular, when $a \le 0$, the intersection is always trivial.

Proposition 2.4.8.

For M a compact Einstein manifold, $-\nabla_k \nabla_k - \frac{n}{n-1} a$, acting on functions modulo constants, is nonnegative.

Proposition 2.4.9.

Zero eigenvalues of $-\nabla_k \nabla_k - \frac{n}{n-1} a$, which correspond to conformal vector fields on M, are possible only when M is the sphere S^n with the standard metric.[28]

The significant spectrum of Δ_β can now be described as:

(1) the spectrum of $(-\nabla_k\nabla_k - 2a)$ on functions, not including the eigenvalue $-2a$, which corresponds to the T- direction, or the eigenvalue $-\frac{n-2}{n-1}$ which corresponds to conformal transformations; and

(2) the spectrum of Δ_β on H_{TT}. The unstable directions, besides possibly T, correspond to negative eigenvalues of Δ_β on H_{TT} and to eigenvalues of the Laplacian on functions in the range

$$\frac{n}{n-1} < -\nabla_k\nabla_k < 2 \,. \tag{2.4.17}$$

The one loop marginal directions correspond to zero eigenvalues of Δ_β on H_{TT} and to the eigenvalue 2 of the Laplacian on functions. Some more detailed information on these spectra is given in section 5.

The foregoing discussion treated only the Einstein solutions to the fixed point equation. The quasi-Einstein fixed points have, by appropriate choice of $v_0(g)$ in (2.1.4), the linearization

$$\frac{d}{dt} k_{ij} = -\frac{1}{2} T\, A(k)_{ij} + O(T^2, Tk^2) \tag{2.4.18}$$

where

$$A(k)_{ij} = \Delta_\beta(k)_{ij} + 2\, v^k\, (\nabla_i k_{jk} + \nabla_j k_{ik} - \nabla_k k_{ij}) \tag{2.4.19}$$

$$+ \ [D_g(v)_{ik} k_{kj} + D_g(v)_{jk} k_{ki}] \ .$$

The operator A is elliptic with positive symbol, so has discrete spectrum and only a finite number of finite dimensional eigenspaces on which its real part is nonpositive. By construction, it preserves $\mathrm{Range}(D_g)$. The eigenspaces in $\mathrm{Range}(D_g)$ and the eigenspace proportional to g_{ij} are discarded. Again, only a finite number of unstable or one-loop marginal directions are possible. Note that A is not necessarily a symmetric operator, so complex eigenvalues are possible. Because no examples of quasi-Einstein manifolds are known, and because the operator A is technically more complicated than Δ_β, the linearization problem for quasi-Einstein fixed points is not discussed further.

3. Homogeneous Spaces

3.1. Introduction

The β- function (1.2) is a natural vector field on the space of metrics, so the renormalization group which it generates preserves isometries. In particular, it carries G- invariant metrics on the homogeneous space $M = G/H$ to G- invariant metrics. In the first part of this section it is shown that the β- function (1.2), restricted to the space $\underline{\tilde{R}}_G$ of G- invariant metrics on the unimodular homogeneous space G/H, is a gradient vector field, up to $O(T^2)$ corrections. As a consequence, the possibility of interesting global topological structure in the renormalization group is severely limited. The second subsection presents β explicitly for a simple example in which $\underline{\tilde{R}}_G$ is a two dimensional space.

Recall, from section 2.3 of Part I, that H is compact, that the Lie algebra $\underline{g}$ of G splits into the Lie algebra $\underline{h}$ of H and a complementary subspace $\underline{m}$, that H acts linearly on $\underline{m}$ by conjugation, and that the G- invariant tensor fields on M are in one to one correspondence with the H- invariant tensors on $\underline{m}$. The representation of H on $\underline{m}$ reduces to a sum of k irreducible representations of multiplicities $n_1 \cdots n_k$. The space of G- invariant metrics on M is the product

$$\underline{\tilde{R}}_G = \underline{\tilde{R}}_G^1 \times \cdots \times \underline{\tilde{R}}_G^k \qquad (3.1.1)$$

where $\underline{\tilde{R}}_G^i$ is the noncompact manifold of positive symmetric forms on a vector space of dimension n_i. $\underline{\tilde{R}}_G$ is therefore a real algebraic manifold of dimension

$$\dim(\underline{\tilde{R}}_g) = \sum_{i=1}^{k} \frac{1}{2} n_i(n_i + 1) . \qquad (3.1.2)$$

3.2. The β- function as a gradient

The vector field β is expressed as the gradient $\tilde{\nabla}\Phi$ of a potential function Φ on $\underline{\tilde{R}}_G$ with respect to a certain Riemannian structure on $\underline{\tilde{R}}_G$. The β- function is meaningful only as a vector field on the space $\underline{R}_G$ of equivalence classes of G- invariant metrics under diffeomorphisms of M, so both potential function and Riemannian structure should be invariant under the action of the diffeomorphisms on $\underline{\tilde{R}}_G$. Recall from section 1 that, in this context, diffeomorphisms of M means diffeomorphisms which commute with G.

A natural metric on $\underline{\tilde{R}}_G$ is

$$(k,k)_g = k_{ij}\, k_{ij} \qquad (3.2.1)$$

where k_{ij} is a tangent vector to $\underline{\tilde{R}}_G$, i.e. an H- invariant symmetric tensor on $\underline{m}$. Contractions are taken with g_{ij}. Clearly this Riemannian structure on $\underline{\tilde{R}}_G$ is invariant under diffeomorphisms of M.

The zero loop term $-\frac{1}{4}g$ in the expansion (1.2) of $\beta(g)$ is the gradient, with respect to the metric (3.2.1), of

$$\Phi_o(g) = -2\frac{1}{4}\log(d_g m/d_o m) \qquad (3.2.2)$$

where $d_g m$ is the metric volume element for g, and $d_o m$ is any fixed G- invariant volume element on M. The ratio is a constant. Φ_o is invariant only under diffeomorphisms of M which preserve $d_o m$. For unimodular spaces, this is all of $\underline{D}_G$.

The one loop term R_{ij} in (1.2) is the gradient of

$$\Phi_1(g) = -R \qquad (3.2.3)$$

(See [21].) The scalar curvature R is G- invariant, therefore constant, so (3.2.3) makes sense. The derivative of the scalar curvature R with respect to the metric is

$$d_g R(k) = -R_{ij}k_{ij} - \nabla_k\nabla_k k_{ii} + \nabla_k\nabla_i k_{ik} . \qquad (3.2.4)$$

The term $-\nabla_k\nabla_k k_{ii}$ vanishes because k_{ii} is G- invariant, so constant. The term $\nabla_k\nabla_i k_{ik}$ vanishes because of the unimodularity of M. Therefore, R_{ij} is the gradient of $-R$ with respect to the metric (3.2.1).

The full two loop approximation to the β- function, (1.2), is the gradient of a potential $\Phi = \Phi_0 + \Phi_1 + \Phi_2$, where

$$\Phi_2(g) = -\frac{1}{4} R_{ijpq} R_{ijpq} , \tag{3.2.5}$$

with respect to a modified metric on $\underline{\tilde{R}}_G$

$$(k,k)_{T^{-1}g} = T^2 k_{ij} k_{ij} \tag{3.2.6}$$

$$+ T^3 (k_{ip} k_{iq} R_{pq} - k_{ij} k_{pq} R_{ipjq} + \nabla_i k_{jk} \nabla_i k_{jk}) + O(T^4) .$$

Explicitly,

$$\leftarrow T^{-1} g_{ij} + R_{ij} + \frac{1}{2} T R_{ij}^2 = \tilde{\nabla}\Phi (T^{-1} g) . \tag{3.2.7}$$

This follows from a direct calculation using standard formulas for the derivative of the curvature tensor with respect to the metric.

It is a trivial observation that

$$\frac{d}{dt} \Phi(g) = (\frac{dg}{dt}, \tilde{\nabla}\Phi)_g$$

or

$$\frac{d}{dt} \Phi(g) = - (\tilde{\nabla}\Phi, \tilde{\nabla}\Phi)_g , \tag{3.2.8}$$

implying that $\tilde{\Phi}$ must decrease along the orbits of the renormalization group. It follows that the only subsets of $\tilde{\underline{R}}_G$ left fixed by the renormalization group are the critical sets, where $\vec{\nabla}\tilde{\Phi} = \beta = 0$.

This observation is strictly useful only when T can be taken small enough that the $O(T^2)$ corrections are of no consequence. This will be possible when the critical sets are isolated points; more precisely, where the zeros of the two loop approximation to β are isolated. If the two loop fixed points are degenerate, then the renormalization group shows gradient-like behavior except in an aysmptotically small neighborhood of ε nontrivial critical set. Then higher order terms in β come into play. When $\leftarrow = 0$, the qualitative topological properties of the renormalization group are not affected by the corrections beyond two loops, so only gradient-like behavior is possible.

3.3. An example

The familiar nonlinear models all have $M = G/H$ an isotropy irreducible space. That is, H acts irreducibly on $\underline{m}$, so that the space $\tilde{\underline{R}}_G$ of G- invariant metrics is one dimensional, described completely by the temperature. To obtain some idea of the possiblities available in more complicated homogeneous models an example is examined here in which $\tilde{\underline{R}}_G$ is two dimensional.

M is taken to be the group manifold $SO(N)$, but the symmetry group is not assumed to be the full $SO(N)\times SO(N)$ of left and right multiplication. G is taken to be $SO(N)\times SO(N-1)$ and H the diagonal

SO(N-1) subgroup. $\underline{m}$ is the Lie algebra so(N), on which H acts by conjugation. The representation of H on $\underline{m}$ decomposes into the standard representation on R^{N-1} and the adjoint representation on so(N-1). A vector in $\underline{m}$ is presented as a pair (v,W) where v is in R^{N-1} and W is in so(N-1).

$\underline{\tilde{R}}_G$ consists of the H- invariant inner products on $\underline{m}$, which are of the form

$$g((v,W),\ (v,W)) = \frac{2}{T_1}|v|^2 + \frac{1}{T_2}\operatorname{tr}(W^t W) . \qquad (3.2.1)$$

The metrics with $T_1 = T_2$ are the bi-invariant metrics on SO(N).

SO(N) should be seen here as a bundle over the quotient $S^{N-1} = SO(N)/SO(N-1)$ with fibers the SO(N-1) cosets. The metric g on SO(N) is a multiple T_1^{-1} of the standard metric on the S^{N-1} cosets combined in the natural way with a multiple T_2^{-1} of the standard metric on SO(N-1). T_1 is the temperature governing fluctuations from coset to coset; T_2 is the temperature governing fluctuations within each coset.

The metric (3.2.1) on $\underline{\tilde{R}}_G$ is, in this case,

$$((k_1,k_2),\ (k_1,k_2))_{(T_1,T_2)} \qquad (3.3.2)$$

$$= (N-1)\ [\ T_1^{-2}\ k_1^2 + \frac{1}{2}(N-2)\ T_2^{-2}\ k_2^2\ ] .$$

(k_1, k_2) is an infinitesmal variation of (T_1, T_2). The scalar curvature is

$$R = \frac{1}{2}(N-1)(N-2)\left[T_1 + \frac{1}{4}(N-3)T_2 - \frac{1}{4}T_2^{-2}T_1^2\right] \qquad (3.3.3)$$

and

$$\log(d_g m/d_o m)$$

$$= -\frac{1}{2}(N-1)\left[\log T_1 + \frac{1}{2}(N-2)\log T_2\right]. \qquad (3.3.4)$$

The one loop renormalization group equations are

$$\frac{d}{dt}T_1 = -\epsilon T_1 + \frac{1}{2}(N-2)T_1^2\left(1 - \frac{1}{2}T_2^{-1}T_1\right) \qquad (3.3.5)$$

$$\frac{d}{dt}T_2 = -\epsilon T_2 + \frac{1}{4}T_2^2\left(N-3 + T_2^{-2}T_1^2\right). \qquad (3.3.6)$$

To exhibit the topological structure it is convenient to change variables to

$$r = \frac{T_2 - T_1}{T_2 + T_1} \qquad -1 \leq r \leq 1 \qquad (3.3.7)$$

$$s = \frac{1}{N-2}(T_1 + T_2) \qquad 0 \leq s. \qquad (3.3.8)$$

The renormalization group equations become

$$\frac{dr}{dt} = - s\, F_1(r) \qquad (3.3.9)$$

$$\frac{ds}{dt} = - \epsilon s + s^2 F_2(r) \qquad (3.3.10)$$

where

$$F_1(r) = \frac{1}{4} r \left(1 - r \right) \left(\frac{1}{N-2} - r \right) \qquad (3.3.11)$$

$$F_2(r) = \frac{1}{8} (1 + r)^{-1} (1 - r)(1 + 3r) + \frac{1}{4} r \left(r - \frac{1}{N-2}\right) . \qquad (3.3.12)$$

The cases $N > 3$ and $N = 3$ are qualitatively different. The flow in (r,s) space is pictured, for $N > 3$, $\epsilon > 0$, in figure 1; for $N > 3$, $\epsilon = 0$, in figure 2; for $N = 3$, $\epsilon > 0$, in figure 3; and, for $N = 3$, $\epsilon = 0$, in figure 4. The lines $r = 0$ describe the $SO(N) \times SO(N)$ invariant models.

First consider the case $N > 3$, $\epsilon > 0$. There are two low temperature phases, governed by the stable gaussian fixed points at $r = 0$, $s = 0$ and $r = 1$, $s = 0$, separated by a critical surface, which is governed by the once unstable gaussian fixed point at $r = \frac{1}{N-2}$, $s = 0$. There are also critical surfaces governed by the nongaussian fixed points at $r = 1$, $s = 4\,\frac{N-2}{N-3}\,\epsilon$ and $r = 0$, $s = 8\,\epsilon$, separated by the multi-critical fixed point at

$r = \frac{1}{N-2}, \quad s = 8\,\frac{(N-1)(N-2)}{(N-3)(N+1)}\,\epsilon.$

There are also two phases in the region: $T_1 + T_2$ immediately above the critical surface. One phase is driven to the line $r = 0$ at long distances, the other to the line $r = 1$. The nongaussian fixed point at $r = \frac{1}{N-2}$ is therefore a quadri-critical point.

The three low temperature phases are all characterized by a manifold of pure equilibrium states equal to M itself. They differ in the symmetry properties of the free energy governing the fluctuations about these states. The line $r = 0$ is the $SO(N)\times SO(N)/SO(N)$ model. The line $r = 1$ is the $SO(N-1)\times SO(N-1)/SO(N-1)$ model, because the limit $T_1 \to 0$ freezes the field into one of the $SO(N-1)$ cosets. As the temperature is increased along the line $r = 0$ it is expected that the system disorders completely at the critical point $s = 8\epsilon$, the space of equilibria becoming a single point. On the other hand, at the critical point on the line $r = 1$ the system should disorder only within one coset, because the temperature T_1 governing fluctuations among the cosets remains zero. The space of equilibria above the critical point should be the space of cosets, S^{N-1}. Between this partially disordered phase at $r = 1$ and the completely disordered phase at $r = 0$ there should be a phase boundary ending at the quadri-critical point.

Note that no analogous partial disordering takes place at $T_2 = 0$ ($\iota = -1$). The curvature of the natural connection in the bundle $SO(N) \to S^{N-1}$ does not permit the system to disorder among cosets while remaining ordered within each coset. Sending $T_2 \to 0$, starting on the high temperature side of the critical surface in figure 1, in an attempt to bring order within the cosets only, actually results in ordering the system completely. There is no way for the system to spontaneously choose, in a continuous fashion, one point in each coset.

In the case $N > 3$, $\epsilon = 0$, pictured in figure 2, the critical surfaces have collapsed to $T = 0$, only the high temperature phases surviving.

The continuum limits of the model are described by the unstable manifolds: the region $0 \leq r \leq 1$, $0 \leq s$ in figures 1 and 2. The boundary $r = 0$ is the one parameter space of $SO(N) \times SO(N)/SO(N)$ models. The boundary $r = 1$ is the one parameter space of $SO(N-1) \times SO(N-1)/SO(N-1)$ models, the rest of the degrees of freedom having been frozen. In two dimensions ($\epsilon = 0$), the limit $r \to 0$ produces two length scales, one for the fluctuations within cosets and another much larger one for the fluctuations between cosets.

When $N = 3$ the phase structure is simpler; there are only the completely ordered and completely disordered phases. Note that, in two dimensions, the line $r = 1$ is the line of fixed points of the $SO(2)/\{e\}$ or XY- model. They are all renormalization group unstable against unfreezing of the fluctuations among the $SO(2)$ cosets in $SO(3)$.

The existence of $r = 1$ fixed points in figures 1, 2 and 4 is suggestive. These are "at ∞" in the language of section 2.1. They are on a part of the boundary of the space of metrics which is not in the interior of the $T = 0$ surface. Here the interior of the $T = 0$ surface is the region $s = 0$, $-1 < r < 1$. The boundary of the space of metrics on a manifold M is a complicated object. It is not clear how to investigate in general the behavior of the renormalization group in the neighborhood of the boundary.

4. Two Dimensional Manifolds

M is assumed in this section to be a compact two dimensional manifold. The one loop β- function for M is shown to be a gradient.

Two properties of two dimensional Riemannian manifolds are used. First, the symmetries of the curvature tensor imply that it is entirely made up of the scalar curvature:

$$R_{ijpq} = \frac{1}{2} R \left(g_{ip}g_{jq} - g_{iq}g_{jp} \right) \tag{4.1}$$

$$R_{ij} = \frac{1}{2} R \, g_{ij} \, . \tag{4.2}$$

Second, every metric is conformal to a metric g^c_{ij} of constant scalar curvature:

$$g_{ij} = e^f \, g^c_{ij} \, , \tag{4.3}$$

where f is some real valued function on M; and R^c the scalar curvature of g^c is constant. The scalar curvature of g is

$$R = e^{-f} \left(R^c + \Delta^c f \right) , \tag{4.4}$$

where

$$\Delta^c = - \nabla^c_i \nabla^c_i \tag{4.5}$$

is the laplacian for g^c.

The renormalization group equation, up to two loops, is

$$\frac{d}{dt} (T^{-1} g)_{ij} = (\leftarrow - \frac{1}{2} T R - \frac{1}{8} T^2 R^2) T^{-1} g_{ij} + O(T^3)) \tag{4.6}$$

Therefore the two loop approximation to the β- function is tangent to the conformal class. There is no reason to suppose that this remains true at higher order; terms like $\nabla_i R \nabla_j R$ might well appear. However, to exhibit the topological properties of the renormalization group, it will only be necessary examine the one loop approximation. Then it does make sense to discuss the action of the renormalization group on the functions f, holding g^c fixed:

$$\frac{df}{dt} = - \beta(f) \tag{4.7}$$

$$\beta(f) = - \leftarrow + \frac{1}{2} R \tag{4.8}$$

$$= - \leftarrow + \frac{1}{2} e^{-f} (R^c + \Delta^c f) . \tag{4.9}$$

The natural metric (2.3.11) on $\underline{\tilde{R}}$ induces a metric on the space $\underline{\tilde{R}}^c$ of functions f:

$$(k,k)_f = \int d_g m \;\; k^2 \tag{4.10}$$

where k is a function on M, an infinitesmal variation of f, and $d_g m$ is the metric volume element for g.

The zero loop term $-\leftarrow$ in β is the gradient of

$$\Phi_0(f) = -\leftarrow \int d_g m \tag{4.11}$$

$$= -\leftarrow \int d^c m \;\; e^f \tag{4.12}$$

where $d^c m$ is the metric volume element for g^c. The derivative of Φ_0 is

$$d_f \Phi_0(k) = -\leftarrow \int d_g m \;\; k \, , \tag{4.13}$$

so, in the metric (4.10),

$$\vec{\nabla}\Phi_0 = -\leftarrow \, . \tag{4.14}$$

The two loop term $\frac{1}{2} R$ in $\beta(f)$ is the gradient of

$$\Phi_1(f) = \frac{1}{2} \int d^c m \;\; f \left(R^c + \frac{1}{2} \Delta^c f \right) . \tag{4.15}$$

The derivative of Φ_1 is

$$d_f \Phi_1(k) = \frac{1}{2} \int d^c m \ (R^c + \Delta^c f) k \tag{4.16}$$

so the gradient with respect to (4.11) is

$$\vec{\nabla} \Phi_1 = \frac{1}{2} e^{-f} (R^c + \Delta^c f)$$

$$= \frac{1}{2} R . \tag{4.17}$$

The one loop renormalization group equation is, writing $\Phi = \Phi_0 + \Phi_1$,

$$\frac{df}{dt} = - \vec{\nabla} \Phi . \tag{4.18}$$

It remains to show that the potential Φ is invariant under the conformal group $\underline{C}$ of diffeomorphisms Ψ of M preserving the conformal class of g^c, These diffeomorphisms satisfy

$$\Psi_* g^c_{ij} = \exp(h_\Psi) g^c_{ij} \tag{4.19}$$

where, for each Ψ in $\underline{C}$, h_Ψ is a real valued function on M. $\underline{C}$ acts on the functions f by

$$\Psi_* (e^f g^c_{ij}) = \exp(\Psi_* f) \Psi_* g^c_{ij} = \exp(\Psi_* f + h_\Psi) g^c_{ij} \tag{4.20}$$

or

$$\Psi f = \Psi_* f + h_\Psi . \tag{4.21}$$

Clearly Φ_0 is $\underline{C}$- invariant, so the problem is to calculate $\Phi_1(\Psi f)$ for Ψ in $\underline{C}$.

Proposition 4.1.

$\Phi_1(\Psi f) = \Phi_1(f) + \Phi_1(h_\Psi)$.

Proposition 4.2.

$\rho: \Psi \to \Phi_1(h_\Psi)$ is a representation of $\underline{C}$ in the additive group of real numbers.

Proposition 4.3.

ρ vanishes on $\underline{C}_0$, the connected component of the identity in $\underline{C}$.

Theorem 4.4.

Φ_1 is $\underline{C}$- invariant.

Propositions 4.1-2 are direct calculations. Proposition 4.3 follows from the fact that the derivative of ρ at the identity is zero, by direct calculation. Theorem 4.4 follows from proposition 4.3 and the fact that, for compact two dimensional manifolds, $\underline{C}/\underline{C}_0$ is finite dimensional.

It has now been shown that the one loop β- function as a vector field on each conformal class of metrics (modulo conformal transformations) is a gradient. This is not exactly to say that β is a gradient on the space of metrics (modulo diffeomorphisms), because the gradient of Φ on the larger space of all metrics has a component which changes the conformal class. Since the one loop β- function preserves the conformal class its topological properties can be studied class by class, so the result obtained is sufficient. It is not clear that an improved result, giving β as a gradient on the space of all metrics, is possible. This point and the nontriviality of the potential function Φ suggest that the β- function on metrics (modulo diffeomorphisms) for the general manifold M might well not be a gradient.

Attention is now directed towards the critical points of the potential Φ. These are the constant functions f, corresponding to the constant curvature metrics themselves. The hessian of Φ at a critical point is easily seen to be positive definite except in the f- directions produced by infinitesmal conformal transformations. In the significant f- directions, therefore, the fixed points are infrared stable. The remaining questions are: is the space of inequivalent constant curvature metrics nontrivial; and, if so, how do higher order corrections to β project onto it.

The two dimensional compact manifolds are: the sphere S^2, the real projective space RP^2 consisting of the sphere with antipodal points identified, the torus T^2, the Klein bottle, and the surfaces of genus greater than one. The sphere and the real projective space each has exactly one constant (positive) curvature metric (up to overall scale). The constant curvature metrics on the torus and Klein bottle are all flat metrics. The manifolds of genus greater than one all possess manifolds of inequivalent constant (negative) curvature metrics. These manifolds of metrics have dimension $6g + 3c - 6$, where g is the genus (the number of handles) and c is the number of crosscaps in M.

The perturbative expansion of the β- function is formed entirely from the curvature and its covariant derivatives. But for a constant curvature metric g^c there are no covariant derivatives. Therefore, to all orders in T,

$$\beta_{ij}(T^{-1}g^c) = f(T)\, g^c_{ij} \tag{4.22}$$

It follows that no perturbative corrections can remove the degeneracy of the fixed points for the manifolds of genus greater than one.

The metrics of constant negative curvature on a given manifold M are all locally, but not globally equivalent. They all have infinite, nonabelian fundamental groups. As discussed in section 6.5 of Part I, perturbative renormalization cannot reliably distinguish among such metrics. It is to be expected that nonperturbative effects enter significantly into the renormalization.

5. Fixed Points (II)

5.1. Introduction

This section is a miscellany of results on the fixed points described in section 2, based on study of the linearization (2.4.8) of the β- function. Except in section 5.6, only Einstein fixed points are discussed. Some of the the results are to be found in [29,30], but those on Kahler-Einstein metrics do not seem to be in the literature.

Two results are most notable. The first is that the linearized one loop β- function at every known (-) type Einstein metric metric has only nonnegative eigenvalues. This implies infrared stability except possibly in the finite number of one loop marginal directions. The second is that for every known (0) type Einstein metric there is, to one loop, except for the infrared instability in the T- direction, only infrared stablity and marginality.

Section 5.2 presents basic estimates of the Bochner type for the Laplacian $-\nabla_k\nabla_k$ on functions and for Δ_β on H_{TT} for Einstein and Kahler-Einstein metrics. The general strategy is to bound the differential operator Δ_β from below by a manifestly nonnegative differential operator plus a zeroth order, algebraic operator formed from the curvature tensor of the metric. Bounds for this algebraic operator are then obtained point by point on M, for the known (0) and (-) type Einstein metrics. Section 5.3 discusses the one loop marginal directions (zero modes of Δ_β) for these metrics. Section 5.4 presents a number of general facts and sample calculations for (+) type Einstein metrics. Section 5.5 contains a number of comments on quasi-Einstein metrics, concentrating on the Kahler ones.

5.2. Bochner estimates

The first proposition of this section estimates the Laplacian on functions (modulo constants) for Einstein-Kahler metrics. It improves the Bochner estimate for Einstein metrics given in Propositions 2.4.8-9. A Kahler-Einstein metric is one satisfying

$$R_{a\bar{b}} - a\, g_{a\bar{b}} = 0 \qquad (5.2.1)$$

$$R_{ab} = R_{\bar{a}\bar{b}} = 0 \,. \qquad (5.2.2)$$

Proposition 5.2.1.

For g_{ij} a Kahler-Einstein manifold,

$$-\nabla_k \nabla_{\bar{k}} - 2a \geq 0 \,, \qquad (5.2.3)$$

on functions modulo constants. Equality is achieved only on functions satisfying $\nabla_a \nabla_b f = 0$.

The next four propositions estimate Δ_β on H_{TT}, the first for Einstein metrics in general, the rest for Einstein-Kahler metrics. Δ_β

is defined in (2.4.3-4), H_{TT} in (2.4.14), L in (2.4.4).

Proposition 5.2.2.

For g_{ij} an Einstein metric, Δ_β on H_{TT} satisfies

$$\Delta_\beta \geq -a - L \qquad (5.2.4)$$

with equality only for k_{ij} satisfying $\nabla_p k_{qi} = \nabla_q k_{pi}$; and

$$\Delta_\beta \geq 2a - 4L \qquad (5.2.5)$$

with equality only for k_{ij} satisfying

$\nabla_p k_{qr} + \nabla_q k_{rp} + \nabla_r k_{pq} = 0.$

In the next three propositions g is a Kahler-Einstein metric $g_{a\bar{b}}$. The space of real, traceless symmetric two tensors on M splits into the real hermition traceless tensors

$$k_{a\bar{b}} = k_{\bar{b}a} = \overline{k_{b\bar{a}}}, \quad k_{ab} = k_{\bar{a}\bar{b}} = 0, \quad k_{a\bar{a}} = 0 \qquad (5.2.6)$$

and the real anti-hermitian tensors

$$k_{ab} = k_{ba} = \overline{k_{\bar{a}\bar{b}}}, \quad k_{a\bar{b}} = k_{\bar{a}b} = 0 . \qquad (5.2.7)$$

Δ_β preserves these subspaces. Therefore, if k is an eigenvector of

Δ_β in H_{TT}, then so are $k_{a\bar{b}}$ and k_{ab}. The vanishing of the divergence of k is

$$\nabla_b k_{a\bar{b}} + \nabla_{\bar{b}} k_{ab} = 0 . \qquad (5.2.8)$$

Proposition 5.2.3.

If k is real, anti-hermitian and symmetric, but not necessarily divergence free, then

$$2 \int \nabla_b k_{\bar{b}\bar{a}} \nabla_{\bar{c}} k_{ca} \leq \int k_{\bar{a}\bar{b}} (\Delta_\beta k)_{ab} \qquad (5.2.9)$$

with equality only for the k_{ab} satisfying $\nabla_a k_{bc} = \nabla_b k_{ac}$.

The object is to find under what conditions Δ_β has nonpositive eigenvalues, so attention is now restricted to the subspace H_{TT}^{-o} of H_{TT} on which Δ_β is nonpositive. By the previous proposition, the anti-hermitian part of a tensor field in H_{TT}^{-o} must be divergence free. Therefore so must be the hermitian part. H_{TT}^{-o} splits into $H_H \oplus H_A$ where H_H consists of the hermitian, traceless divergence free tensor fields in H_{TT}^{-o} and H_A consists of the anti-hermitian divergence free tensor fields in H_{TT}^{-o}. Δ_β respects the splitting.

Proposition 5.2.4.

On traceless divergence free hermitian tensor fields, including those in H_H,

$$\Delta_\beta \geq -2a \qquad (5.2.10)$$

with equality only for $k_{a\bar b}$ satisfying $\nabla_a k_{b\bar c} = \nabla_b k_{a\bar c}$; and

$$\Delta_\beta \geq 2a - 4L \qquad (5.2.11)$$

with equality only for $k_{a\bar b}$ satisfying $\nabla_a k_{b\bar c} = -\nabla_b k_{a\bar c}$.

Proposition 5.2.5.

On divergence free anti-hermitian tensor fields, including those in H_A,

$$\Delta_\beta \geq 0 \qquad (5.2.12)$$

with equality only for $\nabla_a k_{bc} = \nabla_b k_{ac}$,

$$\Delta_\beta \geq -2(a + L) \qquad (5.2.13)$$

with equality only for $\nabla_{\bar c} k_{ab} = 0$, and

$$\Delta_\beta \geq 6(a - L) . \qquad (5.2.14)$$

with equality only for $\nabla_a k_{bc} + \nabla_b k_{ca} + \nabla_c k_{ab} = 0$.

The Bochner estimates are now used to eliminate the possibility of one loop infrared instability for the known (-) and (0) type Einstein metrics. Propositions 2.4.8-9 imply that for these metrics the only significant nonpositive eigenvalues of Δ_β, if any, occur in its action on H_{TT}.

Recall that, given the restriction to compact or homogeneous spaces, the only known (-) type Einstein metrics are the locally symmetric manifolds of noncompact type (which can be compact manifolds) and the Kahler-Einstein metrics of Yau. The next two results, on the locally symmetric manifolds, can be found in [29]. The β- function for Riemannian manifolds which are locally the product of Riemannian manifolds is trivially determined from the β- functions for the factors, so it is assumed here that the Riemannian manifold M is locally irreducible.

<u>Proposition 5.2.6.</u>

For M a locally irreducible, locally symmetric manifold (and therefore necessarily Einstein with $a = \pm 1$),

$$a < L \leq -a \quad \text{if } a = -1 \tag{5.2.15}$$

$$-a \leq L < a \quad \text{if } a = 1 . \tag{5.2.16}$$

<u>Theorem 5.2.7.</u>

If M is a compact manifold which is locally irreducible, locally symmetric of noncompact type $(a = -1)$, and of dimension greater than two, then Δ_β on H_{TT} is positive.

A metric which is locally equivalent to an Einstein metric is obviously Einstein, so theorem 5.2.7 implies that the metrics to which it refers have no locally equivalent metrics infinitesmally close. Therefore the obstruction to renormalizability of equivalence relations discussed in section 6.5 of Part I cannot occur, even though $\pi_1(M)$ is not necessarily finite.

The remaining known (-) or (0) type Einstein metrics are Kahler. The following result is an immediate consequence of (5.2.10) and (5.2.1^).

Theorem 5.2.8.

For M an Einstein-Kahler manifold of (-) or (0) type, $\Delta_\beta \geq 0$ on H_{TT}.

It will be shown below that, in the Einstein-Kahler cases, Δ_β does in general have zero modes in H_{TT}. For a complete portrait of the renormalization group action near a fixed point with one loop marginality (zero modes), a better than linear approximation to the one loop β- function is needed, in addition to higher order corrections. The next section will identify the zero modes, but the discussion will not be carried further.

It is suggestive that the (-) type fixed points, which occur in dimensions $2 + \epsilon \leq 2$ and which are the only known fixed points for which perturbative renormalization cannot be relied on to renormalize equivalent models equivalently, show no infrared instability at all in the one loop approximation.

5.3. Zero modes for known (−) and (0) type Einstein metrics

By theorem 5.2.7, only the Kahler-Einstein metrics of Yau among known (−) and (0) type Einstein metrics can have zero modes. By propositions 5.2.4-5, the zero modes consist of: (1) the symmetric anti-hermitian tensor fields k_{ab} satisfying

$$\nabla_{\bar{a}} k_{ab} = 0, \quad \nabla_a k_{bc} = \nabla_b k_{ac} ; \qquad (5.3.1)$$

and, for (0) type Einstein metrics only, (2) the hermitian tensor fields $k_{a\bar{b}}$ satisfying

$$k_{a\bar{a}} = 0, \quad \nabla_b k_{a\bar{b}} = 0, \quad \text{and} \quad \nabla_a k_{b\bar{c}} = \nabla_b k_{a\bar{c}} . \qquad (5.3.2)$$

Proposition 5.3.1.

The number of independent hermitian zero modes for (0) type Einstein metrics is $p^{1,1}$, the primitive Hodge number of degree (1,1).

The anti-hermitian zero modes for both types of metric are related to the deformations of complex structure. Inequivalent infinitesmal changes in complex structure on M are represented by harmonic (0,1)-forms with values in the complex tangent bundle, i.e. the $k_{\bar{a}}^{c}$ satisfying

$$\nabla_{\bar{a}} k_{\bar{b}}^{c} - \nabla_{\bar{b}} k_{\bar{a}}^{c} = 0, \qquad \nabla_{a} k_{\bar{a}}^{c} = 0 . \tag{5.3.3}$$

These can be regarded as anti-hermitian two tensors, not necessarily symmetric.

Proposition 5.3.2.

For Einstein-Kahler metrics, the space of inequivalent deformations of complex structure split into two subspaces: (1) the symmetric anti-hermitian tensors k_{ab} satisfying (5.3.1), and (2) the harmonic (0,2)- forms. The latter exist only for (0) type metrics, and then are actually covariant constant (0,2)- forms.

Let τ be the natural map from $H^1(T)$, the deformations of complex structure, to $H^2(\underline{O})$, the second cohomology of the sheaf of holomorphic functions. (See [31] for definitions.)

Proposition 5.3.3.

For (-) type metrics, $\tau = 0$. For (0) type metrics, τ is surjective (onto), and $H^2(\underline{O})$ is represented by the covariant constant (0,2)- forms. The number of anti-hermitian zero modes for (-) and (0) type Einstein metrics is twice the complex dimension of the kernel of $\mathcal{T}$.

In the case of the (−) type metrics, the total number of zero modes

$$2\ \dim_C(\mathrm{Ker}\ \mathcal{T}) = 2\ \dim_C(H^1(T)) \qquad (5.3.4)$$

is a local constant. The space of complex structures on M is a manifold.[31] Yau's theorem[16,17] guarantees exactly one Kahler-Einstein metric for each complex structure. Therefore the zero modes are tangent to a true manifold of one loop fixed points.

In the case of the (0) type, or Ricci-flat, metrics, the zero modes are counted by

$$\dim_C(H^{1,1}) - 1 + 2\ (\ \dim_C(H^1(T)) - \dim_C(H^{0,2})\ )\ . \qquad (5.3.5)$$

They include infinitesmal changes in the cohomology class of the fundamental form and infinitesmal changes in the complex structure. Small perturbations of the complex structure remain Kahlerizable.[31] Yau's theorem guarantees existence of a unique Ricci-flat Kahler metric for each cohomology class of the fundamental form and for each Kahlerizable complex structure. But it is not known, in general, if all infinitesmal changes of complex structure can be extended to finite changes, i.e. whether the zero modes are actually tangent to a manifold of one loop fixed points. If there is a nontrivial space of one loop (0) type fixed points, then to find the true fixed points it is necessary to solve the auxiliary fixed point equations which arise from projecting the two loop term in the β- function onto the space of one loop fixed points. (See section 2.1.)

5.4. (+) type Einstein metrics

The general results on instabilities and zero modes for (+) type Einstein metrics are meager. In this section only the simplest examples are discussed.

The one loop unstable and marginal directions are: (1) the functions on which $\frac{n}{n-1} < -\nabla_i\nabla_i \leq 2$, and (2) the symmetric tensors in H_{TT} on which $0 \leq \Delta_\beta$. The marginal directions are those for which equality occurs. By proposition (5.2.1), there are no instabilities of the first kind for Kahler-Einstein metrics.

Proposition 5.4.1.

For Kahler-Einstein manifolds of (+) type, the dimension of the eigenspace of real valued functions on which $-\nabla_i\nabla_i = 2$ is the complex dimension of the space of holomorphic vector fields on M.[32]

Of Kahler-Einstein fixed point metrics, only the (+) type can have nontrivial holomorphic vector fields. Those that do automatically have the one loop marginal directions described in proposition 5.4.1. When $\epsilon = 0$, as discussed in section 2.1, these are true marginalities.

Proposition 5.4.2.

For locally symmetric (+) type Kahler Einstein manifolds with nonzero holomorphic vector fields, the two loop contribution to the β- function, projected onto the space of zero modes described in the previous proposition, gives infrared stability when $\epsilon > 0$.

This is a direct calculation.

A manifold of constant sectional curvature is a Riemannian manifold with curvature tensor

$$R_{ijpq} = \frac{R}{n(n-1)} (g_{ip}g_{jq} - g_{iq}g_{jp}) \, . \qquad (5.4.1)$$

The sectional curvature is positive if $R > 0$, in which case M is the sphere S^n divided by a discrete group of isometries. M is automatically (+) type Einstein.

A Kahler manifold of constant holomorphic sectional curvature is a Kahler manifold for which the non-vanishing part of the curvature tensor is

$$R_{\bar{a}cb\bar{d}} = \frac{2\,R}{n(n+2)} (g_{\bar{a}b}g_{c\bar{d}} + g_{\bar{a}c}g_{b\bar{d}}) \, . \qquad (5.4.2)$$

The holomorphic sectional curvature is positive if $R > 0$, in which case M is the complex projective space $CP^{n/2}$ divided by a discrete group of isometries. M is automatically (+) type Einstein.

Proposition 5.4.3.

For manifolds of constant positive sectional curvature,

$\Delta_\beta > 0$ on H_{TT}.

Proposition 5.4.4.

For Kahler manifolds of constant positive holomorphic sectional curvature, $\Delta_\beta > 0$ on H_{TT}.

Proposition 5.4.5.

For $M = S^n$, the laplacian on functions has eigenvalues

$$- \nabla_i \nabla_i = \frac{r(r+n-1)}{n-1} \qquad r = 0,1,2,\cdots \quad (5.4.3)$$

There are no eigenvalues in the the range $\frac{n}{n-1} < -\nabla_i \nabla_i \leq 2$ which would provide unstable or marginal directions.

Therefore the S^n fixed point is unstable in the T- direction and stable in the rest.[27]

Proposition 5.4.6.

For $M = CP^{n/2}$, the laplacian on functions has eigenvalues

$$- 2 \nabla_a \nabla_{\bar{a}} = \frac{2r(n+2r)}{n+2} \qquad r = 0,1,2,\cdots \quad (5.4.4)$$

In particular, 2 is an eigenvalue. (See proposition 5.4.1.)

The $CP^{n/2}$ fixed point is therefore unstable in the T- direction, marginal in the directions of the form $f\, g_{ij}$ for f an eigenvector of the laplacian with eigenvalue 2, and stable in all other directions. By proposition 5.4.2, the marginality is present only when $\psi = 0$. The functions f giving the marginal directions are of the form

$$f(z) = f_{a\bar{b}}\, z^{a}\, z^{\bar{b}} \tag{5.4.5}$$

for $f_{a\bar{b}}$ some hemitian form on $C^{(n+2)/2}$.

Finally, some information is given on the spectrum of the laplacian for the homogeneous (+) type Einstein manifold $M = SO(N) = SO(N)\times SO(N)/SO(N)$. The functions on M decompose into irreducible $SO(N)$ representations on which the laplacian is proportional to the quadratic Casimir operator. Each representation occurs with multiplicity equal to its dimension. Direct calculation gives

$$-\nabla_i\nabla_i = \frac{N(N-1)}{4(N-2)} \quad \text{on the spinor representation,} \tag{5.4.6}$$

$$-\nabla_i\nabla_i = \frac{2(N-1)}{(N-2)} \quad \text{on the standard representation,} \tag{5.4.7}$$

and larger eigenvalues on the rest of the representations. Since, in this case, the dimension of M is $n = N(N-1)/2$, the spinors represent directions of instability when $N = 5$ and 6.

5.5. Quasi-Einstein metrics

The quasi-Einstein metrics are the solutions of

$$R_{ij} - g_{ij} = \nabla_i v_j + \nabla_j v_i \neq 0 . \qquad (5.5.1)$$

By propositions 2.2.1–4,

$$R \geq 0 \qquad (5.5.2)$$

and

$$- \nabla_j \nabla_j v^i - R_{ij} v^j = 0, \qquad \nabla_i v^i \neq 0 . \qquad (5.5.3)$$

Whether there exist any quasi-Einstein manifolds is unknown. Equations (2.3.2) and (2.4.15) show that infinitesmal variations of a (+) type Einstein metric of the form $f\, g_{ij}$ with $(- \nabla_i \nabla_i - 2)f = 0$ are quasi-Einstein. Proposition 5.4.1 and the comments on the group manifolds SO(5), SO(6) at the end of section 6.4 indicate that such infinitesmal deformations do exist. There is no reason to suppose, however, that there are finite quasi-Einstein deformations corresponding to the infinitesmal ones; the quasi-Einstein equation might not be solvable at some order beyond the first. In any case, these infinitesmal deformations all have v^i a gradient, while the principal interest is in the cases in which v^i is not a gradient.

The final propositions give elementary general information on Kahler quasi-Einstein manifolds.

Proposition 5.5.1.

v^i is the real part of a holomorphic vector field.[32]

Proposition 5.5.2.

The first Chern class is positive and the fundamental form belongs to it.

Proposition 5.5.3.

The first betti number is zero.

Proposition 5.5.4.

There exists a complex valued function F on M such that

$$\nabla_a \nabla_b F = 0 \tag{5.5.4}$$

$$v_a = \frac{1}{2} \nabla_a F, \quad v_{\bar{a}} = \frac{1}{2} \nabla_{\bar{a}} \bar{F} \tag{5.5.5}$$

$$R_{\bar{a}b} - g_{\bar{a}b} = \nabla_{\bar{a}} \nabla_b (\mathrm{Re}\ F) . \tag{5.5.6}$$

References

I. The Low Temperature Expansion

[1] Friedan, D.: Nonlinear Models in 2 + ε Dimensions. To appear in Phys. Rev. Lett.

[2] Meetz, K.: J. Math. Phys. 10, 589 (1969).

[3] Honerkamp, J.: Nucl. Phys. B36, 130 (1972).

[4] Ecker, G., Honerkamp, J.: Nucl. Phys. B35, 481 (1971).

[5] Polyakov, A.M.: Phys. Lett. 59B, 79 (1975).

[6] Migdal, A.A.: Zh. Eksp. Teor. Fiz. 69, 1457 (1975).

[7] Wilson, K.G., Kogut, J.: Phys. Rep. 12C, 75 (1974).

[8] Brezin, E., Zinn-Justin, J.: Phys. Rev. Lett. 36, 691 (1976).

[9] Brezin, E., Zinn-Justin, J.: Phys. Rev. B14, 3110 (1976).

[10] Brezin, E., Zinn-Justin, J., Le Guillou, J.C.: Phys. Rev. D14, 2615 (1976).

[11] Bardeen, W.A., Lee, B.W., Schrock, R.E.: Phys. Rev. D14, 985 (1976).

[12] Zinn-Justin, J.: In: Trends in Elementary Particle Physics. Bonn: Springer 1975.

[13] Nelson, E.: Tensor Analysis. Princeton: Princeton University Press 1967.

[14] Kobayashi,S., Nomizu, K.: Foundations of Differential Geometry Vols. I, II. New York: Interscience 1969.

[15] Elitzur,S.: The Applicability of Perturbation Expansion to Two-Dimensional Goldstone Systems (Revised). IAS preprint 1979.

[16] Ebin, D.G.: In: Global Analysis. Providence: A.M.S. 1970.

[17] Wolf, J.A.: Acta. Math. 120, 59 (1968).

[18] Cartan, E.: Lecons sur la Geometrie Riemannienne.

[19] Becchi, C., Rouet, A., Stora, R.: Ann. Phys. 98, 287 (1976).

[20] Gross, D.: In: Methods in Field Theory. Amsterdam: North Holland 1976.

II. The Renormalization Group Equation

[1] Ebin, D.G.: In: Global Analysis. Providence: A.M.S. 1970.

[2] Kobayashi,S., Nomizu, K.: Foundations of Differential Geometry Vols. I, II. New York: Interscience 1969.

[3] Nelson,E.: Tensor Analysis. Princeton: Princeton University Press 1967.

[4] Berger, M.: Ann. Math. Pur. Appl. 53, 89 (1961).

[5] Berger, M.: C.R.A.S. 260, 1554 (1965).

[6] Berger, M.: In: Comptes Rendus de la III^{e} Reunion du Groupement des Mathematiciens D'Expression Latine. Namur: Librairie Universitaire 1966.

[7] Hawking, S.: Nucl. Phys. B144, 349 (1978).

[8] Hitchen, N.: J. Diff. Geom. 9, 435 (1974).

[9] Donnelly, H.: J. Diff. Geom. 11, 259 (1976).

[10] Bourguignon, J.P.: Ricci Curvature and Einstein Metrics. Talk delivered at Symposium on Global Analysis in honour of E.B. Christoffel at T.U. Berlin 1979.

[11] Myers, S.: Duke Math J. 8, 401 (1941).

[12] Fischer, A.E., Wolf, J.A.: Bull. A.M.S. 80, 92 (1974).

[13] Cheeger, J., Gromoll, D.: J. Diff. Geom. 6, 119 (1971).

[14] Aleksevskii, D.V., Kimel-Feld, B.N.: Funct. Anal. Appl. 9, 5 (1975).

[15] Helgason, S.: Differential Geometry, Lie Groups and Symmetric Spaces. New York: Academic Press 1978.

[16] Yau, S.T.: Proc. Natl. Acad. Sci. 74, 1798 (1977).

[17] Yau, S.T.: Comm. Pur. Appl. Math. xxxi, 339 (1978).

[18] Wolf, J.A.: Acta. Math. 120, 59 (1968).

[19] Kobayashi, S.: Tohoku Math. J. 15, 121 (1963).

[20] Sagle, A.: Nagoya Math. J. 39,

[21] Jensen, G.R.: Indiana Math. J. 20, 1125 (1971).

[22] Jensen, G.R.: J. Diff. Geom. 8, 599 (1973).

[23] Page, D.N.: Phys. Lett. 79B, 235 (1978).

[24] Berard Bergery, L.: Arbeitstagungsbericht, Bonn 1979.

[25] Lichnerowicz, A.: Inst. Hautes Etudes Sci. Publ. Math. no. 10, 293 (1961).

[26] Berger, M.: Ann. Scient. Ec. Norm. Sup. 3, 285 (1970).

[27] Berger, M., Ebin, D.: J. Diff. Geom. 3, 379 (1969).

[28] Yano, K., Nagano, T.: Ann. Math. 69, 451 (1959).

[29] Weil, A.: Ann. Math. 72, 578 (1962).

[30] Koiso, N.: Osaka J. Math. 15, 419 (1978).

[31] Kodaira, K., Morrow, J.: Complex Manifolds. New York: Holt, Rinehart and Winston 1971.

[32] Yano, K.: Differential Geometry on Complex and Almost Complex Spaces. New York: Macmillan 1965.

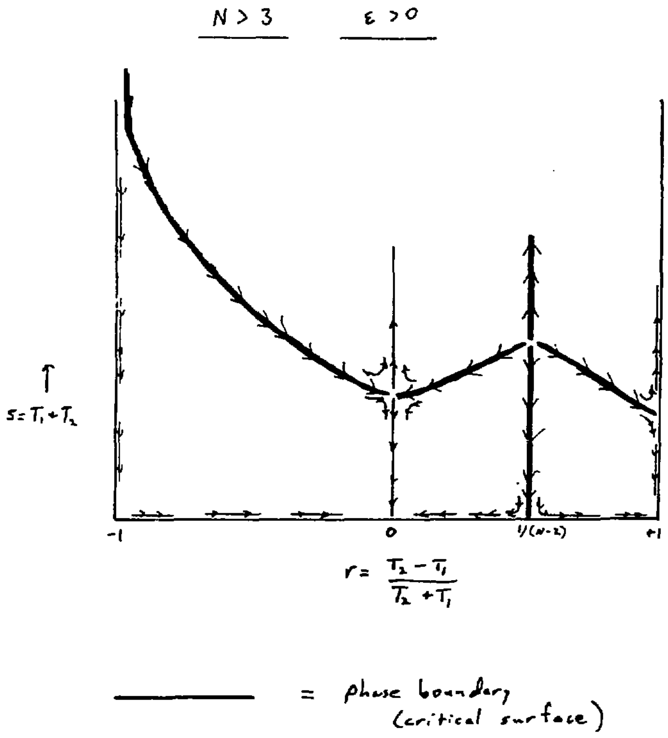
N > 3
ε > 0
s = T1 + T2
-1
0
1/(N-2)
+1
r = (T2 - T1)/(T2 + T1)
= phase boundary
(critical surface)

Fig. 1

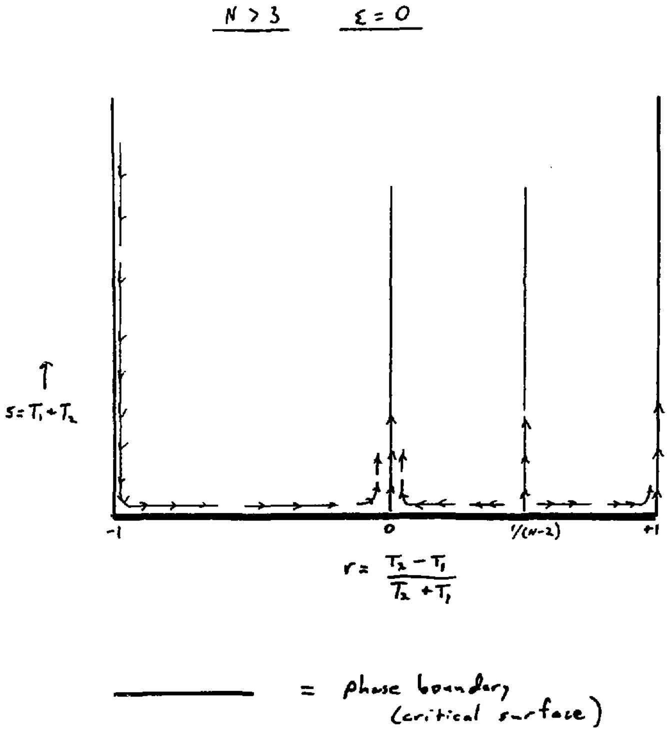
N > 3
ε = 0
s = T₁ + T₂
-1
0
1/(N-2)
+1
r = (T₂ − T₁)/(T₂ + T₁)
= phase boundary
(critical surface)

Fig. 2

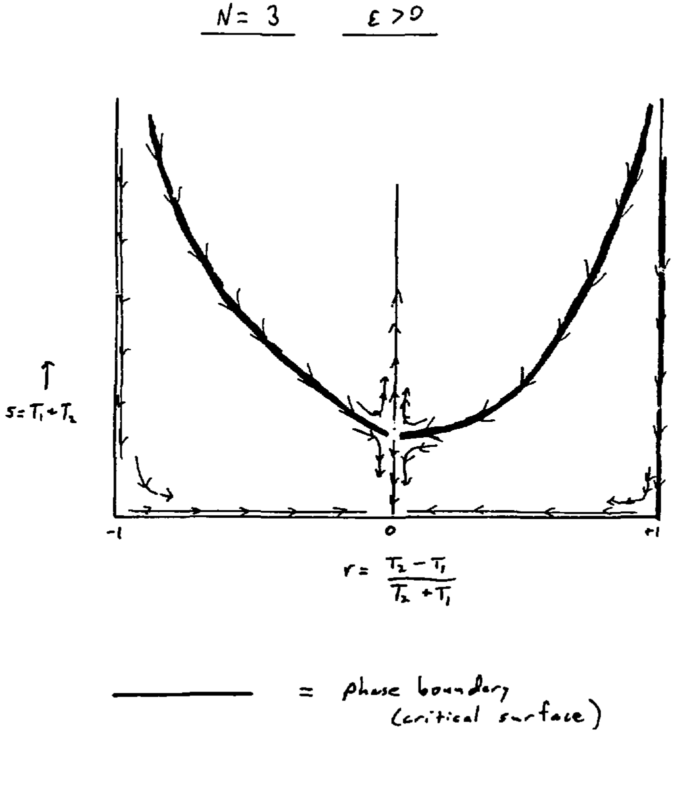

Fig. 3

$N = 3 \quad \varepsilon = 0$

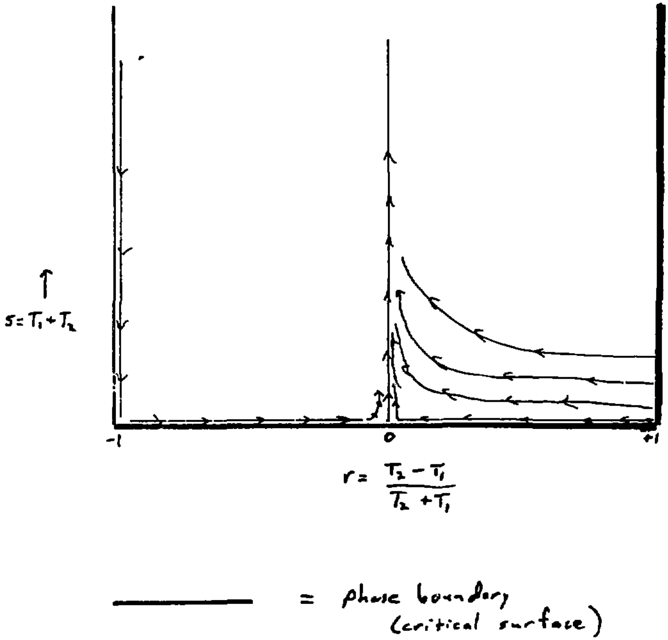

Fig. 4