Nonlinear Multiresolution Signal Decomposition Schemes—Part I: Morphological Pyramids

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***Abstract*—Interest in multiresolution techniques for signal processing and analysis is increasing steadily. An important instance of such a technique is the so-called pyramid decomposition scheme. This paper presents a general theory for constructing linear as well as nonlinear pyramid decomposition schemes for signal analysis and synthesis. The proposed theory is based on the following ingredients: 1) the pyramid consists of a (finite or infinite) number of levels such that the information content decreases toward higher levels and 2) each step toward a higher level is implemented by an (information-reducing) analysis operator, whereas each step toward a lower level is implemented by an (information-preserving) synthesis operator. One basic assumption is necessary: synthesis followed by analysis yields the identity operator, meaning that no information is lost by these two consecutive steps.**

Several examples of pyramid decomposition schemes are shown to be instances of the proposed theory: a particular class of linear pyramids, morphological skeleton decompositions, the morphological Haar pyramid, median pyramids, etc. Furthermore, the paper makes a distinction between single-scale and multiscale decomposition schemes, i.e., schemes without or with sample reduction. Finally, the proposed theory provides the foundation of a general approach to constructing nonlinear wavelet decomposition schemes and filter banks, which will be discussed in a forthcoming paper.

***Index Terms*—Mathematical morphology, morphological adjunction pyramids, morphological operators, multiresolution signal decomposition, pyramid transform.**

I. Introduction

FROM the very early days of signal and image processing, it has been recognized that multiresolution signal decomposition schemes provide convenient and effective ways to process information. Pyramids [1], wavelets [2], multirate filter banks [3], granulometries [4], [5], and skeletons [4], [6] are among the most common tools for constructing multiresolution signal decomposition schemes. Although these tools seem to be built on different paradigms, it is starting to be recognized that they are different instances of the same theory. For example, Rioul established in [7] a clear link between linear pyramids, wavelets, and multirate filter banks for discrete-time signals, by employing a time-domain basis decomposition approach.

A popular way to obtain a multiresolution signal decomposition scheme is to smooth a given signal, by means of a linear lowpass filter, in order to remove high frequencies, and subsample the result in order to obtain a reduced-scale version of the original signal (e.g., see [1] and [3]). By repeating this process, a collection of signals at decreasing scale is thus produced. These signals, stacked on top of each other, form a basic signal decomposition scheme, known as a *multiresolution (signal) pyramid*. A collection of detail signals is also constructed by subtracting from each level of the pyramid an interpolated version of the next coarser level. From a frequency point of view, the resulting detail signals form a signal decomposition in terms of highpass-filtered copies of the original signal. It is not difficult to show that the original signal can be uniquely reconstructed from the detail signals (and the scaled signal residing at the top level). Therefore, the detail signals provide a multiresolution signal representation that guarantees perfect reconstruction. The best-known example of such a scheme is the *Laplacian pyramid* of Burt and Adelson [1].

A linear filtering approach to multiresolution signal decomposition may not be theoretically justified. In particular, the operators used for generating the various levels in a pyramid must crucially depend on the application. The point stressed here is that, scaling an image by means of linear operators may not be compatible with a natural scaling of some image attribute of interest (shape of object, for example), and hence use of linear procedures may be inconsistent in such applications. To address this issue, a number of authors have proposed nonlinear multiresolution signal decomposition schemes based on morphological operators (e.g., [4], [6], and [8]–[26]), median filters (e.g., [14], [27], and [28]), and order statistic filters (e.g., [29] and [30]). These approaches have produced a number of useful nonlinear image processing and analysis tools, such as morphological skeletons [4], [6], morphological subband decompositions and filter banks [13], [16], [20], [25], median and order statistic based subband decompositions and filter banks [29], [30], morphological pyramids [8]–[10], [12], [15], [17], [19], median and order statistic pyramids [28], and, more recently, morphological wavelet decompositions [21], [24]–[26], [31], [32].

A number of interesting questions arise at this point:

1) Are the previous nonlinear multiresolution techniques fundamentally different, or are they all instances of a common theory?
2) What is the foundation of such a theory and how can it be constructed?

Manuscript received October 20, 1998; revised February 7, 2000. This work was supported in part under NATO Collaborative Research Grant CRG.971503. J. Goutsias was also supported by the Office of Naval Research, Mathematical, Computer, and Information Sciences Division under ONR Grant N00014-90-1345, and the National Science Foundation, under NSF Award 9729576. H. J. A. M. Heijmans was also supported by INTAS under Grant 96-785. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. Scott T. Acton.

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Publisher Item Identifier S 1057-7149(00)09390-8.

3) If such a theory exists, does it include the linear multiresolution techniques as a special case?

It is the main purpose of our work, presented in this (and a forthcoming) paper, to lay-down the foundations of such a theory, by employing an axiomatic approach. It will soon become apparent that a general framework can be constructed that allows one to treat linear and nonlinear pyramids, filter banks and wavelets, as well as morphological pyramids, from a unified standpoint.

Toward this goal, we present a general multiresolution scheme which represents a signal, or image, on a pyramid, using a sequence of successively reduced volume signals obtained by applying fixed rules that map one level to the next. In such a scheme, a level is *uniquely* determined by the level below it. Our approach contains the following three ingredients:

1) No assumptions are made on the underlying signal/image space. It may be a linear space (Laplacian pyramid, linear wavelets), it may be a complete lattice (morphological pyramids and wavelets), or any other set.
2) The proposed scheme is constituted by operators between different spaces (the levels of the pyramid). These operators are decomposed into *analysis operators*, representing an upward step reducing the information content of a signal, and *synthesis operators*, representing a downward step which does not (further) reduce the information content.[1]
3) The analysis and synthesis operators are only required to satisfy an elementary condition: synthesis followed by analysis is the identity operator. This condition, to be referred to as the "pyramid condition," plays an important role in the sequel. In fact, pyramidal schemes, based on analysis and synthesis operators that satisfy the pyramid condition, enjoy an intuitive property: repeated application of the analysis/synthesis steps does not modify the decomposition. Moreover, if the pyramid condition is satisfied, and if $V_0^{(j)}$ denotes the space of all signals obtained by applying j analysis steps followed by j synthesis steps on signals $x \in V_0$, then $V_0^{(j+1)} \subseteq V_0^{(j)} \subseteq V_0$, for $j > 0$. This is a basic requirement for a multiresolution signal decomposition scheme [2] that agrees with our intuition that the space $V_0^{(j)}$, which contains the approximations of signals at level 0 of the pyramid, obtained by means of j analysis steps followed by j synthesis steps, contains the approximations of signals at level 0 of the pyramid, obtained by means of $j+1$ analysis steps followed by $j+1$ synthesis steps. However, it is worthwhile mentioning here that, in the literature associated with multiresolution pyramids, this condition is often overlooked. Many proposed linear and nonlinear pyramidal decomposition schemes do not satisfy such a condition.

In our work, we are distinguishing among two types of multiresolution decompositions:

1) *Pyramid Scheme:* Every analysis operator that brings a signal x_j from level j to the next coarser level $j+1$ reduces information. This information can be stored in a detail signal (at level j) which is the difference between x_j and the approximation $\hat{x}_j$ obtained by applying the synthesis operator to x_{j+1}. In general, a representation obtained by means of a pyramid (coarsest signal along with detail signals at all levels) is redundant (in the sense that the decomposition produces more data samples than the original signal). This type of decomposition will be the main subject of this paper.
2) *Wavelet Scheme:* Here, the detail signal resides at level $j+1$ itself, and is obtained from a second family of analysis operators. In this case, the analysis and synthesis operators need to satisfy a condition that is very similar in nature to the pyramid condition, discussed in this paper, and the biorthogonality condition known from the theory of wavelets (note, however, that this condition is formulated in operator terms only, and does not require any sort of linearity assumption or inner product). This type of decomposition will be the main subject of a forthcoming paper [33].

This paper is organized as follows. In Section II, we recall some concepts and notations of mathematical morphology. Section III introduces the main results of our theoretical framework in terms of analysis and synthesis operators and their compositions. Here, we introduce our key assumption, the *pyramid condition*, which plays a major role in our exposition. The remainder of the paper is devoted to examples and applications of our general scheme. Section IV illustrates the fact that a particular class of linear pyramids is a special case of our general framework. This is done by means of an example, which is a nonseparable two-dimensional (2-D) extension of the Burt-Adelson pyramid [1]. Section V is concerned with a class of morphological pyramids based on adjunctions. These pyramids satisfy an interesting property: the detail signals are always nonnegative! In Section V, we also show that a particular type of morphological skeletons fits perfectly within our general framework. An attempt to put Lantuéjoul's skeleton decomposition algorithm [4] into our framework, may lead to an improvement. In Section VI, we discuss more general morphological pyramid decomposition schemes, such as median pyramids and morphological pyramids with quantization. Finally, in Section VII, we end with our conclusions.

The present paper is an extract of our report [22], where one can find some additional results on linear pyramids, multiscale morphological operators, and granulometries.

II. Mathematical Preliminaries

In this section, we provide an overview of basic concepts, notations, and results from mathematical morphology which we need in the sequel. A comprehensive discussion can be found in [5].

A set $\mathcal{L}$ with a partial ordering $\leq$ is called a *complete lattice* if every subset $\mathcal{K}$ of $\mathcal{L}$ has a *supremum* (least upper bound) $\bigvee \mathcal{K}$ and an *infimum* (greatest lower bound) $\bigwedge \mathcal{K}$. We say that $\mathcal{L}$ is a *complete chain* if it is a complete lattice such that $x \leq y$ or $y \leq x$, for every pair $x, y \in \mathcal{L}$. A simple example of a complete chain is the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ with the usual ordering.

Let $\mathcal{L}$ and $\mathcal{M}$ be two complete lattices, and let $\varepsilon: \mathcal{L} \to \mathcal{M}$ and $\delta: \mathcal{M} \to \mathcal{L}$ be two operators. We say that (ε, δ) constitutes an *adjunction* between $\mathcal{L}$ and $\mathcal{M}$ if

$$\delta(y) \leq x \Leftrightarrow y \leq \varepsilon(x), \qquad x \in \mathcal{L},\ y \in \mathcal{M}.$$

[1] We say that an operator "reduces information" if it is not injective, in other words, if the original signal cannot be recovered from the transformed signal; an operator which is injective is said to "preserve information."

If (ε, δ) forms an adjunction between $\mathcal{L}$ and $\mathcal{M}$, then ε has the property

$$\varepsilon\left(\bigwedge_{i\in I} x_i\right) = \bigwedge_{i\in I} \varepsilon(x_i) \tag{1}$$

for any family $\{x_i | i \in I\} \subseteq \mathcal{L}$ of signals. Operator δ has the dual property

$$\delta\left(\bigvee_{i\in I} y_i\right) = \bigvee_{i\in I} \delta(y_i) \tag{2}$$

for any family $\{y_i | i \in I\} \subseteq \mathcal{M}$ of signals. This, in particular, implies that ε and δ are increasing (i.e., monotone) operators. An operator ε that satisfies (1) is called an *erosion*, whereas an operator δ that satisfies (2) is called a *dilation*. We denote the identity operator on $\mathcal{L}$ by $\mathrm{id}_{\mathcal{L}}$, or simply id, when there is no danger for confusion.

With every erosion $\varepsilon: \mathcal{L} \to \mathcal{M}$, there corresponds a unique dilation $\delta: \mathcal{M} \to \mathcal{L}$ such that (ε, δ) constitutes an adjunction. Similarly, with every dilation $\delta: \mathcal{M} \to \mathcal{L}$, there corresponds a unique erosion $\varepsilon: \mathcal{L} \to \mathcal{M}$ such that (ε, δ) constitutes an adjunction. If (ε, δ) is an adjunction between two complete lattices $\mathcal{L}$ and $\mathcal{M}$, then

$$\varepsilon\delta \geq \mathrm{id}, \quad \delta\varepsilon \leq \mathrm{id} \quad \text{and} \quad \varepsilon\delta\varepsilon = \varepsilon, \quad \delta\varepsilon\delta = \delta. \tag{3}$$

If ψ is an operator from a complete lattice $\mathcal{L}$ into itself, then ψ is *idempotent*, if $\psi^2 = \psi\psi = \psi$. If ψ is increasing and idempotent, then ψ is called a *morphological filter*. A morphological filter ψ that satisfies $\psi \leq \mathrm{id}$ (ψ is anti-extensive) is an *opening*, whereas a morphological filter ψ that satisfies $\psi \geq \mathrm{id}$ (ψ is extensive) is a *closing*. If (ε, δ) is an adjunction between two complete lattices $\mathcal{L}$ and $\mathcal{M}$, then, $\varepsilon\delta$ is a closing on $\mathcal{M}$ and $\delta\varepsilon$ is an opening on $\mathcal{L}$. This is a direct consequence of (3).

Given a complete lattice $\mathcal{T}$ and a nonempty set E, the set $\mathrm{Fun}(E, \mathcal{T}) = \mathcal{T}^E$, comprising all functions $x: E \to \mathcal{T}$, is a complete lattice under the pointwise ordering

$$x \leq y \quad \text{if } x(p) \leq y(p), \qquad \forall p \in E.$$

In this paper, $\mathrm{Fun}(E, \mathcal{T})$ represents the signals with domain E and values in $\mathcal{T}$. The least and greatest elements of $\mathcal{T}$ are denoted by $\bot$, $\top$ respectively: $\bigwedge \mathcal{T} = \bot$, $\bigvee \mathcal{T} = \top$. We are mainly interested in the case when E is the d-dimensional discrete space $\mathbb{Z}^d$. Two basic morphological operators on $\mathrm{Fun}(\mathbb{Z}^d, \mathcal{T})$ are the (flat) *dilation* δ_A and the (flat) *erosion* ε_A, given by

$$\delta_A(x)(n) = (x \oplus A)(n) = \bigvee_{k\in A} x(n-k) \tag{4}$$

$$\varepsilon_A(x)(n) = (x \ominus A)(n) = \bigwedge_{k\in A} x(n+k). \tag{5}$$

Here, $A \subseteq \mathbb{Z}^d$ is a given set, the so-called *structuring element*. The pair $(\varepsilon_A, \delta_A)$ constitutes an adjunction on $\mathrm{Fun}(\mathbb{Z}^d, \mathcal{T})$. Thus, we may conclude that the composition $\alpha_A = \delta_A\varepsilon_A$ is an opening whereas the composition $\beta_A = \varepsilon_A\delta_A$ is a closing. We use the following notation: $\alpha_A(x) = x \circ A$ and $\beta_A(x) = x \bullet A$.

III. Multiresolution Signal Decomposition

To obtain a mathematical representation for a multiresolution signal decomposition scheme, we need a sequence of signal domains, assigned at each level of the scheme, and analysis/synthesis operators that map information between different levels. The analysis operators are designed to reduce information in order to simplify signal representation whereas the synthesis operators are designed to undo as much as possible this loss of information. This is a widely accepted approach to multiresolution signal decomposition [2]. Moreover, as discussed in the introduction, the analysis/synthesis operators depend on the application at hand and a sound theory should be able to treat them from a general point of view. Motivated by these reasons, we present in this section a general multiresolution signal decomposition scheme, to be referred to as the *pyramid transform*.

A. Analysis and Synthesis Operators

Let $J \subseteq \mathbb{Z}$ be an index set indicating the levels in a multiresolution signal decomposition scheme. We either consider J to be finite or infinite. In the finite case, we take $J = \{0, 1, \cdots, K\}$, for some $K < \infty$, whereas $J = \{0, 1, \cdots\}$ in the infinite case. A domain V_j of signals is assigned at each level j. No particular assumptions on V_j are made at this point (e.g., it is not necessarily true that V_j is a linear space). In this framework, *signal analysis* consists of decomposing a signal in the direction of increasing j. This task is accomplished by means of *analysis operators* $\psi_j^\uparrow: V_j \to V_{j+1}$. On the other hand, *signal synthesis* proceeds in the direction of decreasing j, by means of *synthesis operators* $\psi_j^\downarrow: V_{j+1} \to V_j$. Here, the upward arrow indicates that the operator $\psi^\uparrow$ maps a signal to a level higher in the pyramid, whereas the downward arrow indicates that the operator $\psi^\downarrow$ maps a signal to a level lower in the pyramid. The analysis operator $\psi_j^\uparrow$ is designed to reduce information in order to simplify signal representation at level $j+1$, whereas the synthesis operator $\psi_j^\downarrow$ is designed to map this information back to level j.

We can travel from any level i in the pyramid to a higher level j by successively composing analysis operators. This gives an operator

$$\psi_{i,j}^\uparrow = \psi_{j-1}^\uparrow \psi_{j-2}^\uparrow \cdots \psi_i^\uparrow, \qquad j > i \tag{6}$$

which maps an element in V_i to an element in V_j. On the other hand, the composed synthesis operator

$$\psi_{j,i}^\downarrow = \psi_i^\downarrow \psi_{i+1}^\downarrow \cdots \psi_{j-1}^\downarrow, \qquad j > i \tag{7}$$

takes us back from level j to level i. Finally, we define the composition

$$\hat{\psi}_{i,j} = \psi_{j,i}^\downarrow \psi_{i,j}^\uparrow, \qquad j > i \tag{8}$$

which takes a signal from level i to level j and back to level i again.

Since the analysis operators $\psi_j^\uparrow$ are designed to reduce signal information, they are not invertible in general, and information loss cannot be recovered by using only the synthesis operators $\psi_j^\downarrow$. Therefore, $\hat{\psi}_{i,j}$ can be regarded as an *approximation operator* that approximates a signal at level i, by mapping (by means of $\psi^{\downarrow}_{j,i}$) the reduced information at level j, incurred by $\psi^{\uparrow}_{i,j}$, back to level i.

The following condition plays a major role in this paper.

Pyramid Condition. The analysis and synthesis operators $\psi^{\uparrow}_j$, $\psi^{\downarrow}_j$ are said to satisfy the *pyramid condition* if $\psi^{\uparrow}_j \psi^{\downarrow}_j = \mathrm{id}$ on V_{j+1}.

This condition has the following important consequences:

- $\psi^{\uparrow}_j$ is surjective;
- $\psi^{\downarrow}_j$ is injective;
- $\psi^{\uparrow}_j \psi^{\downarrow}_j \psi^{\uparrow}_j = \psi^{\uparrow}_j$ and $\psi^{\downarrow}_j \psi^{\uparrow}_j \psi^{\downarrow}_j = \psi^{\downarrow}_j$;
- $\psi^{\downarrow}_j \psi^{\uparrow}_j$ is idempotent, i.e., $\psi^{\downarrow}_j \psi^{\uparrow}_j \psi^{\downarrow}_j \psi^{\uparrow}_j = \psi^{\downarrow}_j \psi^{\uparrow}_j$.

Notice that the injectivity of $\psi^{\downarrow}_j$ means that synthesis does not cause information reduction. To prove, for example, that $\psi^{\uparrow}_j$ is surjective, take $y \in V_{j+1}$ and let $x \in V_j$ be defined as $x = \psi^{\downarrow}_j(y)$. Then $\psi^{\uparrow}_j(x) = \psi^{\uparrow}_j \psi^{\downarrow}_j(y) = y$. In our report [22], we have shown various relationships between the properties above. For example, we have shown that the pyramid condition is satisfied if and only if $\psi^{\uparrow}_j$ is surjective and $\psi^{\uparrow}_j \psi^{\downarrow}_j \psi^{\uparrow}_j = \psi^{\uparrow}_j$ or $\psi^{\downarrow}_j$ is injective and $\psi^{\downarrow}_j \psi^{\uparrow}_j \psi^{\downarrow}_j = \psi^{\downarrow}_j$.

Proposition 1: Assume that the pyramid condition is satisfied. Then

$$\psi^{\uparrow}_{i,j} \psi^{\downarrow}_{j,i} = \mathrm{id} \text{ on } V_j, \quad j > i \tag{9}$$

$$\hat{\psi}_{i,j} \hat{\psi}_{i,k} = \hat{\psi}_{i,j} = \hat{\psi}_{i,k} \hat{\psi}_{i,j}, \qquad j \geq k > i. \tag{10}$$

In particular, $\hat{\psi}_{i,j}$ is idempotent.

Proof: From (6), (7), and the pyramid condition, we have that

$$\begin{aligned}\psi^{\uparrow}_{i,j} \psi^{\downarrow}_{j,i} &= \psi^{\uparrow}_{j-1} \psi^{\uparrow}_{j-2} \cdots \psi^{\uparrow}_{i+1} (\psi^{\uparrow}_i \psi^{\downarrow}_i) \psi^{\downarrow}_{i+1} \cdots \psi^{\downarrow}_{j-1} \\ &= \psi^{\uparrow}_{j-1} \psi^{\uparrow}_{j-2} \cdots (\psi^{\uparrow}_{i+1} \psi^{\downarrow}_{i+1}) \cdots \psi^{\downarrow}_{j-1} \\ &= \cdots = \psi^{\uparrow}_{j-1} \psi^{\downarrow}_{j-1} = \mathrm{id}\end{aligned}$$

which shows (9).

From (6)–(8) and the pyramid condition, we have that

$$\begin{aligned}\hat{\psi}_{i,j} \hat{\psi}_{i,k} &= \psi^{\downarrow}_{j,i} \psi^{\uparrow}_{i,j} \psi^{\downarrow}_{k,i} \psi^{\uparrow}_{i,k} \\ &= \psi^{\downarrow}_i \psi^{\downarrow}_{i+1} \cdots \psi^{\downarrow}_{j-1} \psi^{\uparrow}_{j-1} \psi^{\uparrow}_{j-2} \cdots \psi^{\uparrow}_{i+1} \\ &\quad \cdot (\psi^{\uparrow}_i \psi^{\downarrow}_i) \psi^{\downarrow}_{i+1} \cdots \psi^{\downarrow}_{k-1} \psi^{\uparrow}_{k-1} \psi^{\uparrow}_{k-2} \cdots \psi^{\uparrow}_i \\ &= \psi^{\downarrow}_i \psi^{\downarrow}_{i+1} \cdots \psi^{\downarrow}_{j-1} \psi^{\uparrow}_{j-1} \psi^{\uparrow}_{j-2} \cdots \\ &\quad \cdots (\psi^{\uparrow}_{i+1} \psi^{\downarrow}_{i+1}) \cdots \psi^{\downarrow}_{k-1} \psi^{\uparrow}_{k-1} \psi^{\uparrow}_{k-2} \cdots \psi^{\uparrow}_i \\ &= \cdots = \psi^{\downarrow}_i \psi^{\downarrow}_{i+1} \cdots \psi^{\downarrow}_{j-1} \psi^{\uparrow}_{j-1} \psi^{\uparrow}_{j-2} \cdots \psi^{\uparrow}_k \\ &\quad \cdot (\psi^{\uparrow}_{k-1} \psi^{\downarrow}_{k-1}) \psi^{\uparrow}_{k-1} \psi^{\uparrow}_{k-2} \cdots \psi^{\uparrow}_i \\ &= \psi^{\downarrow}_{j,i} \psi^{\uparrow}_{i,j} = \hat{\psi}_{i,j}\end{aligned}$$

which shows the first equality in (10). From (6)–(8) and the pyramid condition, we also have that

$$\begin{aligned}\hat{\psi}_{i,k} \hat{\psi}_{i,j} &= \psi^{\downarrow}_{k,i} \psi^{\uparrow}_{i,k} \psi^{\downarrow}_{j,i} \psi^{\uparrow}_{i,j} \\ &= \psi^{\downarrow}_i \psi^{\downarrow}_{i+1} \cdots \psi^{\downarrow}_{k-1} \psi^{\uparrow}_{k-1} \psi^{\uparrow}_{k-2} \cdots \psi^{\uparrow}_{i+1} \\ &\quad \cdot (\psi^{\uparrow}_i \psi^{\downarrow}_i) \psi^{\downarrow}_{i+1} \cdots \psi^{\downarrow}_{j-1} \psi^{\uparrow}_{j-1} \psi^{\uparrow}_{j-2} \cdots \psi^{\uparrow}_i \\ &= \cdots = \psi^{\downarrow}_i \psi^{\downarrow}_{i+1} \cdots \psi^{\downarrow}_{k-1} (\psi^{\uparrow}_{k-1} \psi^{\downarrow}_{k-1}) \psi^{\downarrow}_k \\ &\quad \cdot \psi^{\downarrow}_{k+1} \cdots \psi^{\downarrow}_{j-1} \psi^{\uparrow}_{j-1} \psi^{\uparrow}_{j-2} \cdots \psi^{\uparrow}_i \\ &= \psi^{\downarrow}_{j,i} \psi^{\uparrow}_{i,j} = \hat{\psi}_{i,j}\end{aligned}$$

which shows the second equality in (10). ∎

The first equality in (10) simply states that the level k approximation $\hat{\psi}_{i,k}(x)$ of a signal $x \in V_i$ is adequate for determining the level j ($j > k$) approximation $\hat{\psi}_{i,j}(x)$ of x. This agrees with our intuition that higher levels in the decomposition correspond to higher information reduction. The second equality in (10) says that $\hat{\psi}_{i,j}(x)$, $x \in V_i$, is not modified if approximated by means of operator $\hat{\psi}_{i,k}$.

It is worthwhile noticing here that, if $V_i^{(j)} = \mathrm{Ran}(\hat{\psi}_{i,j})$ (i.e., the *range* of the approximation operator $\hat{\psi}_{i,j}$), then $V_i^{(j)} \subseteq V_i$ and the second equality in (10), with $k = j - 1$, results in

$$V_i^{(j)} \subseteq V_i^{(j-1)} \subseteq V_i, \quad j > i + 1. \tag{11}$$

Therefore, operator $\hat{\psi}_{i,j}$ decomposes the signal space V_i into nested subspaces $\cdots \subseteq V_i^{(i+2)} \subseteq V_i^{(i+1)} \subseteq V_i$, each subspace $V_i^{(j)}$ containing all "level j" ($j > i$) approximations of signals in V_i. Equation (11) is a basic requirement for a multiresolution signal decomposition scheme [2] which agrees with our intuition that the space $V_i^{(j-1)}$, which contains the approximations of signals at level i obtained by means of operator $\hat{\psi}_{i,j-1}$, contains the approximations of signals at level i obtained by means of $\hat{\psi}_{i,j}$ as well.

B. Pyramid Transform

Although, as a direct consequence of the pyramid condition, the analysis operator $\psi^{\uparrow}_j$ is the left inverse of the synthesis operator $\psi^{\downarrow}_j$, it is not true in general that it is also the right inverse: $\psi^{\downarrow}_j \psi^{\uparrow}_j(x)$ is only an approximation of $x \in V_j$. Therefore, the analysis step cannot be used by itself for signal representation. This is not a problem however. In fact, this is in agreement with the inherent property of multiresolution signal decomposition of reducing information in the direction of increasing j.

Analysis of a signal $x \in V_j$, followed by synthesis, yields an approximation $\hat{x} = \hat{\psi}_{j,j+1}(x) = \psi^{\downarrow}_j \psi^{\uparrow}_j(x) \in \hat{V}_j$ of x, where $\hat{V}_j = V_j^{(j+1)}$. We assume here that there exists a *subtraction operator* $(x, \hat{x}) \mapsto x \dot{-} \hat{x}$ mapping $V_j \times \hat{V}_j$ into a set Y_j (strictly speaking, we should write $\dot{-}_j$ to denote dependence on level j). Furthermore, we assume that there exists an *addition operator* $(\hat{x}, y) \mapsto \hat{x} \dot{+} y$ mapping $\hat{V}_j \times Y_j$ into V_j. The *detail signal* $y = x \dot{-} \hat{x}$ contains information about x which is not present in $\hat{x}$. It is crucial that x can be reconstructed from its approximation $\hat{x}$ and the detail signal y. Toward this goal, we introduce the following assumption of *perfect reconstruction*:

$$\hat{x} \dot{+} (x \dot{-} \hat{x}) = x, \quad \text{if } x \in V_j \quad \text{and } \hat{x} = \hat{\psi}_{j,j+1}(x).$$

This leads to the following recursive signal analysis scheme:

$$\begin{aligned}x \to \{y_0, x_1\} &\to \{y_0, y_1, x_2\} \to \cdots \\ &\to \{y_0, y_1, \cdots, y_j, x_{j+1}\} \to \cdots\end{aligned} \tag{12}$$

$$\begin{cases} x_0 = x \in V_0 \\ x_{j+1} = \psi^{\uparrow}_j(x_j) \in V_{j+1}, \\ y_j = x_j \dot{-} \psi^{\downarrow}_j(x_{j+1}) \in Y_j \end{cases} \quad j \geq 0. \tag{13}$$

Notice that, because of the perfect reconstruction condition, signal $x \in V_0$ can be *exactly* reconstructed from x_{j+1} and $y_0, y_1, \cdots, y_j$ by means of the backward recursion

$$x = x_0, \quad x_j = \psi_j^{\downarrow}(x_{j+1}) \dot{+} y_j, \quad j \geq 0. \tag{14}$$

Example 1: The specific choice for the subtraction and addition operators depends upon the application at hand. Below, we discuss three alternatives for which the perfect reconstruction condition holds. In all cases, we assume that our signals lie in $\mathrm{Fun}(E, \mathcal{T})$, for some gray-value set $\mathcal{T}$. Now, it suffices to define subtraction and addition operators on $\mathcal{T}$.

1) Assume that $\mathcal{T} \subseteq \mathbb{R}$ and let $\mathcal{T}' = \{t - s | t, s \in \mathcal{T}\}$. We define a subtraction operator $(t, s) \mapsto t - s$ from $\mathcal{T} \times \mathcal{T}$ into $\mathcal{T}'$. Obviously, the perfect reconstruction condition is valid if we choose the standard addition $+$ as the addition operator.
2) Suppose that $\mathcal{T}$ is a complete lattice. If we know that the approximation signal $\hat{x}$ satisfies $\hat{x} \leq x$ pointwise (see Section V for an example), then we can define

$$t \dot{-} s = \begin{cases} t, & \text{if } t > s \\ \perp, & \text{if } t = s \end{cases} \tag{15}$$

where $\perp$ is the least element of $\mathcal{T}$. For $\dot{+}$ we simply take

$$t \dot{+} s = t \vee s. \tag{16}$$

It is easy to verify that $s \dot{+} (t \dot{-} s) = t$, for every $t, s \in \mathcal{T}$ with $s \leq t$.
3) Assume that $\mathcal{T}$ is finite, say $\mathcal{T} = \{0, 1, \cdots, N-1\}$. Define $\dot{+}$ and $\dot{-}$ as the addition and subtraction in the Abelian group $\mathbb{Z}_N$, i.e., $t \dot{+} s = (t + s) \bmod N$ and $t \dot{-} s = (t - s) \bmod N$, where "mod" denotes *modulo*. Observe that, in the binary case, both $\dot{+}$ and $\dot{-}$ correspond to the "exclusive OR" operator. ■

The process of decomposing a signal $x \in V_0$ in terms of (12), (13) will be referred to here as the *pyramid transform* of x, whereas the process of synthesizing x by means of (14) will be referred to as the *inverse pyramid transform*. Block diagrams, illustrating the pyramid transform and its inverse, when $J = \{0, 1, 2\}$, are depicted in Fig. 1.

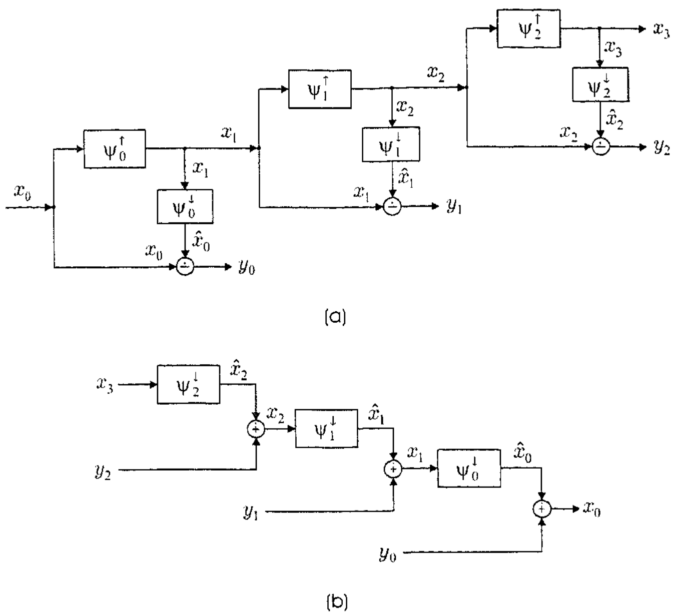

Fig. 1. Illustration of (a) a three-level pyramid transform and (b) its inverse.

IV. Linear Pyramids

A case of particular interest to signal processing and analysis applications is when the analysis/synthesis operators are linear and translation invariant. In this section, we discuss a nonseparable extension of the original one-dimensional (1-D) Burt–Adelson pyramid to two dimensions.

We restrict attention to 2-D discrete-time signals x. We consider pyramid transforms satisfying the following assumptions:

1) all domains V_j are identical;
2) operators $\dot{+}$ and $\dot{-}$ are the usual addition and difference operators $+$ and $-$, respectively;
3) at every level j, we use the same analysis and synthesis operators, i.e., $\psi_j^{\uparrow}$ and $\psi_j^{\downarrow}$ are independent of j; they are denoted by $\psi^{\uparrow}$ and $\psi^{\downarrow}$, respectively;
4) $\psi^{\uparrow}$ and $\psi^{\downarrow}$ are linear operators;
5) $\psi^{\uparrow}$ and $\psi^{\downarrow}$ are translation invariant. The translation invariance condition should hold for every translation operator $\tau = \tau_{(k,l)}$, given by $(\tau_{(k,l)} x)(m, n) = x(m-k, n-l)$.

A straightforward computation shows that there exist *convolution kernels* $\tilde{h}, h$ such that $\psi^{\uparrow}$ and $\psi^{\downarrow}$ are of the following general form (see Rioul [7]):

$$(\psi^{\uparrow} x)(m, n) = \sum_{k,l=-\infty}^{\infty} \tilde{h}(2m - k, 2n - l) x(k, l)$$

$$\psi^{\downarrow}(x)(m, n) = \sum_{k,l=-\infty}^{\infty} h(m - 2k, n - 2l) x(k, l).$$

The pyramid condition $\psi^{\uparrow}\psi^{\downarrow} = \mathrm{id}$ amounts to

$$\sum_{k,l=-\infty}^{\infty} \tilde{h}(2m - k, 2n - l) h(k, l) = \delta(m, n) \tag{17}$$

where δ is the 2-D Dirac-delta sequence, given by $\delta(m, n) = 1$, if $m = n = 0$, and 0 otherwise. This is known as the *biorthogonality condition*.

Let us consider the case when, in the analysis step, a 2×2 pixel block $\{(2m, 2n), (2m+1, 2n), (2m+1, 2n+1), (2m, 2n+1)\}$ at level j is replaced by one pixel (m, n) at level $j+1$. The value of this pixel is a weighted average over 16 pixels at level j, namely the pixels in the 4×4 block surrounding the 2×2 block; see Fig. 2(a). To be precise

$$\begin{aligned} \psi^{\uparrow}(x)(m, n) = {} & a(x(2m, 2n) + x(2m+1, 2n) \\ & + x(2m+1, 2n+1) + x(2m, 2n+1)) \\ & + b(x(2m-1, 2n) + x(2m-1, 2n+1) \\ & + x(2m, 2n-1) + x(2m+1, 2n-1) \\ & + x(2m+2, 2n) + x(2m+2, 2n+1) \\ & + x(2m, 2n+2) + x(2m+1, 2n+2)) \\ & + c(x(2m-1, 2n-1) + x(2m+2, 2n-1) \\ & + x(2m+2, 2n+2) + x(2m-1, 2n+2)). \end{aligned} \tag{18}$$

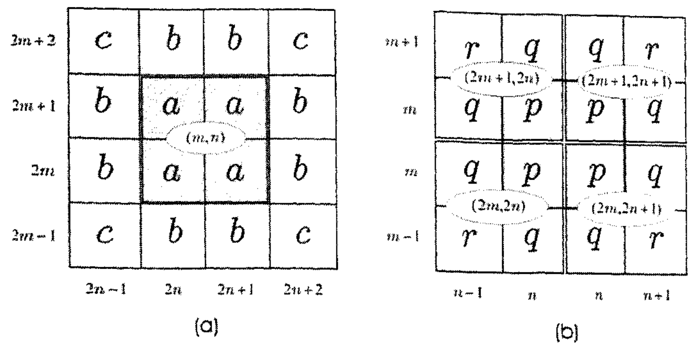

Fig. 2. Stencils for (a) $\psi^{\uparrow}$ in (18) and (b) $\psi^{\downarrow}$ in (19)–(22).

The synthesis step subdivides a pixel (m, n) at level $j+1$ into four pixels $\{(2m, 2n), (2m+1, 2n), (2m+1, 2n+1), (2m, 2n+1)\}$ at level j. The values of $\psi^{\downarrow}(x)$ are given by [see Fig. 2(b)]

$$\begin{aligned}\psi^{\downarrow}(x)(2m, 2n) &= px(m, n) + q(x(m-1, n) + x(m, n-1)) \\ &\quad + rx(m-1, n-1) \qquad (19)\end{aligned}$$

$$\begin{aligned}\psi^{\downarrow}(x)(2m+1, 2n) &= px(m, n) + q(x(m+1, n) + x(m, n-1)) \\ &\quad + rx(m+1, n-1) \qquad (20)\end{aligned}$$

$$\begin{aligned}\psi^{\downarrow}(x)(2m, 2n+1) &= px(m, n) + q(x(m, n+1) + x(m-1, n)) \\ &\quad + rx(m-1, n+1) \qquad (21)\end{aligned}$$

$$\begin{aligned}\psi^{\downarrow}(x)(2m+1, 2n+1) &= px(m, n) + q(x(m+1, n) + x(m, n+1)) \\ &\quad + rx(m+1, n+1). \qquad (22)\end{aligned}$$

The pyramid condition (17) leads to the following relations:

$$\begin{aligned}4ap + 8bq + 4cr &= 1 \\ 2aq + 2bp + 2br + 2cq &= 0 \\ ar + 2bq + cp &= 0.\end{aligned}$$

It is obvious that, due to the symmetry in $\tilde{h}$, $\psi^{\uparrow}$ maps the high-frequency signals $x(m, n) = (-1)^m, (-1)^n, (-1)^{m+n}$ onto the zero signal. We impose the following two normalizing conditions: $\psi^{\uparrow}$ and $\psi^{\downarrow}$ map a constant signal onto the same constant signal (albeit at a different level of the pyramid). This yields the following two conditions:

$$4a + 8b + 4c = 1 \quad \text{and} \quad p + 2q + r = 1.$$

The *unique* solution of the previous system of five equations with six unknowns can be expressed in terms of a as

$$b = a, \quad c = \frac{1}{4} - 3a, \quad p = q = \frac{4a}{16a-1}, \quad r = \frac{4a-1}{16a-1}.$$

Clearly, we must exclude $a = 1/16$ in order to avoid singularities. When $a = 1/4$, we have that

$$a = b = \tfrac{1}{4}, \quad c = -\tfrac{1}{2}, \quad p = q = \tfrac{1}{3}, \quad r = 0. \qquad (23)$$

An example, illustrating the resulting linear pyramid, is depicted in Fig. 3. Due to the calculations associated with (18)–(22), the resulting images will not have integer gray-values between 0–255, as required for computer storage and display, even if the original image is already quantized to these values. To comply with this requirement, all gray-values of the images depicted in Fig. 3 have been mapped to integers between 0–255, with the minimum and maximum values being mapped to 0 and 255, respectively. Finally, and for clarity of presentation, the size of some of the images depicted in Fig. 3 (and later in this paper) is larger than their actual size (e.g., although the size of x_1 should be half the size of x_0, this is not the case in Fig. 3).

It is worth noticing here that most of the linear pyramid transforms used in the literature are 1-D. These transforms, when applied on images, use *separable* analysis and synthesis operators. The linear pyramid transform discussed in this example employs nonseparable analysis and synthesis operators. It is, therefore, an example of a pure (nonseparable) 2-D linear pyramid transform.

V. Morphological Adjunction Pyramids

In this section, we consider the special, but interesting, case when the signal domains are complete lattices and the analysis and synthesis operators between two adjacent levels in the pyramid form an adjunction. More precisely, we make the following assumptions: 1) all domains V_j have the structure of a complete lattice and 2) the pair $(\psi_j^{\uparrow}, \psi_j^{\downarrow})$ is an adjunction between V_j and V_{j+1}. In this case, $\psi_j^{\uparrow}$ is an erosion and $\psi_j^{\downarrow}$ is a dilation. It is easy to see that the pyramid condition is satisfied if and only if $\psi_j^{\downarrow}$ is injective, or, alternatively, if $\psi_j^{\uparrow}$ is surjective. This is a direct consequence of our comment just before Proposition 1, the fact that $(\psi_j^{\uparrow}, \psi_j^{\downarrow})$ is an adjunction, and the last two properties in (3). Notice that $\psi_j^{\downarrow}\psi_j^{\uparrow}$ is an opening and hence $\psi_j^{\downarrow}\psi_j^{\uparrow} \leq \text{id}$, i.e., the approximation operator $\psi_j^{\downarrow}\psi_j^{\uparrow}$ is anti-extensive.

In this section, we distinguish between two types of pyramids: those ones that involve sample reduction (i.e., multiscale pyramids) and those ones that do not (i.e., single-scale pyramids).

A. Multiscale Pyramids

1) Representation: In this subsection, we give a complete characterization of analysis and synthesis operators, between two adjacent levels $j = 0$ and $j = 1$ in a pyramid, under the following general assumptions.

1) $V_0 = V_1 = \text{Fun}(\mathbb{Z}^d, \mathcal{T})$, the complete lattice of functions from $\mathbb{Z}^d$ into a given complete lattice $\mathcal{T}$ of gray-values.
2) The analysis operator $\psi^{\uparrow}: V_0 \to V_1$ and the synthesis operator $\psi^{\downarrow}: V_1 \to V_0$ form an adjunction between V_0 and V_1, i.e.,
$$x_1 \leq \psi^{\uparrow}(x_0) \Leftrightarrow \psi^{\downarrow}(x_1) \leq x_0, \qquad x_0 \in V_0,\ x_1 \in V_1.$$
3) For every translation $\tau = \tau_{(k_1, k_2, \cdots, k_d)}$ of $\mathbb{Z}^d$, where
$$\begin{aligned}(\tau x)(n) &= (\tau x)(n_1, n_2, \cdots, n_d) \\ &= x(n_1 - k_1, n_2 - k_2, \cdots, n_d - k_d) \\ &= x(n - k), \quad n, k \in \mathbb{Z}^d\end{aligned}$$

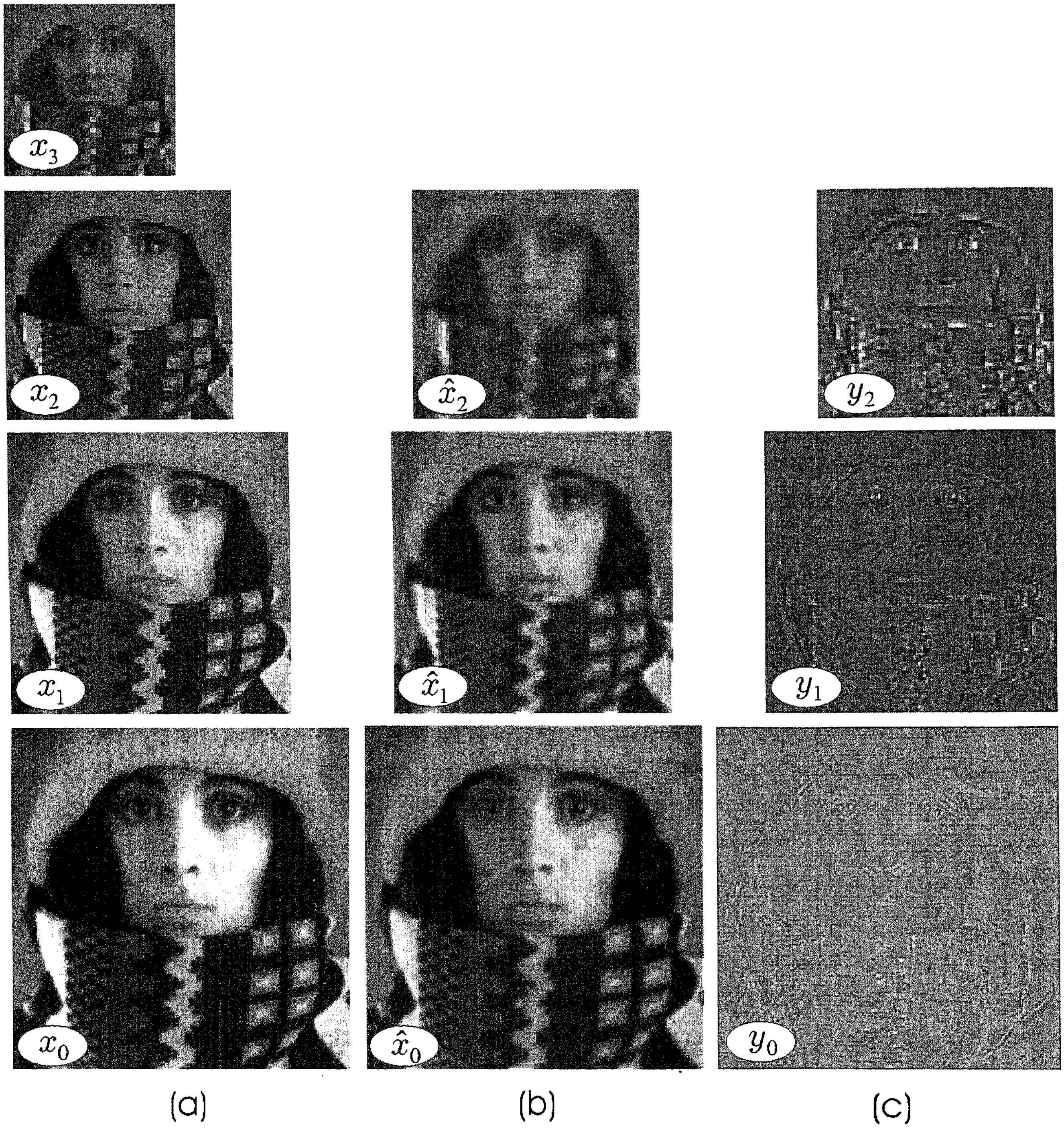

Fig. 3. Multiresolution image decomposition based on the 2-D linear pyramid transform of Section IV. (a) An image x_0 and its decomposition $\{x_0, x_1, x_2, x_3\}$ obtained by means of the analysis operator $\psi^{\uparrow}$ in (18), where a, b, c are given by (23). (b) The approximation images $\{\hat{x}_0, \hat{x}_1, \hat{x}_2\}$ obtained from $\{x_1, x_2, x_3\}$ by means of the synthesis operator $\psi^{\downarrow}$ in (19)–(22), where p, q, r are given by (23). (c) The detail images $\{y_0, y_1, y_2\}$.

we have that

$$\psi^{\uparrow}\tau^2 = \tau\psi^{\uparrow} \quad \text{and} \quad \psi^{\downarrow}\tau = \tau^2\psi^{\downarrow}$$

where $\tau^2 = \tau\tau$ denotes double translation.

Our characterization is given in terms of adjunctions (e, d) on the complete lattice $\mathcal{T}$ and is closely related to the representation of translation invariant adjunctions for grayscale functions in mathematical morphology [5].

Proposition 2: Let $(\psi^{\uparrow}, \psi^{\downarrow})$ be an adjunction on $\mathrm{Fun}(\mathbb{Z}^d, \mathcal{T})$. The translation invariance condition $\psi^{\uparrow}\tau^2 = \tau\psi^{\uparrow}$ implies that $\psi^{\downarrow}\tau = \tau^2\psi^{\downarrow}$ and vice versa. Every adjunction satisfying these equivalent conditions is of the form

$$\psi^{\uparrow}(x)(n) = \bigwedge_{k\in\mathbb{Z}^d} e_{k-2n}(x(k)),$$
$$\psi^{\downarrow}(x)(k) = \bigvee_{n\in\mathbb{Z}^d} d_{k-2n}(x(n)) \qquad (24)$$

where (e_k, d_k) defines an adjunction on $\mathcal{T}$, for every $k \in \mathbb{Z}^d$.

Proof: We show the first part of the assertion concerning translation invariance; the other implication is proved analogously. Assume that $\psi^{\uparrow}\tau^2 = \tau\psi^{\uparrow}$, for every translation τ. For $x_0, x_1 \in \mathrm{Fun}(\mathbb{Z}^d, \mathcal{T})$, we have the following equivalences:

$$\begin{aligned}\psi^{\downarrow}\tau(x_1) \le x_0 &\Leftrightarrow \tau(x_1) \le \psi^{\uparrow}(x_0) \Leftrightarrow x_1 \le \tau^{-1}\psi^{\uparrow}(x_0)\\ &\Leftrightarrow x_1 \le \psi^{\uparrow}\tau^{-2}(x_0) \Leftrightarrow \psi^{\downarrow}(x_1) \le \tau^{-2}(x_0)\\ &\Leftrightarrow \tau^2\psi^{\downarrow}(x_1) \le x_0.\end{aligned}$$

This yields that $\psi^{\downarrow}\tau = \tau^2\psi^{\downarrow}$.

We next prove the identities in (24). From [5, Prop. 5.3] it follows that every adjunction $(\psi^{\uparrow}, \psi^{\downarrow})$ on $\mathrm{Fun}(\mathbb{Z}^d, \mathcal{T})$ is of the form

$$\psi^{\uparrow}(x)(n) = \bigwedge_{k\in\mathbb{Z}^d} e'_{k,n}(x(k)),\ \psi^{\downarrow}(x)(k) = \bigvee_{n\in\mathbb{Z}^d} d'_{n,k}(x(n)) \tag{25}$$

where $(e'_{k,n}, d'_{n,k})$ is an adjunction on $\mathcal{T}$, for every $n, k \in \mathbb{Z}^d$. Equation (25), together with condition $\psi^{\uparrow}\tau^2 = \tau\psi^{\uparrow}$, yields

$$\bigwedge_{k\in\mathbb{Z}^d} e'_{k+2m,n}(x(k)) = \bigwedge_{k\in\mathbb{Z}^d} e'_{k,n-m}(x(k)).$$

Since this identity holds for every $x \in \mathrm{Fun}(\mathbb{Z}^d, \mathcal{T})$ and $n, m \in \mathbb{Z}^d$, we conclude that $e'_{k+2m,n} = e'_{k,n-m}$, for every $k, n, m \in \mathbb{Z}^d$. Similarly, equation (25), together with condition $\psi^{\downarrow}\tau = \tau^2\psi^{\downarrow}$, leads to the identity $d'_{n+m,k} = d'_{n,k-2m}$, for every $k, n, m \in \mathbb{Z}^d$. Set $e_k = e'_{k,0}$ and $d_k = d'_{0,k}$ and observe that (e_k, d_k) constitutes an adjunction on $\mathcal{T}$. A straightforward manipulation shows that $e'_{k,n} = e_{k-2n}$ and $d'_{n,k} = d_{k-2n}$, whence we arrive at the identities in (24). ■

Now that we have found a characterization of analysis and synthesis operators which form adjunctions, we may ask ourselves: for which of these pairs is the pyramid condition satisfied? The next proposition answers this question. In the following, we define the *support* A of the analysis/synthesis pair (24), as being the set of all vectors $k \in \mathbb{Z}^d$ for which the adjunction (e_k, d_k) is nontrivial; i.e., $e_k \not\equiv \top$ and $d_k \not\equiv \bot$, where $\bot, \top$ are the least and greatest element of $\mathcal{T}$, respectively. We introduce the following notation: for $n \in \mathbb{Z}^d$, we define $\mathbb{Z}^d[n] = \{k \in \mathbb{Z}^d | k - n \in 2\mathbb{Z}^d\}$, where $2\mathbb{Z}^d$ denotes all vectors in $\mathbb{Z}^d$ with *even* coordinates. The sets $\mathbb{Z}^d[n]$ yield a disjoint partition of $\mathbb{Z}^d$ into 2^d parts. For $A \subseteq \mathbb{Z}^d$ and $n \in \mathbb{Z}^d$, we set $A[n] = A \cap \mathbb{Z}^d[n]$, which yields a partition of A comprising at most 2^d nonempty and mutually disjoint subsets.

Proposition 3: Consider the analysis/synthesis pair of Proposition 2, and let $A \subseteq \mathbb{Z}^d$ denote its support. Suppose that there exists an $a \in A$ such that 1) $A[a] = \{a\}$ and 2) d_a is injective. Then, the pyramid condition is satisfied.

Proof: Assume that conditions 1) and 2) hold. We show that $\psi^{\downarrow}$ is injective. From (24) notice that

$$\psi^{\downarrow}(x)(k) = \bigvee_{m\in A[k]} d_m\left(x\left(\frac{k-m}{2}\right)\right)$$

for every $k \in \mathbb{Z}^d$. If $x_1 \neq x_2$, then $x_1(n) \neq x_2(n)$, for some $n \in \mathbb{Z}^d$. Let $k = 2n + a$, then $A[k] = A \cap \mathbb{Z}^d[2n + a] = A \cap \mathbb{Z}^d[a] = A[a] = \{a\}$, hence, for $i = 1, 2$, $\psi^{\downarrow}(x_i)(k) = d_a(x_i(n))$. Since d_a is assumed to be injective, we find that $\psi^{\downarrow}(x_1)(k) \neq \psi^{\downarrow}(x_2)(k)$. Therefore, $\psi^{\downarrow}$ is injective and the pyramid condition is satisfied. ■

Observe that d_a is injective if and only if $e_a d_a = \mathrm{id}$. In the following subsection, we consider a particular subclass of analysis and synthesis operators, given by (24), with (e_a, d_a) being either the trivial adjunction $(\top, \bot)$ or the adjunction $(\mathrm{id}, \mathrm{id})$.

2) Pyramids Based on Flat Adjunctions: Let $A \subseteq \mathbb{Z}^d$ be given and assume that $(e_k, d_k) = (\mathrm{id}, \mathrm{id})$, for $k \in A$, and $(e_k, d_k) = (\top, \bot)$ elsewhere. In other words, A is the support of $(\psi^{\uparrow}, \psi^{\downarrow})$. Now, (24) reduces to

$$\begin{aligned}\psi^{\uparrow}(x)(n) &= \bigwedge_{k\in A} x(2n+k),\\ \psi^{\downarrow}(x)(k) &= \bigvee_{n\in A[k]} x\left(\frac{k-n}{2}\right).\end{aligned} \tag{26}$$

In mathematical morphology, these two operators are called *flat operators*, since they transform flat signals ($x(k) = t_0$, for k in the domain of x, and $\bot$ outside) into flat signals; see [5, Ch. 11]. Flatness of an operator means, in particular, that no other gray-values than those present in the signal are created. The resulting pyramids make sense for every gray-value set $\mathcal{T} \subseteq \overline{\mathbb{R}}$ and, in particular, for the binary case $\mathcal{T} = \{0, 1\}$. From Proposition 3, notice that, if there exists an $a \in A$ such that $A[a] = \{a\}$, then the pyramid condition is satisfied.

Since $(\psi^{\uparrow}, \psi^{\downarrow})$ is an adjunction, the approximation signal $\hat{\psi}(x) = \psi^{\downarrow}\psi^{\uparrow}(x)$ satisfies $\hat{\psi}(x) \le x$ (pointwise inequality) and the error signal $y(n) = x(n) - \hat{\psi}(x)(n)$ is nonnegative. The scheme in (26) has been proposed earlier by Heijmans and Toet in their paper on morphological sampling (with the roles of dilation and erosion interchanged) [12].

Example 2 (Morphological Haar Pyramid): Let $A = \{(0, 0), (0, 1), (1, 1), (1, 0)\}$. It is evident that $A[m, n] = \{(m, n)\}$, for $(m, n) \in A$. Hence, the pyramid condition is satisfied. The operators $\psi^{\uparrow}$ and $\psi^{\downarrow}$ are given by

$$\begin{aligned}&\psi^{\uparrow}(x)(m, n)\\ &\quad = x(2m, 2n) \wedge x(2m, 2n+1) \wedge x(2m+1, 2n+1)\\ &\qquad \wedge x(2m+1, 2n)\end{aligned} \tag{27}$$

$$\begin{aligned}&\psi^{\downarrow}(x)(2m, 2n)\\ &\quad = \psi^{\downarrow}(x)(2m, 2n+1) = \psi^{\downarrow}(x)(2m+1, 2n+1)\\ &\quad = \psi^{\downarrow}(x)(2m+1, 2n) = x(m, n).\end{aligned} \tag{28}$$

This leads to a signal decomposition scheme which we call the *morphological Haar pyramid*. The operators in (27) and (28) are the morphological counterparts of that of a linear pyramid where the operators coincide with the lowpass filters associated with the Haar wavelet (see [22] for more details). ■

Example 3: A more interesting example is obtained by taking A to be the 3×3 square centered at the origin; i.e., $A = \{(-1, -1), (-1, 0), (-1, 1), (0, -1), (0, 0), (0, 1), (1, -1), (1, 0), (1, 1)\}$. We have that $A[0, 0] = \{(0, 0)\}$, $A[0, \pm 1] = \{(0, -1), (0, 1)\}$, $A[\pm 1, 0] = \{(-1, 0), (1, 0)\}$,

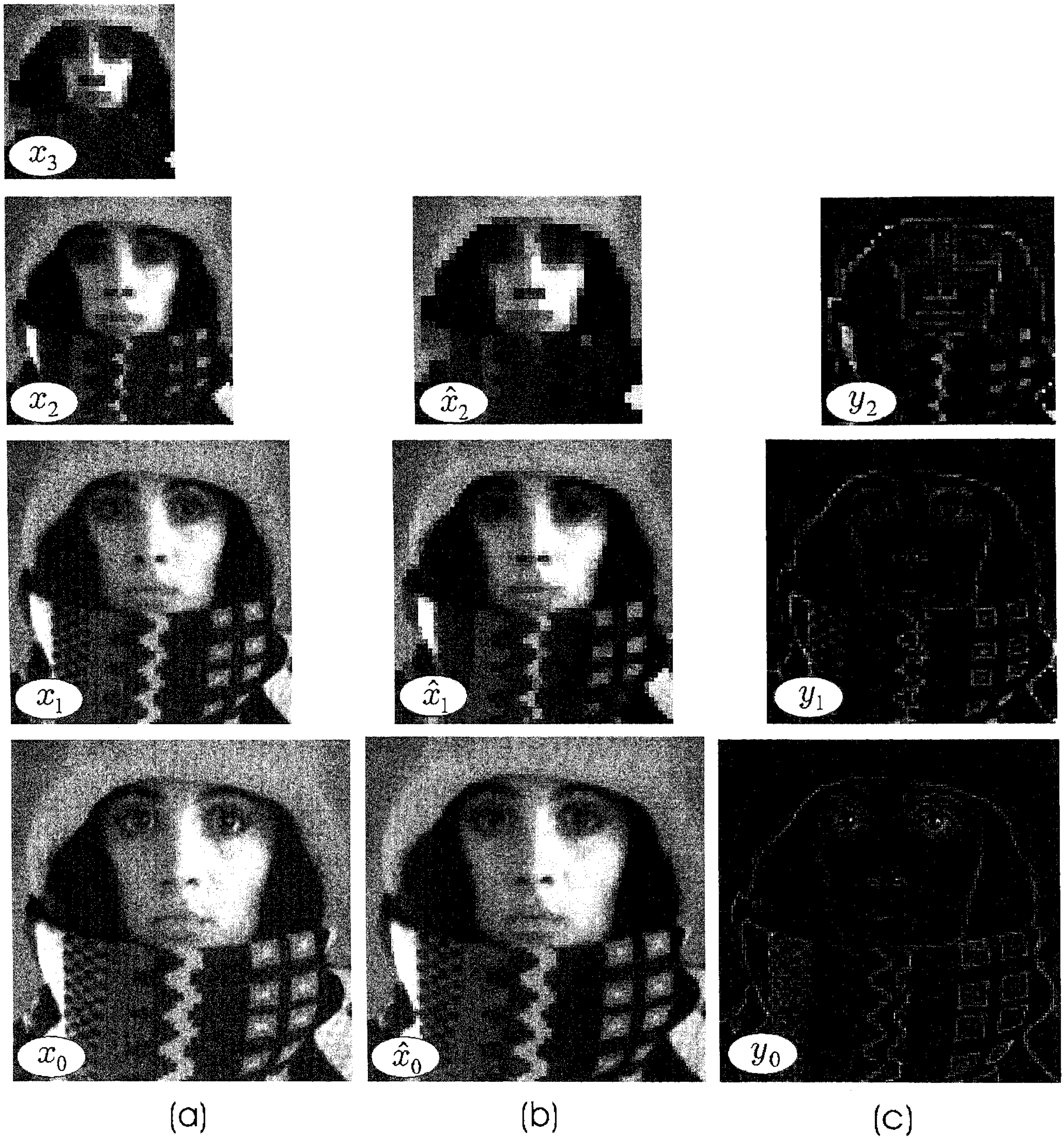

Fig. 4. Multiresolution image decomposition based on the morphological pyramid transform of Example 3. (a) An image x_0 and its decomposition $\{x_0, x_1, x_2, x_3\}$ obtained by means of the analysis operator $\psi^{\uparrow}$ in (29). (b) The approximation images $\{\hat{x}_0, \hat{x}_1, \hat{x}_2\}$ obtained from $\{x_1, x_2, x_3\}$ by means of the synthesis operator $\psi^{\downarrow}$ in (30)–(33). (c) The detail images $\{y_0, y_1, y_2\}$.

and $A[\pm 1, \pm 1] = \{(-1, -1), (-1, 1), (1, -1), (1, 1)\}$. The operators $\psi^{\uparrow}$ and $\psi^{\downarrow}$ are therefore given by

$$\psi^{\uparrow}(x)(m, n) = \bigwedge_{-1 \le k, l \le 1} x(2m + k, 2n + l), \quad (29)$$

$$\psi^{\downarrow}(x)(2m, 2n) = x(m, n) \quad (30)$$

$$\psi^{\downarrow}(x)(2m, 2n + 1) = x(m, n) \vee x(m, n + 1) \quad (31)$$

$$\psi^{\downarrow}(x)(2m + 1, 2n) = x(m, n) \vee x(m + 1, n) \quad (32)$$

$$\begin{aligned}\psi^{\downarrow}(x)&(2m + 1, 2n + 1)\\ &= x(m, n) \vee x(m, n + 1) \vee x(m + 1, n + 1)\\ &\quad \vee x(m + 1, n). \end{aligned} \quad (33)$$

Fig. 4 illustrates such a decomposition. Operators $\dot{+}$ and $\dot{-}$ are taken here to be the usual addition and difference operators $+$ and $-$.

We should point out that the detail signals, depicted in Fig. 4, assume only nonnegative values, which is a direct consequence of the fact that the analysis and synthesis operators are adjunctions. This should be compared to the linear pyramids, in which case the detail signals usually assume both positive and negative values. This may be advantageous in image compression and coding applications, as it has been already discussed in [15]. Finally, it has been experimentally demonstrated in [15] that the pyramid based on (29)–(33) enjoys superior performance in lossy compression, as compared to a number of alternative nonlinear, as well as linear, choices. ■

If we take $A = \{0\}$ and $e_0 = d_0 = \mathrm{id}$, then Proposition 3 is trivially satisfied. Denoting the corresponding analysis/synthesis pair by $\sigma^{\uparrow}$, $\sigma_{\perp}^{\downarrow}$, we have

$$\sigma^{\uparrow}(x)(n) = x(2n) \tag{34}$$

$$\sigma_{\perp}^{\downarrow}(x)(2n) = x(n), \quad \sigma_{\perp}^{\downarrow}(x)(m) = \perp, \quad \text{if } m \notin 2\mathbb{Z}^d. \tag{35}$$

The pair $(\psi^{\uparrow}, \psi^{\downarrow})$ in (26) can be written as

$$\psi^{\uparrow} = \sigma^{\uparrow}\varepsilon_A \quad \text{and} \quad \psi^{\downarrow} = \delta_A\sigma_{\perp}^{\downarrow} \tag{36}$$

where $(\varepsilon_A, \delta_A)$ is the adjunction given by (4) and (5). This shows that the analysis and synthesis operators of pyramids based on flat adjunctions can be implemented by means of flat erosions, followed by dyadic subsampling by means of $\sigma^{\uparrow}$, and flat dilations, following dyadic upsampling by means of $\sigma_{\perp}^{\downarrow}$.

If we replace the erosion ε_A in (36) by the opening $\delta_A\varepsilon_A$, then the pyramid condition is still satisfied, provided that we make an assumption which is slightly stronger than condition 1) in Proposition 3. Indeed, we have the following result.

Proposition 4: Let A be a structuring element such that $A[0] = \{0\}$. The analysis operator $\psi^{\uparrow} = \sigma^{\uparrow}\delta_A\varepsilon_A$ and the synthesis operator $\psi^{\downarrow} = \delta_A\sigma_{\perp}^{\downarrow}$ satisfy the pyramid condition.

Proof: From the fact that $(\varepsilon_A, \delta_A)$ is an adjunction, we get that $\psi^{\uparrow}\psi^{\downarrow} = \sigma^{\uparrow}\delta_A\varepsilon_A\delta_A\sigma_{\perp}^{\downarrow} = \sigma^{\uparrow}\delta_A\sigma_{\perp}^{\downarrow}$. Now

$$\begin{aligned}\sigma^{\uparrow}\delta_A\sigma_{\perp}^{\downarrow}(x)(n) &= \bigvee_{k\in A}\sigma_{\perp}^{\downarrow}(x)(2n-k)\\ &= \bigvee_{k\in A[0]}\sigma_{\perp}^{\downarrow}(x)(2n-k)\\ &= \sigma_{\perp}^{\downarrow}(x)(2n) = x(n).\end{aligned}$$

This yields that $\psi^{\uparrow}\psi^{\downarrow} = \mathrm{id}$ and the result is proved. ■

Notice that the pair $(\psi^{\uparrow}, \psi^{\downarrow})$ in this proposition does not constitute an adjunction. The pyramid decomposition of Sun and Maragos [9] given by $\psi^{\uparrow}(x)(n) = \delta_A\varepsilon_A(x)(2n)$, where $A = \{-1, 0, 1\}$, and $\psi^{\downarrow}(x)(2n) = x(n)$ and $\psi^{\downarrow}(x)(2n+1) = x(n) \vee x(n+1)$, fits within this latter class. Notice that $A[0] = \{0\}$, as required by Proposition 4. It is not difficult to generalize this example to more dimensions.

B. Morphological Skeleton Decomposition

In this subsection, we show that the morphological skeleton decomposition scheme can fit into our pyramidal framework. Recall Lantuéjoul's formula for discrete skeletons, well-known from mathematical morphology [4]. Let $\mathcal{T} \subseteq \overline{\mathbb{R}}$, define $\mathcal{T}' = \{t - s | t, s \in \mathcal{T} \text{ and } s \le t\}$, and consider the set of signals $\mathrm{Fun}(E, \mathcal{T})$. Assume that (ε, δ) is an adjunction on the complete lattice $\mathrm{Fun}(E, \mathcal{T})$. Let $x \in \mathrm{Fun}(E, \mathcal{T})$ and let $K \ge 0$ be such that $\varepsilon^{K+1}(x) = \varepsilon^K(x)$, where $\varepsilon^0 = \mathrm{id}$ and $\varepsilon^j = \varepsilon\varepsilon\cdots\varepsilon$ (j times). Since $\delta\varepsilon$ is an opening, we have that $\varepsilon^j(x) \ge (\delta\varepsilon)\varepsilon^j(x)$. Define $y_j \in \mathrm{Fun}(E, \mathcal{T}')$ by

$$\begin{cases} y_j = \varepsilon^j(x) - (\delta\varepsilon)\varepsilon^j(x), & j = 0, 1, \cdots, K-1,\\ y_K = \varepsilon^K(x). \end{cases} \tag{37}$$

It is possible to reconstruct x from $y_0, y_1, \cdots, y_K$ by means of the (backward) recursion formula

$$\begin{cases} x_K = y_K\\ x_j = \delta(x_{j+1}) + y_j, & j = K-1, K-2, \cdots, 0. \end{cases}$$

It is easy to verify that $x_j = \varepsilon^j(x)$, hence $x_0 = x$.

Our attempt to fit Lantuéjoul's skeleton decomposition into a pyramid framework is not only successful, but even more, it leads to a decomposition, which may be better than Lantuéjoul's, in the sense that it may contain less data.

Assume that $\mathcal{L}$ is a complete lattice and that (ε, δ) is an adjunction on $\mathcal{L}$. Set $V_j = \mathrm{Ran}(\varepsilon^j)$ (i.e., the *range* of operator ε^j), and let $\psi_j^{\uparrow}: V_j \to V_{j+1}$ and $\psi_j^{\downarrow}: V_{j+1} \to V_j$ be given by $\psi_j^{\uparrow} = \varepsilon$ and $\psi_j^{\downarrow} = \varepsilon^j\delta^{j+1}$. We can show the following result [22].

Lemma 1: The pair $(\psi_j^{\uparrow}, \psi_j^{\downarrow})$ defines an adjunction between V_j and V_{j+1}.

It is obvious that $\psi_j^{\uparrow}$ is surjective. We therefore conclude that the pyramid condition holds.

Let us now assume that the underlying lattice $\mathcal{L}$ is of the form $\mathrm{Fun}(E, \mathcal{T})$, where $\mathcal{T} \subseteq \overline{\mathbb{R}}$. We can set $Y_j = \mathrm{Fun}(E, \mathcal{T}')$, where $\mathcal{T}' = \{t - s | t, s \in \mathcal{T} \text{ and } s \le t\}$, and consider $\dot{+}$, $\dot{-}$ to be standard addition and subtraction. Given an input $x_0 = x \in V_0 = \mathrm{Fun}(E, \mathcal{T})$, we arrive at the following signal analysis scheme:

$$\begin{cases} x_0 = x \in V_0\\ x_{j+1} = \varepsilon(x_j) \in V_{j+1}, & j \ge 0\\ y_j = x_j - \varepsilon^j\delta^{j+1}(x_{j+1}). \end{cases}$$

For synthesis, we find

$$x = x_0, \quad x_j = \varepsilon^j\delta^{j+1}(x_{j+1}) + y_j, \quad j \ge 0.$$

Notice that the detail signal y_j can be written as

$$y_j = \varepsilon^j(x) - (\varepsilon^j\delta^j)(\delta\varepsilon)\varepsilon^j(x). \tag{38}$$

Comparing (38) to the original Lantuéjoul formula (37), we see that, in our new decomposition, we have an extra closing $\varepsilon^j\delta^j$. As a result, the detail signal y_j in (38) is never larger than the detail signal in the Lantuéjoul formula (37). It may therefore give rise to a more efficient compression. This skeleton decomposition has been found earlier by Goutsias and Schonfeld [11].

An alternative approach to signal decomposition, suggested by Kresch [17], is to set $Y_j = \mathrm{Fun}(E, \mathcal{T})$ and define $\dot{-}$ by means of (15). In this case, $\dot{+}$ is given by (16). Given an input

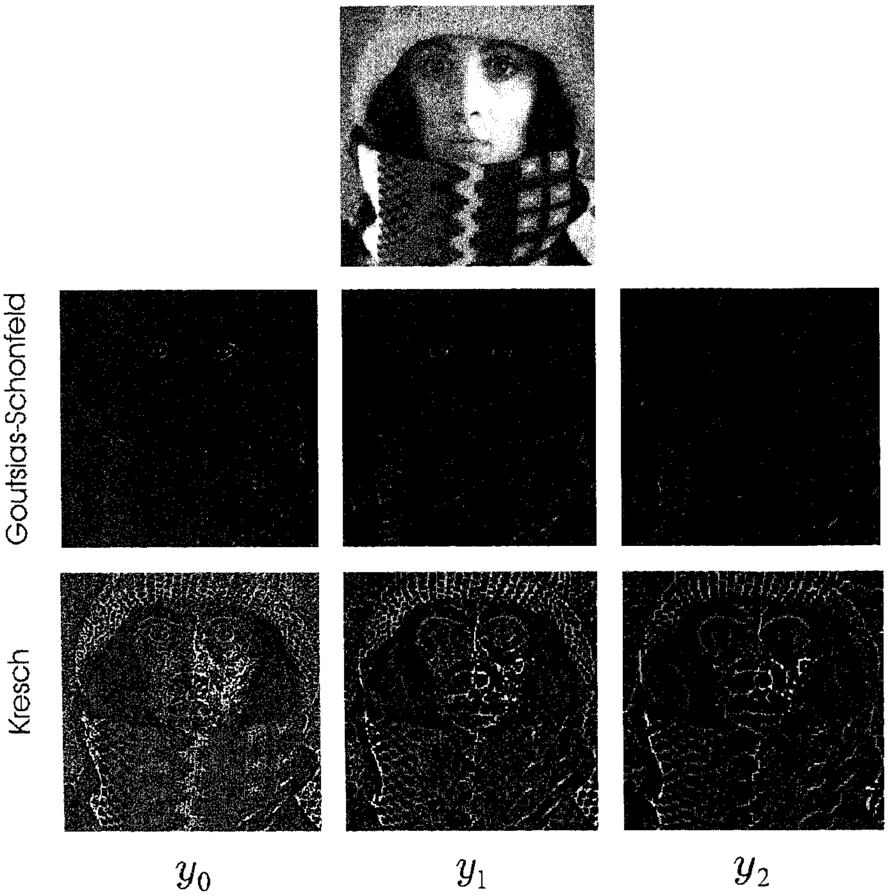

Fig. 5. Grayscale image and the decompositions obtained by means of the Goutsias–Schonfeld and Kresch skeleton transforms.

$x_0 = x \in V_0 = \mathrm{Fun}(E, \mathcal{T})$, we arrive at the following signal analysis scheme:

$$\begin{cases} x_0 = x \in V_0 \\ x_{j+1} = \varepsilon(x_j) \in V_{j+1}, \quad j \geq 0 \\ y_j(n) = \begin{cases} x_j(n), & \text{if } x_j(n) \neq \varepsilon^j \delta^{j+1}(x_{j+1})(n) \\ \bot, & \text{otherwise} \end{cases} \end{cases}.$$

The synthesis scheme is as follows:

$$x = x_0, \quad x_j = \varepsilon^j \delta^{j+1}(x_{j+1}) \vee y_j, \quad j \geq 0. \tag{39}$$

Notice that the detail signal y_j can be written as

$$y_j(n) = \begin{cases} \varepsilon^j(x)(n), & \text{if } \varepsilon^j(x)(n) \neq (\varepsilon^j\delta^j)(\delta\varepsilon)\varepsilon^j(x)(n) \\ \bot, & \text{otherwise.} \end{cases}$$

Assume again that there exists a $K \geq 0$ such that $\varepsilon^{K+1}(x) = \varepsilon^K(x)$ and set $y_K = \varepsilon^K(x)$. Apply δ^j on both sides of (39), and use the fact that δ^j distributes over suprema; we find $\delta^j(x_j) = \delta^{j+1}(x_{j+1}) \vee \delta^j(y_j)$. This implies the following formula:

$$\delta^{K-k}(x_{K-k}) = \bigvee_{j=0}^{k} \delta^{K-j}(y_{K-j}), \qquad k = 0, 1, \cdots, K.$$

Substitution of $k = K$ yields $x_0 = \bigvee_{k=0}^{K} \delta^k(y_k)$. Thus, the original signal can be recovered as a supremum of dilations of the detail signal.

The Goutsias–Schonfeld and Kresch skeleton decomposition schemes are quite different, even though they satisfy the same algebraic description. Fig. 5 depicts the result of applying these decompositions to a grayscale image x. The 3×3 square structuring element A that contains the origin has been used in both cases. In the Goutsias–Schonfeld case, y_0 is the *top-hat transform* [5] of x, since $y_0 = x - x \circ A$. However, the detail signal y_0 in the Kresch case takes value zero (it is black) at all pixels at which $x = x \circ A$ and equals x at all other pixels.

VI. Other Nonlinear Pyramids

The morphological pyramids discussed in the previous section are based on the concept of adjunction and they all satisfy the pyramid condition. In this section, we show that a number of alternative nonlinear pyramids can be constructed, such that the pyramid condition is satisfied as well. We divide this section into three subsections, which present examples of morphological pyramids, median pyramids, and pyramids that employ grayscale quantization.

A. Morphological Pyramids

In most cases, and in order to avoid aliasing, a signal, at level j of a pyramid, is filtered first, by means of a lowpass filter, and then subsampled to obtain the scaled signal at level $j + 1$. In this subsection, we discuss a morphological pyramid scheme in which sampling is done first, followed by filtering.

Let $\mathcal{L} = \text{Fun}(\mathbb{Z}^d, \mathcal{T})$, where $\mathcal{T}$ is a complete chain, and consider the elementary sampling scheme $\sigma^\uparrow$, given by (34), and $\sigma_t^\downarrow$, given by $\sigma_t^\downarrow(x)(2n) = x(n)$ and $\sigma_t^\downarrow(x)(m) = t$, if $m \notin 2\mathbb{Z}^d$. Here, $t \in \mathcal{T}$ is a fixed element; in practice one chooses $t = \perp$ or $\top$ [recall (35)]. Given operators $\phi_j\colon \mathcal{L} \to \mathcal{L}$, we define $V_j = \text{Ran}(\sigma_j)$ (i.e., the *range* of operator ϕ_j) and $\psi_j^\uparrow = \phi_{j+1}\sigma^\uparrow$, $\psi_j^\downarrow = \phi_j\sigma_t^\downarrow$. The pyramid condition can be written as $\phi_{j+1}\sigma^\uparrow\phi_j\sigma_t^\downarrow\phi_{j+1} = \phi_{j+1}$. When all ϕ_j's are identical, say ϕ, the previous condition can be stated as follows:

$$\phi\sigma^\uparrow\phi\sigma_t^\downarrow\phi = \phi \quad \text{on} \quad \mathcal{L}. \tag{40}$$

Example 4 (Toet Pyramid): In this (1-D) example, we use the alternating filter $\phi = \beta\alpha$, where α and β are the opening and closing by the structuring element $A = \{0, 1\}$, and choose $t = \top$. To show the validity of (40), for $\phi = \beta\alpha$, assume that the input signal $x \in V_{j+1}$ has three consecutive values $x(n-1) = r$, $x(n) = s$, $x(n+1) = t$. It is easy to verify that the output value $s' = (\sigma^\uparrow\beta\alpha\sigma_\top^\downarrow)(x)(n)$ is given by $s' = (s \vee t) \wedge (r \vee s)$. Since the input signal x is an element of $\text{Ran}(\beta\alpha)$, it is impossible that $t > s$ and $r > s$. This yields that $s' = s$; hence, the pyramid condition follows.

The resulting pyramidal signal decomposition scheme has been suggested by Toet in [10]. It can be easily extended to the d-dimensional case; there, one chooses $A = \{0, 1\}^d$. ∎

B. Median Pyramids

It has been suggested in [28] that median filtering can be used to obtain a useful nonlinear pyramid that preserves details and produces a decomposition that can be compressed more efficiently than other (linear) hierarchical signal decomposition schemes. We here provide a 2-D example of a pyramid based on median filtering that satisfies the pyramid condition (see also [22], for additional examples).

Assume that $\mathcal{T}$ is a complete chain, and consider a pyramid for which $V_j = \text{Fun}(\mathbb{Z}^2, \mathcal{T})$, for every j, and the same analysis and synthesis operators are used at every level j, given by

$$\psi^\uparrow(x)(m, n) = \text{median}\{x(2m+k, 2n+l) | (k, l) \in A\} \tag{41}$$

where A is the 3×3 square centered at the origin. Take

$$\psi^\downarrow(x)(2m, 2n) = x(m, n) \tag{42}$$
$$\psi^\downarrow(x)(2m, 2n+1) = x(m, n) \wedge x(m, n+1) \tag{43}$$
$$\psi^\downarrow(x)(2m+1, 2n) = x(m, n) \wedge x(m+1, n) \tag{44}$$
$$\psi^\downarrow(x)(2m+1, 2n+1) = x(m, n) \vee x(m, n+1) \vee x(m+1, n+1) \vee x(m+1, n). \tag{45}$$

It is easy to verify that $\psi^\uparrow\psi^\downarrow = \text{id}$, which means that the pyramid condition holds. Fig. 6 illustrates such a decomposition. Operators $\dot{+}$ and $\dot{-}$ are taken here to be the usual addition and difference operators $+$ and $-$. Notice that, in this case, the detail signals may take both positive and negative values, since the resulting operator $\psi^\downarrow\psi^\uparrow$ is not anti-extensive (or extensive).

C. Pyramids with Quantization

An issue that we have not touched upon so far is the topic of quantization. Suppose that the gray-values of the signals at the bottom level of a pyramid are represented by at most N bits. In other words, the gray-value set equals $\mathcal{T}_N = \{0, 1, \cdots, 2^N - 1\}$. The operators involved in a pyramid decomposition scheme may map a signal onto one with values outside this range. In particular, this holds for linear pyramids. In such cases, a quantization step, which reduces the transformed gray-value set, may be indispensable. Also, in cases where the gray-value set does not change by the analysis and synthesis operators (e.g., in the case of flat morphological operators), quantization may be useful in data compression. In this subsection, we briefly discuss the problem of quantization in the context of morphological operators.

Consider the quantization mapping $q\colon \mathcal{T}_N \to \mathcal{T}_{N-1}$, given by $q(t) = \lfloor t/2 \rfloor$, where $\lfloor\cdot\rfloor$ denotes the floor function. For simplicity, we use the same symbol to denote quantization on function spaces, i.e., q can also be considered as the operator from $\text{Fun}(E, \mathcal{T}_N)$ to $\text{Fun}(E, \mathcal{T}_{N-1})$, given by $q(x)(n) = q(x(n))$. There are two different ways of "expanding" a quantized value $t \in \mathcal{T}_{N-1}$ to the original gray-value set $\mathcal{T}_N$, namely by means of mappings $d(t) = 2t$ or $e(t) = 2t + 1$. Again, we use the same notation for their extensions to the corresponding function spaces. The following properties hold:

$$qd(t) = qe(t) = t, \qquad t \in \mathcal{T}_{N-1} \tag{46}$$
$$dq(t) \leq t \leq eq(t), \qquad t \in \mathcal{T}_N. \tag{47}$$

Furthermore, q, d, and e are increasing mappings. It immediately follows that (e, q) is an adjunction from $\mathcal{T}_{N-1}$ to $\mathcal{T}_N$ and that (q, d) is an adjunction from $\mathcal{T}_N$ to $\mathcal{T}_{N-1}$. In what follows, we only use the second adjunction. Similar results can be obtained by using the first one as well.

If we want to emphasize the dependence of q and d on N, we write q_N and d_N. It is obvious how the corresponding operators can be used to construct a single-scale pyramid: remove one bit of information at every analysis step. We can formalize this in the following way: put $V_j = \text{Fun}(E, \mathcal{T}_{N-j})$ and define $\psi_j^\uparrow = q_{N-j}$ and $\psi_j^\downarrow = d_{N-j}$. Notice that (46) and (47) guarantee that the pyramid condition is satisfied. The following example shows that is possible to combine quantization and (morphological) sample reduction into one scheme in such a way that the pyramid condition remains satisfied.

Example 5 (Morphological Pyramid with Quantization): Consider the flat adjunction pyramid, given by (26), where $V_j = \text{Fun}(\mathbb{Z}^d, \mathcal{T}_N)$, in which case

$$\psi^\uparrow(x)(n) = \bigwedge_{k \in A} x(2n + k),$$
$$\psi^\downarrow(x)(k) = \bigvee_{n \in A[k]} x\left(\frac{k-n}{2}\right).$$

Assume that, for some $a \in A$, $A[a] = \{a\}$; this yields that the pyramid condition is satisfied. Put $\overline{V}_j = \text{Fun}(\mathbb{Z}^d, \mathcal{T}_{N-j})$ and

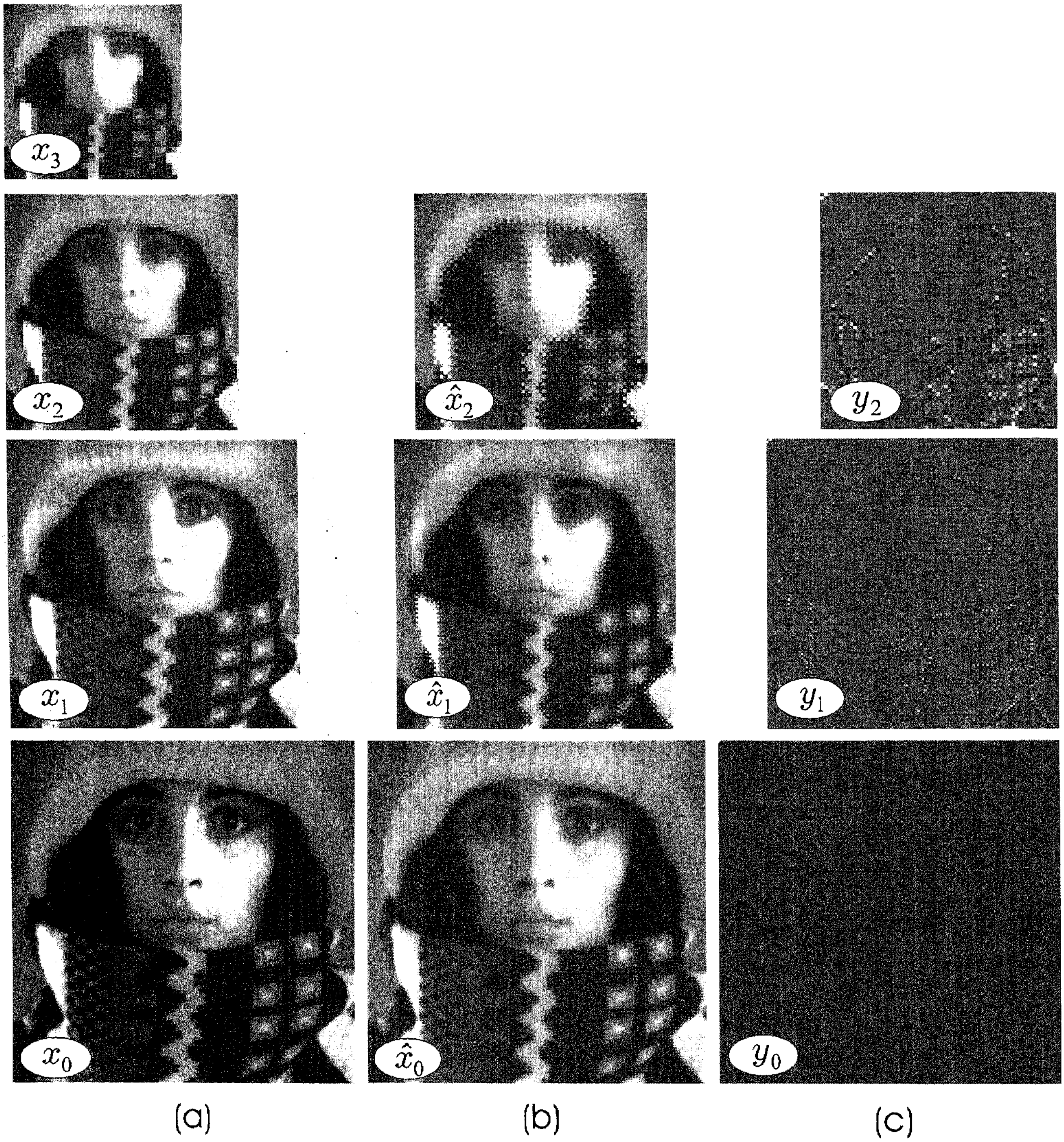

Fig. 6. Multiresolution image decomposition based on a median pyramid. (a) An image x_0 and its decomposition $\{x_0, x_1, x_2, x_3\}$ obtained by means of the analysis operator $\psi^\uparrow$ in (41). (b) The approximation images $\{\hat{x}_0, \hat{x}_1, \hat{x}_2\}$ obtained from $\{x_1, x_2, x_3\}$ by means of the synthesis operator $\psi^\downarrow$ in (42)–(45). (c) The detail images $\{y_0, y_1, y_2\}$.

define quantized analysis and synthesis operators between $\overline{V}_j$ and $\overline{V}_{j+1}$ as follows:

$$\overline{\psi}_j^\uparrow(x)(n) = \left\lfloor \left(\bigwedge_{k \in A} x(2n+k) \right) / 2 \right\rfloor$$

$$\overline{\psi}_j^\downarrow(x)(k) = 2 \left(\bigvee_{n \in A[k]} x\left(\frac{k-n}{2}\right) \right).$$

We can write $\overline{\psi}_j^\uparrow = q_{N-j}\psi_j^\uparrow$ and $\overline{\psi}_j^\downarrow = \psi_j^\downarrow d_{N-j}$. The pair $(\overline{\psi}_j^\uparrow, \overline{\psi}_j^\downarrow)$ defines an adjunction between $\overline{V}_j$ and $\overline{V}_{j+1}$. Furthermore, the pyramid condition is satisfied, since

$$\begin{aligned}\overline{\psi}_j^\uparrow \overline{\psi}_j^\downarrow &= q_{N-j}\psi_j^\uparrow \psi_j^\downarrow d_{N-j} \\ &= q_{N-j} d_{N-j} = \mathrm{id} \quad \text{on } \mathrm{Fun}(\mathbb{Z}^d, \mathcal{T}_{N-j-1}).\end{aligned}$$

By taking $\psi^\uparrow$ and $\psi^\downarrow$ as in (29)–(33), we can construct a morphological pyramid, like the one in Example 3, with the addition of a quantization step at each level.

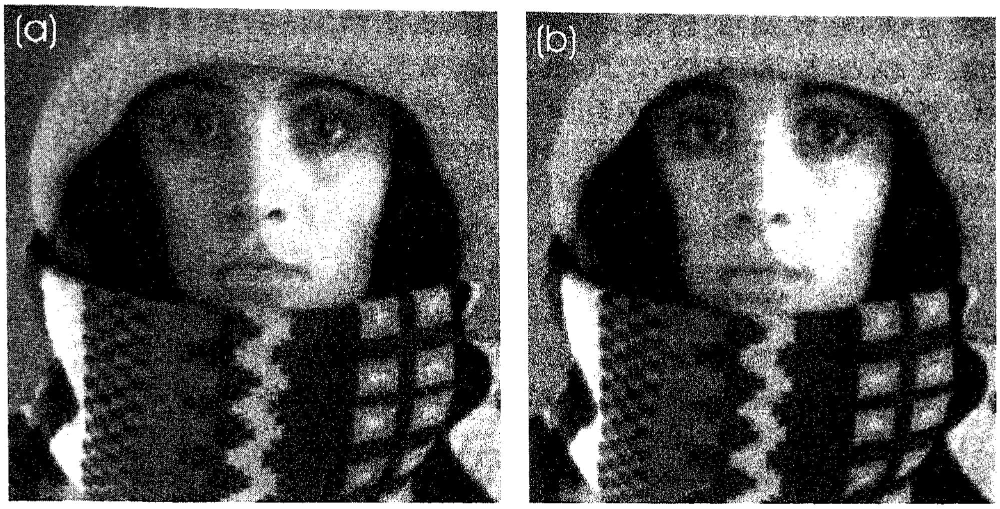

Fig. 7. (a) A 512×512 grayscale image and (b) its partial reconstruction obtained from the detail signals $\{y_1, y_2, \cdots, y_8\}$ calculated by means of the morphological pyramid transform with quantization discussed in Example 5.

When the pyramid transform is used for signal compression, the detail signal y_0 is usually removed from the decomposition (this is due to the overcompleteness of the pyramid transform, e.g., see [15]). In this case, satisfactory compression performance can be achieved at the expense of partially reconstructing the original signal x_0. In fact, given the pyramid decomposition $\{y_1, y_2, \cdots, y_K\}$ of x_0, the inverse pyramid transform reconstructs only an approximation $\hat{x}_0$ of x_0. Fig. 7(b) depicts the partial reconstruction of the 512×512 grayscale image depicted in Fig. 7(a), obtained by means of inverting the decomposition $\{y_1, y_2, \cdots, y_8\}$ based on the previously discussed morphological pyramid transform with quantization. ■

VII. Conclusion

In this first part of our study, on general multiresolution signal decomposition, we presented an axiomatic treatise of *pyramid* decomposition schemes. The basic ingredient of such schemes is the so-called *pyramid condition*, which states that synthesis of a signal followed by analysis returns the original signal. This simple and intuitive condition, which means that synthesis never gives rise to (additional) loss of information, lies at the heart of various linear and nonlinear decomposition schemes.

Our scheme includes various existing pyramids known from the literature, such as the linear Laplacian pyramid, due to Burt and Adelson [1] (but only for a particular choice of the parameters; see Section IV), and the morphological pyramid due to Toet [10]; see also [12] and [15]. Moreover, it includes various morphological shape analysis tools, like morphological skeletons and granulometries.[2]

In [8], Haralick *et al.* develop a theory for morphological sampling in which, among others, they provide relationships between the original and sampled signal. Related work can be found in [12], [20]. Morales *et al.* [19] use the sampling theorem formulated in [8] to construct morphological pyramids based on alternating sequential filters. These pyramids do not fit in our framework, however, as the pyramid condition is not satisfied. In [29], Salembier and Kunt address the problem of size-sensitive multiresolution image decomposition using rank order based filters. Their approach, however, does not include a downsampling stage.

The exposition in this paper is largely theoretical in nature. Our objective was to find a simple axiomatic framework which is flexible enough to allow pyramids based on linear as well as nonlinear filters, but which nevertheless imposes restrictions which are physically meaningful. We believe that the pyramid condition in Section III meets these objectives; this is, among others, reflected by the fact that, in the linear case, every pyramid that satisfies this condition can be extended, in a unique way, to a biorthogonal wavelet. For a proof of this fact, we refer the reader to [26], [33], where we consider nonlinear wavelet decomposition schemes comprising two (or more) analysis and synthesis operators at each level. In that study, we give particular attention to a new family of wavelets, the so-called *morphological wavelets*.

[2]It has been shown in [22] that a given (discrete) *granulometry* [5] generates its own pyramid in terms of adjunctions, which satisfies the pyramid condition.

Acknowledgment

The authors would like to thank the reviewers for their comments, which improved the presentation of the paper.

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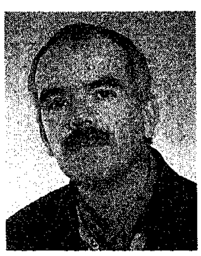

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