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Abstract

Vibrations of a nonlinear coupled parametrically and self-excited oscillators driven by an external harmonic force are presented in the paper. It is shown that if the force excites the system inside the principal parametric resonance then for a small excitation amplitude a resonance curve includes an internal loop. To find the analytical solutions the problem is reduced to one degree of freedom oscillators by applications of Nonlinear Normal Modes (NNMs). The NNMs are formulated on the basis of free vibrations of a nonlinear conservative system as functions of amplitude. The analytical results are validated by numerical simulations and an essential difference between linear and nonlinear modes is pointed out.

Keywords

nonlinear vibrations, nonlinear normal modes, parametric resonances, self-excitation, model reduction

1. Introduction

Self-excited systems are one of the most interesting and common in mechanical engineering theory and practice [1]. The response of this type of systems is represented by periodic limit cycles which can be stable, then the self-excitation is called soft, or unstable, called hard or catastrophic.

Another, well known and intensively studied are parametric vibrations which are generated by periodically changing in time parameters. Dynamics of a parametrically excited system is observed by, so called, parametric resonances where vibration amplitudes increase reaching large values [2].

Interesting dynamical phenomena can be observed when both of mentioned above vibrations meet together at the same time. Then interactions between two different types of vibration take place. Such a situation may arise in many mechanical engineering structures. As an example we can mention dynamics of a helicopter's rotor blade, while during the rotation due to varying blade incidence angle, the stiffness of the blade is changing periodically. Under flutter conditions caused by air flow the interaction of both parametric and self-excitation takes place. Moreover, due to nonlinear coupling various vibration modes are coupled as well.

Tondl [3] and co-authors [4], [5] considerably contributed in the analysis of interactions of parametric and self-excited vibrations. They studied dynamics of self-excited systems of van der Pol's type driven by Mathieu parametric excitation. It has been found there that in wide intervals of excitation frequency the response of the system is quasi-periodic, and observed by quasi-periodic limit cycles on Poincaré maps. However, near the parametric resonances, self-excited frequency is quenched and the response of the system is periodic, represented by singular points on Poincaré maps. Those regions are sometimes called "frequency synchronisation regions", due to the fact of the synchronisation of self- and parametric excitation frequencies. Extension of the works concerning parametric and self excited systems on bifurcation analysis and also chaotic or hyperchaotic dynamics can be found e.g. in [6]-[11].

The behaviour of the parametrically and self-excited system can completely change if the system is additionally forced by external force. In many practical examples it can happen that frequency of the external force is equal to half of the parametric excitation frequency. As an example we can mention a rotating shaft with a rectangular cross-section and unbalanced mass (Fig.1). During the rotation the stiffness of the system changes twice (parametric excitation) while the external inertia force once per revolution.

Fig.1 Example of an externally and parametrically excited system with 1:2 frequencies ratio.

Such a situation may appear also in other examples e.g. in sagged cable dynamics [12]-[14] or gear systems [15], vibrations of bars, beams and plates [16] as well as in micro-electromechanical systems (MEMS) [17]. If the considered structure is additionally damped by a nonlinear force e.g. nonlinear friction or is placed in a fluid flow, which yield in flutter vibration, then we can find a system where also self-excited terms may appear. Differential equation of motion of one degree of freedom system with self-, parametric and external excitation can be written in form

$$\ddot{x} + f\left(\dot{x}\right) + \left(x + \gamma x^3\right)\left(1 - \mu\cos 2\omega t\right) = q\cos\omega t \tag{1}$$

In the considered model the external force excites the system exactly inside the principal parametric resonance, since in this region the response of the system is subharmonic with respect to the parametric excitation frequency. The dynamics of such a system was investigated in papers [18], [19]. The nonlinear damping was there assumed as the Rayleigh's nonlinear function $f\left(\dot{x}\right) = -\alpha\dot{x} + \beta\dot{x}^3$. The results indicated very important quantitative changes in the principal parametric resonance regions. The resonance curves received by the multiple scale of time method for the system without external force ($q = 0$) and driven by external force ($q > 0$) are presented in Fig.2 (a) and (b), respectively.

Fig.2 Resonance curve of a self-excited system driven parametrically (a) and driven parametrically and externally (b); $\alpha = 0.01$, $\beta = 0.05$, $\gamma = 0.1$, $\mu = 0.1$, $q = 0.05$

The presented curves have been determined for the following parameters: nonlinear damping of Rayleigh's type $\alpha = 0.01$, $\beta = 0.05$, nonlinear stiffness $\gamma = 0.1$, parametric excitation $\mu = 0.1$, external force amplitude $q = 0$ (Fig.2a) or $q = 0.05$ (Fig.2b). The response of the self-excited system driven parametrically is mainly quasi-periodic apart from, so called, frequency synchronisation regions where self-excited vibrations are quenched by parametric excitation and the single harmonic oscillations are observed. This phenomenon is visible near the parametric resonance regions. Fig.2(a). Nevertheless, if the self-excited system is driven parametrically and externally then its response changes quantitatively and qualitatively. We see in Fig.2(b) that inside this region the resonance curve possesses an internal loop (red colour). Near the frequency about 1.07 five possible solutions are possible. The stability analysis shows that only two upper among five are stable. Of course to get the stable solution it is necessary to put initial conditions inside their basins of attractions. Possible existence of five equilibria for one DOF parametric system with negative damping has been also presented in [20]. Similar phenomena has been observed in [17] for one DOF MEMS resonator modelled by a van der Pol-Duffing system driven by parametric and external excitation. The bifurcation analysis revealed regions of the five solutions appearance.

The problem of existence of the additional solutions is more complicated for models with many degrees of freedom. For many DOF systems additional interactions between vibrating modes may appear (apart from that resulted from various vibrations types).

The results for two DOF model [21] confirm existence of the internal loop inside the main parametric region. However, to get this result a classical linear normal modes approach has been applied there. It has been shown that there is an essential difference between numerical and analytical results. This linear decoupling, which is usually used for weakly nonlinear system, was not sufficient for the nonlinear system with parametric and self-excitation. Therefore, for a proper decoupling of the system another technique is required. Application of nonlinear normal modes (NNMs) [22], [23] is the promising approach which allows for separation of the structure for single DOF oscillators. Such constructed oscillators should include all information related with the nonlinear system's dynamics.

Various techniques of NNM have been presented and discussed on serious of conferences directly devoted to this topic, the first conference organised by C.H.Lamarque [24], the second by A.Vakais [25] and the third by G.Rega [26]. Nonlinear Normal Modes of a strongly nonlinear autonomous system, conservative or damped, are formulated in [27]. This formulation is based on the centre-manifold approach and it allows for model decoupling by taking into account, both, the displacement and velocity of a chosen set of coordinates, which are so called master coordinates while rest are slave coordinates constituted by modal surfaces. This methodology can be applied for nonlinear but autonomous systems. The centre manifold reduction has been successfully applied for coupled self-excited systems [28]. Complexity of NNM formulation increases if the system is non-autonomous. Then time has to be introduced as an additional coordinate. To get the modal equation it is necessary to apply a special numerical Galerkin-based technique taking into account a time coordinate [29]. This method allows for finding NNMs for every instant of time.

Another proposal of NNM formulation, defined for the nonlinear systems with weak damping, around the resonance region is presented in [30], [31]. The resonant normal modes were found as a function of amplitude for the stationary response instead of functions of displacement and velocity. This approach allows for the system reduction around of resonance zones. The response is represented by a single oscillator with modal mass and frequency, constrained by an amplitude dependent relationship. Similar formulation has been presented in [32] with application to undamped nonlinear and autonomous mechanical systems. The resonant periodic response has been defined by a single degree of freedom system with period of motion depending on the amplitude. The approach presented in [30]-[32], seems be also attractive for the nonlinear the self- and parametrically or/and externally excited system. The study of self- and parametrically excited system without external force influence has been published in [33].

This paper presents resonant nonlinear normal modes based on the approach formulated in [33]. The normal modes are constructed on the same basis of nonlinear natural oscillations. However, in the present case the analysis is extended to the system with two degrees of freedom which includes *self-, parametric and external excitations*. Such a system may exhibit a multi-solutions resonance curve comparing to those presented in [33] where only single classical resonances took place. This study is carried out for the purpose of proper reduction of the multi-degree of freedom model into set of separate single oscillators near the principal parametric resonance region. The separated nonlinear modes should exhibit appearance of the internal loop inside the parametric resonance, similar to whose described for that of one DOF.

2. Model of the system

Let's consider two coupled nonlinear oscillators (I) and (II) composed of mass, nonlinear spring of Duffing type and nonlinear damping which represents self-excitation (Fig.3). The oscillators are coupled parametrically by a spring with periodically changing stiffness. Besides, the first oscillator is driven by a harmonic force.

Fig. 3 Model of coupled self-excited oscillators driven by parametric and external excitations

Motion of the model presented in Fig.3 is described by a set of differential equations

$$\begin{aligned} m_1\ddot{X}_1 + f_{d1}\left(\dot{X}_1\right) + \delta_1 X_1 + \gamma_1 X_1^3 + \left(\delta_{12} - \mu\cos 2\omega t\right)\left(X_1 - X_2\right) = q\cos\Omega t \\ m_2\ddot{X}_2 + f_{d2}\left(\dot{X}_2\right) + \delta_2 X_2 + \gamma_2 X_2^3 - \left(\delta_{12} - \mu\cos 2\omega t\right)\left(X_1 - X_2\right) = 0 \end{aligned} \tag{2}$$

Nonlinear damping is represented by Rayleigh's function $f_{d1} = -\alpha_1\dot{x}_1 + \beta_1\dot{x}_1^3$ and $f_{d2} = -\alpha_2\dot{x}_2 + \beta_2\dot{x}_2^3$. The coupling stiffness is linear of Mathieu type with periodically varying term $f_{12} = \left(\delta_{12} - \mu\cos 2\omega t\right)\left(X_1 - X_2\right)$. In mechanical engineering systems this kind of periodically varied stiffness may appear e.g. in gear boxes due to changing of mesh stiffness during operation.

The main feature of the considered model is that nonlinear damping produces self-excitation represented by limit cycles on the phase plane and the parametric excitation of frequency 2ω yields in instability regions so called parametric resonances. Near the principal parametric resonance the response of the system is subharmonic (1:2) with corresponding frequency ω. Such resonances appear when excitation frequency is close to one of the natural frequencies $\omega \approx \omega_{01}$ or $\omega \approx \omega_{02}$. Because we want the system to be driven by external harmonic force exactly inside the principal parametric resonance, therefore frequency Ω is taken as half of parametric excitation, $\Omega = \omega$. Physical justification of such a model is presented in the introduction.

3. Nonlinear Normal Modes Formulation

The nonlinear normal modes are formulated on the basis of the observation of the physical system response [21]. The interaction between parametric and self-excited vibration results mainly in quasi-periodic motion composed by influence of both vibration types. On the

Poincaré map this kind of motion is represented by quasi-periodic limit cycles. It means that the self-excitation dominates in those regions exhibiting its main features. However, approaching the parametric resonances, near their neighbourhood, after the second kind of Hopf bifurcation (Nejmark-Sacker bifurcation), the self-excitation is quenched, and the system response is periodic.

To formulate the nonlinear normal modes we will concentrate only in the neighbourhood of the main parametric resonance region. Taking into account that the self-excitation (nonlinear damping) is quenched we can assume that the vibration modes are close to the mode of nonlinear conservative system. Therefore, at first we will formulate nonlinear normal modes of a nonlinear conservative system and then we will extend them to the self-excited model driven by parametric and external excitations. The equations of motion of the conservative nonlinear system takes form

$$\begin{aligned} m_1\ddot{X}_1 + \delta_1 X_1 + \gamma_1 X_1^3 + \delta_{12}\left(X_1 - X_2\right) = 0 \\ m_2\ddot{X}_2 + \delta_2 X_2 + \gamma_2 X_2^3 - \delta_{12}\left(X_1 - X_2\right) = 0 \end{aligned} \tag{3}$$

In contrast to the pure linear system we define the vibrations modes as functions of amplitude. One of the mode coefficients can be taken arbitrary and the second is expressed as a function of amplitude. Therefore, the coefficient u_1 is taken as a constant and for convenience is equal to one ($u_1 = 1$) in further analysis, while the second is a function of amplitude $u_2 = u_2\left(a\right)$. The solution of the set (3) is sought by the harmonics balance method (HBM). Only the first harmonic of the response is taken into account therefore the solution is assumed in form

$$\begin{aligned} X_1 &= a\cos\omega_0 t \\ X_2 &= a u_2(a)\cos\omega_0 t \end{aligned} \tag{4}$$

where ω_0 is a natural vibration frequency of the nonlinear system. According to definitions introduced in papers related to NNMs the coordinate X_1 is called “master coordinate” while X_2 is so called “slave” coordinate (see e.g. [27], [23]).

Substituting the solutions (4) into (3), putting and expanding trigonometric functions we get

$$\begin{aligned} \left(-m_1\omega_0^{\,2} + \frac{3}{4}a^2\gamma_1 + \delta_1 + \delta_{12} - u_2\delta_{12}\right)\cos\omega_0 t + \frac{1}{4}a^2\gamma_1\cos 3\omega_0 t = 0 \\ \left(-m_2\omega_0^{\,2}u_2 + \frac{3}{4}a^2\gamma_2 u_2^{\,3} + u_2(\delta_2 + \delta_{12}) - \delta_{12}\right)\cos\omega_0 t + \frac{1}{4}a^2\gamma_2 u_2^{\,3}\cos 3\omega_0 t = 0 \end{aligned} \tag{5}$$

Taking into account the first harmonic approximation, we neglect the terms with $\cos 3\omega_0 t$. We may note that this reduction does not reject nonlinear terms in total, only in the second order approximation. After such simplification we get

$$
\begin{aligned}
&-m_1\omega_0^{\ 2}+\delta_1+\delta_{12}+\frac{3}{4}a^2\gamma_1-u_2\left(a\right)\delta_{12}=0\\
&-\delta_{12}+u_2\left(-m_2\omega_0^{\ 2}+\delta_2+\delta_{12}\right)+\frac{3}{4}a^2\gamma_2 u_2^{\ 3}=0
\end{aligned}
\tag{6}
$$

where coefficient $u_2 \equiv u_2\left(a\right)$ is a function of amplitude. Eliminating natural frequency ω_0 from above equations we get

$$
\frac{3}{4}a^2\left(m_2u_2\gamma_1-m_1\gamma_2u_2^{\ 3}\right)+\left(1-u_2\right)\left(m_1+m_2u_2\right)\delta_{12}+u_2\left(m_2\delta_1-m_1\delta_2\right)=0 \tag{7}
$$

Equation (7) is a modal equation which allows for determination the mode coefficients of the nonlinear conservative system. From (7) we can find two roots for u_2, $u_{21}\left(a\right)$ and $u_{22}\left(a\right)$, for the first and the second mode respectively. However, because it is easier to determine inverse relation we find $a\left(u_2\right)$ which takes form

$$
a=2\sqrt{\frac{\left(u_2-1\right)\left(m_1+m_2u_2\right)\delta_{12}-u_2\left(m_2\delta_1-m_1\delta_2\right)}{3u_2\left(m_2\gamma_1-m_1u_2^2\gamma_2\right)}} \tag{8}
$$

Two real solutions for a represent the nonlinear curves of $a\left(u_2\right)$ for the first and the second nonlinear normal modes, respectively. We have to bear in mind that natural frequency of nonlinear system ω_0 is also amplitude dependent. Eliminating in (6) vibration mode u_2, we can find

$$
\Gamma_3\omega_0^6+\Gamma_2\omega_0^4+\Gamma_1\omega_0^2+\Gamma_0=0 \tag{9}
$$

where

$$
\Gamma_3=-\frac{3a^3m_1^3\gamma_2}{4\delta_{12}^3}, \qquad \Gamma_2=\frac{27a^5m_1^2\gamma_1\gamma_2}{16\delta_{12}^3}+\frac{9a^3m_1^2\gamma_2\left(\delta_1+\delta_{12}\right)}{4\delta_{12}^3}+\frac{am_1m_2}{\delta_{12}},
$$

$$\Gamma_1 = -\frac{81a^7 m_1\gamma_1^2\gamma_2}{64\delta_{12}^3} - \frac{27a^5 m_1\gamma_1\gamma_2\left(\delta_1+\delta_{12}\right)}{8\delta_{12}^3} - \frac{3a^3\left(m_2\gamma_1\delta_{12}^2 + 3m_1\gamma_2\left(\delta_1+\delta_{12}\right)^2\right)}{4\delta_{12}^3} - \frac{a\left(m_2\left(\delta_1+\delta_{12}\right)+m_1\left(\delta_2+\delta_{12}\right)\right)}{\delta_{12}}$$

$$\Gamma_0 = \frac{81a^9\gamma_1^3\gamma_2}{256\delta_{12}^3} + \frac{81a^7\gamma_1^2\gamma_2\left(\delta_1+\delta_{12}\right)}{64\delta_{12}^3} + \frac{27a^5\gamma_1\gamma_2\left(\delta_1+\delta_{12}\right)^2}{16\delta_{12}^3} + \frac{3a^3\left(\gamma_2\left(\delta_1+\delta_{12}\right)^3 + \gamma_1\delta_{12}^2\left(\delta_2+\delta_{12}\right)\right)}{4\delta_{12}^3} + \frac{a\left(\delta_2\delta_{12}+\delta_1\left(\delta_2+\delta_{12}\right)\right)}{\delta_{12}}$$

Equation (9) has three real roots for ω_0^2, however only two of them have physical meaning. The solutions which tend in a limit to frequencies of the linear model, $\lim\limits_{a\to 0}\omega_0^2 = \omega_{01}^2$, and $\lim\limits_{a\to 0}\omega_0^2 = \omega_{02}^2$ will be taken for the further consideration. Thus, we found the formulae for the natural frequencies $\omega_{01} = \omega_{01}\left(a\right)$ and $\omega_{02} = \omega_{02}\left(a\right)$ as functions of amplitude.

Having the modal coefficient $u_{2j}\left(a\right)$, the modal solutions take form

$$\begin{cases} X_1 = Y_j \\ X_2 = u_{2j}(a)Y_j \end{cases} \tag{10}$$

where $j = 1, 2$ corresponds to mode 1 and mode 2, respectively and $Y_j = a\cos\omega_{0j}t$ is a periodic function of time.

The formulated above modes of the nonlinear conservative system will be applied to the system (2) for determination of the resonance curves.

4. Vibrations of a driven parametrically and externally coupled self-excited system

To transform the system from generalized into normal coordinates we make an assumption that inside the principal parametric resonance the vibration modes are close to the nonlinear normal modes formulated in previous chapter for the nonlinear conservative system. Substituting (10) into a set of the original equations (2) and next multiplying the first equation by u_{1j} (which in this case is equal to one) and the second by u_{2j} (*j=1,2*), then adding them, we get

$$\left(m_1 + m_2 u_{2j}^2\right)\ddot{Y}_j + \left[\delta_1 + u_{2j}^2\delta_2 + (1-u_{2j})^2\delta_{12}\right]Y_j + \left(\gamma_1 + \gamma_2 u_{2j}^4\right)Y_j^3 = F_j\left(Y_j, \dot{Y}_j, t\right) \qquad (11)$$

where $F_j = \mu\left(1-u_{2j}\right)^2 Y_j \cos 2\omega t + \left(\alpha_1 + \alpha_2 u_{2j}^2\right)\dot{Y}_j - \left(\beta_1 + \beta_2 u_{2j}^4\right)\dot{Y}_j^3 + q\cos\omega t$, *j=1,2.*

Taking into account that for the conservative system $F_j\left(Y_j, \dot{Y}_j, t\right) = 0$ and then substituting the periodic solution $Y_j = a\cos\omega_{0j}t$ into (11) we get relationship which represents natural frequency of the nonlinear system

$$\omega_{0j}^2 = \frac{1}{m_1 + m_2 u_{2j}^2}\left[\delta_1 + \delta_{12}\left(1-u_{2j}\right)^2 + \delta_2 u_{2j}^2 + \frac{3}{4}a^2\left(\gamma_1 + \gamma_2 u_{2j}^4\right)\right] \qquad (12)$$

This equation corresponds to the frequency found from (6), therefore the equation of motion may be written in nonlinear normal coordinates as

$$M_j(a)\ddot{Y}_j + M_j(a)\omega_{0j}^2(a)Y_j = F_j\left(Y_j, \dot{Y}_j, t\right) \qquad (13)$$

where $M_j(a) = m_1 + m_2 u_{2j}^2$ is the modal mass of the system for the first and the second vibration mode for $j = 1,2$, respectively. Function $F_j\left(Y_j, \dot{Y}_j, t\right)$ can also be expressed as

$$F_j = C_{\mu j} Y_j \cos 2\omega t + C_{\alpha j}\dot{Y}_j - C_{\beta j}\dot{Y}_j^3 + C_{qj}\cos\omega t \qquad (14)$$

where

$$C_{\mu j} = \mu\left(1-u_{2j}\right)^2,\ C_{\alpha j} = \alpha_1 + \alpha_2 u_{2j}^2,\ C_{\beta j} = \beta_1 + \beta_2 u_{2j}^4,\ C_{qj} = q \qquad (15)$$

are modal coefficients representing parametric, self- and external excitation. The essential assumption made to transform the equations (3) into (13) is that the system's response is periodic up to the first balanced harmonic. The modes are called resonant nonlinear normal modes because they can be used only for regions of a single frequency response. For a self-exited system driven by parametric and external excitations they are used for the frequency locking zones ([3]-[5], [17]) characterized by periodic solutions. The formulated modes in approximation decouple the system also if amplitude and phase of the solution are slow functions of time.

To get resonance solution it is necessary to solve a single equation (13) together with constrain equation (7).

5. Analytical Solutions of Self, Parametrically and Externally Driven System

Analytical solutions of Eq.(13) due to its nonlinear nature can be determined by application of approximation methods. The Multiple Scale of time method [1] is used to find the approximate solution. Introducing a formal small parameter ε, equation (13) is written in the form

$$M_j(a)\ddot{Y}_j + M_j(a)\omega_{0j}^2(a)Y_j = \varepsilon\tilde{F}_j(Y_j, \dot{Y}_j, t) \text{ for } j=1,2 \tag{16}$$

Function (14) is expressed by $F_j = \varepsilon\tilde{F}_j$, and parameters: $\mu = \varepsilon\tilde{\mu}$, $\alpha_1 = \varepsilon\tilde{\alpha}_1$, $\beta_1 = \varepsilon\tilde{\beta}_1$, $\alpha_2 = \varepsilon\tilde{\alpha}_2$, $\beta_2 = \varepsilon\tilde{\beta}_2$ and consequently $C_{\mu j} = \varepsilon\tilde{C}_{\mu j}$, $C_{\alpha j} = \varepsilon\tilde{C}_{\alpha j}$, $C_{\beta j} = \varepsilon\tilde{C}_{\beta j}$, $C_{qj} = \varepsilon\tilde{C}_{qj}$.

According to the formulation presented in Chapter 4 it is assumed that the response of the system is periodic. However, assuming that amplitude and phase of the solution of (16) are slow functions of time i.e. if during one period of the fast scale there is a small change of amplitude (the slow scale) the presented approach can be used also in the neighbourhood of the resonance (frequency locking) zones.

Solution of Eq.(16) is sought in form of a series of the small parameter ε

$$Y_j(t,\varepsilon) = Y_{j0}(T_0,T_1) + \varepsilon Y_{j1}(T_0,T_1) + \ldots \tag{17}$$

where $Y_{j0}(T_0,T_1)$, $Y_{j1}(T_0,T_1)$ are zero and first order functions of time. Time is also expressed by the series of the small parameter,

$$t = T_0 + \varepsilon T_1 + \ldots \tag{18}$$

where T_0 and T_1 are respectively fast and slow scale of time. Such time definitions results in the following formulae for the first and the second time derivatives

$$\frac{\mathrm{d}}{\mathrm{d}t}=\frac{\partial}{\partial T_0}+\varepsilon\frac{\partial}{\partial T_1}+\varepsilon^2\frac{\partial}{\partial T_2}=D_0+\varepsilon D_1+\varepsilon^2 D_2\ ... \tag{19}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}=D_0^2+2\mu D_0 D_1+\varepsilon^2(2D_0 D_2+D_1^2)+\ ... \tag{20}$$

where $D_n^m=\dfrac{\partial^m}{\partial T_n}$ means m order partial derivative with respect to n scale of time.

As it has been assumed earlier we sought the solutions around the principal parametric resonance, where the self-excitation is quenched by dominating parametric vibrations, therefore, bearing in mind that parametric excitation frequency is equal to 2ω and external excitation ω, we can write

$$\omega^2=\omega_{0j}^2+\varepsilon\sigma_j \tag{21}$$

where σ_j is frequency detuning parameter around the first and the second natural frequency ω_{0j}, $j=1,2$.

Substituting solution (17) and taking into account the derivatives definitions (19), (20), after grouping terms with respect to ε order we get a set of differential equations in successive perturbation orders

ε^0 - order

$$D_0^2 Y_{j0}+\omega^2 Y_{j0}=0 \tag{22}$$

ε^1 - order

$$\begin{gathered} D_0^2 Y_{j1} + \omega^2 Y_{j1} = \\ \left[\sigma_j Y_{j0} - 2D_0 D_1 Y_{j0} + \tilde{C}_{\alpha j} D_0 D_1 Y_{j0} - \tilde{C}_{\beta j} (D_0 D_1 Y_{j0})^3 + \tilde{C}_{\mu j} Y_{j0} \cos 2\omega t + \tilde{C}_{qj} \cos \omega t \right] / M_j \end{gathered} \tag{23}$$

Solution of Eq.(22) is written in complex form

$$Y_{j0}(T_0, T_1) = A_{j1}(T_1)\exp(i\omega T_0) + \overline{A}_{j1}(T_1)\exp(-i\omega T_0) \tag{24}$$

where $i = \sqrt{-1}$ is an imaginary unit, A_{j1}, $\overline{A}_{j1}$ is complex amplitude and it's conjugate. Solution (24) is substituted into (23) and then, after grouping terms in proper exponential functions, we get

$$\begin{gathered} D_0^2 Y_{j1} + \omega^2 Y_{j1} = e^{3i\omega T_0} \frac{1}{M_j} \left(-i\omega^3 A_{j1}^{\ 3} \tilde{C}_{\beta j} - \frac{A_{j1} \tilde{C}_{\mu j}}{2} \right) \\ + e^{i\omega T_0} \left[2 i \omega D_1 A_{j1} - A_{j1} \sigma_j - \frac{1}{M_j} \left(i\omega A_{j1} \tilde{C}_{\alpha j} + 3 i \omega^3 A_{j1}^{\ 2} \overline{A}_{j1} \tilde{C}_{\beta j} - \frac{\overline{A}_{j1} \tilde{C}_{\mu j}}{2} + \frac{\tilde{C}_{qj}}{2} \right) \right] + cc \end{gathered} \tag{25}$$

where *cc* means complex conjugate functions to those written in the equation. To eliminate secular terms, the components near $exp(i\omega T_0)$ must vanish, thus

$$2 i \omega D_1 A_{j1} - A_{j1} \sigma_j - \frac{1}{M_j} \left(i\omega A_{j1} \tilde{C}_{\alpha j} + 3 i \omega^3 A_{j1}^{\ 2} \overline{A}_{j1} \tilde{C}_{\beta j} - \frac{\overline{A}_{j1} \tilde{C}_{\mu j}}{2} + \frac{\tilde{C}_{qj}}{2} \right) = 0 \tag{26}$$

Expressing complex amplitude A_j by the polar form

$$A_{1j} = a_j e^{i\phi_j} \tag{27}$$

and then separating real and imaginary parts of (26) we get modulation equations for amplitude a_j and phase ϕ_j

$$\begin{aligned} M_j \omega \dot{a}_j &= \frac{1}{2} \tilde{C}_{\alpha j} \omega a_j - \frac{3}{8} \tilde{C}_{\beta j} \omega^3 a_j^3 - \frac{1}{4} \tilde{C}_{\mu j} a \sin 2\phi_j - \frac{1}{2} \tilde{C}_{qj} \sin \phi \\ M_j \omega a_j \dot{\phi}_j &= -\frac{1}{2} \sigma_j M_j a_j - \frac{1}{4} \tilde{C}_{\mu j} a_j \cos 2\phi_j - \frac{1}{2} \tilde{C}_{qj} \cos \phi \end{aligned} \tag{28}$$

In a steady state $\dot{a}_j = 0$, $\dot{\phi}_j = 0$. Therefore, from equations (28) we can find the resonance curve around the principal parametric resonances

$$C_{qj}^2 - a_j^2 C_{\mu j}^2 - \sqrt{\Lambda_j} + C_{qj} \sqrt{2C_{qj}^2 + 4a_j^2 C_{\mu j} \lambda_{2j} - 2\sqrt{\Lambda_j}} = 0 \tag{29}$$

where $\lambda_{1j} = a_j\left(C_{\alpha j}\omega - \frac{3}{4}a_j^2 C_{\beta j}\omega^3\right)$, $\lambda_{2j} = M_j\left(\omega_{0j}^2 - \omega^2\right)$, and

$\Lambda_j = \left(C_{qj}^2 - 2a_j^2 C_{\mu j}\lambda_{2j}\right)^2 + 4a_j^2 C_{\mu j}\left(C_{\mu j}\lambda_{1j}^2 + 2C_{qj}^2\lambda_{2j}\right)$

To get vibration amplitudes in the resonance regions it is necessary to solve equation (29) together with nonlinear modal forms of the considered system formulated by Eq.(7). Introducing the particular solution of (25) into (17) yields

$$Y_j = a_j\cos\left(\omega t + \phi\right) + \varepsilon\frac{a_j}{16M_j}\left[-\tilde{C}_{\mu j}\cos\left(3\omega t + \phi\right) + \frac{a_j^2}{2}\tilde{C}_{\beta j}\omega\sin\left(3\omega t + 3\phi\right)\right] \tag{30}$$

where the amplitude a and the phase ϕ are determined from modulation equations (28) or for a steady state from algebraic equation (29).

6. Numerical Calculations

Numerical calculations are carried out for the nonlinear conservative model data

$$m_1 = 1,\ m_2 = 2,\ \delta_1 = 1,\ \delta_2 = 1,\ \delta_{12} = 0.3,\ \gamma_1 = 0.1,\ \gamma_2 = 0.1 \tag{31}$$

and for self-excitation parameters

$$\alpha_1 = 0.01, \beta_1 = 0.05,\ \alpha_2 = 0.01, \beta_2 = 0.05 \tag{32}$$

while amplitude of parametric and external excitation are

$$\mu = 0.2,\quad 0 \le q \le 0.2 \tag{33}$$

To evaluate results obtained by the proposed method the numerical calculations are also performed for linear normal modes. The physical coordinates $\boldsymbol{X}$ are transformed to normal coordinates $\boldsymbol{Y}$ by linear transformation

$$\mathbf{X} = \mathbf{u}_0\mathbf{Y} \tag{34}$$

where $\mathbf{u_0}$ is a linear modal matrix

$$\mathbf{u}_0 = \begin{bmatrix} u_{110} & u_{120} \\ u_{210} & u_{220} \end{bmatrix} \tag{35}$$

For the assumed data (31) coefficients of modal vectors takes values

$$u_{110} = 1,\ u_{210} = 2.3770,\ u_{210} = 1,\ u_{220} = -0.21035 \tag{36}$$

with corresponding natural frequencies of the linear system $\omega_{01} = 0.766091$, $\omega_{02} = 1.16752$, where index *0* underlines the linear transformation.

Fig. 4. Vibration modes versus amplitude; parameters given by (31)

Taking into account the derived Eq. (7), which constitutes relationship of the mode coefficient u_2 versus vibration amplitude, we can draw the nonlinear modal curves as functions of amplitude. The solid line on the right hand side of the Fig.4 presents the first vibration mode, while on the left hand side the second mode is plotted. Of course, for amplitude equal zero $a = 0$ value of the vibration mode corresponds to linear mode coefficients u_{210} or u_{220} (36). The vertical dashed lines arising from these points represent linear modal lines which are amplitude independent. To solve the equation (16) apart from modal coefficients also natural frequencies of the nonlinear model have to be determined from Eq.(12).

Fig. 5. Vibration natural frequencies versus amplitude; parameters given by (31)

Solid lines in Fig.5 presents course of frequency for free vibration of a nonlinear system with respect to amplitude, dashed line points out amplitude independent linear natural frequencies.

To construct the response based on nonlinear normal modes and related with it nonlinear resonant curves we have to solve Eq.(29) together with modal relationships (7) and nonlinear frequencies (12). Instead of Eqs, (7), (12) the set of algebraic equations (6) may be taken for numerical convenience.

At first, correctness of the nonlinear normal modes is demonstrated for a nonlinear conservative system without any excitations or damping. In Fig.6 we see free vibrations of coupled nonlinear oscillators in physical (generalised) and normal coordinates determined by linear transformation (34). Initial conditions have been selected to activate only the first normal coordinate. Therefore, for simulations, initial conditions have values: $X_1(0) = 10.0$,

$\dot{X}_1(0)=0$, $X_2(0)=23.7701$, $\dot{X}_2(0)=0$. These conditions correspond to normal coordinates $Y_1(0)=10.0$, $\dot{Y}_1(0)=0$, $Y_2(0)=0$, $\dot{Y}_2(0)=0$. We note that assumed values are high in the aim to expose the nonlinear effects (Fig.6a,b). For a linear system only the first mode is activated while the second is equal to zero. Application of the linear normal modes to the nonlinear model is acceptable only for relatively small amplitudes.

Fig. 6. Free vibrations in physical coordinates X_1, X_2 and linear normal coordinates Y_1, Y_2; initial conditions $X_1(0)=10.0$, $\dot{X}_1(0)=0$, $X_2(0)=23.7701$, $\dot{X}_2(0)=0$; initial values for normal coordinates $Y_1(0)=10.0$, $\dot{Y}_1(0)=0$, $Y_2(0)=0$, $\dot{Y}_2(0)=0$; parameters given by (31).

For the considered example the linear transformation leads to large differences. Fig.6(c) presents amplitude of Y_1 coordinate. The second coordinate Y_2 should stay quiet (Y_2=0). However, after a few seconds the coordinate increases and goes to very high values (Fig.6d). The linear transformation leads definitely to wrong results. Therefore, for large displacements the nonlinear transformation is required.

The same analysis has been repeated but applying nonlinear transformation (10). It means that modal coefficient u_{21} has been calculated taking into account vibration amplitude. Initial condition for physical coordinates take values $X_1(0)=10.0$, $\dot{X}_1(0)=0$, $X_2(0)=14.4413$, $\dot{X}_2(0)=0$, which result in nonlinear normal coordinates $Y_1(0)=10.0$, $\dot{Y}_1(0)=0$, $Y_2(0)=0$, $\dot{Y}_2(0)=0$. Note that linear transformation gives different normal coordinates $Y_1(0)=6.3945$ $\dot{Y}_1(0)=0$, $Y_2(0)=3.6055$, $\dot{Y}_2(0)=0$.

Fig. 7. Free vibrations in physical coordinates X_1, X_2 (a),(b) and nonlinear normal coordinates Y_1, Y_2 (c), (d); initial conditions $X_1(0)=10.0$, $\dot{X}_1(0)=0$, $X_2(0)=14.4413$, $\dot{X}_2(0)=0$; initial values for nonlinear normal coordinates $Y_1(0)=10.0$, $\dot{Y}_1(0)=0$, $Y_2(0)=0$, $\dot{Y}_2(0)=0$, parameters given by (31).

Time histories of X_1 and X_2 generated for initial conditions $X_1(0)=10.0$, $\dot{X}_1(0)=0$, $X_2(0)=14.4413$, $\dot{X}_2(0)=0$ are presented in Fig.7(a) and (b). After nonlinear transformation we get nonlinear normal coordinates Y_1 and Y_2 which are presented in Fig.7(c) and (d) respectively. Coordinates Y_2 is very close to zero value, for comparison Y_2 is plotted in Fig.7(c) in the same scale, too. This result demonstrates usefulness of the NNM formulation for nonlinear free vibrations.

Presented above formulation is used for resonant curves determination of the self-, parametrically and externally excited system near the main parametric resonances. The solutions are received from (13) by using analytical solutions (29) assuming that in the resonance the vibrations modes are close to the mentioned above NNMs found for the nonlinear conservative system. Solving together (29) and Eq.(7) we get the resonance curve presented in Fig.8(a).

Fig.8 Resonance curve with internal loop around the second principal parametric resonance (a), vibration mode coefficient (b), and the second natural frequency (c); nonlinear normal modes, $\alpha_1=0.01, \beta_1=0.05, \alpha_2=0.01, \beta_2=0.05$, $\mu=0.2, q=0.1$

Fig.9 Resonance curve with internal loop around the second principal parametric resonance; linear normal modes, $\alpha_1=0.01, \beta_1=0.05, \alpha_2=0.01, \beta_2=0.05$, $\mu=0.2, q=0.1$

This curve has been plotted for the principal parametric resonance region near the second natural frequency. The resonance is chosen because the nonlinear effects are very well demonstrated there. In the Fig. 8 we can see behaviour of the system similar to that of one degree of freedom model presented in the introduction and papers [17], [19]. External force acting together with parametric and self-excitation changes classical resonance curve of the considered model. We see that in the main parametric resonance additional internal loop arises. In the frequency region, near $\omega=1.25$, five solutions are possible. However, numerical check up shows that only two upper solutions are stable, while three lower are unstable. It is worth to mention that apart from steady state response, the sixth quasi-periodic motion also takes place. This solution is not presented in the figure because it has not been determined by the proposed analytical approach. The quasi-periodic motion requires a special treatment. The curves presented in Fig.8(a) have been found by the nonlinear normal mode transformation. Therefore, each amplitude has been determined together with modal and frequency dependencies presented in Fig.8(b) and (c) respectively. For amplitude equal to zero the modal coefficient takes values equal to the linear modes, however for $a > 0$ the modes is a nonlinear function (Fig.8b).

The equivalent resonance curve got by linear normal modes approach is shown in Fig.9 for comparison. We see that solutions differ essentially. The internal loop is located on the left hand side of the main resonance branch and is much larger than that got by nonlinear normal mode formulation.

Fig.10 Bifurcation diagram of the physical coordinate X_1 versus excitation frequency; $\alpha_1 = 0.01, \beta_1 = 0.05, \alpha_2 = 0.01, \beta_2 = 0.05$, $\mu = 0.2, q = 0.1$

To validate the received results the original Eqs.(2) have been solved directly by numerical methods. The bifurcation diagram of the physical coordinate X_1 versus excitation frequency is presented in Fig.10. The phenomenon of internal loop appearance is clearly visible near the principal parametric resonance around the second natural frequency for $\omega \in (1.1, 1.3)$. The frequency intervals of the main branch of the resonance curve as well as internal loop fit very well to the resonance regions obtained by nonlinear normal modes approach.

5. Conclusions

The paper presents a technique for resonant nonlinear normal modes formulation with application to a self- and parametrically excited system. It is shown that NNMs defined on the basis of free vibrations of a nonlinear system give results which are much closer to direct numerical simulation, comparing with classical LNMs. A width of synchronisation regions, near the principal parametric resonance fits very well to the regions found by numerical simulations, presented in the bifurcation diagram. Resonance curves, obtained by nonlinear and linear normal modes are essentially different. Correctness of the applied technique is confirmed by time histories analysis for various set of parameters. An effort of numerical solving of a set of nonlinear algebraic equations is a cost which is paid for the accuracy increase. Out of the synchronisation regions, due to strong influence of self-excitation, another method for NNMs formulation is required. The method should take into account also influence of velocity on the vibration modes.

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Figures Caption

Fig.1 Example of an externally and parametrically excited system with 1:2 frequencies ratio.

Fig.2 Resonance curve of a self-excited system driven parametrically (a) and driven parametrically and externally (b); $\alpha = 0.01$, $\beta = 0.05$, $\gamma = 0.1$, $\mu = 0.1$, $q = 0.05$

Fig. 3 Model of coupled self-excited oscillators driven by parametric and external excitations

Fig. 4. Vibration modes versus amplitude; parameters given by (31)

Fig. 5. Vibration natural frequencies versus amplitude; parameters given by (31)

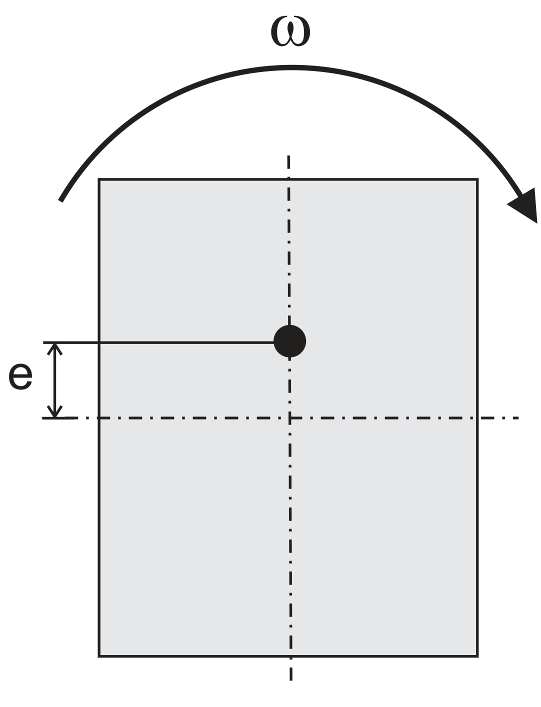
Fig. 6. Free vibrations in physical coordinates X_1, X_2 and linear normal coordinates Y_1, Y_2; initial conditions $X_1(0) = 10.0$, $\dot{X}_1(0) = 0$, $X_2(0) = 23.7701$, $\dot{X}_2(0) = 0$; initial values for normal coordinates $Y_1(0) = 10.0$, $\dot{Y}_1(0) = 0$, $Y_2(0) = 0$, $\dot{Y}_2(0) = 0$; parameters given by (31).

Fig. 7. Free vibrations in physical coordinates X_1, X_2 (a),(b) and nonlinear normal coordinates Y_1, Y_2 (c), (d); initial conditions $X_1(0) = 10.0$, $\dot{X}_1(0) = 0$, $X_2(0) = 14.4413$, $\dot{X}_2(0) = 0$; initial values for nonlinear normal coordinates $Y_1(0) = 10.0$, $\dot{Y}_1(0) = 0$, $Y_2(0) = 0$, $\dot{Y}_2(0) = 0$; parameters given by (13).

Fig.8 Resonance curve with internal loop around the second principal parametric resonance (a), vibration mode coefficient (b), and the second natural frequency (c); nonlinear normal modes, $\alpha_1 = 0.01, \beta_1 = 0.05, \alpha_2 = 0.01, \beta_2 = 0.05$, $\mu = 0.2, q = 0.1$.

Fig.9 Resonance curve with internal loop around the second principal parametric resonance; linear normal modes, $\alpha_1 = 0.01, \beta_1 = 0.05, \alpha_2 = 0.01, \beta_2 = 0.05$, $\mu = 0.2, q = 0.1$

Fig.10 Bifurcation diagram of the physical coordinate X_1 versus excitation frequency; $\alpha_1 = 0.01, \beta_1 = 0.05, \alpha_2 = 0.01, \beta_2 = 0.05$, $\mu = 0.2, q = 0.1$



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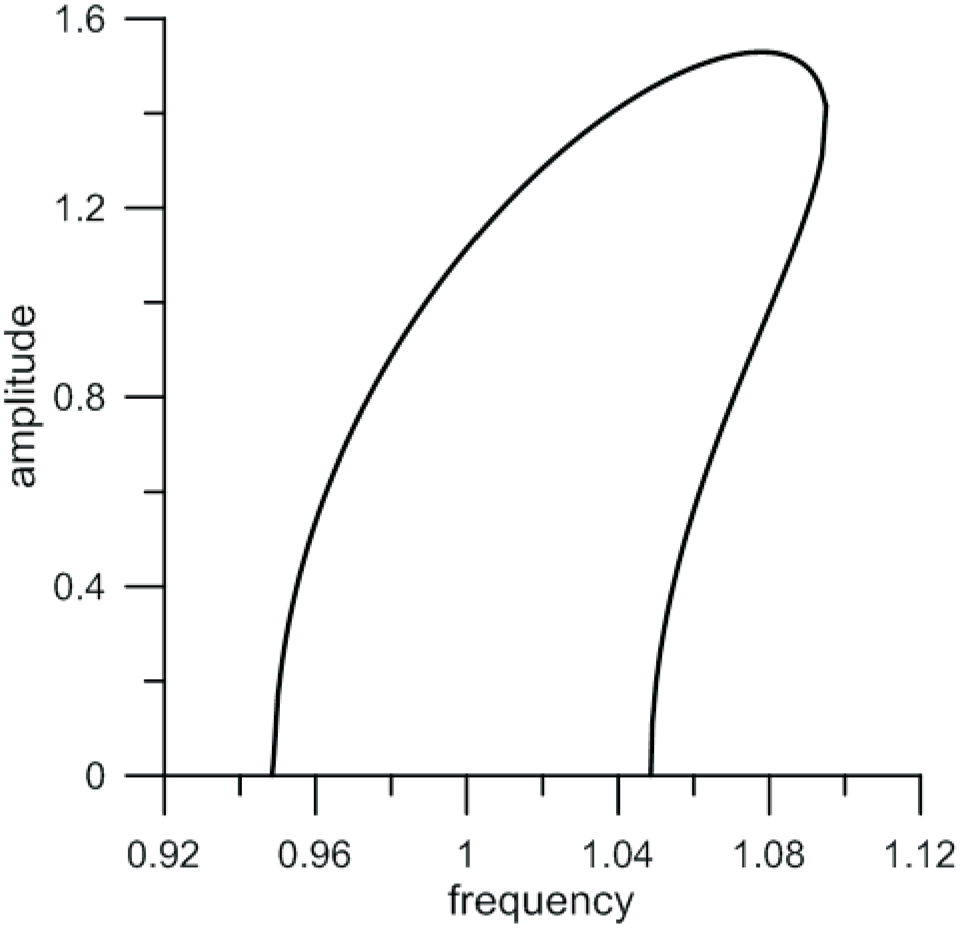

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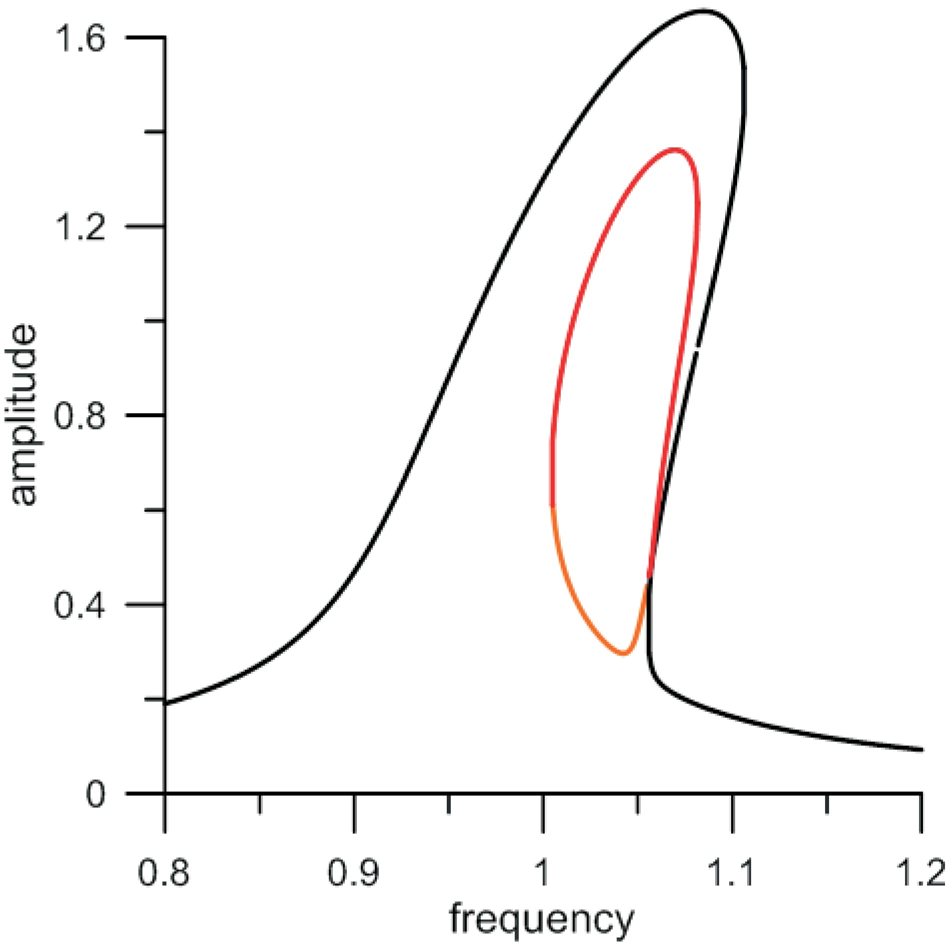
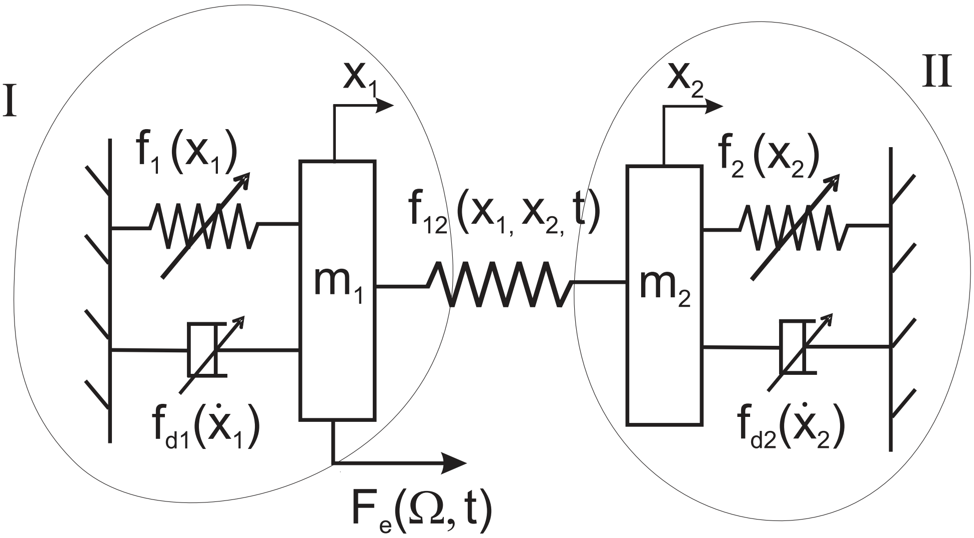
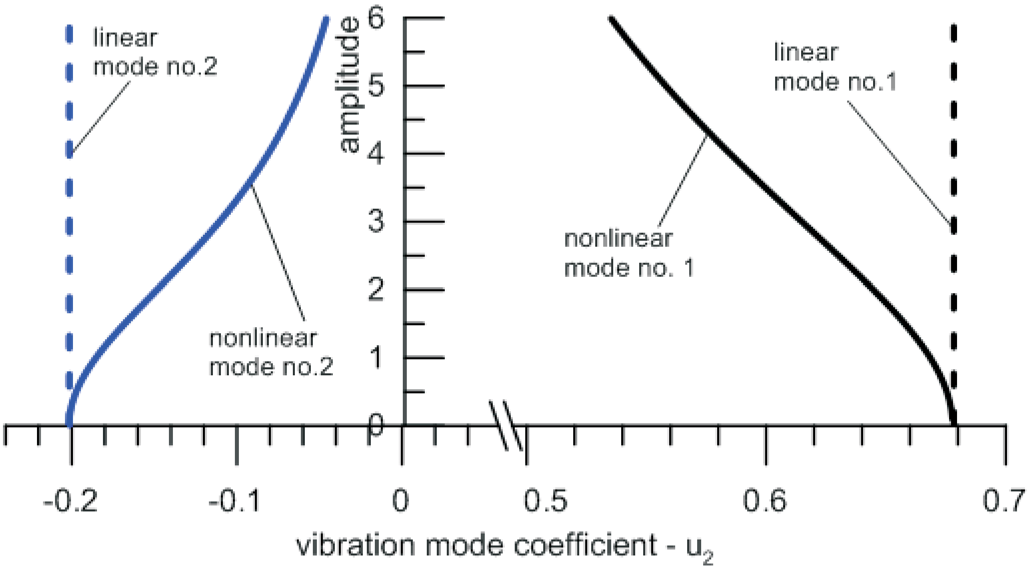


Fig2b



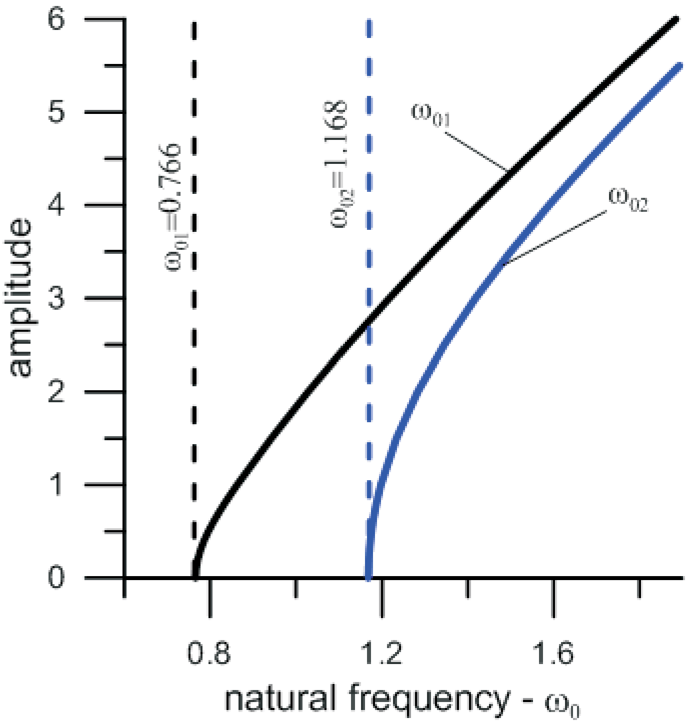
I
II
x_1
x_2
$f_1\,(x_1)$
$f_2\,(x_2)$
$f_{12}\,(x_1, x_2, t)$
m_1
m_2
$f_{d1}(\dot{x}_1)$
$f_{d2}(\dot{x}_2)$
$F_e(\Omega, t)$

Fig3



linear
mode no.2
nonlinear
mode no.2
amplitude
6
5
4
3
2
1
0
linear
mode no.1
nonlinear
mode no. 1
-0.2
-0.1
0
0.5
0.6
0.7
vibration mode coefficient - u_2

Fig4


6
5
4
3
2
1
0
amplitude
0.8
1.2
1.6
natural frequency - ω_0
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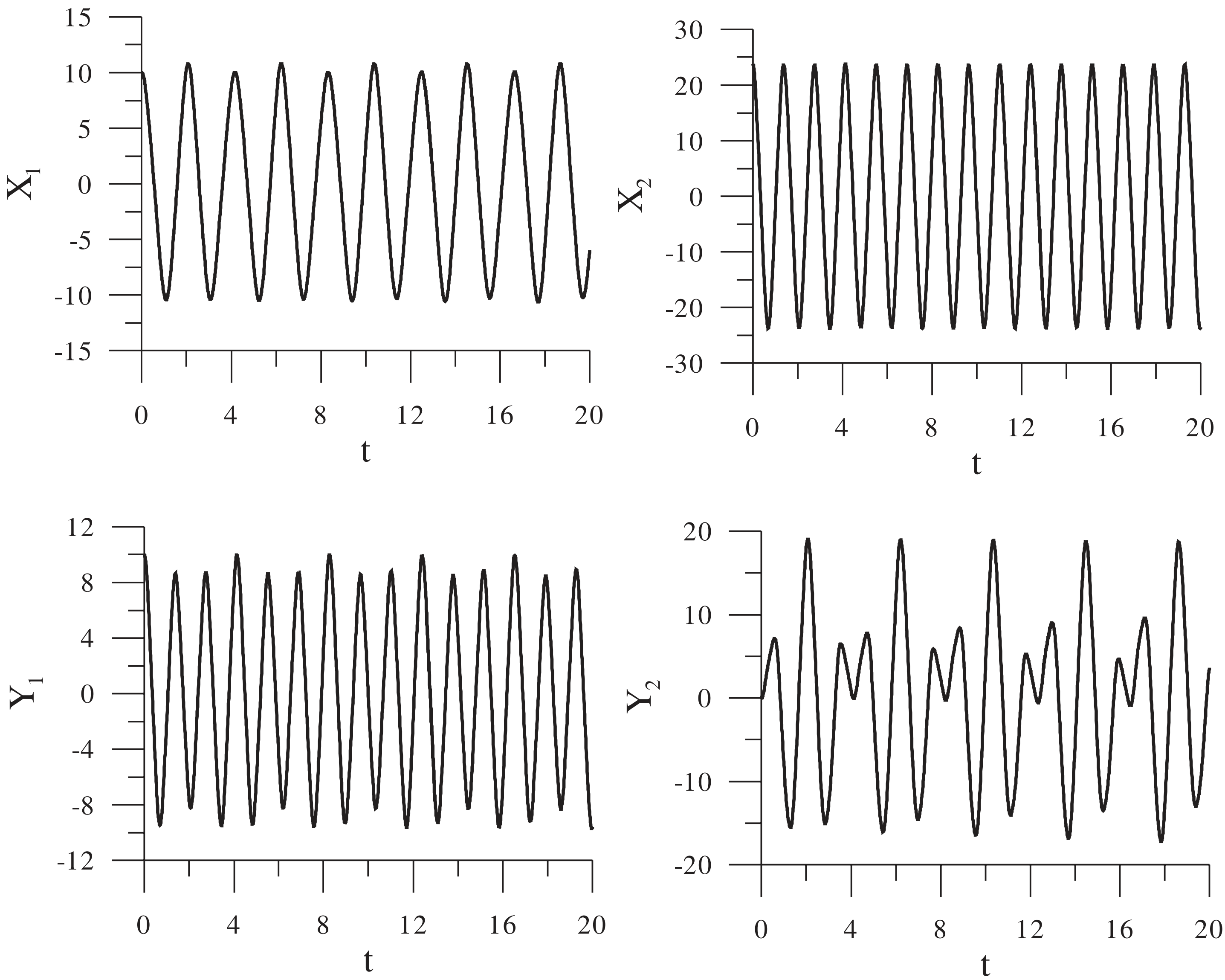
ω_{02}=1.168
ω_{01}
ω_{02}

Fig5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5
0
-5
-10
-15
X1
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t
30
20
10
0
-10
-20
-30
X2
0 4 8 12 16 20
t
12
8
4
0
-4
-8
-12
Y1
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t
20
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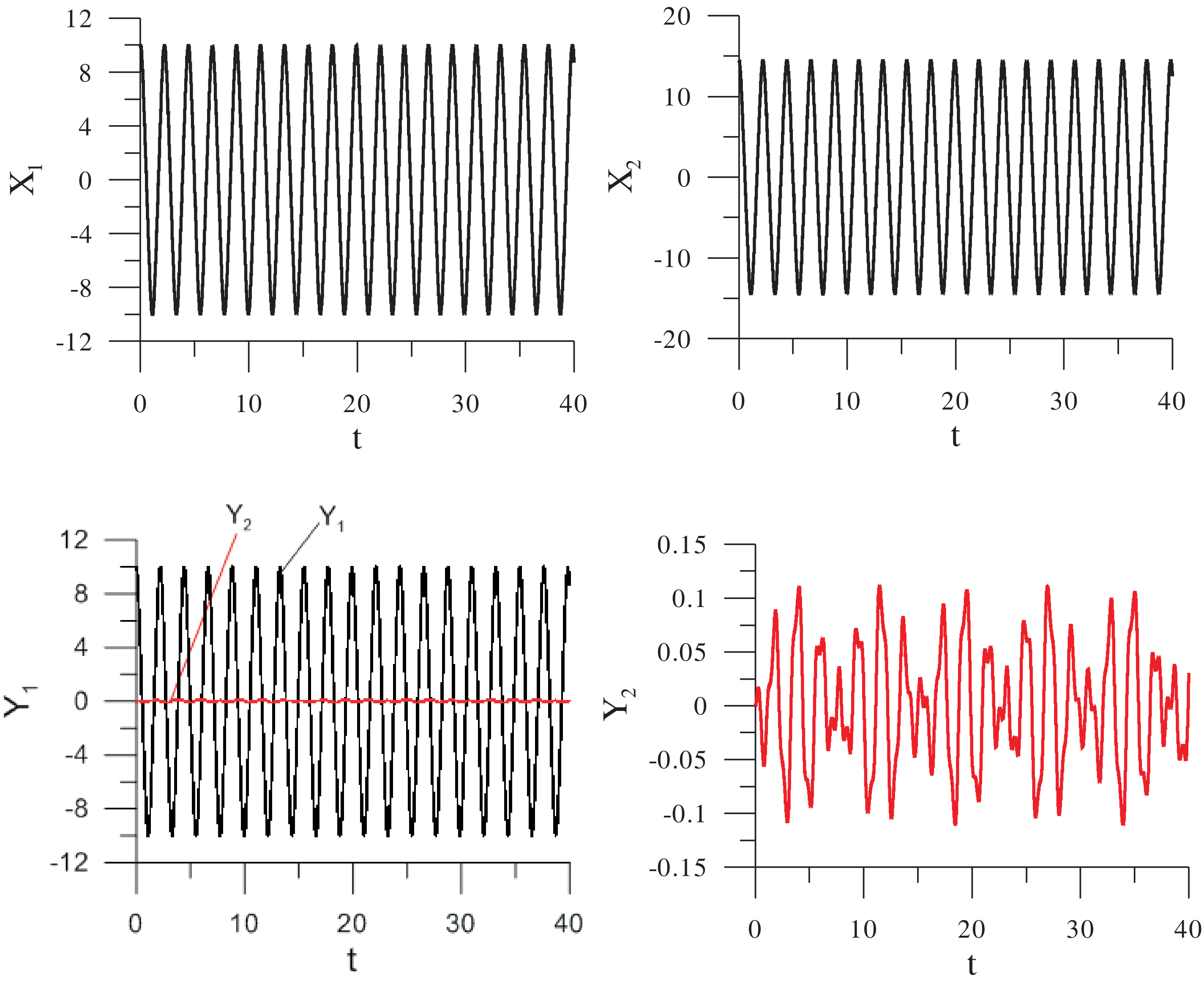
0
-10
-20
Y2
0 4 8 12 16 20
t

Fig6


12
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4
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-12
X_1
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20
30
40
t
20
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0
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X_2
0
10
20
30
40
t
Y_2
Y_1
12
8
4
0
-4
-8
-12
Y_1
0
10
20
30
40
t
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0
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-0.1
-0.15
Y_2
0
10
20
30
40
t

Fig7

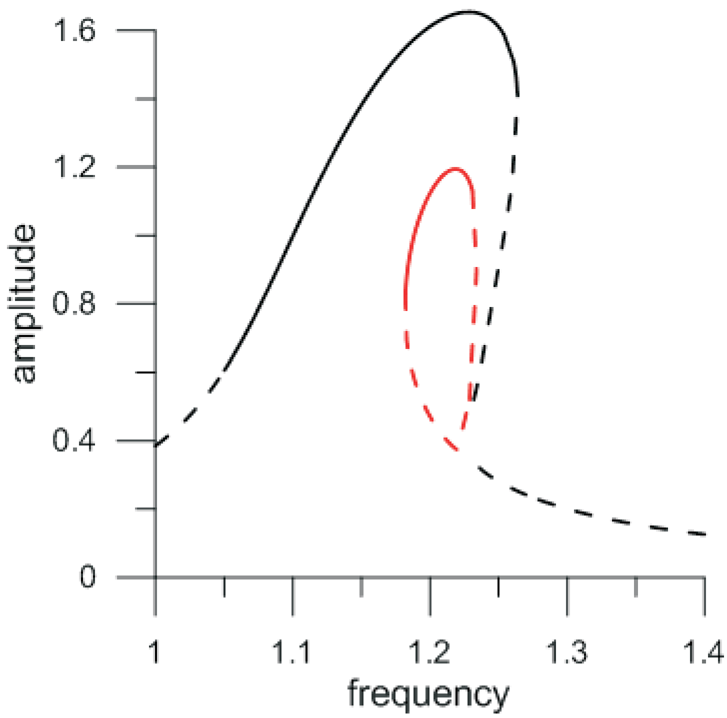

Fig8a

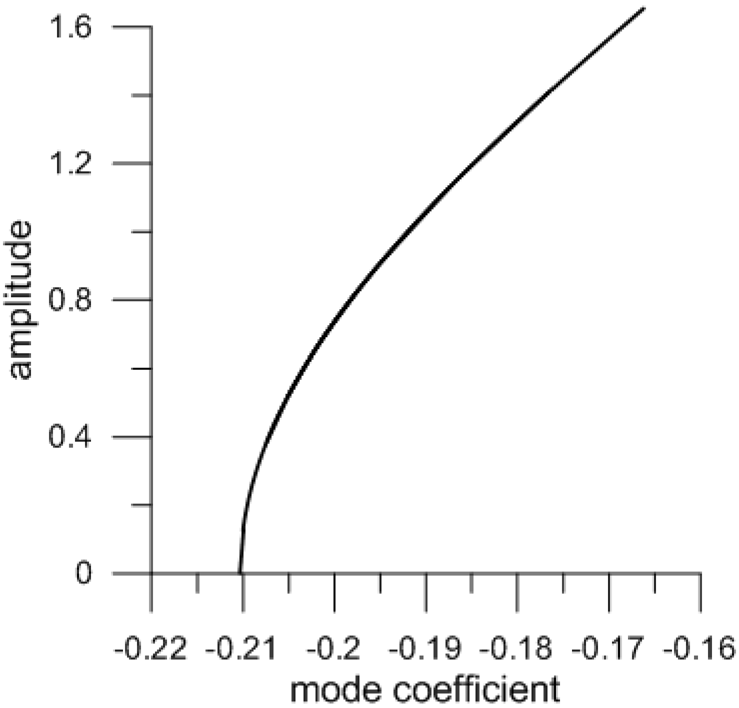
1.6
1.2
0.8
0.4
0
amplitude
-0.22
-0.21
-0.2
-0.19
-0.18
-0.17
-0.16
mode coefficient

Fig8b

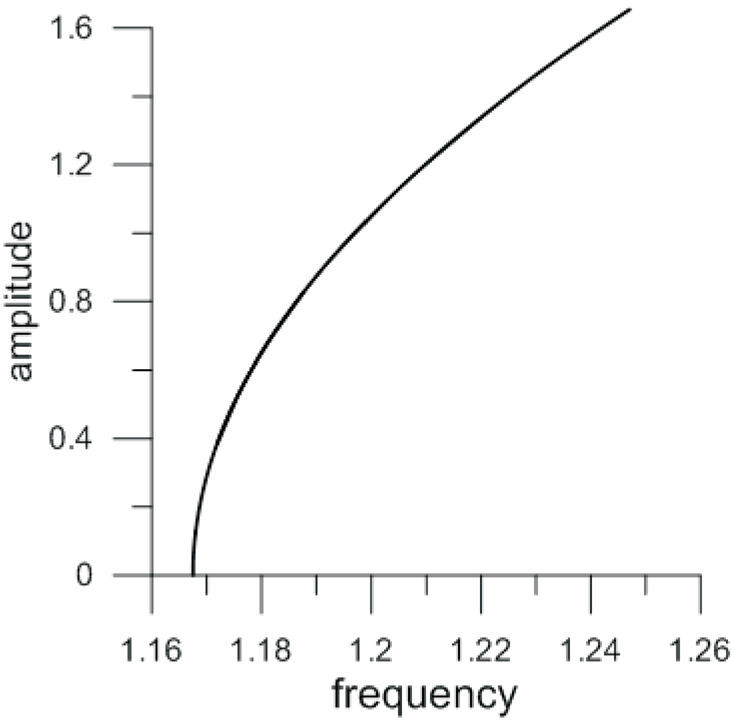

Fig8c

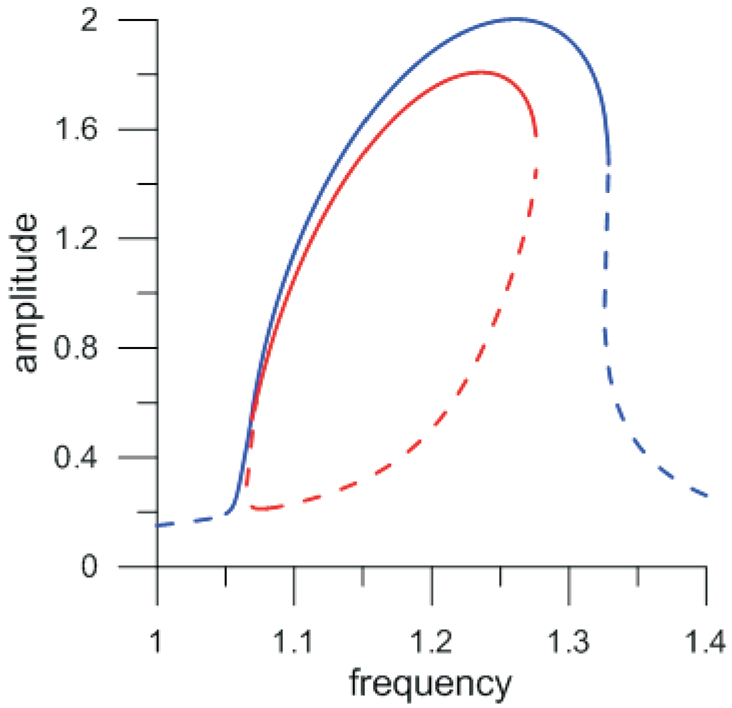

Fig9

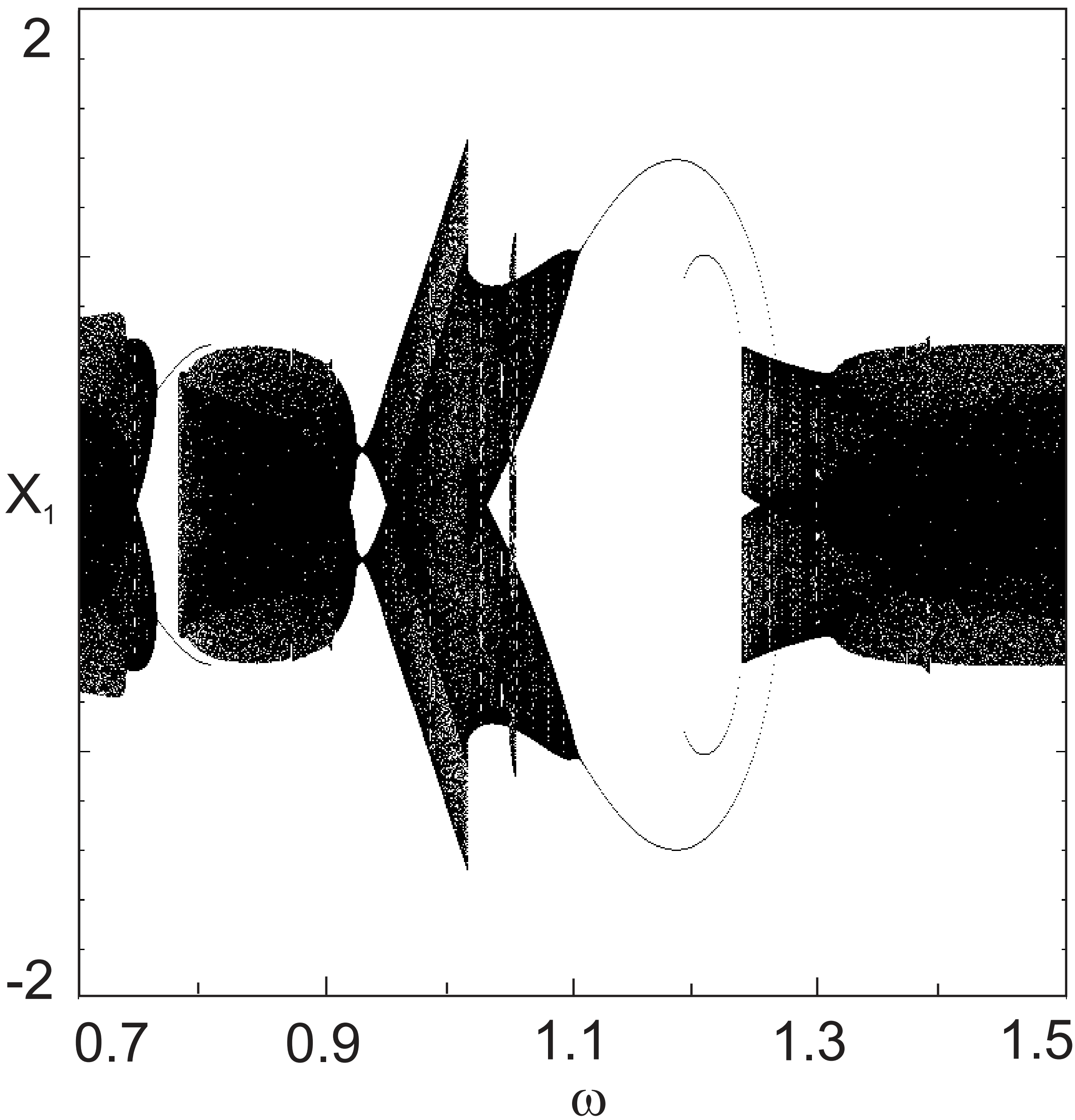

Fig10