Nonlinear Observer Design to Synchronize Hyperchaotic Systems via a Scalar Signal

Giuseppe Grassi and Saverio Mascolo

***Abstract*—In this brief control theory is used to formalize hyperchaos synchronization as a nonlinear observer design issue. Following this approach, a new *systematic* tool to synchronize a class of hyperchaotic systems via a *scalar* transmitted signal is developed. The proposed technique has been applied to synchronize two well-known hyperchaotic systems.**

***Index Terms*—Hyperchaotic circuits and systems, synchronization theory.**

I. Introduction

In recent years, synchronization of chaotic systems and its potential application to secure communications have received ever increasing attention [1]–[13]. The possibility of two or more chaotic systems oscillating in a synchronized way is not an obvious one. In fact, as chaos is characterized by a sensitive dependence on initial conditions, one could conclude that synchronization is not obtainable, since even infinitesimal change will eventually result in divergence of nearby starting orbits [1]. To overcome this problem, different approaches have been developed. In [2], [3] the suggested scheme consists in taking a chaotic system, duplicating some subsystem and driving the duplicate and the original subsystem with signals from the unduplicated part. When all the Lyapunov exponents of the driven subsystem (response system) are less then zero, the response system synchronizes with the drive system, assuming that both systems start in the same basin of attraction [2]. Instead of searching for a stable subsystem, in [4]–[7] a linear feedback of the error signals is used as control input into one of the chaotic systems. In these cases, synchronization is achieved by computing proper elements of a coupling matrix in order to get negative Lyapunov exponents, provided that the initial conditions of both systems are very close to each other [6].

It should be noted that the abovementioned methods mainly concern the synchronization of low dimensional systems with only one positive Lyapunov exponent. This feature limits the complexity of the chaotic dynamics and suggests the adoption of higher dimensional chaotic systems for applications to secure communications [14]–[16]. In fact, the presence of more than one positive Lyapunov exponent clearly improves security by generating more complex dynamics. However, this approach raises the question of whether synchronization can still be achieved by transmitting a scalar signal. Until now, only some attempts have been made to give an answer to this question. In [17] the conjecture that the number of synchronizing signals had to be equal to the number of positive Lyapunov exponents led to the adoption of two scalar signals to synchronize Rössler's hyperchaotic system. Recently, some interesting results have been reported in [14]–[16]. In particular, in [14] a scalar signal represented by a linear combination of the original state variables is used to synchronize hyperchaos in Rössler's systems. However, this approach cannot be considered a systematic technique for synchronization, because the coefficients of the linear combination are somewhat arbitrary. Furthermore, the computation of the conditional Lyapunov exponents is still required in order to verify the synchronization [14].

In this brief a new method is developed to synchronize hyperchaotic systems via a scalar transmitted signal. The proposed technique is based on nonlinear control theory and has several advantages over the existing methods. In particular

1) it enables synchronization be achieved in a systematic way;
2) it can be successfully applied to several well-known hyperchaotic systems;
3) it does not require the computation of any Lyapunov exponent;
4) it does not require initial conditions belonging to the same basin of attraction.

The brief is organized as follows. In Section II, the synchronization of chaotic systems is restated as a nonlinear observer design issue. Following this approach, a linear and time-invariant synchronization error system is obtained, for which a necessary and sufficient condition can be given in order to asymptotically stabilize its dynamics. Finally, in Section III the proposed method is applied to synchronize two well-known examples of hyperchaotic systems.

II. Hyperchaos Synchronization as a Nonlinear Observer Design Issue

Definition 1: Given two chaotic systems, the dynamics of which are described by the following two sets of differential equations:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \tag{1}$$

$$\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}) \tag{2}$$

where $\boldsymbol{x} \in \Re^n, \boldsymbol{y} \in \Re^n$, and $\boldsymbol{f}: \Re^n \rightarrow \Re^n$ is a nonlinear vector field, systems (1) and (2) are said to be synchronized if

$$\boldsymbol{e}(t) = (\boldsymbol{y}(t) - \boldsymbol{x}(t)) \rightarrow \boldsymbol{0} \quad \text{as} \quad t \rightarrow \infty \tag{3}$$

where $\boldsymbol{e}$ represents the synchronization error [1].

In order to obtain synchronization, system (2) has to receive a proper synchronizing signal from system (1). From a control theory point of view, this signal can be considered as an observed quantity feeding a nonlinear observer for the state $\boldsymbol{x}$ of the system (1) [18]–[21]. Informally, an observer is a dynamic system designed to be driven by the output of another dynamic system (*plant*) and having the property that the state of the observer converges to the state of the plant. More precisely, the following definition is given.

Definition 2: Given dynamic system (1) with output $z = \boldsymbol{s}(\boldsymbol{x}) \in \Re^n$, the dynamic system

$$\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}) + \boldsymbol{g}(z - \boldsymbol{s}(\boldsymbol{y})) \tag{4}$$

is said to be a nonlinear observer of system (1) if $\boldsymbol{y}$ converges to state $\boldsymbol{x}$ as $t \rightarrow \infty$, where $\boldsymbol{g}: \Re^n \rightarrow \Re^n$ is a suitably chosen nonlinear function [18]. Moreover, system (4) is said to be a global observer of system (1) if $\boldsymbol{y} \rightarrow \boldsymbol{x}$ as $t \rightarrow \infty$ for any initial condition $\boldsymbol{y}(0), \boldsymbol{x}(0)$ [20].

A block diagram of a nonlinear observer for the state $\boldsymbol{x}$ of system (1) is reported in Fig. 1.

Remark 1: System (4) is a (global) observer of system (1) if the error system

$$\begin{aligned}\dot{\boldsymbol{e}} &= \boldsymbol{f}(\boldsymbol{y}) + \boldsymbol{g}(\boldsymbol{s}(\boldsymbol{x}) - \boldsymbol{s}(\boldsymbol{y})) - \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x} + \boldsymbol{e}) \\ &\quad + \boldsymbol{g}(\boldsymbol{s}(\boldsymbol{x}) - \boldsymbol{s}(\boldsymbol{x} + \boldsymbol{e})) - \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{h}(\boldsymbol{e}, t)\end{aligned} \tag{5}$$

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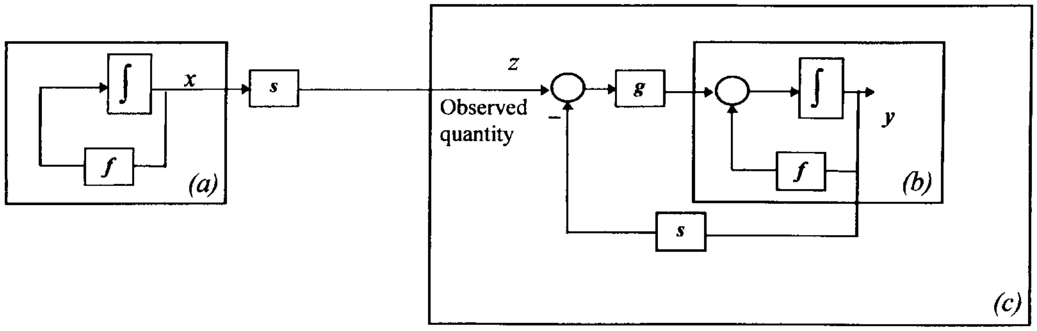

Fig. 1. Synchronization as a nonlinear observer issue. (a) System (1). (b) System (2). (c) Structure of the observer (4).

has a (globally) asymptotically stable equilibrium point for $\boldsymbol{e} = \boldsymbol{0}$ [18], [20].

It is known that control theory offers no general method to choose a function $\boldsymbol{g}(z - \boldsymbol{s}(\boldsymbol{y}))$ such that the nonlinear and nonautonomous system (5) has a (globally) asymptotically stable equilibrium point for $\boldsymbol{e} = \boldsymbol{0}$. In the following, a proposition will be stated in order to give a function $\boldsymbol{g}$ for synchronizing a class of hyperchaotic systems. To this purpose, an assumption is made.

Assumption 1: The dynamic system (1) can be written as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}f(\boldsymbol{x}) + \boldsymbol{c} \tag{6}$$

where $\boldsymbol{A} \in \Re^{n \times n}$, $\boldsymbol{b} \in \Re^{n \times 1}$, $\boldsymbol{c} \in \Re^{n \times 1}$ and $f\colon \Re^n \rightarrow \Re$.

Remark 2: Several well-known chaotic systems belong to the class individualized by (6). For example, Chua's circuit [1], Rössler's hyperchaotic system [23], the hyperchaotic circuits in [24], [25] and the n-dimensional Chua's circuit in [26], [27] all satisfy Assumption 1.

Regarding the synchronizing signal, it is worth noting that $\boldsymbol{s}(\boldsymbol{x})$ is an artificial output of the system (1) which can be properly designed to feed the nonlinear observer (4). Since the adoption of a scalar signal is a suitable feature for secure communications applications, it is assumed that $z = s(\boldsymbol{x}) \in \Re$.

Now, a proposition is given, so that the error system (5) becomes linear and time-invariant when $s(\boldsymbol{x})$ and $\boldsymbol{g}(z - s(\boldsymbol{y}))$ are properly chosen.

Proposition 1: Given a dynamic system (1) satisfying Assumption 1, let

$$s(\boldsymbol{x}) = f(\boldsymbol{x}) + \boldsymbol{k}\boldsymbol{x} \tag{7}$$

be the scalar synchronizing signal with $\boldsymbol{k} = [k_1, k_2, \cdots k_n] \in \Re^{1 \times n}$, and let

$$\boldsymbol{g}(s(\boldsymbol{x}) - s(\boldsymbol{y})) = \boldsymbol{b}(s(\boldsymbol{x}) - s(\boldsymbol{y})) \tag{8}$$

be the function $\boldsymbol{g}$ in (4). Then the error system (5) becomes linear and time-invariant, and can be expressed as

$$\dot{\boldsymbol{e}} = \boldsymbol{A}\boldsymbol{e} - \boldsymbol{b}\boldsymbol{k}\boldsymbol{e} = \boldsymbol{a}\boldsymbol{e} + \boldsymbol{b}u \tag{9}$$

where $u = -\boldsymbol{k}\boldsymbol{e}$ plays the role of a state feedback.

Proof: By substituting (7) and (8) in (5), the error system becomes:

$$\begin{aligned}\dot{e} &= \boldsymbol{f}(\boldsymbol{y}) + \boldsymbol{g}(s(\boldsymbol{x}) - s(\boldsymbol{y})) - \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{y} + \boldsymbol{b}f(\boldsymbol{y}) + \boldsymbol{c}\\ &\quad + \boldsymbol{b}(s(x) - s(\boldsymbol{y})) - (\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}f(\boldsymbol{x}) + c)\\ &= \boldsymbol{A}\boldsymbol{e} + \boldsymbol{b}(f(\boldsymbol{y}) - f(\boldsymbol{x})) + \boldsymbol{b}(f(\boldsymbol{x}) + \boldsymbol{k}\boldsymbol{x} - f(\boldsymbol{y}) - \boldsymbol{k}\boldsymbol{y})\\ &= \boldsymbol{A}\boldsymbol{e} - \boldsymbol{b}\boldsymbol{k}\boldsymbol{e} = \boldsymbol{A}\boldsymbol{e} + \boldsymbol{b}u\end{aligned}$$

This completes the proof.

Now, by exploiting linear control theory [22], [28], the following result can be stated:

Proposition 2: Given a dynamic system satisfying Assumption 1, and the functions $s(\boldsymbol{x})$ and $\boldsymbol{g}(z - s(\boldsymbol{y}))$ defined by (7) and (8), respectively, a necessary and sufficient condition for the existence of a feedback gain vector $\boldsymbol{k}$ such that system (4) becomes a global observer of system (1) is that all the uncontrollable eigenvalues of the error system (9), if any, have negative real parts.

Proof: For linear system (9) a proper coordinate transformation $\boldsymbol{e} = [\boldsymbol{T}_1\ \boldsymbol{T}_2]\overline{\boldsymbol{e}}$ can be found, where the columns of $\boldsymbol{T}_1$ form a set of basis vector for the controllable state subspace and the columns of $\boldsymbol{T}_2$ are orthogonal to these [22], [28]. Since the orthogonal basis set gives $\boldsymbol{T}^{-1} = \boldsymbol{T}^T$, system (9) can be transformed to the following Kalman controllable canonical form [28]:

$$\begin{aligned}\begin{bmatrix}\dot{\overline{\boldsymbol{e}}}_c\\ \dot{\overline{\boldsymbol{e}}}_{nc}\end{bmatrix} &= \begin{bmatrix}\boldsymbol{T}_1^T\boldsymbol{A}\boldsymbol{T}_1 & \boldsymbol{T}_1^T\boldsymbol{A}\boldsymbol{T}_2\\ \boldsymbol{0} & \boldsymbol{T}_2^T\boldsymbol{A}\boldsymbol{T}_2\end{bmatrix}\begin{bmatrix}\overline{\boldsymbol{e}}_c\\ \overline{\boldsymbol{e}}_{nc}\end{bmatrix} + \begin{bmatrix}\boldsymbol{T}_1^T\boldsymbol{b}\\ \boldsymbol{0}\end{bmatrix}u\\ &= \begin{bmatrix}\overline{\boldsymbol{A}}_{\boldsymbol{c}} & \overline{\boldsymbol{A}}_{\boldsymbol{12}}\\ \boldsymbol{0} & \overline{\boldsymbol{A}}_{\boldsymbol{nc}}\end{bmatrix}\begin{bmatrix}\overline{\boldsymbol{e}}_{\boldsymbol{c}}\\ \overline{\boldsymbol{e}}_{\boldsymbol{nc}}\end{bmatrix} + \begin{bmatrix}\overline{\boldsymbol{b}}_{\boldsymbol{c}}\\ \boldsymbol{0}\end{bmatrix}u\end{aligned} \tag{10}$$

where the eigenvalues of $\overline{\boldsymbol{A}}_c$ are controllable, i.e., they can be placed anywhere by proper state feedback $u = -\boldsymbol{k}\boldsymbol{e}$, whereas the eigenvalues of $\overline{\boldsymbol{A}}_{nc}$ are uncontrollable, i.e., they are not affected by the introduction of any state feedback. Therefore a necessary and sufficient condition to globally asymptotically stabilize system (10) is that the eigenvalues of $\overline{\boldsymbol{A}}_{nc}$ lie in the left half plane [22], [28]. Since $\overline{e} \rightarrow \boldsymbol{0}$ implies $\boldsymbol{e} \rightarrow \boldsymbol{0}$, this completes the proof.

Remark 3: If system (4) becomes an observer of system (1), then $\boldsymbol{y} \rightarrow \boldsymbol{x}, s(\boldsymbol{y}) \rightarrow s(\boldsymbol{x})$ and $\boldsymbol{g} \rightarrow \boldsymbol{0}$ as $t \rightarrow \infty$ (see (8)). As a consequence, the dynamics of systems (1) and (2) are identical.

Remark 4: If system (9) is controllable, then all the modes can be arbitrarily assigned and, consequently, synchronization can be achieved according to any specified feature.

Remark 5: A technique similar to the one developed herein has been proposed in [11]. Both the methods generate an error system which is linear and time-invariant. However, since the error system in [11] is $\dot{\boldsymbol{e}} = \boldsymbol{A}\boldsymbol{e}$, its eigenvalues cannot be moved by any state feedback and, consequently, synchronization can be achieved only if the eigenvalues of $\boldsymbol{A}$ have negative real part. In this brief less restrictive conditions are given, because the controllable eigenvalues of the error system $\dot{e} = \boldsymbol{A}\boldsymbol{e} + \boldsymbol{b}u$ can be shifted via a state feedback $u = -\boldsymbol{k}\boldsymbol{e}$.

Remark 6: An interesting approach to *chaos* synchronization, based on the concept of observer design, has been proposed in [13]. In particular, synchronization is achieved by considering a linear output for the drive system, whereas for the response one a Luenberger observer is chosen. This leads to a nonlinear and nonautonomous synchronization error system for which it is not easy to obtain the stability properties of the origin. Thus, the conclusion of the analysis developed in [13] is that local synchronization is possible under relatively mild conditions, whereas global synchronization can be achieved only if the system can be transformed to Brunowsky canonical form. Unlike the method just mentioned, the technique developed herein chooses a nonlinear output for the drive system and a nonlinear observer for the response one, so that *global* synchronization can be easily achieved if the conditions of Proposition 2 are satisfied.

III. Examples

In this section the proposed tool is applied to synchronize two examples of nonlinear systems which exhibit hyperchaotic dynamics.

A. Synchronization of Rössler's System

Rössler's system [23] can be written in the form of Assumption 1 as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & -1 & -1 & 0\\ 1 & 0.25 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & -0.5 & 0.05\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\\ 0\end{bmatrix} x_1x_3 + \begin{bmatrix}0\\ 0\\ 3\\ 0\end{bmatrix}. \tag{11}$$

This system exhibits a hyperchaotic behavior starting from proper initial conditions [23]. Proposition 1 gives

$$s(\boldsymbol{x}) = x_1x_3 + \sum_{j=1}^{4} k_jx_j$$

$$\boldsymbol{g}(s(\boldsymbol{x}) - s(\boldsymbol{y})) = [0 \quad 0 \quad 1 \quad 0]^T[s(\boldsymbol{x}) - s(\boldsymbol{y})]$$

whereas (4) becomes

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\\ \dot{y}_4\end{bmatrix} = \begin{bmatrix}0 & -1 & -1 & 0\\ 1 & 0.25 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & -0.5 & 0.05\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\\ y_4\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\\ 0\end{bmatrix} y_1y_3 + \begin{bmatrix}0\\ 0\\ 3\\ 0\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\\ 0\end{bmatrix}(s(\boldsymbol{x}) - s(\boldsymbol{y})) \tag{12}$$

with the error system given by

$$\begin{bmatrix}\dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\\ \dot{e}_4\end{bmatrix} = \left(\begin{bmatrix}0 & -1 & -1 & 0\\ 1 & 0.25 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & -0.5 & 0.05\end{bmatrix} - \begin{bmatrix}0\\ 0\\ 1\\ 0\end{bmatrix}[k_1 \quad k_2 \quad k_3 \quad k_4]\right)\begin{bmatrix}e_1\\ e_2\\ e_3\\ e_4\end{bmatrix}. \tag{13}$$

Since the controllability matrix of system (13) is full rank, Proposition 2 assures that there exists a gain vector $\boldsymbol{k}$ such that system (12) becomes a global observer of system (11), i.e., $\boldsymbol{y} \to \boldsymbol{x}$ as $t \to \infty$ for any initial state. For example, all eigenvalues of (13) can be placed in -1 for $\boldsymbol{k} = [-3.3712\ -0.9561\ 4.3000\ -5.8126]$.

B. Synchronization of a Fourth-Order Circuit

In 1986 hyperchaos has been observed, for the first time, from a *real physical system*: a fourth-order electrical circuit [24]. This simple circuit is autonomous and contains only one nonlinear element, a three-segment piecewise-linear resistor. All other elements are linear and passive, except an active resistor, which has negative resistance. By considering the circuit parameters reported in [24], the dynamics can be written as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 0 & -2 & 0\\ 0 & 0 & 0 & -20\\ 1 & 0 & 1 & 0\\ 0 & 1.5 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}2\\ -20\\ 0\\ 0\end{bmatrix} g(x_2 - x_1) \tag{14}$$

where $g(\cdot)$ is the piecewise-linear function given by

$$g(x_2 - x_1) = 3(x_2 - x_1) - 1.6(|x_2 - x_1 - 1| - |x_2 - x_1 + 1|).$$

From Proposition 1, it follows:

$$s(\boldsymbol{x}) = g(x_2 - x_1) + \sum_{i=1}^{n} k_ix_i$$

$$\boldsymbol{g}(s(\boldsymbol{x}) - s(\boldsymbol{y})) = [2 \quad -20 \quad 0 \quad 0]^T(s(\boldsymbol{x}) - s(\boldsymbol{y}))$$

whereas (4) becomes

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\\ \dot{y}_4\end{bmatrix} = \begin{bmatrix}0 & 0 & -2 & 0\\ 0 & 0 & 0 & -20\\ 1 & 0 & 1 & 0\\ 0 & 1.5 & 0 & 0\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\\ y_4\end{bmatrix} + \begin{bmatrix}2\\ -20\\ 0\\ 0\end{bmatrix} g(y_2 - y_1) + \begin{bmatrix}2\\ -20\\ 0\\ 0\end{bmatrix}(s(\boldsymbol{x}) - s(\boldsymbol{y})). \tag{15}$$

Since the controllability matrix of the error system is full rank, its eigenvalues can be moved anywhere. By placing them in -2, it results $\boldsymbol{k} = [0.8022\ -0.3698\ 0.0381\ -0.0308]$ and system (15) becomes a global observer of system (14).

Remark 7: In [11] the attention is focused on synchronization of *chaotic* systems. When dealing with hyperchaos, the hypothesis in [11] (that is, eigenvalues of $\boldsymbol{A}$ in the open left half plane) seems hard to be satisfied. In fact, by examining the systems considered herein, it can be pointed out that the matrix $\boldsymbol{A}$ of Rössler's system has three eigenvalues with positive real part whereas the one of Example B has two eigenvalues with positive real part. The same consideration can be made for other examples of hyperchaotic systems [25].

IV. Conclusion

In this brief a new technique to synchronize a class of hyperchaotic systems via a scalar transmitted signal has been developed. The proposed approach exploits the concept of nonlinear observer and represents a *systematic* tool which can be successfully applied to obtain global synchronization of nonlinear systems in the form (6) if structural properties on $\boldsymbol{A}$ and $\boldsymbol{b}$ hold.

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References

[1] M. Ogorzalek, "Taming chaos-Part I: Synchronization," *IEEE Trans. Circuits Syst. I*, vol. 40, pp. 693–699, Oct. 1993.
[2] T. L. Carroll and L. M. Pecora, "Synchronizing chaotic circuits," *IEEE Trans. Circuits Syst.*, vol. 38, pp. 453–456, Apr. 1991.
[3] ——, "A circuit for studying the synchronization of chaotic systems," *Int. J. Bifurc. Chaos*, vol. 2, no. 3, pp. 659–667, 1992.
[4] T. Kapitaniak, "Synchronization of chaos using continuos control," *Phys. Rev. E*, vol. 50, no. 2, pp. 1642–1644, 1994.

[5] T. Kapitaniak, M. Sekieta, and M. Ogorzalek, "Monotone synchronization of chaos," *Int. J. Bifurc. Chaos*, vol. 6, no. 1, pp. 211–217, 1996.

[6] L. Kocarev, A. Shang, and L. O. Chua, "Transitions in dynamical regimes by driving: a unified method of control and synchronization of chaos," *Int. J. Bifurc. Chaos*, vol. 3, no. 2, pp. 479–483, 1993.

[7] T. Kapitaniak, "Controlling chaos." San Diego, CA: Academic, 1996.

[8] K. M. Cuomo, A. V. Oppenheim, and S. H. Strogatz, "Synchronization of Lorenz-based chaotic circuits with applications to communications," *IEEE Trans. Circuits Syst. II*, vol. 40, pp. 626–633, Oct. 1993.

[9] L. Kocarev, K. S. Halle, K. Eckert, L. O. Chua, and U. Parlitz, "Experimental demonstration of secure communications via chaotic synchronization," *Int. J. Bifurc. Chaos*, vol. 2, no. 3, pp. 709–713, 1992.

[10] U. Feldmann, M. Hasler, and W. Schwarz, "Communication by chaotic signals: The inverse system approach," *Int. J. Circuit Theory Applicat.*, vol. 24, no. 4, pp. 551–580, 1996.

[11] C. W. Wu and L. O. Chua, "A simple way to synchronize chaotic systems with applications to secure communication systems," *Int. J. Bifurc. Chaos*, vol. 3, no. 6, pp. 1619–1627, 1993.

[12] C. W. Wu and L. O. Chua, "A unified framework for synchronization and control of dynamical systems," *Int. J. Bifurc. Chaos*, vol. 4, no. 4, pp. 979–998, 1994.

[13] Ö. Morgül and E. Solak, "Observed based synchronization of chaotic systems," *Phys. Rev. E*, vol. 54, no. 5, pp. 4803–4811, 1996.

[14] J. H. Peng, E. J. Ding, M. Ding, and W. Yang, "Synchronizing hyperchaos with a scalar transmitted signal," *Phys. Rev. Lett.*, vol. 76. no. 6, pp. 904–907, 1996.

[15] L. Kocarev and U. Parlitz, "General approach for chaotic synchronization with applications to communication," *Phys. Rev. Lett.*, vol. 74. no. 25, pp. 5028–5031, 1995.

[16] U. Parlitz, L. Kocarev, T. Stojanovski, and H. Preckel, "Encoding messages using chaotic synchronization," *Phys. Rev. E*, vol. 53, no. 5, pp. 4351–4361, 1996.

[17] K. Pyragas, "Predictable chaos in slightly perturbed unpredictable chaotic systems," *Phys. Lett. A*, vol. 181, no. 3, pp. 203–210, 1993.

[18] B. Friedland, *Advanced Control System Design*. Englewood Cliffs, NJ: Prentice Hall, 1996.

[19] A. Isidori, *Nonlinear Control Systems*. New York: Springer, 1995.

[20] R. Marino and P. Tomei, *Nonlinear Control Design*. Englewood Cliffs, NJ: Prentice-Hall, 1995.

[21] J. J. Slotine and W. Lie, *Applied Nonlinear Control.* Englewood Cliffs, NJ: Prentice-Hall, 1991.

[22] C. T. Chen, *Linear System Theory and Design.* Holt, Rinehart and Winston, 1984.

[23] O. E. Rössler, "An equation for hyperchaos," *Phys. Lett.*, vol. 71A, no. 2–3, pp. 155–157, 1979.

[24] T. Matsumoto, L. O. Chua, and K.Kobayashi, "Hyperchaos: laboratory experiment and numerical confirmation," *IEEE Trans. Circuits Syst.*, vol. 33, pp. 1143–1147, Nov. 1986.

[25] A. Tamasevicius, A. Namajunas, and A. Cenys, "Simple 4D chaotic oscillator," *Inst. Elect. Eng. Electron. Lett.*, vol. 32, no. 11, pp. 957–958, 1996.

[26] M. Gotz, U. Feldmann, and W. Schwarz, "Synthesis of higher dimensional Chua circuits," *IEEE Trans. Circuits Syst. I*, vol. 40, pp. 854–860, Nov. 1993.

[27] A. Rogers, P. Sheridan, and O. Feely, "Synthesis of higher order Chua circuits," in *Proc. Fourth Int. Workshop on Nonlinear Dynamics of Electron. Syst. (NDES'96)*, Seville, Spain, 1996, pp. 265–270.

[28] W. L. Brogan, "*Modern Control Theory*. Englewood Cliffs, NJ: Prentice-Hall, 1991.

Applications of Symbolic Dynamics in Chaos Synchronization

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***Abstract*— We give the relationship between symbolic dynamics and chaos synchronization. If the capacity of the channel which one-directionally connects two chaotic systems with the aim of synchronizing them is larger than Kolmogorov–Sinai entropy of the driving system, then the synchronization error can be made arbitrarily small.**

***Index Terms*— Chaos, channel capacity, symbolic dynamics, synchronization.**

I. Introduction

In this brief, we analyze nonlinear discrete-time dynamical systems whose chaotic evolution is governed by *deterministic* equations [1]. Despite of the absence of stochastic terms in the governing equations it is usually said that the long-term behavior of chaotic systems is unpredictable. Such an unpredictability is due to a unique property of chaotic systems, namely—exponential sensitivity to changes in initial states. Any uncertainty in the knowledge of the initial state gets amplified by the chaotic nature of the dynamical system, and eventually reaches the chaotic attractor's size thus preventing the long-term prediction.

Still, two or more chaotic systems when suitably coupled can successfully synchronize [2]–[4], that is, their trajectories tend to each other. As early as in one of the pioneering works on chaos synchronization [4] it was pointed out that the reproducibility of chaotic trajectories through synchronized chaotic motion in addition to the unpredictability and random-like appearance of chaotic trajectories might be interesting for secure communications applications. Indeed, numerous papers have been published on the subject since then. Making a complete reference is almost impossible and we only point to several papers with extensive references [5]–[9].

The issue of influence of the capacity of a communication channel on the synchronization between two chaotic systems connected by the channel has not been addressed yet. Channel capacity is equal to the maximal amount of information that can be conveyed through a channel per unit time where the maximization is done over all possible channel input signals. In simple terms, both analog and digital communication channels have *finite* capacity, while chaos synchronization methods require that the driving signal is transmitted to the response circuit without any distortions including noise addition or amplitude quantization. Having in mind that chaotic signals take values from a continuous set, this is virtually a requirement for a channel with infinite capacity which is impossible to be satisfied.

When a coarse-graining of the state space is introduced, for example, by a measurement process, then the deterministic behavior of a chaotic system on a microscopic (continuous) scale is turned into a stochastic behavior on a macroscopic (coarse-grained) scale [1],

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