Nonlinear PID Predictive Controller

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Abstract

A new class of nonlinear PID controllers are derived for nonlinear systems using a Nonlinear Generalised Predictive Control (NGPC) approach. First the disturbance decoupling ability of the nonlinear generalised predictive controller is discussed. For a nonlinear system where the disturbance cannot be decoupled, a nonlinear observer is designed to estimate the offset. By selecting the nonlinear gain function in the observer, it is shown that the closed-loop system under optimal generalised predictive control with the nonlinear observer is asymptotically stable. It is pointed out that this composite controller is equivalent to a nonlinear controller with integral action. As a special case, for a nonlinear system with a low relative degree, the proposed nonlinear controller reduces to a nonlinear PI or PID predictive controller, which consists of a nonlinear PI or PID controller and a prediction controller. The design method is illustrated by an example nonlinear mechanical system.

Indexing terms: Optimal control, nonlinear systems, predictive control, disturbance, PID controller, Nonlinear Generalised Predictive Control.

1   Introduction

It is well known that the elimination of offset is a major objective in the control of industrial systems. Offset in control systems may arise from many sources including loads, friction, intrinsic steady-state nonlinearity, uncertainties in modelling, etc.. It can also arise from nonzero-mean noise. For linear systems, many approaches have been proposed to eliminate offset. Introducing integral action is the main method. There are many ways to embed the integral action in the control structures [1, 2]. In general they can be divided into two classes. One is to enforce integral action in a controller. The other is to appropriately model the disturbance and design a controller and then the integral action arises naturally. This paper falls into the latter class. That is, we believe it should not be necessary to force a controller to have integral action, but rather this structure should arise naturally.

The PID controller has been widely applied in engineering. Apart from its simple structure and relatively easy tuning, one of the main reasons for its popularity is that it provides the ability to remove offset by using integral action. It improves the performance robustness in the steady state against noise and uncertainties. Moreover since PID (PI) controllers are so widely used, one might expect that the structure should arise naturally given reasonable assumptions on system internal dynamics and control performance specifications. For linear systems, it has been discussed by many authors; for example, see [2, 3, 4].

Generalised Predictive Control (GPC) is one of the most promising control methods in engineering. It has received considerable attention in the last decade. Much effort has been made to extend GPC to nonlinear systems. In general a nonlinear optimisation problem, which is rarely convex, requires to be solved on-line. To avoid this, several closed-form nonlinear model predictive control laws have been developed in [5, 6] and [7, 8] where the control order is selected as the relative degree of the nonlinear systems and the relationship between the derived control laws and feedback linearisation technique has been explored. Recently, following the continuous-time GPC approach introduced by Gawthrop and his co-workers [9, 10, 11, 12], a new Nonlinear Generalised Predictive Control (NGPC) method has been proposed by Chen et al. [13] and a closed-form solution for optimal NGPC has been given. Its main features are that no on-line optimisation is required and the stability of the closed-loop system is guaranteed. Moreover it is shown that the design parameters, the control order and the predictive time, can be determined according to desired time-domain transient specifications by look-up tables. As such, to design an optimal nonlinear controller for a nonlinear system in the predictive control frame is, in many aspects, easier and more transparent than to design a Linear Quadratic Regulator (LQR) for a linear system. This paper is an extension of the work in [13] to handle system disturbance using a nonlinear observer.

The original GPC was proposed in the incremental control formulation by Clarke et al. [14, 15]. It has integral action and can remove offset. However, the approach cannot be extended to the nonlinear GPC problem considered in this paper. In many cases, it is impossible to put

a *nonlinear* system with a constant disturbance into an incremental control formulation with no disturbance. Basically, linear GPC is an output feedback control strategy and was initially proposed in a transfer function setting although GPC has also been discussed extensively in the state space setting [16, 17, 18]. In general, nonlinear GPC is considered in a state space setting. Although the state-space observer/ state feedback approach for GPC advocated by Bitmead, Gevers and Wertz [16] can be adopted, nonlinear GPC at present is a state feedback control strategy since, in general, to design an observer for a nonlinear system is not trivial [19, 20].

The optimal tracking problem for nonlinear systems with disturbance is considered in this paper. Section 2 discusses the disturbance decoupling ability of NGPC. It is pointed out that when the relative degree of the disturbance is higher than that of the control, NGPC can completely decouple the disturbance. For a nonlinear system where the disturbance cannot be decoupled, a nonlinear disturbance observer is designed in Section 3 and the global stability of the observer is established. The composite controller consists of a NGPC and a nonlinear disturbance observer. Section 4 further discusses the composite controller structure. It is shown that by selecting the nonlinear design function, integral action arises in the composite controller and the explicit observer disappears. NGPC with a nonlinear disturbance observer is shown to be equivalent to an optimal nonlinear predictive controller with integral action. The stability of the closed-loop system under the nonlinear controller is investigated in Section 5. It is proved that the closed-loop system is globally asymptotically stable. For a nonlinear system with a low relative degree, Section 6 indicates that the optimal nonlinear predictive control reduces to a nonlinear PID or PI controller. This partly explains why a PID or PI structure is widely used in engineering. However, the proposed controller is superior to a nonlinear PID/PI controller since it includes a predictive part which can predict the future of the system's output caused by the system dynamics. The new design method is applied to the control of a nonlinear mechanical system in Section 7. The paper ends with concluding remarks in Section 8.

2 Nonlinear GPC under disturbances

2.1 Problem and Notation

For the sake of simplicity, the standard Lie derivative notation is used in this paper (for example see [21]).

A single-input single-output (SISO) nonlinear system under disturbance considered in this paper is given by

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}_1(\boldsymbol{x})u + \boldsymbol{g}_2(\boldsymbol{x})b \\ y = h(\boldsymbol{x}) \end{cases} \tag{1}$$

where $\boldsymbol{x} \in R^n$, $u \in R$, $y \in R$ and $b \in R$ are a state vector, control, output and disturbance respectively. The functions $\boldsymbol{f}(\boldsymbol{x})$ and $h(\boldsymbol{x})$ are assumed to be continuously differentiable a sufficient number of times, and $\boldsymbol{g}_1(\boldsymbol{x})$ and $\boldsymbol{g}_2(\boldsymbol{x})$ are continuous functions of $\boldsymbol{x}$.

It is well known that to obtain a good performance it is helpful to model the disturbance. The disturbance will be treated in a non-stochastic setting in this paper. Although, for the sake of simplicity, it is assumed that the disturbance is an unknown constant, it can be a piece-wise constant. This represents a wide class of disturbances such as load variation, friction, nonzero-mean noise, etc.. More importantly, it will be shown that integral action then arises naturally. It is well known that integral action can improve tracking and regulation performance in steady state and also improves robustness against uncertainties and noise.

Definition 1: The relative degree of the mapping from the input $u(t)$ to the output $y(t)$ at x^0 is defined as the minimum order of the derivative of the output $y(t)$ where the control $u(t)$ first appears at x^0. For simplicity it is referred to the input relative degree at x^0. The input relative degree is well-defined when the input relative degree has the same value ρ for all $x^0 \in R^n$. Similarly, the relative degree of the mapping from the disturbance $b(t)$ to the output $y(t)$ is defined as the disturbance relative degree.

In this paper the following assumptions are imposed on the system (1):

$A$1: The zero dynamics are stable;

$A$2: The input relative degree ρ is well-defined;

$A$3: All states are available.

The first assumption implies that the nonlinear system is minimum phase. A nonlinear system satisfying $A1$–$A3$ is input-output feedback linearisable. These assumptions are made for most of the existing nonlinear control methods [21, 22]. It is important to note that Grizzle et al [23] have indicated that Assumptions $A1$ and $A2$ are necessary for the existence of an analytical compensator which yields asymptotic tracking for an open set of output trajectories.

The optimal tracking problem considered is to design a controller such that the output can optimally track a desired command signal $y_d(t)$, in presence of the disturbance b, so as to minimise the generalised predictive control performance index

$$J = \frac{1}{2} \int_0^T (y(t+\tau) - y_d(t+\tau))^2 d\tau \qquad (2)$$

where T is the predictive time. It is not necessary to include a control weight in this cost function since the control is effectively weighted by use of control order constraints.

Similar to other existing GPC methods [9, 12, 14, 15] in the moving time frame, the control signal is constrained and limited to be a polynomial function of the time. That is, the order of the highest derivative of the control signal $\hat{u}(t+\tau)$, which is called *control order*, is specified. This defines an allowable set of the control signal in the moving time frame. For example when the control order r is set to be zero, the allowable control signal in the moving time frame is constant. There is no limitation on the control order r in this paper. It is important to note that the constraint is only imposed in the control $\hat{u}(t+\tau), 0 \le \tau \le T$ and the real control $u(t)$ has no such constraint. To emphasise this, the hatted variables are used for the moving time

frame in this paper. Moreover when $\tau = 0$, $u(t) = \hat{u}(t + \tau)$. Explicit control input constraint is not directly handled in this paper. [13] has discussed how to select the predictive time and the control order to adjust the control effort. When input and/or state-related constraints need to be considered, Command Governors for the nonlinear GPC developed in this paper can be designed by the technique in [24].

2.2 Disturbance decoupling

For a nonlinear system without disturbance, this nonlinear predictive control problem has been solved in [13] without the use of an observer. For a nonlinear system with disturbance, we have the following results.

Theorem 1: If the disturbance relative degree is higher than the input relative degree in the nonlinear system (1), then the optimal NGPC for the system (1) with the performance index (2) is given by

$$u(t) = -G_1(\boldsymbol{x})^{-1}(\sum_{i=0}^{\rho} k_i(L_f^i h(\boldsymbol{x}) - y_d^{[i]}(t))) \tag{3}$$

where from Assumption $A2$, $G_1(\boldsymbol{x}) \neq 0$, and $y_d^{[i]}$ denotes the ith derivative of y_d,

$$G_1(\boldsymbol{x}) = L_{\boldsymbol{g_1}} L_{\boldsymbol{f}}^{\rho-1} h(\boldsymbol{x}), \tag{4}$$

$$k_i = z_i T^{-\rho+i} \frac{\rho!}{i!}, i = 0, \ldots, \rho - 1, \ k_\rho = 1, \tag{5}$$

$$\begin{bmatrix} z_0 & z_1 & \cdots & z_{\rho-1} \end{bmatrix} = \text{First column of the matrix } \boldsymbol{M_1}^{-1}\boldsymbol{M_2}, \tag{6}$$

$$\boldsymbol{M_1} = \left\{ \frac{1}{i + j + 2\rho - 1} \right\}_{i,j=1,\ldots,r+1}, \tag{7}$$

$$\boldsymbol{M_2} = \left\{ \frac{1}{i + j + \rho - 1} \right\}_{i=1,\ldots,r+1, j=1,\ldots,\rho}. \tag{8}$$

and r is the control order. Moreover the optimal NGPC (3) completely rejects the disturbance b.

Proof: See Appendix.

Theorem 1 implies that when the disturbance relative degree is higher than the control relative degree, the disturbance is completely rejected by nonlinear GPC. This is the same as the conclusion in geometric nonlinear control theory (for example see [21]) since the closed-loop system is feedback linearised by the optimal NGPC (3) (see Appendix).

Theorem 2: For a nonlinear system (1) with the same control and disturbance relative degree ρ, the optimal NGPC is given by

$$u(t) = -G_1(\boldsymbol{x})^{-1} \left(\sum_{i=0}^{\rho} k_i(L_{\boldsymbol{f}}^i h(\boldsymbol{x}) - y_d^{[i]}(t)) + G_2(\boldsymbol{x})b \right) \tag{9}$$

where

$$G_2(\boldsymbol{x}) = L_{\boldsymbol{g_2}} L_{\boldsymbol{f}}^{\rho-1} h(\boldsymbol{x}) \tag{10}$$

and all other variables are defined as in Theorem 1.

Proof: See Appendix.

In the remainder of this paper, it is assumed that the disturbance relative degree is the same as the input relative degree since under the optimal NGPC the disturbance is completely decoupled for a nonlinear system with the disturbance relative degree higher than the control relative degree. The case where the disturbance relative degree is less than the input relative degree is not considered in this paper but it will be discussed in Section 8.

3   A disturbance Observer

The optimal NGPC (9) is a static state feedback control law. To implement it, information about the disturbance b is required. In many cases, it is difficult to measure a disturbance like load variation, friction etc. In this section, a nonlinear disturbance observer is designed. Since in general there is no information about the derivative of the disturbance, the simplest assumption is

$$\dot{b} = 0 \tag{11}$$

When the offset b is considered as a state, then the augmented system with the state $[\boldsymbol{x}^T; b]^T$ is given. To obtain information about the offset b, one approach is to design a full-order observer for this augmented nonlinear system. However, nonlinear observer design for a nonlinear system is non-trivial and, in general, the stability of such nonlinear observers is not guaranteed [19, 25]. Furthermore, since all states in (1) are available, only the disturbance b needs to be estimated. In this paper, a nonlinear reduced-order observer is designed to estimate the disturbance and, as will be shown, the stability not only of the observer but also of the closed-loop system can be guaranteed.

It follows from (1) that

$$\boldsymbol{g_2}(\boldsymbol{x})b = \dot{\boldsymbol{x}} - \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{g_1}(\boldsymbol{x})u \tag{12}$$

In this paper an initial disturbance observer is given by

$$\dot{\hat{b}} = -\boldsymbol{l}(\boldsymbol{x})\boldsymbol{g_2}(\boldsymbol{x})\hat{b} + \boldsymbol{l}(\boldsymbol{x})\left(\dot{\boldsymbol{x}} - \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{g_1}(\boldsymbol{x})u\right) \tag{13}$$

where $\boldsymbol{l}(\boldsymbol{x})$ is a gain matrix to be designed. Let

$$e_1 = b - \hat{b}$$

Then

$$\dot{e}_1(t) = \dot{b} - \dot{\hat{b}} = \boldsymbol{l}(\boldsymbol{x})\boldsymbol{g_2}(\boldsymbol{x})(\hat{b} - b) = -\boldsymbol{l}(\boldsymbol{x})\boldsymbol{g_2}(\boldsymbol{x})e_1(t) \tag{14}$$

$\boldsymbol{l}(\boldsymbol{x})$ can be chosen such that the system

$$\dot{e}_1(t) + \boldsymbol{l}(\boldsymbol{x})\boldsymbol{g_2}(\boldsymbol{x})e_1(t) = 0 \tag{15}$$

is exponentially stable for all $\boldsymbol{x} \in R^n$. Consequently, $\hat{b}(t)$ approaches $b(t)$ exponentially as $t \to \infty$. One possible candidate for $\boldsymbol{l}(\boldsymbol{x})$ is $q(h(\boldsymbol{x})\boldsymbol{g_2}(\boldsymbol{x}))^{-1}h(\boldsymbol{x})$ for all $h(\boldsymbol{x})$ such that $h(\boldsymbol{x})\boldsymbol{g_2}(\boldsymbol{x})$ is invertible for all $\boldsymbol{x} \in R^n$. In this case, the observer error equation becomes

$$\dot{e}_1(t) + qe_1(t) = 0 \tag{16}$$

The convergence rate can be specified by the parameter $q > 0$. However since $\dot{\boldsymbol{x}}$ is not available, the observer (13) is not practically implementable.

To overcome this define an auxiliary variable

$$z = \hat{b} - p(\boldsymbol{x}) \tag{17}$$

where $p(\boldsymbol{x})$ is a function to be designed, and let

$$\boldsymbol{l}(\boldsymbol{x}) = \frac{\partial p(\boldsymbol{x})}{\partial \boldsymbol{x}} \tag{18}$$

Combining the above equations with (13) yields

$$\begin{aligned}\dot{z} &= \dot{\hat{b}} - \frac{\partial p(\boldsymbol{x})}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} \\ &= -h(\boldsymbol{x})\boldsymbol{g_2}(\boldsymbol{x})(z + p(\boldsymbol{x})) - h(\boldsymbol{x})(\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g_1}(\boldsymbol{x})u)\end{aligned} \tag{19}$$

That is,

$$\dot{z} = -h(\boldsymbol{x})\boldsymbol{g_2}(\boldsymbol{x})z - h(\boldsymbol{x})(\boldsymbol{g_2}(\boldsymbol{x})p(\boldsymbol{x}) + \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g_1}(\boldsymbol{x})u) \tag{20}$$

The offset estimation is given by

$$\hat{b} = z + p(\boldsymbol{x}) \tag{21}$$

The whole observer consists of the equations (20) and (21). It can be shown the $\hat{b}(t)$ approaches $b(t)$ exponentially if $p(\boldsymbol{x})$ is chosen such that

$$\dot{e}_1(t) + \frac{\partial p(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{g_2}(\boldsymbol{x})e_1(t) = 0 \tag{22}$$

is globally exponentially stable for all $\boldsymbol{x} \in R^n$. As far as the stability of the estimation error is concerned, any non-linear vector valued function $\boldsymbol{l}(\boldsymbol{x}) = \frac{\partial p}{\partial \boldsymbol{x}}$, such that (22) is asymptotically stable, can be chosen. Having chosen $\boldsymbol{l}(\boldsymbol{x})$, $p(\boldsymbol{x})$ is found by integration. An alternative method is to start from $p(\boldsymbol{x})$ and choose $p(\boldsymbol{x})$ such that the estimation error approaches zero. However in general both of these methods are not trivial. In addition, there exists considerable design freedom in the choice of $p(\boldsymbol{x})$. A systematic method for design of $p(\boldsymbol{x})$ will be developed in the next section by requiring that integral action should exist in the controller.

4 The controller structure with integral action

In this section it will be shown that by choosing the design function, $p(\boldsymbol{x})$, in the observer, integral action arises in the closed-loop systems with NGPC. More importantly, it will be shown
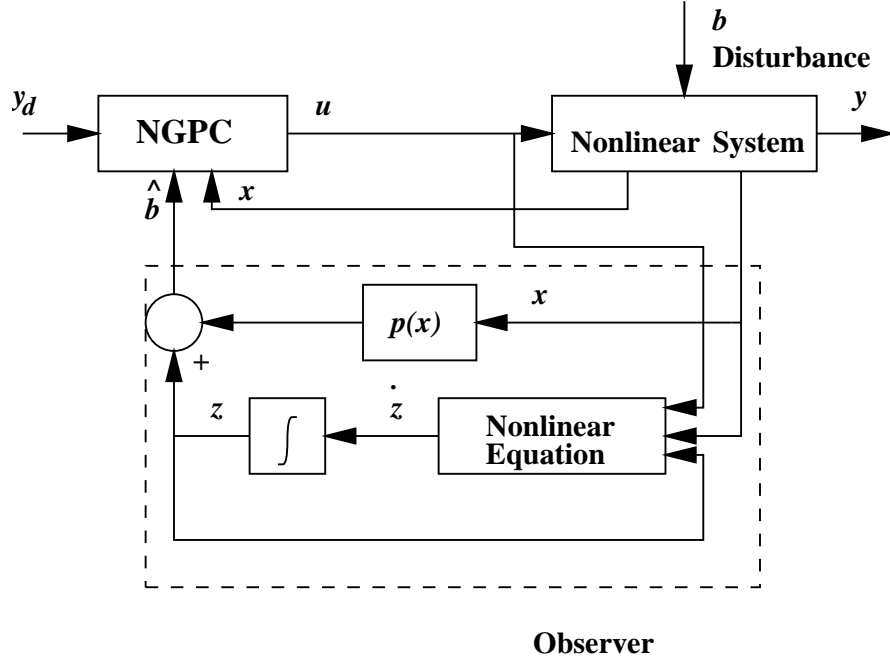
Figure 1: NGPC with nonlinear disturbance observer

that the closed-loop system under composite controller with the chosen design function, $p(\boldsymbol{x})$, is globally asymptotically stable.

Once the offset estimation is given by the nonlinear observer (20) and (21), the offset b in the NGPC (9) is replaced by its estimate $\hat{b}$. Then the NGPC (9) becomes

$$u(t) = -G_1(\boldsymbol{x})^{-1} \sum_{i=0}^{\rho} k_i \left(L_{\boldsymbol{f}}^i h(\boldsymbol{x}) - y_d^{[i]}(t) + G_2(\boldsymbol{x})\hat{b} \right) \tag{23}$$

The NGPC with the observer (20) and (21) is depicted in Figure 1.

Substituting (23) and (21) into (19) yields

$$\begin{aligned}
\dot{\hat{b}} &= \boldsymbol{l}(\boldsymbol{x})(\dot{\boldsymbol{x}} - \boldsymbol{f}(\boldsymbol{x})) - \boldsymbol{l}(\boldsymbol{x})\boldsymbol{g_2}(\boldsymbol{x})\hat{b} - \boldsymbol{l}(\boldsymbol{x})\boldsymbol{g_1}(\boldsymbol{x})u \\
&= \boldsymbol{l}(\boldsymbol{x})(\dot{\boldsymbol{x}} - \boldsymbol{f}(\boldsymbol{x})) - \boldsymbol{l}(\boldsymbol{x})\boldsymbol{g_2}(\boldsymbol{x})\hat{b} + \\
&\quad + \boldsymbol{l}(\boldsymbol{x})\boldsymbol{g_1}(\boldsymbol{x}) \left(G_1(\boldsymbol{x})^{-1} (\sum_{i=0}^{\rho} k_i (L_{\boldsymbol{f}}^i h(\boldsymbol{x}) - y_d^{[i]}) + G_2(\boldsymbol{x})\hat{b} \right)
\end{aligned} \tag{24}$$

We choose $p(\boldsymbol{x})$ as

$$p(\boldsymbol{x}) = p_0 L_{\boldsymbol{f}}^{\rho-1} h(\boldsymbol{x}) \tag{25}$$

where $p_0 \neq 0$ is a constant. Then it follows from (18) that $\boldsymbol{l}(\boldsymbol{x})$ is given by

$$\boldsymbol{l}(\boldsymbol{x}) = p_0 \frac{\partial L_{\boldsymbol{f}}^{\rho-1} h(\boldsymbol{x})}{\partial \boldsymbol{x}}, \tag{26}$$

Section 5 will show there exists such a constant p_0 that the observer with the design function (25) is asymptotically stable.

Using the Lie notation and taking account of the definitions of $G_1(\boldsymbol{x})$ in (4) and $G_2(\boldsymbol{x})$ in (10), gives

$$G_1(\boldsymbol{x}) = \frac{\partial L_{\boldsymbol{f}}^{\rho-1} h(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{g_1}(\boldsymbol{x}) = \boldsymbol{l}(\boldsymbol{x}) \boldsymbol{g_1}(\boldsymbol{x})/p_0,$$

$$G_2(\boldsymbol{x}) = \frac{\partial L_{\boldsymbol{f}}^{\rho-1} h(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{g_2}(\boldsymbol{x}) = \boldsymbol{l}(\boldsymbol{x}) \boldsymbol{g_2}(\boldsymbol{x})/p_0,$$

$$L_{\boldsymbol{f}}^{\rho} h(\boldsymbol{x}) = \frac{\partial L_{\boldsymbol{f}}^{\rho-1} h(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{l}(\boldsymbol{x}) \boldsymbol{f}(\boldsymbol{x})/p_0$$

and

$$\boldsymbol{l}(\boldsymbol{x})\dot{\boldsymbol{x}} = p_0 \frac{\partial L_{\boldsymbol{f}}^{\rho-1} h(\boldsymbol{x})}{\partial \boldsymbol{x}} \dot{\boldsymbol{x}} = p_0 \frac{\partial y^{[\rho-1]}}{\partial \boldsymbol{x}} \dot{\boldsymbol{x}} = p_0 y^{[\rho]}$$

Thus (24) can be written as

$$
\begin{aligned}
\dot{\hat{b}} &= p_0 y^{[\rho]} - \boldsymbol{l}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x}) + p_0 \left(\sum_{i=0}^{\rho-1} k_i (L_{\boldsymbol{f}}^{[i]} h(\boldsymbol{x}) - y_d^{[\rho]}) + \boldsymbol{l}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})/p_0 - y_d^{[\rho]} \right) \\
&= p_0 \left(\sum_{i=0}^{\rho-1} k_i (y^{[i]} - y_d^{[i]}(t)) + y^{[\rho]} - y_d^{[\rho]} \right) \\
&= -p_0 \sum_{i=0}^{\rho} k_i e^{[i]}
\end{aligned}
$$ (27)

where

$$e = y_d - y$$ (28)

Integrating (27) yields

$$\hat{b}(t) = -p_0 \left(k_0 \int_0^t e(\tau)d\tau + \sum_{i=1}^{\rho} k_i e^{[i-1]}(t) \right) + \bar{b}(0)$$ (29)

where

$$\bar{b}(0) = \hat{b}(0) + p_0 \sum_{i=1}^{\rho} k_i e^{[i-1]}(0)$$ (30)

Substituting the estimation of the offset (29) into the NGPC law (23) yields

$$
\begin{aligned}
u(t) = G_1(\boldsymbol{x})^{-1} \Bigg(&\sum_{i=1}^{\rho} (k_{i-1} + p_0 k_i G_2(\boldsymbol{x})) e^{[i-1]}(t) - L_{\boldsymbol{f}}^{\rho} h(\boldsymbol{x}) + y_d^{[\rho]} \\
&+ G_2(\boldsymbol{x}) p_0 k_0 \int_0^t e(\tau)d\tau - \bar{b}(0) \Bigg)
\end{aligned}
$$ (31)

Hence the optimal NGPC for nonlinear systems with disturbance is given by

$$
\begin{aligned}
u(t) = G_1(\boldsymbol{x})^{-1} \Bigg(&\sum_{i=1}^{\rho} (k_{i-1} + p_0 k_i G_2(\boldsymbol{x})) (y_d^{[i-1]} - L_{\boldsymbol{f}}^{i-1} h(\boldsymbol{x})) - y_d^{[\rho]} + L_{\boldsymbol{f}}^{\rho} h(\boldsymbol{x}) \\
&+ G_2(\boldsymbol{x}) p_0 k_0 \int_0^t (y_d(\tau) - y(\tau))d\tau \Bigg)
\end{aligned}
$$ (32)

It implies that $\hat{b}(0)$ is chosen as

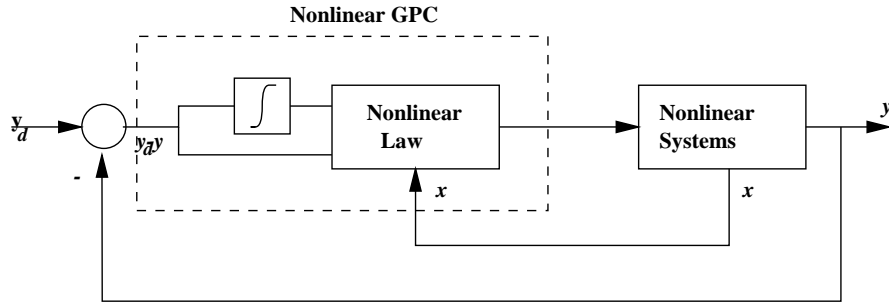$$\hat{b}(0) = -p_0 \sum_{i=1}^{\rho} k_i e^{[i-1]}(0)$$ (33)

Figure 2: NGPC with integral action

in this controller. The Lie notation rather than the derivatives of the output is used in (32) to emphasise that the NGPC is a state feedback controller and the derivatives of the output can be constructed from the states. The controller structure is depicted in Figure 2. In this structure, the explicit observer, shown in Figure 1, disappears. However, the integral action arises. It is important to notice that the closed-loop system under the optimal NGPC (3) is feedback linearised but the closed-loop system under the nonlinear controller (23) is not feedback linearised. The nonlinear error dynamics of the observer are involved.

5 Stability of the closed-loop system under NGPC

When the disturbance is measurable and it is not necessary to use an observer [13], it can be shown that the closed-loop system is globally exponentially stable provided the control order, r, is chosen correctly. That is, when the relative degree is less than or equal to 4, the closed-loop system is globally exponentially stable for any control order. When the relative degree is larger than 4, the stability of the closed-loop system can be recovered by appropriate choice of the control order (for detail see [13]). In the following, it is supposed that the design parameters in NGPC have been chosen such that when all states and the offset are measurable, the closed-loop system is globally exponentially stable. We will show that the closed-loop system under the NGPC (9) with the observer (20)–(21) is also globally exponentially stable with appropriate choice of design function $p(\boldsymbol{x})$ in (25).

The whole closed-loop system consists of the nonlinear observer (20) and (21), the NGPC (9) and the nonlinear system (1). Substituting the NGPC (9) into the nonlinear system (1) and after a series of manipulations, the closed-loop system error equation is given by

$$e^{[\rho]}(t) + k_{\rho-1}e^{[\rho-1]}(t) + \cdots + k_0 e(t) + G_2(\boldsymbol{x})e_1(t) = 0 \tag{34}$$

where the observer error is governed by the equation

$$\dot{e}_1(t) + \frac{\partial p(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{g_2}(\boldsymbol{x})e_1(t) = 0 \tag{35}$$

First it will be shown that the observer error subsystem is asymptotically stable by the choice

of $p(\boldsymbol{x})$ in (25). It is important to note that the observer error equation (35) is not decoupled from the system equation.

Lemma 1: If the disturbance relative degree, ρ, is well-defined, then there exists a constant, p_0, such that the estimation, $\hat{b}(t)$, yielded by the observer (20) and (21) with the design function (25) approaches $b(t)$ exponentially.

Proof: Let a Lyapunov function candidate for the observer error subsystem be given by

$$V(e_1) = e_1^2$$

Then along an arbitrary system state trajectory $\boldsymbol{x} \in R^n$,

$$\dot{V}(e_1) = -2e_1 \left(\frac{\partial p(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{g_2}(\boldsymbol{x}) \right) e_1 = -2p_0 G_2(\boldsymbol{x}) e_1^2 \tag{36}$$

Since the disturbance relative degree is well-defined, it implies that $G_2(\boldsymbol{x}) \neq 0$ for any $\boldsymbol{x} \in R^n$. Moreover, since $\boldsymbol{g_2}(\boldsymbol{x})$ is a continuous function, and $\boldsymbol{f}(\boldsymbol{x})$ and $h(\boldsymbol{x})$ is continuously differentiable a sufficient number of times, $G_2(\boldsymbol{x})$ is a continuous function of $\boldsymbol{x}$. Hence $G_2(\boldsymbol{x}$ has the same sign for all $\boldsymbol{x} \in R^n$. There exists a constant p_0 such that for all $\boldsymbol{x}(t) \in R^n$

$$\dot{V}(e_1) \leq -\alpha e_1^2 \tag{37}$$

where $\alpha > 0$ is a prescribed scalar. Hence the result.

Algorithm:

Step 1: Calculate the input and disturbance relative degrees, ρ;

Step 2: Determine the control order and the predictive time according to the stability and the time-domain specification by look-up tables in [13];

Step 3: Design NGPC law (9) from Theorem 2;

Step 4: Choose the constant p_0 according to the sign of $G_2(\boldsymbol{x})$ and the desired estimation convergence rate, and calculate the design function $p(\boldsymbol{x}) = p_0 L_{\boldsymbol{f}}^{\rho} h(\boldsymbol{x})$.

Step 5: Design the disturbance observer by (20) and (21) and replace the disturbance b in (9) by the estimate given by the observer, or directly calculate the nonlinear control law (32).

Theorem 3: For the nonlinear system (1) satisfying Assumptions A1–A3, the closed-loop system under the optimal nonlinear generalised predictive controller (NGPC) designed by the above procedure is globally exponentially stable.

Proof: This result is based on [26]. First the closed-loop system under the optimal NGPC designed in Steps 1, 2 and 3 is globally exponentially stable when there is no observer and the disturbance is measurable. From Lemma 1, the observer designed in Steps 4 and 5 is globally stable under an arbitrary state trajectory $\boldsymbol{x} \in R^n$. It can be proved that the observer is a "weak detector" for the nonlinear system (1) by using the Lyapunov function $V(e_1)$. Moreover it follows from (37) that

$$|b - \hat{b}| \leq |b(0) - \hat{b}(0)| e^{-\alpha t}.$$

In addition, the various Jacobian matrices in Theorem 4.1 in [26] are bounded due to the assumptions on differentiability and continuity of $\boldsymbol{f}(\boldsymbol{x})$, $h(\boldsymbol{x})$ and $\boldsymbol{g_2}(\boldsymbol{x})$. It can be shown that all conditions in Theorem 4.1 in [26] are satisfied. Hence the closed-loop system under the optimal NGPC with a nonlinear disturbance observer (20)and (21) is globally exponentially stable. QED.

Theorem 3 establishes the stability result of the closed-loop system under the composite controller consisting of the nonlinear controller (23) and the nonlinear observer (20) and (21). Since this composite controller is equivalent to the nonlinear controller (32) with integral action, this result also implies the stability of the closed-loop system under the nonlinear controller (32). It should be noted that this result only holds for SISO nonlinear systems. It cannot be extended to the multivariable case directly.

6   Nonlinear PID predictive controllers

In many physical systems, the relative degree is 1 and 2. For example in many mechanical systems, the input is force or torque and the output is displacement. The relative degree for those systems is 2. In this section we will show that the optimal NGPC for a nonlinear system with a low relative degree has a PI or PID structure. To compare with the traditional PID regulator structure, only the regulation problem will be considered in this section. That is the derivative of the reference signal is set to be zero.

6.1   Nonlinear PI predictive controller

For a nonlinear system with relative degree $\rho = 1$, the optimal NGPC (32) becomes

$$u(t) = G_1(\boldsymbol{x})^{-1}\left(G_2(\boldsymbol{x})k_0 p_0 \int_0^t \big(y_d(\tau) - y(\tau)\big)d\tau + \big(k_0 + p_0 G_2(\boldsymbol{x})\big)(y_d - y)\right) - N(\boldsymbol{x}) \quad (38)$$

where

$$N(\boldsymbol{x}) = G_1(\boldsymbol{x})^{-1}\frac{\partial h(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x}) \quad (39)$$

$$G_1(\boldsymbol{x}) = \frac{\partial h(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{g_1}(\boldsymbol{x})$$

$$G_2(\boldsymbol{x}) = \frac{\partial h(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{g_2}(\boldsymbol{x})$$

This NGPC can also be written as

$$u(t) = P(\boldsymbol{x})e(t) + I(\boldsymbol{x})\int_0^t e(\tau)d\tau - N(\boldsymbol{x}) \quad (40)$$

with the coefficients

$$
\begin{aligned}
P(\boldsymbol{x}) &= G_1(\boldsymbol{x})^{-1}\big(k_0 + p_0 G_2(\boldsymbol{x})\big); \\
I(\boldsymbol{x}) &= G_1(\boldsymbol{x})^{-1}G_2(\boldsymbol{x})k_0 p_0
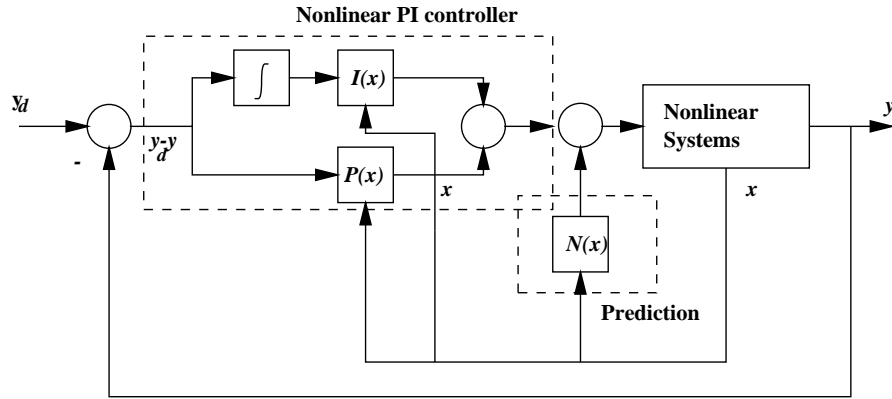\end{aligned}
$$

Figure 3: Optimal Nonlinear PI predictive controller

The structure of this controller is as shown in Figure 3. The nonlinear generalised predictive controller is reduced to a PI-like controller. It consists of two parts, a nonlinear PI controller and a prediction part, $N(\boldsymbol{x})$. Hence in addition to the nonlinear PI controller function, this controller takes into account the future of the system's output by prediction. The controller predicts the first derivative of the system's output based on the knowledge of the plant and the current system states. Therefore this controller is called *nonlinear PI predictive controller*.

Corollary 1: Consider a nonlinear system (1) with input relative degree of 1 and satisfying Assumptions $A1$–$A3$. Then the nonlinear PI predictive controller (40) is an optimal controller in term of the performance index (2) with a nonlinear observer. Moreover when the design parameter, p_0, in the observer is chosen according to Step 4 in Section 5, the closed-loop system under the nonlinear PI predictive controller (40) is globally exponentially stable.

6.2   Nonlinear PID predictive controller

Similarly, for a nonlinear system with relative degree of 2, the NGPC (32) reduces to a nonlinear PID predictive controller

$$u(t) = P(\boldsymbol{x})e(t) + I(\boldsymbol{x}) \int_0^t e(\tau)d\tau + D(\boldsymbol{x})\dot{e}(t) - N(\boldsymbol{x}) \tag{41}$$

where

$$P(\boldsymbol{x}) = G_1(\boldsymbol{x})^{-1}(k_0 + G_2(\boldsymbol{x})k_1 p_0)$$

$$I(\boldsymbol{x}) = G_1(\boldsymbol{x})^{-1}G_2(\boldsymbol{x})k_0 p_0$$

$$D(\boldsymbol{x}) = G_1(\boldsymbol{x})^{-1}(k_1 + p_0 G_2(\boldsymbol{x}))$$

$$N(\boldsymbol{x}) = G_1(\boldsymbol{x})^{-1}L_{\boldsymbol{f}}^2 h(\boldsymbol{x})$$

The structure of this controller is as depicted in Figure 4. Similarly, this controller also consists of two parts, a nonlinear PID controller and a prediction part $N(\boldsymbol{x})$, which predicts the second derivative of the system's output using the knowledge of the plant and the current
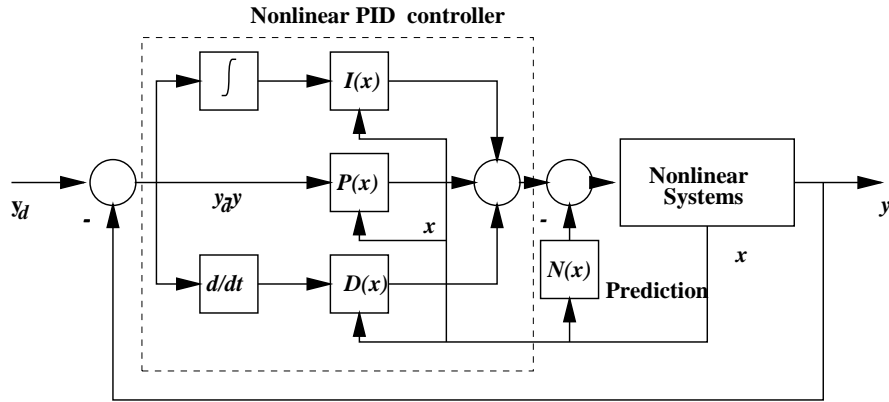
Figure 4: Optimal nonlinear PID predictive controller

states. In the nonlinear PID predictive control, the velocity of the system output is measured and is provided to the nonlinear PID controller. The trend of the velocity due to the system dynamics is predicted by $N(\boldsymbol{x})$ and then compensated for by the controller.

Corollary 2: Consider a nonlinear system (1) with input relative degree of 2 and satisfying Assumptions $A1$–$A3$. The nonlinear PID predictive controller (41) is an optimal controller in term of the performance index (2) with a nonlinear observer. Moreover when the design parameter p_0 in the observer is chosen according to Step 4 in Section 5, the closed-loop system under the nonlinear PID predictive controller (41) is globally exponentially stable.

7   Application Example

An electro-mechanical system with a nonlinear spring can be modelled as a controlled Duffing's equation. The equation of motion is described by [27]

$$m\ddot{x} + c\dot{x} + f_1 x + f_3 x^3 = k_t u + b \tag{42}$$

where x, u and b are displacement, controller force and disturbance respectively. m, c and k_t are the mass, damping and the torque constant respectively. The characteristics of the nonlinear spring are represented by the parameters f_1 and f_3. The disturbance b may arise from friction, or unmodelled dynamics. In this system, the output is the displacement

$$y = x$$

This is a nonlinear system with a well-defined relative degree of 2 and no zero dynamics. Assumptions $A1$–$A3$ are satisfied. The control order, r, is chosen as 3 and the predictive time, T, is selected as 1 second in this paper. Then a nonlinear generalised predictive controller can be designed with the parameters
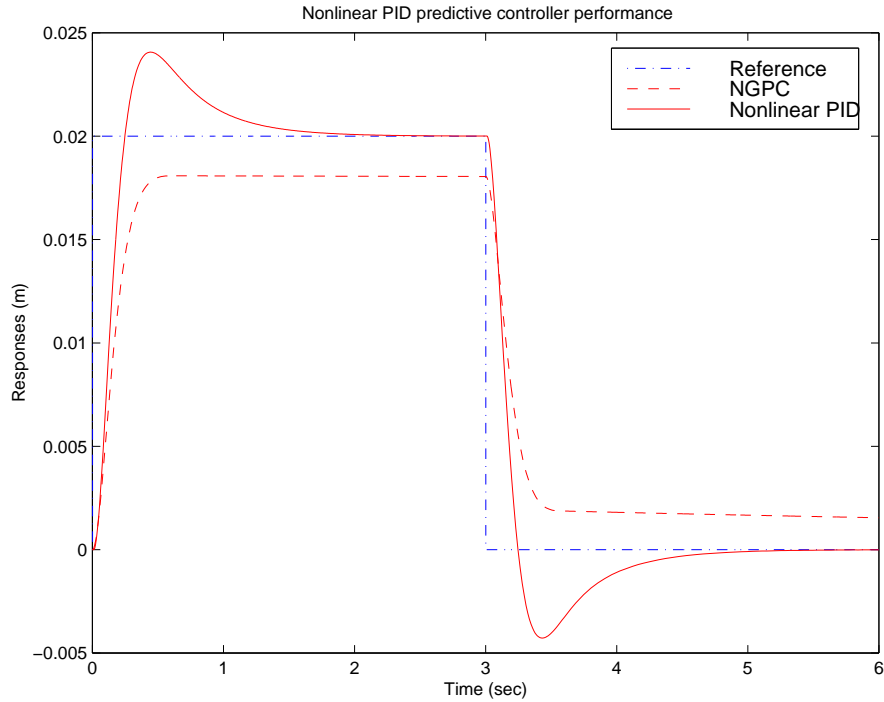
$$k_0 = 16; \ k_1 = 99.333.$$

Figure 5: Responses of the nonlinear mechanical system with friction disturbance

In order to overcome the effect of the disturbance and eliminate the offset, a disturbance observer is added in the NGPC. Since this is a nonlinear system with second order relative degree, a nonlinear PID predictive controller (41) results, given by

$$
\begin{aligned}
u(t) &= m \left(k_1 p_0 \int_0^t (y_d - y) d\tau + (k_1 + k_0 p_0)(y_d - y) - (k_0 + p_0)\dot{x} \right) / k_t + \\
&\quad + (cx + f_1 y + f_3 y^3)/k_t
\end{aligned}
\tag{43}
$$

The parameters for the simulation has the following values: $m = 1$ kg, $c = 5$ Ns/m, $f_1 = 100$ N/m, $f_3 = 500,000$ N/m^3, and $k_t = 1$.

In the first case, it is assumed that the disturbance is caused by the Coulomb friction. That is

$$
b = b_0 \, \text{sign}(\dot{x})
$$

where b_0 is chosen as 0.2 Nm in simulation.

The comparative performance of the NGPC with and without the integral action is shown in Figure 5.

Now we consider the system (42) has modelling error. In this case b represents the uncertainties in the modelling or unmodelled dynamics. It also might arise due to the variation of the plant. It is assumed that the actual plant has the parameters $m = 1.2$ kg, $c = 6$, $f_1 = 120$, $f_3 = 600,000$. That is, all parameters are changed by 20% of their nominal values. It should be noted that in this case the assumption that the disturbance is an unknown constant or piece-
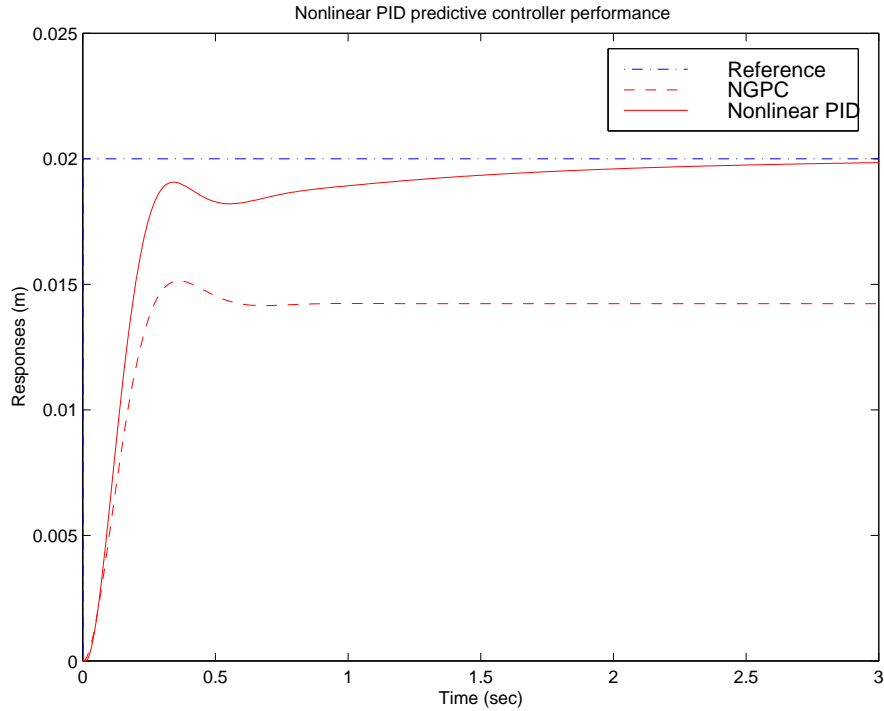
Figure 6: Responses of the nonlinear mechanical system with 20% modelling error

wise constant is not satisfied. The controller is designed according to the nominal parameters and thus is the same as before. The responses under NGPC and the Nonlinear PID predictive controller are shown in Figure 6. This shows the nonlinear PID controller improves the performance robustness of the plant against the disturbance and the uncertainties especially in the steady-state.

8 Conclusions

A new class of nonlinear controllers which have offset removal ability are presented in this paper. The work is formulated in the nonlinear generalised predictive control framework. It makes the nonlinear generalised predictive control more practical since elimination of offset is one of the main objectives in the control of industrial systems. Rather than immediately assuming the integral action in a nonlinear controller, a nonlinear observer is designed to estimate the offset. The composite controller consists of an optimal NGPC and a nonlinear disturbance observer. After choosing the design function, a nonlinear control with integral action arises and the explicit observer disappears. The stability of the closed-loop system was also established. For nonlinear systems with low relative degree, this nonlinear controller has a nonlinear PID/PI like structure. This partly explains why the PID controllers are so widely used and gives insight into the mechanism of PID-like controllers. The special feature of the controller developed in this paper lies in that, in addition to a PID structure, it has a prediction part. It can predict the trend of

the system's output using the knowledge of the system and the current states and compensate for it.

The case where the disturbance relative degree is higher than or equal to the input relative degree is discussed in this paper. When the disturbance relative degree is less than the input relative degree, the analysis of the closed-loop system under the NGPC with integral action becomes very complicated. However, it is well known that if the disturbance/uncertainty does not destroy the stability of the closed-loop system, a controller with integral action always removes the offset in the steady-state. Hence the NGPC developed in this paper also works for the case where the disturbance relative degree is less than the input relative degree.

This work can be considered as an extension of some self-tuning results for linear systems (for example see [28, 29, 4, 2]) to nonlinear GPC systems. It should be emphasised that the nonlinear case is much more complicated than the linear case. The conclusion that the composite controller consisting of the optimal NGPC and a nonlinear observer is equivalent to a nonlinear system with integral action only holds for the specially chosen nonlinear design function $p(\boldsymbol{x})$, e.g., (25), as does the global stability result. The behaviour of the closed-loop system under the proposed controller is nonlinear. Although, in principle, the method for SISO nonlinear system in this paper could be extended to MIMO systems, there are some technical difficulties. For example, it is not easy to design a globally stable nonlinear disturbance observer for general MIMO nonlinear systems. For a MIMO nonlinear system with a special structure, Chen et al. [30] give a method for design of a nonlinear disturbance observer for robotic manipulators.

Acknowledgements

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Appendix

Proof of Theorem 1 and 2:

The proof of Theorem 1 is similar to that of Theorem 2. Hence only the proof of Theorem 2 is given.

Repeated differentiation up to ρ times of the output y with respect to time, together with repeated substitution of the system (1) gives

$$\dot{y}(t) = L_{\boldsymbol{f}} h(\boldsymbol{x})$$

$$\vdots$$

$$y^{[\rho-1]}(t) = L_{\boldsymbol{f}}^{\rho-1} h(\boldsymbol{x})$$

and

$$y^{[\rho]}(t) = L_{\boldsymbol{f}}^{\rho} h(\boldsymbol{x}) + G_1(\boldsymbol{x}) u(t) + G_2(\boldsymbol{x}) b \tag{44}$$

When the control order is chosen as r, then in order to predict the output y accurately, the order of Taylor expansion of the output $y(t + \tau)$ should at least be $\rho + r$. Differentiating the equation (44) with respect to time yields

$$y^{[\rho+1]}(t) = L_{\boldsymbol{f}}^{\rho+1} h(\boldsymbol{x}) + p_{11}(u(t), \boldsymbol{x}(t)) + L_{\boldsymbol{g_1}} L_{\boldsymbol{f}}^{\rho-1} h(\boldsymbol{x}) \dot{u}(t) \tag{45}$$

where

$$\begin{aligned} p_{11}(u(t), b, \boldsymbol{x}(t)) &= L_{\boldsymbol{g_1}} L_{\boldsymbol{f}}^{\rho} h(\boldsymbol{x}) u + \frac{\partial G_1(\boldsymbol{x})}{\partial \boldsymbol{x}} (\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g_1}(\boldsymbol{x}) u + \boldsymbol{g_2}(\boldsymbol{x}) b) u + \\ &\quad + L_{\boldsymbol{g_2}} L_{\boldsymbol{f}}^{\rho} h(\boldsymbol{x}) b + \frac{\partial G_2(\boldsymbol{x})}{\partial \boldsymbol{x}} (\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g_1}(\boldsymbol{x}) u + \boldsymbol{g_2}(\boldsymbol{x}) b) b \end{aligned} \tag{46}$$

It should be noted that u^2, ub and b^2 arise in p_{11} and hence p_{11} is a nonlinear function of u and b.

Similarly it can be shown that the higher derivatives of the output, $y(t)$, are given by

$$y^{[\rho+2]}(t) = L_{\boldsymbol{f}}^{\rho+2} h(\boldsymbol{x}) + p_{21}(u(t), b, \boldsymbol{x}(t)) + p_{22}(u(t), \dot{u}(t), b, \boldsymbol{x}(t)) + G_1(\boldsymbol{x}) u^{[2]}(t) \tag{47}$$

$$\vdots$$

$$\begin{aligned} y^{[\rho+r]}(t) &= L_{\boldsymbol{f}}^{\rho+r} h(\boldsymbol{x}) + p_{r1}(u(t), b, \boldsymbol{x}(t)) + p_{r2}(u(t), \dot{u}(t), b, \boldsymbol{x}(t)) + \\ &\quad + \cdots + p_{rr}(u(t), \ldots, u^{[r-1]}(t), b, \boldsymbol{x}(t)) + G_1(\boldsymbol{x}) u^{[r]}(t) \end{aligned} \tag{48}$$

That is

$$\bar{\boldsymbol{y}}(t) = \begin{bmatrix} y^{[0]} \\ y^{[1]} \\ \vdots \\ y^{[\rho]} \\ y^{[\rho+1]} \\ \vdots \\ y^{[\rho+r]} \end{bmatrix} = \begin{bmatrix} h(\boldsymbol{x}) \\ L_{\boldsymbol{f}}^1 h(\boldsymbol{x}) \\ \vdots \\ L_{\boldsymbol{f}}^{\rho} h(\boldsymbol{x}) \\ L_{\boldsymbol{f}}^{\rho+1} h(\boldsymbol{x}) \\ \vdots \\ L_{\boldsymbol{f}}^{\rho+r} h(\boldsymbol{x}) \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \boldsymbol{ld} \end{bmatrix} \tag{49}$$

where $\boldsymbol{ld} \in R^{r+1}$ is given by

$$\boldsymbol{ld} = \begin{bmatrix} G_1(\boldsymbol{x}) u(t) \\ p_{11}(u(t), b, \boldsymbol{x}(t)) + G_1(\boldsymbol{x}) \dot{u}(t) \\ \vdots \\ p_{r1}(u(t), b, \boldsymbol{x}(t)) + \cdots + p_{rr}(u(t), \ldots, u^{[r-1]}(t), b, \boldsymbol{x}(t)) + G_1(\boldsymbol{x}) u^{[r]}(t) \end{bmatrix} \tag{50}$$

Within the moving time frame, the predictive values of the output $y(t + \tau)$ at the time τ is given by

$$\hat{y}(t + \tau) = \boldsymbol{t}(\tau)\hat{\bar{\boldsymbol{y}}}(t) \tag{51}$$

where $\hat{\bar{\boldsymbol{y}}}$ is given in (49) by replacing $u(t)$ and the derivatives of $u(t)$ with $\hat{u}(t)$ and that of $\hat{u}(t)$ respectively. $\boldsymbol{t}(\tau) \in R^{\rho+r+1}$ is given by

$$\boldsymbol{t}(\tau) = \begin{bmatrix} 1 & \tau & \cdots & \frac{\tau^{(\rho+r)}}{(\rho+r)!} \end{bmatrix} \tag{52}$$

respectively.

Suppose the reference input $y_d(t)$ is differentiable up to $\rho+r$. In the moving time frame, the reference $y_d(t+\tau)$ at the time τ is approximated by the Taylor expansion of $y_d(t)$ at the time t up to $(\rho + r)$th order, given by

$$\hat{y}_d(t + \tau) = \boldsymbol{t}(\tau)\bar{\boldsymbol{y}}_d(t) \tag{53}$$

where

$$\bar{\boldsymbol{y}}_d(t) = \begin{bmatrix} y_d(t) & \dot{y}_d(t) & \cdots & y_d^{[\rho+r]}(t) \end{bmatrix} \tag{54}$$

Using (51) and (53), the performance index (2) can be written as

$$
\begin{aligned}
J &= \frac{1}{2}\int_0^T (\hat{y}(t+\tau) - \hat{y}_d(t+\tau))^T(\hat{y}(t+\tau) - \hat{y}_d(t+\tau))d\tau \\
&= \frac{1}{2}\int_0^T (\hat{\bar{\boldsymbol{y}}}(t) - \bar{\boldsymbol{y}}_d(t))^T \boldsymbol{t}(\tau)^T \boldsymbol{t}(\tau)(\hat{\bar{\boldsymbol{y}}}(t) - \bar{\boldsymbol{y}}_d(t))d\tau \\
&= \frac{1}{2}(\hat{\bar{\boldsymbol{y}}}(t) - \bar{\boldsymbol{y}}_d(t))^T \boldsymbol{T}(\hat{\bar{\boldsymbol{y}}}(t) - \bar{\boldsymbol{y}}_d(t))
\end{aligned} \tag{55}
$$

where $\boldsymbol{T}$ is the $(\rho + r + 1) \times (\rho + r + 1)$ matrix, defined by

$$\boldsymbol{T} = \int_0^T \boldsymbol{t}(\tau)^T \boldsymbol{t}(\tau)d\tau \tag{56}$$

The optimal tracking problem now can be reformulated as,

$$\min_{\bar{\boldsymbol{u}}(t)=[\ \hat{u}(t)\quad \dot{\hat{u}}(t)\quad \cdots\quad \hat{u}^{[r]}(t)\ ]\in R^{\rho+r+1}} J(\bar{\boldsymbol{u}}, b, \boldsymbol{x}, y_d) \tag{57}$$

The necessary condition for the optimal control $\bar{\boldsymbol{u}}$ is given by

$$\frac{\partial J}{\partial \bar{\boldsymbol{u}}} = 0 \tag{58}$$

Remember that $u(t) = \hat{u}(t + \tau)$ when $\tau = 0$ in GPC, that is, only the first element in $\bar{\boldsymbol{u}}$ is implemented. Similar to [13], it can be derived that the first element in $\bar{\boldsymbol{u}}(t)$ which satisfies the condition (58) is given by (9). Hence the result.

Furthermore, for Theorem 1, substituting the optimal NGPC (3) into the plant (1) yields the closed-loop system error equation

$$e^{[\rho]}(t) + k_{\rho-1}e^{[\rho-1]}(t) + \cdots + k_0 e(t) = 0 \tag{59}$$

Hence the disturbance is completely decoupled under the NGPC (9).

List of Figures